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ON CYCLIC VECTORS AND THIN VON NEUMANN ALGEBRAS

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Abstract. We prove that certain classes of von Neumann algebras with regular, injective subalgebras are thin. As a consequence, all Hochschild cohomology groups of these algebras are zero.

Introduction

The notion of thinness for von Neumann algebras can be traced back to the pioneering work of Feldman and Moore [5] on type II_1 factors with Cartan subalgebras. If $\mathscr{M} \subset B(L^2(\mathscr{M},\tau))$ is a separable type II_1 factor, a Cartan subalgebra is a maximal abelian *-subalgebra (m.a.s.a, for short) $\mathscr{A} \subset \mathscr{M}$ whose normalizer $\mathscr{N}_{\mathscr{M}}(\mathscr{A}) = \{u \in \mathscr{M} \text{ unitary} \mid u\mathscr{A}u^* = \mathscr{A}\}$ generates $\mathscr{M}$ as a von Neumann algebra.

Feldman and Moore proved the remarkable fact that the von Neumann algebra $W^*(\mathscr{A}, J\mathscr{A}J)$ generated by $\mathscr{A}$ and $J\mathscr{A}J$ (J being the canonical conjugation $Jx = x^*$ on $L^2(\mathscr{M},\tau)$) is maximal abelian in $B(L^2(\mathscr{M},\tau))$. In particular, $W^*(\mathscr{A}, J\mathscr{A}J)$ has a cyclic vector, thus $L^2(\mathscr{M},\tau)$ equals the 2-norm closure of the linear span of $\mathscr{A}\xi\mathscr{A}$.

A weaker condition requires two injective subalgebras $\mathscr{A}$ and $\mathscr{B}$ in $\mathscr{M}$ such that $L^2(\mathscr{M},\tau)$ equals the 2-norm closure of the linear span of $\mathscr{A}\xi\mathscr{B}$, and $\mathscr{B}$ is often assumed to be abelian. This represents, generally speaking, the notion of thinness, and Paragraph 1.1 in Section 1 describes it in more detail.

These ideas turned out to be very useful in proving automatic complete boundedness for certain linear and multilinear maps on von Neumann algebras, which in turn established the vanishing of the Hochschild cohomology groups $H^n(\mathscr{M},\mathscr{M})$. We refer the reader to [1], [3], [4], [10], [11], [14], as well as to the monograph [13], for more information and for the current state of knowledge in the area.

In this note we add a few more examples to the list of known thin algebras (sections 2 and 3) and consequently obtain new examples of von Neumann algebras for which all cohomology groups $H^n(\mathscr{M},\mathscr{M})$ are trivial (section 4).

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1. Background results

Throughout this paper separability of a von Neumann algebra is understood as the separability of its predual. If $\mathscr{M}$ is of type II_1 with a faithful, normal trace τ, this fact is equivalent to the separability of the Hilbert space $L^2(\mathscr{M},\tau)$. For any non-empty subset $X \subset L^2(\mathscr{M},\tau)$, we denote by $sp\,X$ and $\overline{sp}\,X$ the linear span of X and, respectively, the 2-norm closure in $L^2(\mathscr{M},\tau)$ of the linear span of X.

1.1. Following [6], let $\mathscr{M}$ be a separable, finite von Neumann algebra in standard representation on $L^2(\mathscr{M},\tau)$. Suppose that there exist two subalgebras $\mathscr{A},\mathscr{B} \subset \mathscr{M}$ and a vector $\xi \in L^2(\mathscr{M},\tau)$ such that $L^2(\mathscr{M},\tau) = \overline{sp}\ \mathscr{A}\xi\mathscr{B}$. If both $\mathscr{A}$ and $\mathscr{B}$ are injective, then $\mathscr{M}$ is called weakly h.h.-thin. If $\mathscr{A}$ is injective and $\mathscr{B}$ is abelian, then $\mathscr{M}$ is called weakly a.h.-thin. Finally, if both $\mathscr{A}$ and $\mathscr{B}$ are abelian, then $\mathscr{M}$ is called weakly a.a.-thin. The letters "a" and "h" stand for "abelian", respectively "hyperfinite". We refer the reader to [6] for various other notions of thinness and further results.

Of course, saying that $L^2(\mathscr{M},\tau) = \overline{sp}\ \mathscr{A}\xi\mathscr{B}$ means precisely that ξ is a cyclic vector for the algebra $W^*(\mathscr{A},J\mathscr{B}J)$, but we shall see throughout this paper that both concepts have merit and, for technical reasons, neither one is redundant.

1.2. Let $\mathscr{M}$ be a separable type II_1 factor with a Cartan subalgebra $\mathscr{A} \subset \mathscr{M}$. While it is well-known [5] that $W^*(\mathscr{A},J\mathscr{A}J)$ has a cyclic vector in $L^2(\mathscr{M},\tau)$, it turns out that this cyclic vector can be chosen to be in $\mathscr{M}$. By Corollary 2.5 in [12], there exists a sequence $(u_n)_{n\geqslant 0}$ in the normalizer $\mathscr{N}_{\mathscr{M}}(\mathscr{A})$ such that the spaces $(\mathscr{A}u_n)_{n\geqslant 0}$ are mutually orthogonal and $\sum_{n\geqslant 0}\mathscr{A}u_n = \mathscr{M}$.

LEMMA 1.1. *If* $m = \sum_{n\geqslant 0} 2^{-n}u_n \in \mathscr{M}$, *then* $\overline{sp}\ \mathscr{A}m\mathscr{A} = L^2(\mathscr{M},\tau)$.

Proof. It suffices to show that every vector of the form $\sum_{n\geqslant 0} a_n u_n$ belongs to $\overline{sp}\ \mathscr{A}m\mathscr{A}$. By Lemma 3.1 in [12], the orthogonal projection onto the space $\mathscr{A}u_n$ belongs to $W^*(\mathscr{A},J\mathscr{A}J)$, for all $n \geqslant 0$. Consequently, for every fixed $n \geqslant 0$ and $\varepsilon > 0$, there exist $a_1,\dots,a_k,b_1,\dots,b_k \in \mathscr{A}$ such that $||\sum_{i=1}^{k}\ a_i m b_i - u_n||_2 < \varepsilon$. Then u_n (and hence $a_n u_n$) belongs to $\overline{sp}\ \mathscr{A}m\mathscr{A}$, which concludes the proof. $\square$

1.3. Let $\mathscr{M}$ be a separable type II_1 algebra with a regular subalgebra $\mathscr{A} \subset \mathscr{M}$. The algebra $\mathscr{B} = \{\sum \lambda_i u_i\}$ of finite linear combinations of unitaries in $\mathscr{N}_{\mathscr{M}}(\mathscr{A})$ is ultraweakly dense in $\mathscr{M}$, and the same is true if the unitaries are chosen from a countable, strongly dense subset of $\mathscr{N}_{\mathscr{M}}(\mathscr{A})$. Therefore, in order to prove that a certain subspace $\mathscr{S} \subset \mathscr{M}$ is dense in $L^2(\mathscr{M},\tau)$, it suffices to prove that $\overline{\mathscr{S}}$ (closure in the 2-norm) contains all unitaries from a countable, strongly dense subset of $\mathscr{N}_{\mathscr{M}}(\mathscr{A})$. This simple argument will be used several times in Section 3, so we thought it appropriate to state it here and use it subsequently without additional explanations.

1.4. We recall that, if $\mathscr{M} \subset B(H)$ is a von Neumann factor and $\mathscr{B} \subset \mathscr{M}'$ is a unital, abelian C^*-subalgebra in the commutant of $\mathscr{M}$, then $C^*(\mathscr{M},\mathscr{B})$, the C^*-algebra generated by $\mathscr{M}$ and $\mathscr{B}$, is isomorphic to the spatial tensor product $\mathscr{M} \otimes \mathscr{B}$.

The proof of this fact represents the first part of the proof of Theorem 2.1 in [10]. As a consequence, if $\varphi : \mathscr{M} \to \mathscr{M}$ is a completely positive map, then $\varphi \otimes id : C^*(\mathscr{M},\mathscr{B}) \to C^*(\mathscr{M},\mathscr{B})$ is a completely positive extension of φ.

2. Factors with property Γ

In [6] Ge and Popa showed that separable type II_1 factors with property Γ are weakly h.h.-thin. In this section we take one step forward and prove that they are weakly a.h.-thin. We recall that a type II_1 factor $\mathscr{M}$ has the property Γ of Murray and von Neumann if, given $x_1,\ldots,x_n \in \mathscr{M}$ and $\varepsilon > 0$, there exists a unitary $u \in \mathscr{M}$ of trace $\tau(u) = 0$, such that $||ux_j - x_ju||_2 < \varepsilon$, $1 \leqslant j \leqslant n$.

PROPOSITION 2.1. *Let $\mathscr{M}$ be a separable type II_1 factor with property Γ. Then there exists a hyperfinite subfactor $\mathscr{R} \subset \mathscr{M}$ with trivial relative commutant, a Cartan subalgebra $\mathscr{A} \subset \mathscr{R}$ and a vector $\eta \in L^2(\mathscr{M},\tau)$ such that $L^2(\mathscr{M},\tau) = \overline{sp}\, \mathscr{R}\eta\mathscr{A}$.*

Proof. By Theorem 5.3 in [4] there exists a hyperfinite subfactor $\mathscr{R} \subset \mathscr{M}$ with a Cartan subalgebra $\mathscr{A}$ such that: for every $x_1,x_2,\ldots,x_n \in \mathscr{M}, r \geqslant 1$ and $\varepsilon > 0$, there exist mutually orthogonal projections $p_1,p_2,\ldots,p_r \in \mathscr{A}$, $\tau(p_i) = 1/r$, $\sum_{i=1}^r p_i = I$, such that $||p_ix_j - x_jp_i||_2 < \varepsilon$ for all $1 \leqslant i \leqslant r$, $1 \leqslant j \leqslant n$.

Fix $x_1,x_2,\ldots,x_n \in \mathscr{M}$ and apply the above for $r = n$ and $2^{-1}n^{-2}\varepsilon$ instead of ε. Consider an $n \times n$ matrix subalgebra of $\mathscr{R}$ with matrix units e_{ij}, $1 \leqslant i,j \leqslant n$, such that $e_{ii} = p_i$. Intuitively, the x_i's are almost diagonal with respect to $p_1,\ldots,p_n$ and we define the vector ξ to be the $n \times n$ matrix whose i^{th} row consists of the diagonal pieces of x_i.

More precisely, let $\xi \in \mathscr{M}$ be

$$\xi = \sum_{i=1}^n e_{i1}x_ie_{11} + e_{i2}x_ie_{22} + \ldots + e_{in}x_ie_{nn}$$

Then, for every $1 \leqslant i \leqslant n$ we have

$$\sum_{j=1}^n p_jx_ip_j = e_{1i}\xi e_{11} + e_{2i}\xi e_{22} + \ldots + e_{ni}\xi e_{nn}$$

so $\sum_{j=1}^n p_jx_ip_j$ belongs to the linear space $sp\,\mathscr{R}\xi\mathscr{A}$. Moreover,

$$\begin{aligned}||x_i - \sum_{j=1}^n p_jx_ip_j||_2 &= ||\sum_{k\neq l} p_kx_ip_l||_2 = ||\sum_{k\neq l}(p_kx_i - x_ip_k)p_l||_2 \\ &\leqslant (n-1)\sum_{k=1}^n ||p_kx_i - x_ip_k||_2 \leqslant (n^2-n)2^{-1}n^{-2}\varepsilon < \varepsilon/2\end{aligned}$$

hence x_i is within $\varepsilon/2$ from $sp\,\mathscr{R}\xi\mathscr{A}$. If we fix $\xi_1,\xi_2,\ldots,\xi_n \in L^2(\mathscr{M},\tau)$, there exist $x_1,x_2,\ldots,x_n \in \mathscr{M}$ such that $||x_i - \xi_i||_2 < \varepsilon/2$ and $dist(x_i, sp\,\mathscr{R}\xi\mathscr{A}) < \varepsilon/2$, hence $dist(\xi_i, sp\,\mathscr{R}\xi\mathscr{A}) < \varepsilon$.

If J denotes the canonical conjugation on $L^2(\mathscr{M},\tau)$, then Lemma 2.7 in [7] implies that the von Neumann algebra $W^*(\mathscr{R},J\mathscr{A}J) \subset B(L^2(\mathscr{M},\tau))$ has a cyclic vector, which concludes the proof. $\square$

REMARK 2.2. The case when $\mathscr{M}$ is a McDuff factor is more straightforward. Let $\mathscr{M} = \mathscr{N}\overline{\otimes}\mathscr{R}$ be the tensor product of type II_1 factors, where $\mathscr{R}$ is hyperfinite and $\mathscr{N}$ has separable predual. Then there exists a Cartan subalgebra $\mathscr{A} \subset \mathscr{R}$ and a vector $\xi \in \mathscr{N}\overline{\otimes}\mathscr{R}$ such that $L^2(\mathscr{N}\overline{\otimes}\mathscr{R}) = \overline{sp}\,\mathscr{R}\xi\mathscr{A}$.

Proof. The argument is very close to the construction in [6]. Fix $\mathscr{A}$ a Cartan subalgebra of $\mathscr{R}$ and consider a sequence (e_n) of mutually orthogonal projections in $\mathscr{A}$, with $\tau(e_n) = 2^{-n}$. Let $u_1, u_2, \ldots$ be a sequence of unitary operators in $\mathscr{N}$, strongly dense in the set of unitaries of $\mathscr{N}$. Define the unitary operator $U = u_1 \otimes e_1 + u_2 \otimes e_2 + \ldots \in \mathscr{M}$, which is easily seen to commute with $\mathscr{A}$. Finally, recall that $L^2(\mathscr{R},\tau) = \overline{sp}\,\mathscr{A}m\mathscr{A}$ and denote $\xi = Um$. It is easy to see that there exist finitely many unitary operators $v_i \in \mathscr{R}$ such that $\sum v_i e_{i_0} v_i^* = I$ and this fact will be used repeatedly throughout the paper. It follows that $\sum v_i (U e_{i_0}) v_i^* = u_{i_0}$, which shows that $\overline{sp}\,\mathscr{R}U\mathscr{R}$ contains $L^2(\mathscr{N},\tau)$. Then, clearly, $\overline{sp}\,\mathscr{R}U\mathscr{R} = L^2(\mathscr{N}\overline{\otimes}\mathscr{R},\tau)$. We have $L^2(\mathscr{N}\overline{\otimes}\mathscr{R}) = \overline{sp}\,\mathscr{R}U\mathscr{R} = \overline{sp}\,\mathscr{R}U\mathscr{A}m\mathscr{A} = \overline{sp}\,\mathscr{R}\mathscr{A}Um\mathscr{A} = \overline{sp}\,\mathscr{R}\xi\mathscr{A}$. $\square$

3. Algebras with regular, injective subalgebras

In this section we study several classes of type II_1 von Neumann algebras which contain regular, injective subalgebras.

PROPOSITION 3.1. *Let $\mathscr{M}$ be a separable type II_1 von Neumann algebra with a regular type II_1 subalgebra $\mathscr{P} \subset \mathscr{M}$. Then there exists a vector $\xi \in \mathscr{M}$ such that $L^2(\mathscr{M},\tau) = \overline{sp}\,\mathscr{P}\xi\mathscr{P}$. In particular, if $\mathscr{P}$ is injective, then $\mathscr{M}$ is weakly h.h.-thin.*

Proof. By Proposition 2.15 in [18], $\mathscr{P}$ contains a hyperfinite subfactor $\mathscr{R}$. Let $\mathscr{A} \subset \mathscr{R}$ be a Cartan subalgebra with a vector $m \in \mathscr{R}$ such that $L^2(\mathscr{R},\tau) = \overline{sp}\,\mathscr{A}m\mathscr{A}$. Consider a sequence (e_n) of mutually orthogonal projections in $\mathscr{A}$, with $\tau(e_n) = 2^{-n}$. Let (v_n) be a sequence of unitary operators, strongly dense in the normalizer $\mathscr{N}_{\mathscr{M}}(\mathscr{P})$. Define the vector $\xi \in \mathscr{M}$ by

$$\xi = \sum_{i \geqslant 1} 2^{-i} e_i m v_i.$$

By left multiplying ξ by $2^{i_0} e_{i_0}$, we get $e_{i_0} m v_{i_0}$. Left multiplication by $\mathscr{A}$ leads to $e_{i_0}\mathscr{A} m v_{i_0}$, while subsequent right multiplication by $v_{i_0}^*\mathscr{A} v_{i_0} \subset \mathscr{P}$ obtains $e_{i_0}\mathscr{A}m\mathscr{A}v_{i_0}$, hence $e_{i_0}\mathscr{R}v_{i_0}$ by taking closures. One more left multiplication by $\mathscr{R}$ leads to $\mathscr{R}e_{i_0}\mathscr{R}v_{i_0}$. Since the identity operator can be written as a finite sum in $sp\,\mathscr{R}e_{i_0}\mathscr{R}$, we see that $\overline{sp}\,\mathscr{P}\xi\mathscr{P}$ contains every unitary operator v_n. $\square$

PROPOSITION 3.2. *Let $\mathscr{M}$ be a separable type II_1 factor with a regular type II_1 subalgebra $\mathscr{P} \subset \mathscr{M}$ satisfying $\mathscr{P}' \cap \mathscr{M} \subset \mathscr{P}$. Then there exists an abelian subalgebra*

$\mathscr{A} \subset \mathscr{P}$ and a vector $\xi \in L^2(\mathscr{M},\tau)$ such that $L^2(\mathscr{M},\tau) = \overline{sp}\ \mathscr{P}\xi\mathscr{A}$. In particular, if $\mathscr{P}$ is injective, then $\mathscr{M}$ is weakly a.h.-thin.

Proof. Since $\mathscr{P}' \cap \mathscr{M} \subset \mathscr{P}$, there exists a m.a.s.a $\mathscr{A} \subset \mathscr{M}$ contained in $\mathscr{P}$ (Theorem 12.2.4 in [15]). Following the discussion in 1.4, denote by $E : \mathscr{M} \to \mathscr{P}$ a trace-preserving, normal conditional expectation and consider the unital, completely positive map $\varphi : C^*(\mathscr{M}, J\mathscr{A}J) \to C^*(\mathscr{P}, J\mathscr{A}J)$ defined by $\varphi(mJaJ) = E(m)JaJ$. If $u \in \mathscr{N}_{\mathscr{M}}(\mathscr{P})$ then $u\mathscr{A}u^* \subset u\mathscr{P}u^* = \mathscr{P}$ and we have

$$\begin{aligned}||\varphi(\sum_{i=1}^{n} m_iJa_iJ)(u)||_2 &= ||\sum_{i=1}^{n} E(m_i)ua_i^*||_2 = ||\sum_{i=1}^{n} E(m_i)(ua_i^*u^*)u||_2 \\ &= ||E(\sum_{i=1}^{n} m_iua_i^*u^*)u||_2 = ||E(\sum_{i=1}^{n} m_iua_i^*u^*)||_2 \leqslant ||\sum_{i=1}^{n} m_iua_i^*u^*||_2 \\ &= ||\sum_{i=1}^{n} m_iua_i^*||_2 = ||(\sum_{i=1}^{n} m_iJa_iJ)(u)||_2.\end{aligned}$$

The above computation shows that $||\varphi(x)(u)||_2 \leqslant ||x(u)||_2$ for all $x \in C^*(\mathscr{M}, J\mathscr{A}J)$. Fix $x \in W^*(\mathscr{M}, J\mathscr{A}J)$ of norm 1 and choose, by Kaplansky's density theorem, a net (x_α) in the unit ball of $C^*(\mathscr{M}, J\mathscr{A}J)$ strongly convergent to x. Since $(x_\alpha(u))$ is Cauchy in $L^2(\mathscr{M},\tau)$, it follows that $(\varphi(x_\alpha)(u))$ is Cauchy for every $u \in \mathscr{N}_{\mathscr{M}}(\mathscr{P})$. Then $(\varphi(x_\alpha)(\eta))$ is Cauchy for every η in the linear span of $\mathscr{N}_{\mathscr{M}}(\mathscr{P})$, which is dense in $L^2(\mathscr{M},\tau)$. Since $||\varphi(x_\alpha)|| \leqslant 1$, $(\varphi(x_\alpha)(\xi))$ is Cauchy for all $\xi \in L^2(\mathscr{M},\tau)$, and consequently we denote by $\varphi(x)$ the strong limit of $\varphi(x_\alpha)$. Thus φ extends to a unital, completely positive map (still denoted by φ) from $W^*(\mathscr{M}, J\mathscr{A}J)$ to $W^*(\mathscr{P}, J\mathscr{A}J)$ which acts identically on $C^*(\mathscr{P}, J\mathscr{A}J)$ and satisfies $||\varphi(x)(u)||_2 \leqslant ||x(u)||_2$ for all $x \in W^*(\mathscr{M}, J\mathscr{A}J)$ and all $u \in \mathscr{N}_{\mathscr{M}}(\mathscr{P})$. An argument based on Cauchy nets like in the first part of the proof shows that φ is strong operator to strong operator continuous on the unit ball of $W^*(\mathscr{M}, J\mathscr{A}J)$. This implies that φ is normal (i.e. ultraweakly to ultraweakly continuous) by virtue of a classic argument, see for example [17], II.2.6.

We have proved that φ extends to a normal conditional expectation from $W^*(\mathscr{M}, J\mathscr{A}J)$ to $W^*(\mathscr{P}, J\mathscr{A}J)$. Since $W^*(\mathscr{M}, J\mathscr{A}J) = (J\mathscr{A}J)'$ is of type I, it follows that $W^*(\mathscr{P}, J\mathscr{A}J)$ must also be of type I (Proposition 10.21 in [16]), and necessarily of type I_∞, since $\mathscr{P}$ has no non-trivial finite dimensional representations. But properly infinite von Neumann algebras on separable Hilbert spaces have cyclic vectors, which concludes the proof. □

PROPOSITION 3.3. *Let $\mathscr{M}$ be a separable type II_1 von Neumann algebra with a regular type II_1 subfactor $\mathscr{P} \subset \mathscr{M}$. Then there exists an abelian subalgebra $\mathscr{A} \subset \mathscr{P}$ and a vector $\xi \in \mathscr{M}$ such that $L^2(\mathscr{M},\tau) = \overline{sp}\ \mathscr{P}\xi\mathscr{A}$. In particular, if $\mathscr{P}$ is injective, then $\mathscr{M}$ is weakly a.h.-thin.*

Proof. Let $\mathscr{R} \subset \mathscr{P}$ be a hyperfinite subfactor with a Cartan subalgebra $\mathscr{A} \subset \mathscr{R}$ and a vector $m \in \mathscr{R}$ such that $L^2(\mathscr{R},\tau) = \overline{sp}\ \mathscr{A}m\mathscr{A}$. Consider a sequence (e_n) of mutually orthogonal projections in $\mathscr{A}$, with $\tau(e_n) = 2^{-n}$. Let (v_n) be a sequence of unitary operators, strongly dense in the normalizer $\mathscr{N}_{\mathscr{M}}(\mathscr{P})$. If we denote $f_i = v_i e_i v_i^* \in \mathscr{P}$, the fact that $\mathscr{P}$ is a factor ensures the existence of a unitary operator $u_i \in \mathscr{P}$ such that $u_i f_i u_i^* = e_i$. To this end, define the vector $\xi \in \mathscr{M}$ by

$$\xi = \sum_{i \geqslant 1} 2^{-i} u_i v_i e_i m.$$

Note that $u_i v_i e_i m = u_i(v_i e_i v_i^*) v_i e_i m = u_i f_i v_i e_i m = (u_i f_i u_i^*) u_i v_i e_i m = e_i u_i v_i e_i m.$

As a consequence, left multiplication of ξ by $2^{i_0} e_{i_0}$ leads to $u_{i_0} v_{i_0} e_{i_0} m$, while a subsequent left multiplication by $u_{i_0}^* \in \mathscr{P}$ gives $v_{i_0} e_{i_0} m$. One more left multiplication by $v_{i_0} \mathscr{A} v_{i_0}^* \in \mathscr{P}$ gets $v_{i_0} \mathscr{A} e_{i_0} m = v_{i_0} e_{i_0} \mathscr{A} m$. Right multiplication by $\mathscr{A}$ leads to $v_{i_0} e_{i_0} \mathscr{A} m \mathscr{A}$, hence to $v_{i_0} e_{i_0} \mathscr{R}$ by taking closures. Next we left multiply by $v_{i_0} \mathscr{R} v_{i_0}^* \in \mathscr{P}$ to get $v_{i_0} \mathscr{R} e_{i_0} \mathscr{R}$, hence v_{i_0}, since $I \in sp\, \mathscr{R} e_{i_0} \mathscr{R}$. $\square$

4. Applications

In this section we prove that von Neumann algebras satisfying the hypotheses of Propositions 3.2 and 3.3 have trivial cohomology groups $H^n(\mathscr{M}, \mathscr{M})$.

We take a moment to recall the definitions. If M is a von Neumann algebra and X is a Banach M-bimodule, let $L^k(M,X)$ be the Banach space of k-linear bounded maps from M^k into X, $k \geqslant 1$. For $k = 0$, $L^0(M,X)$ is taken to be X. The coboundary operator $\partial^k : L^k(M,X) \to L^{k+1}(M,X)$ (usually abbreviated to just ∂) is defined, for $k \geqslant 1$, by

$$\partial\varphi(x_1,\dots,x_{k+1}) = x_1 \varphi(x_2,\dots,x_{k+1})$$

$$+\sum_{i=1}^{k} (-1)^i \varphi(x_1,\dots,x_{i-1},x_i x_{i+1},x_{i+2},\dots,x_{k+1}) + (-1)^{k+1}\varphi(x_1,\dots,x_k)x_{k+1},$$

for $x_1,\dots,x_{k+1} \in M$. When $k = 0$, we define $\partial\xi$, for $\xi \in X$, by

$$\partial\xi(x) = x\xi - \xi x, \qquad x \in M$$

The coboundary operator satisfies $\partial^{k+1}\partial^k = 0$, and, consequently, Im ∂^k (the space of coboundaries) is a subspace of Ker ∂^{k+1} (the space of cocycles). The continuous Hochschild cohomology groups $H^k(M,X)$ are then defined to be the quotient vector spaces Ker ∂^k/Im ∂^{k-1}, $k \geqslant 1$.

Next, we recall from [11] that, for an inclusion $\mathscr{B} \subset \mathscr{A}$ of C^*-algebras, $\mathscr{B}$ is said to be norming for $\mathscr{A}$ if, for every $n \geqslant 1$ and for every $n \times n$ matrix X over $\mathscr{A}$, we have $||X|| = sup\, ||RXC||$, the supremum being taken over all $1 \times n$ rows R and all $n \times 1$ columns C with entries in $\mathscr{B}$ satisfying $||R||, ||C|| \leqslant 1$. The following is Proposition 4.1 in [14].

PROPOSITION 4.1. *Let $\mathscr{M}$ be a von Neumann algebra with subalgebras $\mathscr{A}$ and $\mathscr{B}$ satisfying*

(i) $\mathscr{B}$ is abelian.

(ii) There exists a vector $\xi \in L^2(\mathscr{M}, \tau)$ such that $L^2(\mathscr{M}, \tau) = \overline{sp}\, \mathscr{A}\xi\mathscr{B}$.

Then $\mathscr{A}$ is norming for $\mathscr{M}$.

The combination of Proposition 4.1 and Propositions 3.2, respectively 3.3, leads to

COROLLARY 4.2. *(1) Let $\mathscr{M}$ be a separable type* II_1 *factor with a regular type* II_1 *subalgebra $\mathscr{P} \subset \mathscr{M}$ satisfying $\mathscr{P}' \cap \mathscr{M} \subset \mathscr{P}$. Then $\mathscr{P}$ is norming for $\mathscr{M}$.*

(2) Let $\mathscr{M}$ be a separable type II_1 *von Neumann algebra with a regular type* II_1 *subfactor $\mathscr{P} \subset \mathscr{M}$. Then $\mathscr{P}$ is norming for $\mathscr{M}$.*

The statement in part (2) in the case of a type II_1 factor $\mathscr{M}$ was first proved by Cameron [2] by completely different methods.

The conjunction of regularity and norming leads to the vanishing of all cohomology groups $H^n(\mathscr{M},\mathscr{M})$, as shown in [14] for type II_1 factors with Cartan subalgebras (see also [1] for the non-separable case). The essential ideas from [14] were collected in Theorem 6.1(3) in [11], which we restate here and take the opportunity to remove an unnecessary hypothesis.

PROPOSITION 4.3. *If $\mathscr{M}$ is a separable von Neumann algebra with an injective, regular, norming subalgebra $\mathscr{A} \subset \mathscr{M}$, then $H^n(\mathscr{M},\mathscr{M}) = 0$, $n \geqslant 1$.*

Proof. The proof is essentially the same as the proof of Theorem 5.1 in [14], with one modification. The row boundedness of bounded, $\mathscr{A}$-modular maps on $\mathscr{M}$ is an immediate consequence of the norming properties of $\mathscr{A}$ and does not require any additional conditions, like the existence of an intermediate injective, irreducible subalgebra $\mathscr{A} \subset \mathscr{R} \subset \mathscr{M}$ as in [11]. Indeed, if $\varphi : \mathscr{M} \to \mathscr{M}$ is bounded and right $\mathscr{A}$-modular, then for any row $X = (x_1,\dots,x_n)$ with entries in $\mathscr{M}$ and every $\varepsilon > 0$, there exists a contractive column $C = (a_1,\dots,a_n)^T$ with entries in $\mathscr{A}$ such that $||(\varphi(x_1),\dots,\varphi(x_n)) \cdot (a_1,\dots,a_n)^T|| \geqslant ||(\varphi(x_1),\dots,\varphi(x_n))|| - \varepsilon$.

Then, by right modularity, we get $||(\varphi(x_1),\dots,\varphi(x_n))|| \leqslant \varepsilon + ||\varphi(XC)|| \leqslant \varepsilon + ||\varphi|| \cdot ||X|| \cdot ||C||$, and therefore $||(\varphi(x_1),\dots,\varphi(x_n))|| \leqslant ||\varphi|| \cdot ||(x_1,\dots,x_n)||$. $\square$

From Proposition 4.3, together with Corollary 4.2, we obtain

COROLLARY 4.4. *Let $\mathscr{M}$ be a separable type* II_1 *von Neumann algebra with a regular, injective, type* II_1 *subalgebra $\mathscr{P}$. If either $\mathscr{M}$ is a factor and $\mathscr{P}' \cap \mathscr{M} \subset \mathscr{P}$, or if $\mathscr{P}$ is a factor, then $H^n(\mathscr{M},\mathscr{M}) = 0$, $n \geqslant 1$.*

REMARK 4.5. (1) Corollary 4.4 applies, for instance, to the crossed product type II_1 factors $\mathscr{R} \times G$, where G is a discrete group acting on $\mathscr{R}$ by outer automorphisms. It also applies to the tensor products $(\mathscr{R} \times G)\overline{\otimes}\mathscr{N}$, for arbitrary separable type II_1 factors $\mathscr{N}$. Consequently, we have $H^n(\mathscr{R} \times G, \mathscr{R} \times G) = H^n((\mathscr{R} \times G)\overline{\otimes}\mathscr{N}, (\mathscr{R} \times G)\overline{\otimes}\mathscr{N}) = 0$.

(2) Since the free group factors $L(F_n)$ are not weakly h.h.-thin ([6], see also [8]), we also obtained alternative proofs of particular cases of a deep result of Ozawa and Popa ([9]): $L(F_n)$ does not contain regular, injective, diffuse subalgebras.

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