RELATIVE OSCILLATION THEORY
FOR JACOBI MATRICES EXTENDED

KERSTIN AMMANN

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Abstract. We present a comprehensive treatment of relative oscillation theory for finite Jacobi matrices. We show that the difference of the number of eigenvalues of two Jacobi matrices in an interval equals the number of weighted sign-changes of the Wronskian of suitable solutions of the two underlying difference equations. Until now only the case of perturbations of the main diagonal was known. We extend the known results to arbitrary perturbations, allow any (half-) open and closed spectral intervals, simplify the proof, and establish the comparison theorem.

1. Introduction

Jacobi operators appear at numerous occasions in mathematics as well as in physical models. For example, they are intimately related to the theory of orthogonal polynomials or constitute a simple one-band tight binding model in quantum mechanics. They can be viewed as the discrete counterpart of Sturm–Liouville operators and their investigation has many similarities with Sturm–Liouville theory. Moreover, spectral and inverse spectral theory for Jacobi operators plays a fundamental role in the investigation of the Toda lattice and its modified counterpart, the Kac–van Moerbeke lattice. For a comprehensive introduction we refer to [18].

Let $a, b \in \ell(\mathbb{Z}) = \{ \varphi \mid \varphi : \mathbb{Z} \to \mathbb{R} \}$, where $a(n) < 0$ holds for all $n \in \mathbb{Z}$. Then, the *Jacobi matrix*

$$J = \begin{pmatrix} b(1) & a(1) & 0 & 0 & 0 \\ a(1) & b(2) & \ddots & 0 & 0 \\ 0 & & \ddots & & 0 \\ 0 & 0 & \ddots & b(N-2) & a(N-2) \\ 0 & 0 & 0 & a(N-2) & b(N-1) \end{pmatrix} \tag{1.1}$$

is self-adjoint and $\sigma(J)$ is real and simple. The corresponding *Jacobi difference equation* is given by

$$\tau u = z u, \tag{1.2}$$

where

$$\tau : \ell(\mathbb{Z}) \rightarrow \ell(\mathbb{Z})$$
$$u(n) \mapsto (\tau u)(n) = a(n)u(n+1) + a(n-1)u(n-1) + b(n)u(n) \tag{1.3}$$
$$= \partial(a(n-1)\partial u(n-1)) + (b(n) + a(n) + a(n-1))u(n),$$

$z \in \mathbb{R}$, and $\partial u(n) = u(n+1) - u(n)$ is the usual forward difference operator.

We call $u(z)$ a *solution* of (1.2) if $(\tau - z)u(z) = 0$ and $u(z) \not\equiv 0$. Whenever the spectral parameter is evident from the context we abbreviate $u = u(z)$. For any two initial values $u(n_0), u(n_0+1), n_0 \in \mathbb{Z}$, there exists a unique 'solution' u of (1.2) which vanishes if and only if $(u(n_0), u(n_0+1)) = (0,0)$. We exclude this case and thus, a solution of (1.2) cannot have two consecutive zeros. We call n a *node* (sign-change) of u if

$$u(n) = 0 \quad \text{or} \quad a(n)u(n)u(n+1) > 0 \tag{1.4}$$

and say that a node n of u lies between m and l if either $m < n < l$ or if $n = m$ and $u(m) \neq 0$. The *number of nodes of u between m and l* is denoted as $\#_{(m,l)}(u)$.

From classical oscillation theory originating in the seminal work of Sturm from 1836 [17] we know that the n-th eigenfunction of a Sturm–Liouville operator has $n-1$ nodes. This also holds for eigensequences of Jacobi operators, see [7, 8, 20] and also [14]. Our aim now is to show that the number of nodes of the Wronskian determinant of two (suitable) solutions $u_j(z_j)$ of $(\tau_j - z_j)u_j = 0$, $j = 0, 1$, equals the difference of the number of eigenvalues of J_0 and J_1 in (z_0, z_1).

In [3] (confer also [2]) Teschl and myself considered the special case $a_0 = a_1$ which is now generalized to arbitrary perturbations (see also [1]). We still assume $a_0, a_1 < 0$. This is no restriction since altering the sign of one or more elements of a does not affect the spectrum of the corresponding matrices, their similarity can easily be shown. Nevertheless, the signs of the solutions of the underlying difference equations depend on the signs of a and therefore we assume $a < 0$ to simplify (1.6), confer also [1].

The Wronskian is given by $W(u_0, u_1) \in \ell(\mathbb{Z})$, where

$$W_n(u_0, u_1) = u_0(n)a_1(n)u_1(n+1) - u_1(n)a_0(n)u_0(n+1). \tag{1.5}$$

We set

$$\#_n(u_0, u_1) = \begin{cases} 1 & \text{if } W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \text{ and} \\ & \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \text{or } W_n(u_0,u_1) = 0 \text{ and } W_{n+1}(u_0,u_1) \neq 0 \\ -1 & \text{if } W_n(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \text{ and} \\ & \text{either } W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \\ & \text{or } W_n(u_0,u_1) \neq 0 \text{ and } W_{n+1}(u_0,u_1) = 0 \\ 0 & \text{otherwise} \end{cases} \tag{1.6}$$

and say the Wronskian has a *(weighted) node* at n if $\#_n(u_0,u_1) \neq 0$. We denote the *number of weighted nodes of the Wronskian between m and n, $m < n$*, by

$$\#_{[m,n]}(u_0,u_1) = \sum_{j=m}^{n-1} \#_j(u_0,u_1) \tag{1.7}$$

and set

$$\#_{(m,n]}(u_0,u_1) = \#_{[m,n]}(u_0,u_1) - \begin{cases} 1 & \text{if } W_m(u_0,u_1) = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.8}$$

$$\#_{[m,n)}(u_0,u_1) = \#_{[m,n]}(u_0,u_1) + \begin{cases} 1 & \text{if } W_n(u_0,u_1) = 0 \\ 0 & \text{otherwise,} \end{cases} \tag{1.9}$$

and

$$\#_{(m,n)}(u_0,u_1) \tag{1.10}$$
$$= \#_{[m,n]}(u_0,u_1) - \begin{cases} 1 & \text{if } W_m(u_0,u_1) = 0 \\ 0 & \text{otherwise} \end{cases} + \begin{cases} 1 & \text{if } W_n(u_0,u_1) = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Here we slightly changed the notation compared to [3]: $\#_{(m,n)}$ from [3] is now denoted as $\#_{(m,n]}$. That (1.6) is a generalization of the counting method established in [3, (1.8)], where $a_0 = a_1$ holds, follows from (2.6), see also [1, Lemma 3.21].

In the Sections 2–4 we prove our main theorem:

THEOREM 1.1. (Relative Oscillation Theorem) *Let $E_\Omega(J_j)$, $j = 0,1$, be the number of eigenvalues of J_j in $\Omega \subseteq \mathbb{R}$ and let $u_{j,\pm}(z_j)$ be solutions of $(\tau_j - z_j)u = 0$ fulfilling the right/left Dirichlet boundary condition of J_j, i.e. $u_{j,+}(z_j,N) = u_{j,-}(z_j,0) = 0$.*
  If $a_0, a_1 < 0$, then,

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0)$$
$$= \#_{(0,N-1]}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{(0,N-1]}(u_{0,-}(z_0),u_{1,+}(z_1)) \tag{1.11}$$

and

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0)$$
$$\qquad = \#_{[0,N-1]}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{(0,N-1)}(u_{0,-}(z_0),u_{1,+}(z_1)),$$
$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0]}(J_0)$$
$$\qquad = \#_{(0,N-1)}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{[0,N-1]}(u_{0,-}(z_0),u_{1,+}(z_1)), \tag{1.12}$$
$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0)}(J_0)$$
$$\qquad = \#_{[0,N-1)}(u_{0,+}(z_0),u_{1,-}(z_1)) = \#_{[0,N-1)}(u_{0,-}(z_0),u_{1,+}(z_1))$$

if we set $a_0(N-1) = a_1(N-1)$ to compute $u_{j,-}(N)$.

To simplify the previous theorem we've set $a_0(N-1) = a_1(N-1)$. This doesn't influence J and $\sigma(J)$, but the value $u_-(N)$ depends on it. However, if we drop this assumption, then we have to take the weight at $N-1$ into account. We state that case in Theorem 4.10. For a computation of $u_+(0)$ any negative values $a_0(0)$ and $a_1(0)$ will do the job.

Equation (1.11) generalizes Theorem 1.2 from [3] to different a's. In the continuous case it has been established by Krüger and Teschl in [11]. For the case of Dirac operators see Stadler and Teschl in [16] and for extensions to symplectic eigenvalue problems see Elyseeva [4–6].

In the sequel (Sections 2–4) we prove Theorem 1.1 using the discrete Prüfer transformation. Compared to [2, 3, 11, 12, 16] we also present a simplified proof which eliminates the need to interpolate between operators. This is of particular importance in the present case, since $a_0 < a_1$ does not imply the corresponding relation for the operators, which would make the interpolation step more difficult. In addition, (1.12) is new. The proofs for regular Sturm–Liouville operators [11, Theorem 2.3] (confer also [12]) and regular Dirac operators [16, Theorem 3.3] can be shortened in the same manner and both theorems can be extended to (half-)open and closed spectral intervals analogously to (1.12) (for the first case cf. also [19]).

An extension of Sturm's classical comparison theorem for nodes of solutions to nodes of Wronskians is established in Section 5. We show that it holds analogously to the continuous case [12] if $a_0 = a_1$, therefore confer also [2]. Moreover, we give Sturm-type comparison theorems for arbitrary perturbations of Jacobi matrices, where, unlike the case of Sturm–Liouville operators [11], we do not obtain a direct dependence on the coefficients of the operators because $a_0 \leqslant a_1$ doesn't imply $J_0 \leqslant J_1$.

An extension of Theorem 1.1 to Jacobi operators on the half-line and on the line is in preparation, see [1]. This will fill the gap that classical oscillation theory is only applicable below the essential spectrum, while relative oscillation theory works perfectly inside gaps of the essential spectrum. We hope that this will stimulate further research, e.g. to find new relative oscillation criteria as in the continuous case, see [10, 9].

We'd be remiss not to mention that several other extensions of relative oscillation theory are thinkable, e.g. to CMV matrices. Only recently, Šimon Hilscher pointed out in [15] that an extension to the case of Jacobi difference equations with a nonlinear dependence on the spectral parameter would be of particular interest.

2. The Wronskian

At first we look at the Wronskian and its 'derivative' (2.6) along the $\mathbb{Z}$-axis.

DEFINITION 2.1. We define the (modified) *Wronskian* (also referred to as Wronski determinant or Casorati determinant) by

$$W : \mathbb{D}^2 \times \ell(\mathbb{Z})^2 \to \ell(\mathbb{Z}) \tag{2.1}$$
$$(\tau_0, \tau_1, \varphi, \psi) \mapsto W^{\tau_0, \tau_1}(\varphi, \psi),$$

where $\mathbb{D}$ denotes the space of difference equations, such that

$$W_n^{\tau_0,\tau_1}(\varphi,\psi) = \varphi(n)a_1(n)\psi(n+1) - \psi(n)a_0(n)\varphi(n+1) \tag{2.2}$$

$$= \begin{vmatrix} \varphi(n) & \psi(n) \\ a_0(n)\varphi(n+1) & a_1(n)\psi(n+1) \end{vmatrix}.$$

We abbreviate $\Delta b = b_0 - b_1$, $\Delta a = a_0 - a_1$, and $W_n(\varphi,\psi) = W_n^{\tau_0,\tau_1}(\varphi,\psi)$ whenever the corresponding difference equations are evident from the context. Clearly, if $a_0 = a_1$ holds, then W equals the Wronskian from [3]. We have

$$\begin{aligned}
W^{\tau_0,\tau_0}(\varphi,\varphi) &\equiv 0, \\
W^{\tau_0,\tau_1}(\varphi,\psi) &= -W^{\tau_1,\tau_0}(\psi,\varphi), \\
W^{\tau_0,\tau_1}(c\,\varphi,\psi) &= W^{\tau_0,\tau_1}(\varphi,c\,\psi) = c\,W^{\tau_0,\tau_1}(\varphi,\psi), \\
W^{\tau_0,\tau_1}(\varphi+\tilde{\varphi},\psi) &= W^{\tau_0,\tau_1}(\varphi,\psi) + W^{\tau_0,\tau_1}(\tilde{\varphi},\psi), \\
W^{\tau_0,\tau_1}(\varphi,\psi+\tilde{\psi}) &= W^{\tau_0,\tau_1}(\varphi,\psi) + W^{\tau_0,\tau_1}(\varphi,\tilde{\psi})
\end{aligned} \tag{2.3}$$

for all $c \in \mathbb{R}$ and $\varphi, \tilde{\varphi}, \psi, \tilde{\psi} \in \ell(\mathbb{Z})$.

LEMMA 2.2. *Green's Formula. We find*

$$\sum_{j=n}^{m}(\varphi(\tau_1\psi) - \psi(\tau_0\varphi))(j) = W_m(\varphi,\psi) - W_{n-1}(\varphi,\psi) \tag{2.4}$$

$$-\sum_{j=n-1}^{m-1}\Delta a(j)(\varphi(j+1)\psi(j) + \varphi(j)\psi(j+1)) - \sum_{j=n}^{m}\Delta b(j)\varphi(j)\psi(j).$$

Proof. Just a short calculation. $\square$

COROLLARY 2.3. *Let* $(\tau_j - z)u_j = 0$, *then*

$$W_m(u_0,u_1) - W_{n-1}(u_0,u_1) \tag{2.5}$$

$$= \sum_{j=n-1}^{m-1}\Delta a(j)(u_0(j+1)u_1(j) + u_0(j)u_1(j+1)) + \sum_{j=n}^{m}\Delta b(j)u_0(j)u_1(j)$$

and

$$W_n(u_0,u_1) - W_{n-1}(u_0,u_1) \tag{2.6}$$

$$= \Delta a(n-1)(u_0(n)u_1(n-1) + u_0(n-1)u_1(n)) + \Delta b(n)u_0(n)u_1(n).$$

If u and $\tilde{u}$ solve $\tau u = zu$, then $W(u,\tilde{u})$ is constant (and vanishes if and only if u and $\tilde{u}$ are linearly dependent).

3. Prüfer transformation

In Lemma 3.8 we'll establish the connection between the spectra of two different Jacobi matrices and the difference of the Prüfer angles of suitable solutions of the corresponding Jacobi difference equations. Therefore, from now on let u be a solution of $\tau u = zu$ and let $u_{-/+}$ moreover fulfill the left/right Dirichlet boundary condition of J. In order to keep this presentation self-contained at first we recall a few well-known findings which can e.g. be found in [18]:

LEMMA 3.1. *Confer* [18]. *The Jacobi matrix J has $N-1$ real and simple eigenvalues. Moreover,*

$$z \in \sigma(J) \iff u_-(z,N) = 0 \iff u_+(z,0) = 0. \tag{3.1}$$

Proof. Since J is Hermitian all eigenvalues are real: let $z \in \sigma(J)$, $Jv = zv$, and $\|v\| = 1$. Then $z = \langle v, zv \rangle = \langle v, Jv \rangle = \langle Jv, v \rangle = \overline{z}$. It can easily be seen that every eigenvector u corresponding to z fullfills $\tau u = zu$ and $u(0) = 0$. Hence, by $W_0(u_-(z), u) = 0$, $u_-(z)$ and u are linearly dependent. $\square$

THEOREM 3.2. Confer [8], [18, Theorem 4.7]. *For all $z \in \mathbb{R}$*

$$E_{(-\infty,z)}(J) = \#_{(0,N)}(u_-(z)) = \#_{(0,N)}(u_+(z)) \tag{3.2}$$

holds.

LEMMA 3.3. *If $u(n) = 0$, then $u(n-1)u(n+1) < 0$.*

Proof. Since (1.2) is a three-term-recursion and $u \not\equiv 0$, all zeros of u are simple and

$$u(n+1) = \underbrace{-a(n)^{-1}}_{>0}(\underbrace{a(n-1)}_{<0}u(n-1) + \underbrace{(b(n)-z)u(n)}_{=0}) \neq 0 \tag{3.3}$$

holds. $\square$

By $(u(n), u(n+1)) \neq (0,0)$ for all $n \in \mathbb{Z}$ we can introduce *Prüfer variables*: let $\rho_u, \theta_u \in \ell(\mathbb{Z})$ denote sequences so that

$$u(n) = \rho_u(n)\sin\theta_u(n), \tag{3.4}$$
$$-a(n)u(n+1) = \rho_u(n)\cos\theta_u(n),$$

and $\rho_u(n) > 0$ holds for all $n \in \mathbb{Z}$. Choose $\theta_u(n_0) \in (-\pi, \pi]$ at the initial position n_0 and assume

$$\lceil \theta_u(n)/\pi \rceil \leqslant \lceil \theta_u(n+1)/\pi \rceil \leqslant \lceil \theta_u(n)/\pi \rceil + 1 \tag{3.5}$$

for all $n \in \mathbb{Z}$, then both sequences are well-defined and unique. Here, $x \mapsto \lceil x \rceil = \min\{n \in \mathbb{Z} \mid n \geqslant x\}$ denotes the ceiling function, a left-continuous analog to the well-known floor function $x \mapsto \lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \leqslant x\}$ which itself is a right-continuous step function.

We follow [11] and use the slightly refined (compared to [3, 18, 20]) definition of Prüfer variables by taking the secondary diagonals a into account. By $-a > 0$ this will not influence the herein recalled claims on the nodes of solutions, but it will simplify our calculations as soon as we look at the nodes of the Wronskian.

LEMMA 3.4. *Fix some $n \in \mathbb{Z}$, then $\exists\, k \in \mathbb{Z}$ such that $\theta_u(n) = k\pi + \gamma$ and $\theta_u(n+1) = k\pi + \Gamma$, where*

$$\gamma \in \left(0, \frac{\pi}{2}\right], \quad \Gamma \in (0, \pi] \quad \Longleftrightarrow \quad n \text{ is not a node of } u, \tag{3.6}$$

$$\gamma \in \left(\frac{\pi}{2}, \pi\right], \quad \Gamma \in (\pi, 2\pi) \quad \Longleftrightarrow \quad n \text{ is a node of } u \tag{3.7}$$

holds. Moreover,

$$\theta_u(n) = k\pi + \frac{\pi}{2} \quad \Longleftrightarrow \quad \theta_u(n+1) = (k+1)\pi. \tag{3.8}$$

Proof. Abbreviate $\theta = \theta_u$. Choose $k \in \mathbb{Z}$ such that $\theta(n) = k\pi + \gamma$, $\gamma \in (0, \pi]$ holds. By (3.5) we have $\Gamma \in (0, 2\pi]$. If $u(n)u(n+1) \neq 0$, then $\sin\gamma\cos\gamma > 0$ iff n is not a node of u and $\sin\gamma\cos\gamma < 0$ iff n is a node of u, hence (3.6) clearly holds for γ. By $\sin\Gamma\cos\gamma > 0$ we have $\sin\Gamma > 0$ iff n is not a node of u and $\sin\Gamma < 0$ iff n is a node of u, thus, (3.6) also holds for Γ.

Now, suppose we have $u(n+1) = 0$, then n is not a node of u and either $\Gamma = \pi$ or $\Gamma = 2\pi$ holds. By Lemma 3.3 we have $u(n)u(n+2) < 0$, hence $\sin\theta(n)\cos\theta(n+1) = (-1)^k\sin\gamma(-1)^k\cos\Gamma < 0$. Thus, by $\cos\Gamma < 0$, we have $\Gamma = \pi$. From $-a(n)u(n+1) = \rho(n)\cos\theta(n) = 0$ we conclude that $(-1)^k\cos\gamma = 0$, thus $\gamma = \frac{\pi}{2}$ and hence (3.6) and (3.8) hold. If $u(n) = 0$, then n is a node of u, $\gamma = \pi$, and (3.6) holds by $\sin\theta(n+1)\cos\theta(n) > 0$, i.e. $(-1)^k\sin\Gamma(-1)^k\cos\gamma > 0$. $\square$

COROLLARY 3.5. *For all $n \in \mathbb{Z}$ we have*

$$\lceil \theta_u(n+1)/\pi \rceil = \begin{cases} \lceil \theta_u(n)/\pi \rceil + 1 & \text{if } n \text{ is a node of } u \\ \lceil \theta_u(n)/\pi \rceil & \text{otherwise.} \end{cases} \tag{3.9}$$

Now we are able to count nodes of solutions of the Jacobi difference equation using Prüfer variables and the number of nodes in an interval (m, n) is given by

THEOREM 3.6. *Confer [20, Lemma 2.5]. We have*

$$\#_{(m,n)}(u) = \lceil \theta_u(n)/\pi \rceil - \lfloor \theta_u(m)/\pi \rfloor - 1. \tag{3.10}$$

Proof. We use mathematical induction: let $n = m+1$, then if $u(m) = 0$, $u(n) \neq 0$ we have $\#_{(m,n)}(u) = 0$ and by Corollary 3.5

$$\lceil \theta_u(n)/\pi \rceil = \lceil \theta_u(m+1)/\pi \rceil = \underbrace{\lceil \theta_u(m)/\pi \rceil}_{\in \mathbb{Z}} + 1 = \lfloor \theta_u(m)/\pi \rfloor + 1 \tag{3.11}$$

holds. If $u(m) \neq 0$ holds, then by Corollary 3.5 we have

$$\underbrace{\lfloor \theta_u(m)/\pi \rfloor}_{\notin \mathbb{Z}} = \lceil \theta_u(m)/\pi \rceil - 1 = \begin{cases} \lceil \theta_u(n)/\pi \rceil - 2 & \text{if } m \text{ is a node} \\ \lceil \theta_u(n)/\pi \rceil - 1 & \text{otherwise.} \end{cases} \tag{3.12}$$

The inductive step follows again from Corollary 3.5. $\square$

Let $s_{-/+}(z)$ denote the solution of $\tau s = z s$ fulfilling

$$s_-(z,0) = 0, s_-(z,1) = 1, \quad \text{resp. } s_+(z,N) = 0, s_+(z,N+1) = 1 \tag{3.13}$$

and let n_0 denote the base point, i.e. $n_0 = 0$, resp. $n_0 = N$.

Then, by $s_\pm(n_0) = 0$ we have $\sin\theta_\pm(n_0) = 0$ and by $s_\pm(n_0+1) = 1$ we have $-a(n_0)s_\pm(n_0+1) = \rho_s(n_0)\cos\theta_\pm(n_0) > 0$, hence $\theta_\pm(n_0) = 0$ holds by $\theta_\pm(n_0) \in (-\pi,\pi]$.

COROLLARY 3.7. *We have*

$$\#_{(0,N)}(s_-) = \lceil \theta_{s_-}(N)/\pi \rceil - 1 \quad and \quad \#_{(0,N)}(s_+) = -\lfloor \theta_{s_+}(0)/\pi \rfloor - 1. \tag{3.14}$$

In the last step we now introduce the difference Δ of two Prüfer angles on which our subsequent considerations rely:

LEMMA 3.8. *We find*

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0) \tag{3.15}$$
$$= \lceil \Delta_{s_{0,+}(z_0),s_{1,-}(z_1)}(N)/\pi \rceil - \lceil \Delta_{s_{0,+}(z_0),s_{1,-}(z_1)}(0)/\pi \rceil$$
$$= \lfloor \Delta_{s_{0,-}(z_0),s_{1,+}(z_1)}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,-}(z_0),s_{1,+}(z_1)}(0)/\pi \rfloor,$$

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0) \tag{3.16}$$
$$= \lceil \Delta_{s_{0,\pm}(z_0),s_{1,\mp}(z_1)}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm}(z_0),s_{1,\mp}(z_1)}(0)/\pi \rfloor - 1,$$

$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0)}(J_0) \tag{3.17}$$
$$= \lfloor \Delta_{s_{0,\pm}(z_0),s_{1,\mp}(z_1)}(N)/\pi \rfloor - \lceil \Delta_{s_{0,\pm}(z_0),s_{1,\mp}(z_1)}(0)/\pi \rceil + 1, and$$

$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0]}(J_0) \tag{3.18}$$
$$= \lceil \Delta_{s_{0,-}(z_0),s_{1,+}(z_1)}(N)/\pi \rceil - \lceil \Delta_{s_{0,-}(z_0),s_{1,+}(z_1)}(0)/\pi \rceil$$
$$= \lfloor \Delta_{s_{0,+}(z_0),s_{1,-}(z_1)}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,+}(z_0),s_{1,-}(z_1)}(0)/\pi \rfloor,$$

where $\Delta_{u,v} = \theta_v - \theta_u \in \ell(\mathbb{Z})$.

Proof. Abbreviate $s_{j,\pm} = s_{j,\pm}(z_j)$. By Theorem 3.2, Corollary 3.7, and $-\lceil x \rceil = \lfloor -x \rfloor$ for all $x \in \mathbb{R}$ we have

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0) = \#_{(0,N)}(s_{1,-}) - \#_{(0,N)}(s_{0,+}) \tag{3.19}$$

$$= \lceil \theta_{s_{1,-}}(N)/\pi \rceil - \lceil -\theta_{s_{0,+}}(0)/\pi \rceil = \lceil \Delta_{s_{0,+},s_{1,-}}(N)/\pi \rceil - \lceil \Delta_{s_{0,+},s_{1,-}}(0)/\pi \rceil$$
$$= -(E_{(-\infty,z_0)}(J_0) - E_{(-\infty,z_1)}(J_1)) = \lfloor \Delta_{s_{0,-},s_{1,+}}(N)/\pi \rfloor - \lfloor \Delta_{s_{0,-},s_{1,+}}(0)/\pi \rfloor.$$

By Lemma 3.1 and (3.4) we have

$$z_0 \in \sigma(J_0) \quad \Longleftrightarrow \quad \Delta_{s_{0,-},s_{1,+}}(N)/\pi \in \mathbb{Z} \quad \Longleftrightarrow \quad \Delta_{s_{0,+},s_{1,-}}(0)/\pi \in \mathbb{Z}, \quad (3.20)$$

$$z_1 \in \sigma(J_1) \quad \Longleftrightarrow \quad \Delta_{s_{0,+},s_{1,-}}(N)/\pi \in \mathbb{Z} \quad \Longleftrightarrow \quad \Delta_{s_{0,-},s_{1,+}}(0)/\pi \in \mathbb{Z} \quad (3.21)$$

and hence

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0) = \lceil \Delta_{s_{0,\pm},s_{1,\mp}}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm},s_{1,\mp}}(0)/\pi \rfloor - 1 \quad (3.22)$$

holds by

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0) \quad (3.23)$$

$$= \lceil \Delta_{s_{0,\pm},s_{1,\mp}}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm},s_{1,\mp}}(0)/\pi \rfloor - \begin{cases} 1 & \text{if } z_0 \notin \sigma(J_0) \\ 0 & \text{if } z_0 \in \sigma(J_0). \end{cases}$$

The rest now follows analogously. $\quad \square$

4. Nodes of the Wronskian

It remains to investigate the sign-changes of $W(u_0,u_1)$. We will express them in terms of the difference Δ of Prüfer angles of the involved solutions to finally gain their connection to the difference of the spectra of the corresponding matrices (which is Theorem 1.1) by Lemma 3.8.

Therefore let u_j be solutions of $\tau_j - z, j = 0,1$, where $\rho_j, \theta_j \in \ell(\mathbb{Z})$ are their Prüfer variables from (3.4). They correspond to the same spectral parameter z, which is no restriction, since we can always replace b_1 by $b_1 - (z_1 - z_0)$. We abbreviate

$$\Delta = \Delta_{u_0,u_1} = \theta_1 - \theta_0 \in \ell(\mathbb{Z}) \quad (4.1)$$

and adopt Lemma 4.1 and Lemma 4.2 from [3]:

LEMMA 4.1. *Confer* [3]. *Fix some* $n \in \mathbb{Z}$, *then* $\exists k_j \in \mathbb{Z}, j = 0,1$, *such that*

$$\theta_j(n) = k_j\pi + \gamma_j, \quad \gamma_j \in (0,\pi], \quad (4.2)$$
$$\theta_j(n+1) = k_j\pi + \Gamma_j, \quad \Gamma_j \in (0,2\pi), \quad (4.3)$$

and one of the following holds:

(1) *either* u_0 *and* u_1 *have a node at* n *or both do not have a node at* n, *then*

$$\gamma_1 - \gamma_0 \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \quad \text{and} \quad \Gamma_1 - \Gamma_0 \in (-\pi,\pi). \quad (4.4)$$

(2) u_1 *has no node at* n, *but* u_0 *has a node at* n, *then*

$$\gamma_1 - \gamma_0 \in (-\pi, 0) \quad and \quad \Gamma_1 - \Gamma_0 \in (-2\pi, 0). \tag{4.5}$$

(3) u_1 *has a node at* n, *but* u_0 *has no node at* n, *then*

$$\gamma_1 - \gamma_0 \in (0, \pi) \quad and \quad \Gamma_1 - \Gamma_0 \in (0, 2\pi). \tag{4.6}$$

Proof. Use Lemma 3.4.   □

LEMMA 4.2. *Confer* [3]. *We have*

$$\lceil \Delta(n)/\pi \rceil - 1 \leqslant \lceil \Delta(n+1)/\pi \rceil \leqslant \lceil \Delta(n)/\pi \rceil + 1. \tag{4.7}$$

Proof. Let $k = k_1 - k_0$, $n \in \mathbb{Z}$. By Lemma 4.1 we have either

$$\Delta(n) \in \left(k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2} \right) \quad and \quad \Delta(n+1) \in (k\pi - \pi, k\pi + \pi), \tag{4.8}$$

$$\Delta(n) \in (k\pi - \pi, k\pi) \quad and \quad \Delta(n+1) \in (k\pi - 2\pi, k\pi), \text{ or} \tag{4.9}$$

$$\Delta(n) \in (k\pi, k\pi + \pi) \quad and \quad \Delta(n+1) \in (k\pi, k\pi + 2\pi). \tag{4.10}$$

In each case the lemma holds.   □

The rest of this section is new, see also [1].

LEMMA 4.3. *We have*

$$W_n(u_0, u_1) = \rho_0(n)\rho_1(n)\sin\Delta(n), \tag{4.11}$$

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) = p\sin(\gamma_1 - \gamma_0)\cos\gamma_0\cos\gamma_1, \tag{4.12}$$

$$W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) = \tilde{p}\sin(\Gamma_1 - \Gamma_0)\cos\gamma_0\cos\gamma_1, \tag{4.13}$$

where $p, \tilde{p} > 0$.

Proof. Consider

$$\begin{aligned} W_n(u_0, u_1) &= u_0(n)a_1(n)u_1(n+1) - u_1(n)a_0(n)u_0(n+1) \\ &= \rho_0(n)\rho_1(n)\sin(\theta_1(n) - \theta_0(n)) \\ &= \rho_0(n)\rho_1(n)(-1)^{k_1 - k_0}\sin(\gamma_1(n) - \gamma_0(n)) \end{aligned} \tag{4.14}$$

and set $p = \frac{\rho_0(n)^2\rho_1(n)^2}{a_0(n)a_1(n)}$ and $\tilde{p} = \frac{\rho_0(n)\rho_1(n)\rho_0(n+1)\rho_1(n+1)}{a_0(n)a_1(n)}$.   □

LEMMA 4.4. *We have*

$$u_0(n+1) = u_1(n+1) = 0 \implies W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0, \tag{4.15}$$

$$u_0(n+1) = 0, u_1(n+1) \neq 0 \implies W_n(u_0, u_1)W_{n+1}(u_0, u_1) > 0, \tag{4.16}$$

$$u_0(n+1) \neq 0, u_1(n+1) = 0 \implies W_n(u_0, u_1)W_{n+1}(u_0, u_1) > 0. \tag{4.17}$$

Proof. The first claim holds obviously. For the second claim just observe that by Lemma 3.3

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) = -u_0(n)u_0(n+2)a_0(n+1)a_1(n)u_1(n+1)^2 > 0 \qquad (4.18)$$

holds if $u_0(n+1) = 0, u_1(n+1) \neq 0$ and

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) = -u_1(n)u_1(n+2)a_0(n)a_1(n+1)u_0(n+1)^2 > 0 \qquad (4.19)$$

holds if $u_0(n+1) \neq 0, u_1(n+1) = 0$. $\square$

COROLLARY 4.5. *If* $W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0$ *holds or if* $W_n(u_0,u_1) = 0$ *and* $W_{n+1}(u_0,u_1) \neq 0$ *holds, or if* $W_n(u_0,u_1) \neq 0$ *and* $W_{n+1}(u_0,u_1) = 0$ *holds, then*

$$u_0(n+1)u_1(n+1) \neq 0 \qquad (4.20)$$

and moreover $\Delta a(n) \neq 0$ *or* $\Delta b(n+1) \neq 0$ *holds.*

To shorten notation we denote

$$\begin{aligned}
(+1) \quad &\text{if} \quad \lceil \Delta(n+1)/\pi \rceil = \lceil \Delta(n)/\pi \rceil + 1, &(4.21)\\
(0) \quad &\text{if} \quad \lceil \Delta(n+1)/\pi \rceil = \lceil \Delta(n)/\pi \rceil, \text{ and} &(4.22)\\
(-1) \quad &\text{if} \quad \lceil \Delta(n+1)/\pi \rceil = \lceil \Delta(n)/\pi \rceil - 1. &(4.23)
\end{aligned}$$

LEMMA 4.6. *Let* $n \in \mathbb{Z}$, *then*

$$\begin{aligned}
(+1) \Longleftrightarrow \ &W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \text{ and}\\
&\text{either} \quad W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 &(4.24)\\
&\text{or} \quad W_n(u_0,u_1) = 0, W_{n+1}(u_0,u_1) \neq 0,\\
(-1) \Longleftrightarrow \ &W_n(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \text{ and}\\
&\text{either} \quad W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 &(4.25)\\
&\text{or} \quad W_n(u_0,u_1) \neq 0, W_{n+1}(u_0,u_1) = 0,\\
(0) \Longleftrightarrow \ &\text{otherwise.} &(4.26)
\end{aligned}$$

Proof. If $(+1)$, then we either have case (1) of Lemma 4.1 and $\gamma_1 - \gamma_0 \in (-\frac{\pi}{2},0]$, $\Gamma_1 - \Gamma_0 \in (0,\pi)$ or we have case (3) of Lemma 4.1 and $\gamma_1 - \gamma_0 \in (0,\pi), \Gamma_1 - \Gamma_0 \in (\pi,2\pi)$. Clearly, by (4.11), in either case we have

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \quad \text{or} \quad W_n(u_0,u_1) = 0, W_{n+1}(u_0,u_1) \neq 0. \qquad (4.27)$$

Hence, by Corollary 4.5 we have $u_0(n+1)u_1(n+1) \neq 0$ and thus $\cos\gamma_0 \cos\gamma_1 \neq 0$. In case (1) of Lemma 4.1 we have $\sin(\Gamma_1 - \Gamma_0) > 0$ and $\cos\gamma_0 \cos\gamma_1 > 0$ by Lemma 3.4. Hence, by (4.13) $W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0$ holds. In case (3) of Lemma 4.1 we have $\sin(\Gamma_1 - \Gamma_0) < 0$ and $\cos\gamma_0 \cos\gamma_1 < 0$ by Lemma 3.4. Hence, by (4.13)

$$W_{n+1}(u_0,u_1)u_0(n+1)u_1(n+1) > 0 \qquad (4.28)$$

holds.

If (-1), then we either have case (1) of Lemma 4.1 and $\gamma_1 - \gamma_0 \in (0, \frac{\pi}{2}), \Gamma_1 - \Gamma_0 \in (-\pi, 0]$ or we have case (2) of Lemma 4.1 and $\gamma_1 - \gamma_0 \in (-\pi, 0), \Gamma_1 - \Gamma_0 \in (-2\pi, -\pi]$. Clearly, by (4.11), in either case we have

$$W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \quad \text{or} \quad W_n(u_0, u_1) \neq 0, W_{n+1}(u_0, u_1) = 0. \tag{4.29}$$

Hence, by Corollary 4.5 we have $u_0(n+1)u_1(n+1) \neq 0$ and thus $\cos\gamma_0\cos\gamma_1 \neq 0$. In case (1) of Lemma 4.1 we have $\sin(\gamma_1 - \gamma_0) > 0$ and $\cos\gamma_0\cos\gamma_1 > 0$ by Lemma 3.4. Hence, by (4.12) $W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0$ holds. In case (2) of Lemma 4.1 we have $\sin(\gamma_1 - \gamma_0) < 0$ and $\cos\gamma_0\cos\gamma_1 < 0$ by Lemma 3.4. Hence, by (4.12)

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0 \tag{4.30}$$

holds.

On the other hand, if $W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0$ by (4.11) we have either $(+1)$ or (-1). If, use (4.12),

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) = p\sin(\gamma_1 - \gamma_0)\cos\gamma_0\cos\gamma_1 > 0, \tag{4.31}$$

then we have either case (1) or case (2) of Lemma 4.1 and in each case we have (0) or (-1). Hence,

$$W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \text{ and } W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0 \implies (-1).$$

If, use (4.12),

$$W_n(u_0, u_1)u_0(n+1)u_1(n+1) = p\sin(\gamma_1 - \gamma_0)\cos\gamma_0\cos\gamma_1 < 0, \tag{4.32}$$

then we have either case (1) or case (3) of Lemma 4.1 and in each case we have (0) or $(+1)$. Hence,

$$W_n(u_0, u_1)W_{n+1}(u_0, u_1) < 0 \text{ and } W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) > 0 \implies (+1).$$

If $W_n(u_0, u_1) = 0, W_{n+1}(u_0, u_1) \neq 0$, then we have case (1) of Lemma 4.1 and by Corollary 4.5 we have $\cos\gamma_0\cos\gamma_1 > 0$. Hence, if $W_{n+1}(u_0, u_1)u_0(n+1)u_1(n+1) > 0$, then (4.13) implies $\sin(\Gamma_1 - \Gamma_0) > 0$, thus, $(+1)$ holds by case (1) of Lemma 4.1.

If $W_n(u_0, u_1) \neq 0, W_{n+1}(u_0, u_1) = 0$, then by Corollary 4.5 we have $\cos\gamma_0\cos\gamma_1 \neq 0$. If additionally $W_n(u_0, u_1)u_0(n+1)u_1(n+1) > 0$ holds, then by (4.12) $\cos\gamma_0\cos\gamma_1$ and $\sin(\gamma_1 - \gamma_0)$ are of the same sign. Hence, we have case (1) of Lemma 4.1 and (-1) or case (2) of Lemma 4.1 and (-1).

Thus, (4.24) and (4.25) hold and clearly by Lemma 4.2 we have (0) otherwise. $\square$

REMARK 4.7. Consider (1.6), then

$$W_n(u_0, u_1)W_{n+1}(u_0, u_1) \neq 0 \quad \text{or} \quad W_n(u_0, u_1) = W_{n+1}(u_0, u_1) = 0 \tag{4.33}$$
$$\implies \#_n(u_0, u_1) = -\#_n(u_1, u_0)$$

and

$$W_n(u_0,u_1)W_{n+1}(u_0,u_1) < 0 \implies \#_n(u_0,u_1) \neq 0$$

by Corollary 4.5. Moreover, if $W_n(u_0,u_1) = 0$ and $W_{n+1}(u_0,u_1) \neq 0$ holds, then

$$u_0(n) = 0 \iff u_1(n) = 0. \tag{4.34}$$

From Lemma 4.6 we conclude

$$\#_n(u_0,u_1) = \lceil \Delta(n+1)/\pi \rceil - \lceil \Delta(n)/\pi \rceil, \tag{4.35}$$

$$\#_{[m,n]}(u_0,u_1) = \lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil. \tag{4.36}$$

LEMMA 4.8. *We have*

$$\#_{(m,n]}(u_0,u_1) = \lceil \Delta(n)/\pi \rceil - \lfloor \Delta(m)/\pi \rfloor - 1, \tag{4.37}$$

$$\#_{[m,n)}(u_0,u_1) = \lfloor \Delta(n)/\pi \rfloor - \lceil \Delta(m)/\pi \rceil + 1, \text{ and} \tag{4.38}$$

$$\#_{(m,n)}(u_0,u_1) = \lfloor \Delta(n)/\pi \rfloor - \lfloor \Delta(m)/\pi \rfloor. \tag{4.39}$$

Proof. By (4.11) we have $W_j(u_0,u_1) = 0 \iff \Delta(j)/\pi \in \mathbb{Z}$ and hence by (4.36)

$$\#_{(m,n]}(u_0,u_1) = \lceil \Delta(n)/\pi \rceil - \lceil \Delta(m)/\pi \rceil - \begin{cases} 0 & \text{if } W_m(u_0,u_1) \neq 0 \\ 1 & \text{if } W_m(u_0,u_1) = 0 \end{cases} \tag{4.40}$$

$$= \lceil \Delta(n)/\pi \rceil - \lfloor \Delta(m)/\pi \rfloor - 1 \tag{4.41}$$

holds. The second and the third claim follow analogously. □

LEMMA 4.9. *We have*

$$\#_{[m,n]}(u_0,u_1) = -\#_{(m,n)}(u_1,u_0), \quad \#_{(m,n]}(u_0,u_1) = -\#_{[m,n)}(u_1,u_0). \tag{4.42}$$

If $W_m(u_0,u_1) \neq 0$ and $W_n(u_0,u_1) \neq 0$, then

$$\#_{[m,n]}(u_0,u_1) = -\#_{[m,n]}(u_1,u_0). \tag{4.43}$$

Proof. Use $\lceil x \rceil = -\lfloor -x \rfloor$ and Lemma 4.8. □

THEOREM 4.10. *Let $a_0, a_1 < 0$, then*

$$\begin{aligned} & E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0) \\ & = \#_{(0,N]}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{(0,N]}(u_{0,-}(z_0), u_{1,+}(z_1)) \end{aligned} \tag{4.44}$$

and

$$\begin{aligned} & E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0)}(J_0) \\ & = \#_{[0,N]}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{(0,N)}(u_{0,-}(z_0), u_{1,+}(z_1)), \end{aligned}$$

$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0]}(J_0)$$
$$= \#_{(0,N)}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{[0,N]}(u_{0,-}(z_0), u_{1,+}(z_1)), \quad (4.45)$$
$$E_{(-\infty,z_1]}(J_1) - E_{(-\infty,z_0)}(J_0)$$
$$= \#_{[0,N)}(u_{0,+}(z_0), u_{1,-}(z_1)) = \#_{[0,N]}(u_{0,-}(z_0), u_{1,+}(z_1)),$$

where $u_{j,\pm}(z_j), j = 0,1$, are solutions fulfilling the right/left Dirichlet boundary condition of J_j, i.e. $u_{j,-}(z_j,0) = u_{j,+}(z_j,N) = 0$.

Proof. By Lemma 3.8 and Lemma 4.8 we have

$$E_{(-\infty,z_1)}(J_1) - E_{(-\infty,z_0]}(J_0)$$
$$= \lceil \Delta_{s_{0,\pm}(z_0),s_{1,\mp}(z_1)}(N)/\pi \rceil - \lfloor \Delta_{s_{0,\pm}(z_0),s_{1,\mp}(z_1)}(0)/\pi \rfloor - 1 \quad (4.46)$$
$$= \#_{(0,N]}(s_{0,\pm}(z_0), s_{1,\mp}(z_1)) = \#_{(0,N]}(u_{0,\pm}(z_0), u_{1,\mp}(z_1))$$

and the equations (4.45) can be shown analogously. □

Proof of Theorem 1.1. In Theorem 4.10 the solutions $u_{j,\pm}(z_j), j = 0,1$, depend on the coefficients $a_j(0)$ and $a_j(N-1)$ of τ_j, although J_j (and hence also $\sigma(J_j)$) doesn't depend on them. If $a_0(N-1) = a_1(N-1)$, then by (2.6) we have

$$W_N(u_{0,\pm}(z_0)), u_{1,\mp}(z_1)) - W_{N-1}(u_{0,\pm}(z_0), u_{1,\mp}(z_1)) \quad (4.47)$$
$$= (b_0(N) - z_0 - b_1(N) + z_1)u_{0,\pm}(z_0,N)u_{1,\mp}(z_1,N) = 0$$

and hence there's no node at $N-1$. □

We close the proof of our main theorem with the following

REMARK 4.11. By Theorem 1.1 we have

$$\#_{[0,N]}(u_{0,\pm}(z), u_{1,\mp}(z_1)) = -\#_{[0,N]}(u_{1,\pm}(z), u_{0,\mp}(z)), \quad (4.48)$$
$$\#_{[0,N]}(u_{0,+}(z), u_{3,-}(z)) \quad (4.49)$$
$$= \#_{[0,N)}(u_{0,+}(z), u_{1,-}(z)) + \#_{[0,N]}(u_{1,-}(z), u_{2,+}(z)) + \#_{(0,N]}(u_{2,+}(z), u_{3,-}(z))$$

and

$$\#_{[0,N]}(u_{0,-}(z), u_{3,+}(z)) \quad (4.50)$$
$$= \#_{(0,N]}(u_{0,-}(z), u_{1,+}(z)) + \#_{[0,N]}(u_{1,+}(z), u_{2,-}(z)) + \#_{[0,N)}(u_{2,-}(z), u_{3,+}(z)).$$

5. Triangle inequality and comparison Theorem

In the last section we now establish the Triangle Inequality and the Comparison Theorem for Wronskians which generalize Theorem 5.12 and Theorem 5.13 from [2] to different a's. Moreover, Theorem 5.3 generalizes and sharpens Theorem 5.11 from [2]. We refer also to [1].

THEOREM 5.1. (Comparison Theorem for Wronskians I) *Let* $J_1 \geqslant J_2$, *then*

$$\#_{[0,N]}(u_{0,\pm}(z),u_{2,\mp}(z)) \geqslant \#_{[0,N]}(u_{0,\pm}(z),u_{1,\mp}(z)), \tag{5.1}$$

where $\#_{[0,N]}$ *can be replaced by* $\#_{(0,N]}$, $\#_{[0,N)}$, *or* $\#_{(0,N)}$.

Proof. Let $\sigma(J_1) = \{z_1,\ldots,z_{N-1}\}$ and $\sigma(J_2) = \{\tilde{z}_1,\ldots,\tilde{z}_{N-1}\}$, then $z_i \geqslant \tilde{z}_i$ for all i by $J_1 \geqslant J_2$, confer [13, Theorem 8.7.1], and hence we have $E_{(-\infty,z)}(J_2) \geqslant E_{(-\infty,z)}(J_1)$. Thus, by Theorem 1.1

$$\begin{aligned}
\#_{[0,N]}(u_{0,+}(z),u_{2,-}(z)) &= E_{(-\infty,z)}(J_2) - E_{(-\infty,z)}(J_0) \\
&\geqslant E_{(-\infty,z)}(J_1) - E_{(-\infty,z)}(J_0) = \#_{[0,N]}(u_{0,+}(z),u_{1,-}(z)).
\end{aligned} \tag{5.2}$$

The rest follows analogously from $E_{(-\infty,z]}(J_2) \geqslant E_{(-\infty,z]}(J_1)$ and Theorem 1.1. □

COROLLARY 5.2. *Let* $a_0 = a_1 = a_2$ *and moreover let* $b_0(j) \geqslant b_1(j) \geqslant b_2(j)$ *for all* $j = 1,\ldots,N-1$.
If 0 *and* $N-1$ *are positive nodes of* $W(u_{0,\pm}(z),u_{1,\mp}(z))$, *then the Wronskian* $W(u_{0,\pm}(z),u_{2,\mp}(z))$ *has at least two positive nodes at* $0,\ldots,N-1$.

THEOREM 5.3. *Let* $m < n$, *then*

$$\left| \#_{[m,n]}(u_0,u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0)) \right| \leqslant 1, \tag{5.3}$$

where $\#_{[m,n]}$ *can be replaced by* $\#_{(m,n]}$ *or* $\#_{[m,n)}$.

Proof. For all $x,y \in \mathbb{R}$ we have

$$0 \leqslant \lceil x-y \rceil - (\lceil x \rceil - \lceil y \rceil) \leqslant 1 \quad \text{and} \quad -1 \leqslant \lfloor x-y \rfloor - (\lfloor x \rfloor - \lfloor y \rfloor) \leqslant 0. \tag{5.4}$$

Hence, by (4.36), Theorem 3.6, and $-\lceil x \rceil = \lfloor -x \rfloor$ we have

$$\begin{aligned}
&\left| \#_{[m,n]}(u_0,u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0)) \right| \\
&= \left| \lceil (\theta_1(n) - \theta_0(n))/\pi \rceil - (\lceil \theta_1(n)/\pi \rceil - \lceil \theta_0(n)/\pi \rceil) \right. \\
&\quad \left. + \lfloor (\theta_0(m) - \theta_1(m))/\pi \rfloor - (\lfloor \theta_0(m)/\pi \rfloor - \lfloor \theta_1(m)/\pi \rfloor) \right| \leqslant 1.
\end{aligned} \tag{5.5}$$

Moreover, by Lemma 4.8 and Theorem 3.6 we have

$$\begin{aligned}
&\#_{(m,n]}(u_0,u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0)) \\
&= \lceil \Delta(n)/\pi \rceil - (\lceil \theta_1(n)/\pi \rceil - \lceil \theta_0(n)/\pi \rceil) \\
&\quad - (\lfloor \Delta(m)/\pi \rfloor - (\lfloor \theta_1(m)/\pi \rfloor - \lfloor \theta_0(m)/\pi \rfloor)) - 1
\end{aligned} \tag{5.6}$$

and

$$\begin{aligned}
&\#_{[m,n)}(u_0,u_1) - (\#_{(m,n)}(u_1) - \#_{(m,n)}(u_0)) \\
&= 1 + \lfloor (\theta_0(m) - \theta_1(m))/\pi \rfloor - (\lfloor \theta_0(m)/\pi \rfloor - \lfloor \theta_1(m)/\pi \rfloor) \\
&\quad - (\lceil (\theta_0(n) - \theta_1(n))/\pi \rceil - (\lceil \theta_0(n)/\pi \rceil - \lceil \theta_1(n)/\pi \rceil)). \quad □
\end{aligned} \tag{5.7}$$

THEOREM 5.4. (Triangle Inequality for Wronskians) *We have*

$$|\#_{[m,n]}(u_0,u_2) - (\#_{[m,n]}(u_0,u_1) + \#_{[m,n]}(u_1,u_2))| \leqslant 1, \tag{5.8}$$

where $\#_{[m,n]}$ *can be replaced by* $\#_{(m,n]}$.

Proof. Abbreviate $\Delta_{i,j} = \Delta_{u_i,u_j}$, then $\Delta_{0,1} + \Delta_{1,2} = \Delta_{0,2}$ holds.
By (4.36) we have $\#_{[m,n]}(u_0,u_2) = \lceil \Delta_{0,2}(n)/\pi \rceil - \lceil \Delta_{0,2}(m)/\pi \rceil$ and hence

$$\#_{[m,n]}(u_0,u_1) + \#_{[m,n]}(u_1,u_2) \tag{5.9}$$
$$\leqslant \lceil \Delta_{0,2}(n)/\pi \rceil + 1 - \lceil \Delta_{0,2}(m)/\pi \rceil = \#_{[m,n]}(u_0,u_2) + 1$$

and

$$\#_{[m,n]}(u_0,u_1) + \#_{[m,n]}(u_1,u_2) \tag{5.10}$$
$$\geqslant \lceil \Delta_{0,2}(n)/\pi \rceil - (\lceil \Delta_{0,2}(m)/\pi \rceil + 1) = \#_{[m,n]}(u_0,u_2) - 1$$

holds by $\lceil x+y \rceil \leqslant \lceil x \rceil + \lceil y \rceil \leqslant \lceil x+y \rceil + 1$ for all $x,y \in \mathbb{R}$. Further, by Lemma 4.8 and $\lfloor x+y \rfloor - 1 \leqslant \lfloor x \rfloor + \lfloor y \rfloor \leqslant \lfloor x+y \rfloor$ we have

$$\#_{(m,n]}(u_0,u_1) + \#_{(m,n]}(u_1,u_2) \tag{5.11}$$
$$\leqslant \lceil \Delta_{0,2}(n)/\pi \rceil - \lfloor \Delta_{0,2}(m)/\pi \rfloor = \#_{(m,n]}(u_0,u_2) + 1$$

and $\#_{(m,n]}(u_0,u_2) \leqslant \#_{(m,n]}(u_0,u_1) + \#_{(m,n]}(u_1,u_2) + 1$. $\square$

THEOREM 5.5. (Comparison Theorem for Wronskians II) *If either*

(A) $W_j(u_0,u_1)u_0(j+1)u_1(j+1) \leqslant 0$ *and* $W_j(u_1,u_2)u_1(j+1)u_2(j+1) \leqslant 0$
for all $j = 0,\ldots,N-2$ *or*

(B) $a_0 = a_1 = a_2$ *and* $b_0(j) \geqslant b_1(j) \geqslant b_2(j)$ *for all* $j = 1,\ldots N-1$

holds and 0 and $N-2$ are (positive) nodes of $W(u_0,u_1)$, then $W(u_0,u_2)$ has at least one positive node at $0,\ldots,N-2$.

Proof. In either case for all $j = 0,\ldots,N-2$ we have $\#_j(u_0,u_1) \geqslant 0$ and also $\#_j(u_1,u_2) \geqslant 0$, hence from Theorem 5.4 we conclude

$$\#_{[0,N-1]}(u_0,u_2) \geqslant \underbrace{\#_{[0,N-1]}(u_0,u_1)}_{\geqslant 2} + \underbrace{\#_{[0,N-1]}(u_1,u_2)}_{\geqslant 0} - 1. \quad \square \tag{5.12}$$

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Kerstin Ammann
Faculty of Mathematics, University of Vienna
Nordbergstrasse 15
1090 Wien, Austria
e-mail: `Kerstin.Ammann@univie.ac.at`
`http://www.mat.univie.ac.at/~kerstin`

Operators and Matrices
`www.ele-math.com`
`oam@ele-math.com`