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ON DIFFERENT CONCEPTS OF CLOSEDNESS OF LINEAR OPERATORS

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Abstract. The purpose of this paper is to introduce, by means of the extensions of almost closed operators, the notion of almost closable linear operator acting in a Hilbert or Banach space. This class of operators is strictly included in the class of all unbounded linear operators, it contains the set of all closable operators and that of all almost closed operators and is invariant under finite and countable sums, finite products, limits and integrals. We also present some fundamental properties relative to almost closability and we define a locally convex Hausdorff topology in the set of all almost closable operators.

1. Introduction

Let H be an infinite dimensional complex Hilbert space. $\langle .,. \rangle_H$ denote the inner product on H and $\|x\|_H = \sqrt{\langle x,x \rangle_H}$ the associated norm. If M is a subset of H, $M^\perp$ is the orthogonal complement of M with respect to the inner product of H. For a linear operator A defined on H, the domain, null space and the range space of A are denoted respectively by $D(A)$, $N(A)$ and $R(A)$. A^* is the adjoint operator of A. The graph $G(A)$ of A is the subset of $H \times H$ defined by $G(A) = \{(x,Ax) \ ; \ x \in D(A)\}$. The operator A is said to be closed if its graph $G(A)$ is a closed subspace of $H \times H$. We denote by $B(H,K)$ the Banach space of bounded linear operators from H to another Hilbert space K and we put $B(H,H) = B(H)$. Let $C(H)$ denote the set of all closed, densely defined linear operators in H. If $A \in C(H)$, then A^* is closed. In particular, selfadjoint operators are closed. We write $A \subset B$ when B is an extension of A, in the sense that $D(A) \subset D(B)$ and the restriction of B to $D(A)$ agrees with A. In particular, $A \subset B$ is equivalent to $G(A) \subset G(B)$. It is interesting to recall in the beginning the well-known procedure of making a closed linear operator A bounded on H by renorming it domain with the graph norm $\|x\|_{G(A)} = (\|x\|_H^2 + \|Ax\|_H^2)^{1/2}$ defined by the graph inner product $\langle x,y \rangle_{G(A)} = \langle x,y \rangle_H + \langle Ax,Ay \rangle_H$ for all $x,y \in D(A)$. Indeed, A is closed if and only if $(D(A), \langle .,. \rangle_{G(A)})$ is a Hilbert space. Note that by closed graph theorem we have $B(H) \subset C(H)$.

The notion of closability is an important generalisation of that of closedness, in that closable operators may, in many respects, be treated similarly to closed ones. A is closable if the closure $\overline{G(A)}$ of $G(A)$ in $H \times H$ is the graph of a linear operator

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$\overline{A}$. It follows immediately that the closure $\overline{A}$ of A is the smallest closed extension of A. Every closed operator is closable on H, but the converse is not true. If A is not closable then the closure $\overline{G(A)}$ is not a graph (it may contain some $(0,y)$, $y \neq 0$), and A has no closed extension. Let us remind the reader that a densely defined linear operator A is closable if and only if A^* is densely defined, in which case $\overline{A} = A^{**}$. In the mathematical literature there are many typical examples of closed operators, and examples of operators with nondensely defined adjoint. Thus, a linear operator A loses its closability as soon as $D(A^*)$ being rather small, and it can happen that $D(A^*) = \{0\}$. Indeed, we have for example the following result:

LEMMA 1. *If A is an arbitrary linear operator and $N(A)$ is dense, then $D(A^*) = N(A^*)$. If $R(A)$ is also dense, then $D(A^*) = \{0\}$.*

The natural operations sum, product and limits are well defined on $B(H)$, however, one has to be careful with those manipulations when dealing with unbounded operators, this is essentially due to the domains: $D(A+B) = D(A) \cap D(B)$ or $D(AB) = B^{-1}(D(A))$ can be trivial, that is, equals to zero set. On the other hand if $A, B \in C(H)$, then $A+B$ and AB are not generally closed on H even if strong conditions are imposed on A and B [3]. To avoid the problems with closures of sums, products and limits, some authors have tried to weaken the closedness of operators ([5], [6], [23]), other authors gave sufficient topological conditions on the graph of two operators of $C(H)$ so that their sums and products remain in $C(H)$ ([21], [22], [3]). Nevertheless, the sums, products and limits of closed operators are necessarely almost closed operators or quotient of bounded linear operators, these notions will be defined in the next paragraph (for more details, the interested reader can consult [16], [23]). The class of almost closed operators contains $C(H)$, but there exists almost closed operators which are not closable and closable operators which are not almost closed [23].

On one hand, $C(H)$ equiped with the metric g called "gap" metric becomes a non complete metric space

$$g(A,B) = \left\| P_{G(A)} - P_{G(B)} \right\|_{B(H \times H)} , A, B \in C(H)$$

$P_{G(A)}$ and $P_{G(B)}$ denote respectively the orthogonal projection in $H \times H$ on the graph $G(A)$ of A and the graph $G(B)$ of B. Fernandez Miranda and Labrousse [9] have shown that the completion of $C(H)$ for the topology defined by the metric g intersects the set of closed linear relations on H (ie the set of closed linear subspaces of $H \times H$ of infinite dimension and codimension), where closed linear operators are identified as linear relations via their graphs. The topology induced by g on $C(H)$ has good properties concerning the stability of the index of operators with index, but the results are not as good as regards the stability of the spectrum of an operator. For application, it is shown that it is necessary to have other metrics on $C(H)$, which are more practical. Indeed, to be able to refine the completion of $C(H)$ from g, several metrics strictly stronger than the gap metric were defined on the space of almost closed operators on H by using in particular the quotient representation of bounded operators [15]. It becomes interesting to compare the completion of $C(H)$ for these metrics with the space of almost closed operators on H. In other words, is it possible to determine a metric on

$C(H)$ such as the completion of $C(H)$ with respect to this metric coincide with the space of almost closed operaors on H ?.

On the other hand, some boundary and initial value problems in the partial differential equations theory lead to an abstract Cauchy problem of type $\frac{du}{dt} = Au(t)$, $t \in [0,T[$, $T \leqslant \infty$, $u(0) = x$ in function spaces on $[0,T[\times\Omega$ $(\Omega \subseteq \mathbb{R}^n)$, $u(t) = u(t,.) \in E$ a complex Banach space, such that the operator A with domain $D(A)$ is not closable on E (see e.g. [4] for Feedback control equations and [2, 8, 12, 20]). It is known that a lot of information on the abstract Cauchy problem is contained in E and in particular in the domain of A, so it means that the change of the structure of E or that of $D(A)$ can strongly influence the character of the study of these problems. It is rather suitable to consider the graph of A in an auxiliary Banach space $E_B \times E$, where $A \subset B$, B is bounded from the Banach space E_B into E and $D(A) \subset E_B \hookrightarrow E$ (E_B is continuously embedded in E).

Another way of treating these questions consists in defining a space of unbounded linear operators on H (or E) containing the set of all closable linear operators on H. We introduce in this paper the notion of almost closable linear operators on Hilbert and Banach spaces by almost closed extensions. This new class of operators is strictly included in the set of all unbounded linear operators, it is closed under addition, composition, inversion, restriction, limits and integrals. We give some interesting characterizations of these operators and we represent them by means of the products of linear operators. Finally we introduce a locally convex Hausdorff topology in the set of all almost closable operators and investigate the topological structure by using the decomposition method of almost closable operators. This topology on $C(H)$ is strictly stronger than that induced from the gap metric g and it coincide with the uniform operator topology on $B(H)$.

Our paper is organized as follows:

In section 2, we recall the concept of almost closed linear operators.

In section 3, we give some preliminary results on almost closed extensions in which our investigation will be done. Afterward, we define and characterize almost closable linear operators on a Hilbert space.

In section 4, we generalize the concept of almost closable linear operators to Banach spaces. Here it is shown that the class of almost closable linear operators with respect to a fixed Banach space is invariant under restriction, finite compositions, finite and infinite sums, limits and integrals.

In section 5, we introduce a locally convex Hausdorff topology on the set of almost closable linear operators. The sum and the product are particularly continuous mappings on the set of almost closable operators.

2. Almost closed operators

In this section, we mention the basic results about almost closed operators. These operators were studied by several authors, they were often called by different names. Dixmier uses in [6] the term “Julia operators”, Agmon and Nirenberg introduce them in [1] under the name “relatively closed operators”, they are called respectively “operator ranges" by Fillmore and Williams in [10], "paracomplete operators" by Labrousse in [17] and "semiclosed operators" by Foias [11], Caradus [5] and Kaufman [16]. Kaufman uses also the term "quotient of bounded operators", finally Messirdi and al. named them "almost closed operators" in [23].

DEFINITION 1. A linear operator A with domain $D(A)$ is called almost closed on H if there exists an inner product $[.,.]_A$ on $D(A)$ such that the auxiliary space $H_A = (D(A),[.,.]_A)$ is complete and that the inclusion mapping $i : H_A \to H$ is continuous with respect to the norm $\|.\|_A$ induced by $[.,.]_A$ (we write $H_A \hookrightarrow H$) and $A \in B(H_A,H)$.

Obviously, if H_A is a Hilbert space, then A is almost closed if and only if the graph $G(A)$ of A is closed in $H_A \times H$, thus if $(x_n)_n$ converges to x in H_A and $(Ax_n)_n$ converges to y in H, then $x \in D(A)$ and $y = Ax$.

An almost closed operator can be also characterized by means of the almost closed subspaces or operator ranges. Let M be a subspace of H, M is said to be an almost closed subspace in H, if there exists an inner product $\langle .,. \rangle_M$ on M such that M is complete with respect to $\langle .,. \rangle_M$ and that $(M, \langle .,. \rangle_M)$ is continuously embedded in $(H, \langle .,. \rangle_H)$. Fillmore and Williams established the relationship between almost closed subspaces and operator ranges, they showed in [10] that M is almost closed in H if and only if M is the range of a member of $B(H)$. Hence we have the following result:

THEOREM 1. *Let $A\colon D(A) \to H$ be a linear operator with a domain $D(A) \subseteq H$. Then the following conditions are equivalent.*

1) A is almost closed operator on H.

2) $D(A)$ is an almost closed subspace of H such that A is bounded with respect to some Hilbert space norm on $D(A)$.

3) The graph $G(A)$ of A is an almost closed subspace in $H \times H$.

Let $AC(H)$ be the set of all almost closed operators on H. It is known that $AC(H)$ is the smallest family containing the closed operators on H that is closed under products, limits, and at most countable sums. Each almost closed operator on H with closed domain is bounded, this is the almost closed theorem. Furthermore, if $A \in AC(H)$, then $N(A)$ is a closed linear subspace of the auxiliary Hilbert space H_A; in particular, if $R(A) = H$ and A is invertible then $A^{-1} \in B(H)$.

Nevertheless, Messirdi and al. showed in [23], by means of typical examples, that there is no link between closable operators and those almost closed. Effectively, there exists almost closed operators which are not closable and closable operators which are not almost closed.

By using the following lemma of Mac Nearney [19],

LEMMA 2. ([19]) *Let $A \in AC(H)$. Then, there exist a unique linear bounded and positive operator B on H and a inner product $\langle .,. \rangle'$ on the domain $D(A)$ of A for which $(D(A), \langle .,. \rangle')$ is complete and continuously included in H, such that $R(B) = D(A)$ and $\langle x,y \rangle' = \langle B^{-1}x, B^{-1}y \rangle_H$, for all $x,y \in D(A)$.*

Kaufman obtained in [15] a quotient representation of almost closed operators.

THEOREM 2. *Let A be a linear operator in H with domain $D(A)$. Then $A \in AC(H)$ if and only if there is a member (B,C) of $B(H) \times B(H)$ such that A is the quotient C/B on H (in other words, under the kernel condition $N(B) \subset N(C)$, C/B is the unbounded linear operator defined on $R(B)$ by the mapping $Bx \to Cx$, $x \in H$).*

Note that $AC(H)$ is not a vector space because the uniqueness of the zero element as quotient operator does not hold. Almost closedness is invariant under sums, products and limits, and this characteristic persists in other useful ways as follows (see [5], [16], and [23]):

THEOREM 3. *Let $A \in AC(H)$. Then,*

1) $R(A)$ is almost closed subspace of H.

2) The image and the inverse image under A of every almost closed subspace of H is an almost closed subspace of H.

3) The restriction of A to every almost closed subspace of H included in $D(A)$ is an almost closed operator on H.

4) $(A+B) \in AC(H)$ if and only if B is A-relatively bounded with A-bound smaller than 1.

5) If A is invertible its inverse A^{-1} is in $AC(H)$.

6) There is $B,C \in B(H)$ such that $R(B) = R(C) = D(A)$ and $A = B^{-1} + C^{-1}$

7) If B is a closed operator on H with $D(B) \subset D(A)$, $B + \lambda A$ is an analytic family of closed operators on H for sufficiently small complex number λ.

We recall that the operator B is A-relatively bounded if $D(A) \subset D(B)$ and there are nonnegative constants a and b so that

$$\|Bx\|_H \leqslant a\,\|Ax\|_H + b\,\|x\|_H \quad \text{for all } x \in D(A) \tag{2.1}$$

We call the greatest lower bound a_0 of all possible a for which (2.1) holds the A-bound of B. We say that B is directly A-bounded if (2.1) is satisfied with $b = 0$.

To end this section we recall the topological structure defined on $AC(H)$ by Caradus [5], we examine in particular this structure on the space $C(H)$. We know that if $A \in AC(H)$ then A is represented by a quotient B/A_+ of bounded operators on H where A_+ is unique and positive on H and $\|A\|_{B(H_A,H)} = \|B\|_{B(H)}$, H_A is the auxiliary space of A.

Consider α be the correpondence between almost closed operators A and the associated positive bounded operators A_+. Such operator $A \in AC(H)$ is uniquely represented up to α by a quotient B/A_+, so that we denote $A \overset{\alpha}{=} B/A_+$. Let's define a $\delta-$neighborhood for $\delta > 0$ of an almost closed operator $A \overset{\alpha}{=} B/A_+$ on H by:

$$\begin{aligned}\mathscr{V}(A,\alpha,\delta) &= \{T \in AC(H)\,;\, T \overset{\alpha}{=} C/A_+\,,\ \|B-C\|_{B(H)} < \delta\} \\ &= \{T \in AC(H)\,;\, D(A) = D(T)\,,\ \|A-T\|_{B(H_A,H)} < \delta\}\end{aligned} \tag{2.2}$$

and consider the topology τ induced from the neighborhood system as above. τ is a locally convex Hausdorff topology in the set $AC(H)$ and is independent from the correspondence α. In fact, $AC(H)$ becomes metrizable by means of the metric

$$m(A,T) = \begin{cases} 1 \text{ if } D(A) \neq D(T) \\ \frac{\|A-T\|_{B(H_A,H)}}{1+\|A-T\|_{B(H_A,H)}} \text{ if } D(A) = D(T) \end{cases} \tag{2.3}$$

A sequence $(A_n)_n$ converges to A in $AC(H)$ for the metric m if and only if the domain of A_n coincides with the domain of A for sufficiently large n and

$$\lim_{n\to\infty} \|A_n - A\|_{B(H_A,H)} = 0,$$

H_A is the auxiliary space concerning the common domain.

THEOREM 4. ([7]) *In $C(H)$, the topology induced from the metric m is strictly stronger than that induced from the gap metric g. $B(H)$ is a connected component of $AC(H)$ and $C(H)$ is open in $AC(H)$.*

In particular, the addition and the scalar multiplication in the set $AC(H)$ are continuous, and that the multiplication from the left side is continuous.

Let $A \in C(H)$. In [15] Kaufman showed that $A = \Gamma(B) = B/(I-B^*B)^{1/2}$ with a unique positive pure contraction $B \in C_0(H) = \{S \in B(H) \colon \|S\| \leqslant 1 \text{ and } N(I-B^*B) = \{0\}\}$, where Γ is a reversible function from $C_0(H)$ onto $C(H)$ with inverse function defined by $\Gamma^{-1}(A) = A(I+A^*A)^{-1/2}$. The related convergence in the space $C(H)$, called quotient-convergence, is defined as follows: $A_n = B_n/(I-B_n^*B_n)^{1/2}$ converges to $A = B/(I-B^*B)^{1/2}$ if B_n converges to B in $B(H)$ where $B_n, B \in C_0(H)$.

The orthogonal projection $P_{G(A)} \colon H \times H \to H \times H$ on the graph $G(A)$ of the quotient operator $A = B/(I-B^*B)^{1/2}$ can be described through the following matrix:

$$P_{G(A)} = \begin{pmatrix} (I-B^*B) & (I-B^*B)^{1/2}B^* \\ B(I-B^*B)^{1/2} & BB^* \end{pmatrix} \tag{2.4}$$

Consequently, if B_n converges to B in $B(H)$, then we have

$$(I-B_n^*B_n)^{1/2}B_n^* \to (I-B^*B)^{1/2}B^*,$$

and this assures the convergence $P_{G(A_n)} \to P_{G(A)}$ as $n \to \infty$, or $\lim_{n\to\infty} g(A,A_n) = 0$.

On the other hand, let us remind the reader that if $A \in C(H)$ then $R_A = (1+A^*A)^{-1}$ exists as a bounded self-adjoint positive operator with domain $D(R_A) = H$, $AR_Ax = R_{A^*}Ax$ for all $x \in D(A)$, $\|R_A\|_{B(H)} \leqslant 1$ and $\|AR_A\|_{B(H)} \leqslant 1$. $P_{G(A)}$ is also represented by the matrix (see Proposition 1.3, [18]):

$$P_{G(A)} = \begin{pmatrix} R_A & A^*R_{A^*} \\ AR_A & I-R_{A^*} \end{pmatrix} \tag{2.5}$$

Let us define the operators A_n on the space l^2 of square-summable complex sequences by:

$$[A_n(x)]_k = \begin{cases} kx_k & \text{if } k < n \\ -kx_k & \text{if } k \geqslant n \end{cases} \tag{2.6}$$

and

$$[A(x)]_k = kx_k \,,\, k \in \mathbb{N} \tag{2.7}$$

on the natural domain $D(A) = \{x \in l^2 : \sum_{k=0}^{\infty} k^2 |x_k|^2 < +\infty\}$. Then, $A, A_n \in C(l^2)$, $A^* = A$, $A_n^* = A_n$ and $[R_A(x)]_k = [R_{A^*}(x)]_k = [R_{A_n}(x)]_k = [R_{A_n^*}(x)]_k = (1+k^2)^{-1} x_k$ for all $n \in \mathbb{N}$.

$$[A_n R_{A_n}(x)]_k = [A_n^* R_{A_n^*}(x)]_k = \begin{cases} k(1+k^2)^{-1} x_k & \text{if } k < n \\ -k(1+k^2)^{-1} x_k & \text{if } k \geqslant n \end{cases}$$

We see from (2.5) that $g(A, A_n) = \|AR_A - A_n R_{A_n}\|_{B(H)} \leqslant 2n(1+n^2)^{-1}$. Thus, the sequence $(A_n)_n$ converges to A for g but does not converge to the same limit for the quotient-convergence. Indeed, $A_n = \Gamma(B_n)$ and $A = \Gamma(B)$, where $B_n = \Gamma^{-1}(A_n) = A_n \sqrt{R_{A_n}}$ and $B = \Gamma^{-1}(A) = A\sqrt{R_A}$ are contractions corresponding to A_n and A respectively. Let us notice that $\|B_n\|_{B(H)} \leqslant 1$, $\|B\|_{B(H)} \leqslant 1$, $N(I - B_n^* B_n) = N(I - A_n^* \sqrt{R_{A_n^*}} A_n \sqrt{R_{A_n}}) = N(R_{A_n}) = \{0\}$ and $N(I - B^* B) = N(I - A^* \sqrt{R_{A^*}} A \sqrt{R_A}) = N(R_A) = \{0\}$. Thus, $B, B_n \in C_0(l^2)$, for all $n \in \mathbb{N}$.

$$[B_n(x)]_k = [B_n^*(x)]_k = \begin{cases} k(1+k^2)^{-1/2} x_k & \text{if } k < n \\ -k(1+k^2)^{-1/2} x_k & \text{if } k \geqslant n \end{cases}$$

and

$$[B(x)]_k = k(1+k^2)^{-1/2} x_k \,,\, k \in \mathbb{N}$$

As, $\|B_n - B\|_{B(H)} = \left\|A_n\sqrt{R_{A_n}} - A\sqrt{R_A}\right\|_{B(H)} \geqslant 2n(1+n^2)^{-1/2} \to 2$, then $(A_n)_n$ does not converge to the same limit for quotient-convergence.

We have then shown the following fundamental result:

THEOREM 5. *The topology induced on $C(H)$ by quotient-convergence is strictly stronger than the topolgy induced from the gap metric g.*

3. Almost closable operators

Let A be an unbounded operator on H. Is it possible to extend A to an almost closed linear operator on H ? This question was essentially raised by Caradus [5] when the Hilbert space H is supposed separable, it was afterward studied by Kaufman [16] in a more general situation.

THEOREM 6. ([5]) *If A is almost closed on a separable Hilbert space H, then A has a densely defined almost closed extension.*

If H is not necessarily separable Hilbert space, Kaufman claimed that with adequate assumptions on A the above question is affirmative.

THEOREM 7. ([16]) *Let A be an operator on H. Then the following conditions are equivalent.*

1) A has an almost closed extension on H.

2) A is directly B^{-1}-bounded where $B \in B(H)$.
3) A is directly B-bounded where B is a selfadjoint operator on H.
4) A is directly B-bounded where B is a closed operator on H.
5) A is directly B-bounded where $B \in AC(H)$.

Technically, Kaufman proved that given any countable linearly independent subset $\mathscr{M}$ of H and any linear operator A defined from the subspace $span\mathscr{M}$ of H, spanned by $\mathscr{M}$, to H there is an extension of A which is in $AC(H)$. This last result is a consequence of the following more general theorem of existence of almost closed extensions of linear operators.

THEOREM 8. ([16]) *Let A be an unbounded linear operator on H with domain $D(A)$. Suppose that there exists a monotonic collection $\mathscr{M}$ of closed subspaces of H such that $\mathscr{M}$ covers $D(A)$ and for each M in $\mathscr{M}$, the restriction of A to $M \cap D(A)$ is bounded. Then A has an almost closed extension on H.*

Let us observe now that the linear span of a countable linearly independent subset $(e_k)_{k\in\mathbb{N}}$ of H is the union of monotonically increasing sequence of the finite dimensional spaces $M_k = span\{e_j : \ j \leqslant k\}$, $k \in \mathbb{N}$. As immediate consequence of the Theorem 8, we deduct that each linear operator from the linear span of a countable linearly independent subset of H into H has an extension in $AC(H)$.

REMARK 1. It is an interesting problem to construct an operator on H with no almost closed extension on H. We shall treat this question in the following section by using the well-known fact that there is an unbounded linear functional defined on each infinite dimensional Hilbert space. The construction of such a linear functional may use a Hamel basis for the infinite dimensional Hilbert space. The existence of Hamel bases uses a Zorn's Lemma argument.

The idea developed now consists in defining, by means of the extensions of almost closed operators, another class of unbounded linear operators on H called almost closable operators, containing the class $AC(H)$ and the set of all closable operators on H. This new class of operators is closed under restriction, addition and composition. We give some interesting characterizations of almost closable operators and we represent them by means of the product of closable operators.

DEFINITION 2. Let $A : D(A) \longrightarrow H$ be a linear operator with a domain $D(A) \subseteq H$. A is said to be almost closable operator on H if A admits an almost closed extension on H.

Thus, A is almost closable on H if and only if there exists $B \in AC(H)$ such that $A \subset B$ or equivalently $G(A) \subset G(B)$. As $G(B)$ is almost closed in $H \times H$, we can say in this case that $G(A)$ is almost closable subspace of $H \times H$. In other words, A is almost closable on H if and only if $A \subset B$ such that there exists an inner product $[.,.]_B$ on $D(B)$ for which the auxiliary space $H_B = (D(B), [.,.]_B)$ is complete, $H_B \hookrightarrow H$ and $B \in B(H_B, H)$. In particular if $(D(A), [.,.]_B)$ is complete then A is almost closed on

H. Consequently, if A is almost closable with an extension B in $AC(H)$ and auxiliary space H_B such that $(D(A), [.,.]_B)$ is complete in H_B, then for every sequence $(x_n)_{n\in\mathbb{N}}$ of elements in $D(A)$ such that $x_n \to x$ in H_B and $Ax_n \to y$ in H_B, we have $x \in D(A)$ and $Ax = y$. As the topology induced by H_B is finer than that induced by that of H, one can see that every set of H_B closed for the topology of H is also closed in H_B for the topology of H_B and $\overline{G(A)}^{H_B\times H} \subset \overline{G(A)}^{H\times H}$. Thus, almost closability consists of refining the closure of the graph $G(A)$ of A by renorming $D(A)$ using the Hilbertian structure of the auxiliary space. This change of topology makes it possible to extract from $\overline{G(A)}^{H\times H}$ all singular vectors $(0,v)$, $v \in H$ and $v \neq 0$.

Almost closable operators are unbounded operators A on H on which one imposes a topological condition that is finer than that in H, inspired from almost closed operators. This condition allows these operators to have bounded extensions on intermediate Hilbert spaces between the domain $D(A)$ of A and H.

Since every closed operator is almost closed, it is clear that all closable operators are almost closable but there exists almost closable operators A on H which are not closable if, for example, the graph $G(A)$ of A is dense in $H \oplus H$. As an example of almost closable operator but not closable, we can consider a separable infinite dimensional Hilbert space with an orthonormal basis $(e_n)_{n\in\mathbb{N}}$ and

$$D = \{x \in H \; ; \; \sum_{n=1}^{\infty} n^4 \, |\langle x, e_n \rangle|^2 < +\infty\} \, , y = \sum_{n=2}^{\infty} n^{-1} e_n$$

Define the operators B in $B(H)$ by $Bx = \langle x, y \rangle \, y$; and A with the domain D, which is dense in H, by

$$Ax = \sum_{n=2}^{\infty} n^2 \, \langle x, e_n \rangle \, e_n \qquad (x \in D)$$

Then A is a closed densely defined linear operator in H, BA isn't closable (cf. Problem 2.8.43 of [13]) but BA is almost closable in H. As mentioned above, we can also construct closable (then almost closable) linear operators which are not almost closed [23].

We denote by $ACl(H)$ (resp. $Cl(H)$) the set of all almost closable (resp. the set of all closable) operators on H. We have by definition the following inclusions:

$$B(H) \subset C(H) \subset AC(H) \subset ACl(H) \text{ and } B(H) \subset C(H) \subset Cl(H) \subset ACl(H)$$

An important class of almost closable operators are sums and products of closable operators on a Hilbert space. In general, the sum $A+B$ and the product AB of closable operators need not be closable in H. However, $A+B$ and AB are almost closable when $D(A)\cap D(B)$ and $D(AB)$ are not trivial, if we take the auxiliary spaces $H_{\overline{A}+\overline{B}} = (D(\overline{A})\cap D(\overline{B}), [.,.]_{\overline{A}+\overline{B}})$ and $H_{\overline{A}\overline{B}} = (D(\overline{A}\overline{B}), [.,.]_{\overline{A}\overline{B}})$, where

$$[x,y]_{\overline{A}+\overline{B}} = \langle x,y \rangle_{G(\overline{A})} + \langle x,y \rangle_{G(\overline{B})} \ , x,y \in D(\overline{A})\cap D(\overline{B})$$

$$[x,y]_{\overline{A}\overline{B}} = \langle x,y \rangle_{G(\overline{B})} + \left\langle \overline{B}x, \overline{B}y \right\rangle_{G(\overline{A})} \ , x,y \in D(\overline{A}\overline{B}) \qquad (3.1)$$

Clearly, $H_{\overline{A}+\overline{B}}$ and $H_{\overline{AB}}$ are Hilbert spaces, $H_{\overline{A}+\overline{B}} \hookrightarrow H$, $H_{\overline{AB}} \hookrightarrow H$ and $\overline{A}+\overline{B} \in B(H_{\overline{A}+\overline{B}}, H)$, $\overline{AB} \in B(H_{\overline{AB}}, H)$. Thus, $\overline{A}+\overline{B}$ and $\overline{AB}$ are respectively almost closed extensions of $A+B$ and AB on H, what shows that $A+B$, $AB \in ACl(H)$.

There are also unbounded linear operators which are not almost closable, we so show the following general result:

THEOREM 9. *On every infinite dimensional complex Hilbert space we can define linear operators who are not almost closable.*

Proof. Let H be an infinite dimensional complex Hilbert space, we can always find an unbounded linear operator A on H with domain $D(A) = H$. Indeed, it is well-known that there exists an unbounded linear functional f defined on H, that is, there is a linear functional $f : H \longrightarrow \mathbb{C}$ such that f is unbounded. Let $\omega \in H$, $\omega \neq 0$, and let $Ax = f(x)\omega$ for each x in H. It is clear that A is an unbounded linear operator and $D(A) = H$. Now, if A is an unbounded linear operator defined on H with domain $D(A) = H$, then A cannot be extended to an almost closed operator on H. Indeed, if A' is an almost closed extension of A then $D(A) = D(A') = H$ and A' is bounded from $H_{A'}$ to H where $H_{A'}$ is the auxiliary Hilbert space of A'. Thus, $H_{A'}$ and H are isometrically isomorphic Hilbert spaces because the injection from $H_{A'}$ to H is necessarily bicontinuous by the inverse mapping theorem and then $A \in B(H)$ which is a contradiction. □

Some fundamental properties related to almost closability are an immediate consequence of definition 2 which are summarized as follows:

THEOREM 10. *Let $A \in ACl(H)$ with the associated almost closed extension B of A and the auxiliary Hilbert space H_B. Then,*

1. $\overline{N(A)}^{H_B} \subset N(B) \cap \overline{N(A)}$ *where* $\overline{N(A)}^{H_B}$ *is the closure of $N(A)$ with respect to the topology of H_B.*
2. *If $D(A) = H$, then $A = B \in B(H)$.*
3. *If B is invertible then $A^{-1} \in ACl(H)$. Furthermore, if $R(A) = H$ then $A^{-1} \in B(H)$.*
4. *There exists $A_0 \in B(H_B, H)$ and C closable on H_B such that $A = A_0 C$ on $D(A)$.*
5. *If H_0 is a Hilbert space such that $D(A) \subseteq H_0 \hookrightarrow H_B$, then A is almost closable from H_0 to H.*

Proof. 1) The assertion is true since B is bounded from H_B to H.

2) Indeed, the topologies induced on H by the norms of H and H_B are equivalent, then $A \in B(H)$ since $B \in B(H_B, H)$.

3) Since B is an invertible extension of A, then A is also invertible and $D(B^{-1}) = R(B)$ is a Hilbert space, denoted H_{-1}, for the inner product

$$[y,z]_{-1} = \langle y,z \rangle_H + [B^{-1}y, B^{-1}z]_B \tag{3.2}$$

where $[.,.]_B$ is the inner product of H_B. B^{-1} is bounded from H_{-1} to H and thus is an almost closed extension of A^{-1} which means that A^{-1} is almost closable on H.

4) We adopt here the same idea used in ([16], p. 69) by means of the Lemma 2. Since B is bounded from H_B to H, let

$$\langle x,y\rangle' = [x,y]_B + \langle Bx,By\rangle_H \ , x,y \in D(B) \tag{3.3}$$

$H'_B = (D(B),\langle .,.\rangle')$ is a Hilbert space and we have for all $x \in D(B)$,

$$\max([x,x]_B; \|Bx\|_H^2) \leqslant \langle x,x\rangle',$$

thus $H'_B \hookrightarrow H_B \hookrightarrow H$ and $B \in B(H'_B,H)$. It follows that there exists a nonnegative operator $B_0 \in B(H_B)$ such that $R(B_0) = D(B)$ and for all $x,y \in D(B)$,

$$\langle x,y\rangle' = [B_0^{-1}x, B_0^{-1}y]_B \tag{3.4}$$

Let $C = B_0^{-1}$ (C is positive and bounded in H_B), and $A_0 = BB_0$. Then $B = A_0C$. For all x in H_B we have $\|A_0x\|_H^2 = \|BB_0x\|_H^2 \leqslant \langle B_0x,B_0x\rangle' \leqslant [x,x]_B$. Thus, $A_0 \in B(H_B,H)$, $A = B_{|D(A)} = A_0C_{|D(A)} = A_0(C_{|D(A)})$ where the restriction $C_{|D(A)}$ is closable from $D(A)$ to H_B with closure equal to C.

5) Let i the identity operator from H_0 to H_B. Since $Bi = B_{|H_0}$ is bounded on H_0, then $A \in ACl(H_0)$. □

REMARK 2. By virtue of 4) Theorem 10, we can represent every almost closable operator by a quotient of unbounded linear operators and conversely. We shall treat this question in a forthcoming paper.

Our first main result consists of verifying that sums and products of almost closable operators are also almost closable operators.

THEOREM 11. *Let $A,B \in ACl(H)$ such that $D(A)\cap D(B)$ and $D(AB)$ are not trivial sets, then $A+B$, $AB \in ACl(H)$.*

Proof. If A' and B' are respectively the almost closed extensions of A and B on H, using the fact that A' and B' are bounded from the corresponding auxiliary Hilbert spaces $H_{A'} = (D(A'),[.,.]_{A'})$ and $H_{B'} = (D(B'),[.,.]_{B'})$, we take $H_{A'+B'} = (D(A')\cap D(B'),[.,.]_{A'+B'})$ and $H_{A'B'} = (D(A'B'),[.,.]_{A'B'})$ where

$$[x,y]_{A'+B'} = [x,y]_{A'} + [x,y]_{B'} + \langle A'x,A'y\rangle_H + \langle B'x,B'y\rangle_H,\ x,y \in D(A')\cap D(B'),$$

$$[x,y]_{A'B'} = [x,y]_{B'} + [B'x,B'y]_{A'} + \langle A'B'x,A'B'y\rangle_H\ x,y \in D(A'B') \tag{3.5}$$

Clearly $H_{A'+B'}$ and $H_{A'B'}$ are Hilbert spaces continuously embedded in H. $A'+B'$ and $A'B'$ are respectively almost closed extensions of $A+B$ and AB (see [23]), which means that $A+B$ and AB are almost closable on H. □

4. Almost closable operators on Banach spaces

The definition of almost closable operator on an infinite dimensional complex Banach space $(E, \|.\|_E)$ is similar to that given, in the previous section, in Hilbert space case. A linear operator defined on E with domain $D(A)$ is said almost closable on E if and only if it possesses an almost closed extension B on E. B almost closed on E means that there exists a norm $\|.\|_B$ on $D(B)$ such that $E_B = (D(B), \|.\|_B)$ is a Banach space, $E_B \hookrightarrow E$ and B is bounded from E_B to E. All properties satisfied by almost closable operators on a Hilbert space stay true in the Banach space's case. However, some new results are technically valid on Banach spaces and that we have to use in certain applications [12]. We use in this section all of the previous notations with the Hilbert space H replaced by a Banach space E.

REMARK 3. Let A' and B' be the respective almost closed extensions of linear unbounded operators A and B on E such that B' is A'-relatively bounded with A'-bound smaller than 1. Then, from a result of [23], A is almost closable on E if and only if $(A+B)$ is almost closable on E. The relative boundedness of B with respect to A is not sufficient to insure this result.

We establish in what follows other main results of this paper, that the class of almost closable operators with respect to a fixed Banach space is also invariant under limits, infinite sums and integrals.

THEOREM 12. *For all* $\varepsilon > 0$, *let* $A_\varepsilon \in ACl(E)$ *with the extension* $B_\varepsilon \in AC(E)$ *and the auxiliary Banach space* $E_\varepsilon = (D(B_\varepsilon), \|.\|_{B_\varepsilon})$. *Assume that there exists a Banach space* L *such that* $L \hookrightarrow E_\varepsilon$ *for all* $\varepsilon > 0$ *and* $\sup\limits_{\varepsilon>0} \|A_\varepsilon x\|_E < +\infty$, *for all* $x \in L$. *Then,* $Ax = \lim\limits_{\varepsilon\to 0} A_\varepsilon x$ *with domain*

$$D(A) = \left\{ x \in \left(\bigcap_{\varepsilon>0} D(A_\varepsilon) \right) \cap L\colon \lim_{\varepsilon\to 0} A_\varepsilon x \text{ exists in } E \right\} \tag{4.1}$$

is almost closable linear operator on E.

Proof. Let us define $Bx = \lim\limits_{\varepsilon\to 0} B_\varepsilon x$ on

$$D(B) = \left\{ x \in \left(\bigcap_{\varepsilon>0} D(B_\varepsilon) \right) \cap L\colon \lim_{\varepsilon\to 0} B_\varepsilon x \text{ exists in } E \text{ and } \sup_{\varepsilon>0} \|B_\varepsilon x\|_E < +\infty \right\}$$

Then clearly, $\|x\|_B = \|x\|_L + \sup\limits_{\varepsilon>0} \|B_\varepsilon x\|_E$ is a norm on $D(B)$, we show that $F = (D(B), \|.\|_B)$ is complete. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in F. Then $(x_n)_{n\in\mathbb{N}}$ is Cauchy in L, E and E_ε for each $\varepsilon > 0$, and $(B_\varepsilon x_n)_{n\in\mathbb{N}}$ is Cauchy in E for all $\varepsilon > 0$. Thus, $(x_n)_{n\in\mathbb{N}}$ converges to x in L, E and E_ε for each $\varepsilon > 0$, and $(B_\varepsilon x_n)_{n\in\mathbb{N}}$ converges in E for each $\varepsilon > 0$. Since $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in F, there exists a real number M such that $\|x_n\|_B \leqslant M$ for each n in $\mathbb{N}$. Hence $\|B_\varepsilon x_n\|_E \leqslant \|x_n\|_B \leqslant M$ for each n in $\mathbb{N}$ and for each $\varepsilon > 0$. Then, since $x_n \to x$ in E_ε, $\|B_\varepsilon x\|_E = \lim_{n\to\infty} \|B_\varepsilon x_n\|_E \leqslant M$ for each $\varepsilon > 0$. It follows that $\sup_{\varepsilon>0} \|B_\varepsilon x\|_E \leqslant M$.

Let $y_n = \lim_{\varepsilon\to 0} B_\varepsilon x_n$ for each n in $\mathbb{N}$. Let $\delta > 0$. There is a natural number N such that $\|x_m - x_n\|_B < \frac{\delta}{3}$ for $m,n \geqslant N$. Hence $\|B_\varepsilon x_m - B_\varepsilon x_n\|_E \leqslant \|x_m - x_n\|_B < \frac{\delta}{3}$ for $m,n \geqslant N$ and for each $\varepsilon > 0$. Fix $m \geqslant N$ and $n \geqslant N$. There exists $\delta_0 > 0$ such that $\|y_m - B_\varepsilon x_m\|_E < \frac{\delta}{3}$ and $\|y_n - B_\varepsilon x_n\|_E < \frac{\delta}{3}$ for $0 < \varepsilon < \delta_0$. Fix ε, $0 < \varepsilon < \delta_0$. Then, for each $m,n \geqslant N$,

$$\begin{aligned}\|y_m - y_n\|_E &\leqslant \|y_m - B_\varepsilon x_m\|_E + \|B_\varepsilon x_m - B_\varepsilon x_n\|_E + \|B_\varepsilon x_n - y_n\|_E \\ &< \frac{\delta}{3} + \frac{\delta}{3} + \frac{\delta}{3} = \delta\end{aligned}$$

Hence $(y_n)_{n\in\mathbb{N}}$ is Cauchy in E. There exists y in E such that $y_n \to y$ in E.

Let $\delta > 0$. Then, there exists a natural number N such that $\|x_m - x_n\|_B < \frac{\delta}{3}$ for all $m,n \geqslant N$. Thus $\|B_\varepsilon x_m - B_\varepsilon x_n\|_E \leqslant \|x_m - x_n\|_B < \frac{\delta}{3}$ for $m,n \geqslant N$ and for all $\varepsilon > 0$. Hence $\|B_\varepsilon x - B_\varepsilon x_n\|_E = \lim_{m\to\infty} \|B_\varepsilon x_m - B_\varepsilon x_n\|_E \leqslant \frac{\delta}{3}$ for each $n \geqslant N$ and for each $\varepsilon > 0$. So, for each $n \geqslant N$,

$$\begin{aligned}\|B_\varepsilon x - y\|_E &\leqslant \|B_\varepsilon x - B_\varepsilon x_n\|_E + \|B_\varepsilon x_n - y_n\|_E + \|y_n - y\|_E \\ &\leqslant \frac{\delta}{3} + \|B_\varepsilon x_n - y_n\|_E + \|y_n - y\|_E\end{aligned}$$

Choose $n \geqslant N$ such that $\|y_n - y\|_E < \frac{\delta}{3}$. There exists $t > 0$ such that $\|B_\varepsilon x_n - y_n\|_E < \frac{\delta}{3}$ for $0 < \varepsilon < t$. So $\|B_\varepsilon x - y\|_E < \delta$ for all $\varepsilon > 0$ satisfying $0 < \varepsilon < t$. Hence $\lim_{\varepsilon\to 0} B_\varepsilon x = y$ in E. It follows that $x \in F$. It is straight forward to show that $\|x - x_n\|_B \to 0$. Consequently, F is a Banach space and $F \hookrightarrow E$. It remains to show that B is bounded from F to E. Let $(x_n)_{n\in\mathbb{N}} \subset D(B)$ converge to 0 in F. Then, $(x_n)_{n\in\mathbb{N}}$ converges to 0 in L, E_ε and E for all $\varepsilon > 0$, in particular $(B_\varepsilon x_n)_{n\in\mathbb{N}}$ converges to 0 in E uniformly with respect to $\varepsilon > 0$. Thus, we can permute the limits and obtain $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} (\lim_{\varepsilon\to 0} B_\varepsilon x_n) = \lim_{\varepsilon\to 0} (\lim_{n\to\infty} B_\varepsilon x_n) = 0$ since B_ε is bounded from E_ε to E. Consequently, B is an almost closed extension on E of A. $\square$

THEOREM 13. *Let $A_n \in ACl(E)$ with the extension $B_n \in AC(E)$ and the auxiliary Banach space $E_n = (D(B_n), \|.\|_{B_n})$ for all $n \in \mathbb{N}$. Suppose that L is a Banach space such that $L \hookrightarrow E_n$ for all $n \in \mathbb{N}$. Then, $Ax = \sum_{n=0}^{\infty} A_n x$ with domain*

$$D(A) = \left\{ x \in \left(\bigcap_{n\in\mathbb{N}} D(A_n) \right) \cap L\colon \sum_{n=0}^{\infty} A_n x \text{ exists in } E \right\} \tag{4.2}$$

is almost closable linear operator on E.

Proof. Let $Bx = \sum_{n=0}^{\infty} B_n x$ with domain

$$D(B) = \left\{ x \in \left(\bigcap_{n\in\mathbb{N}} D(B_n) \right) \cap L \colon \sum_{n=0}^{\infty} B_n x \text{ exists in } E \right\}$$

and $\|x\|_B = \|x\|_L + \sup_{N\in\mathbb{N}} \left\| \sum_{n=0}^{N} B_n x \right\|_E$. Define $S_N = \sum_{n=0}^{N} B_n$ with domain

$$D(S_N) = \left(\bigcap_{n=0}^{N} D(B_n) \right) \cap L, \quad N \in \mathbb{N}.$$

It follows from Theorem 11, that S_N is almost closed on E if $\left(\bigcap_{n=0}^{N} D(B_n) \right) \cap L$ is the associated auxiliary Banach space with the norm $\|x\|_{S_N} = \|x\|_L + \|S_N x\|_E$, $N \in \mathbb{N}$. First we show that $F = (D(B), \|x\|_B)$ is a Banach space. Clearly $\|x\|_B$ is a norm on $D(B)$. On the other hand if $(x_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in F, then $(x_k)_{k\in\mathbb{N}}$ converges to x in L, E_n and E for all $n \in \mathbb{N}$. Hence, $x \in D(B_n) \cap L$ and $\lim_{k\to+\infty} B_n x_k = B_n x$, by virtue of the boundedness of B_n from E_n to E for all $n \in \mathbb{N}$. $(x_k)_{k\in\mathbb{N}}$ is in particular bounded in F, then there exists $M > 0$ such that for all $k \in \mathbb{N}$,

$$\sup_{N\in\mathbb{N}} \|S_N x\|_E \leqslant \|x_k\|_B \leqslant M$$

One needs to show now that $\sum_{i=0}^{\infty} B_i x$ exists in E. Let $\varepsilon > 0$. There exists a natural number N such that $\|S_n x_m - S_n x_k\|_E \leqslant \|x_m - x_k\|_B < \frac{\varepsilon}{3}$ for $m,k \geqslant N$ for all n in $\mathbb{N}$. Hence $\|S_n x - S_n x_k\|_E = \lim_{m\to\infty} \|S_n x_m - S_n x_k\|_E \leqslant \frac{\varepsilon}{3}$ for $k \geqslant N$ and for every natural number n. Fix $k \geqslant N$. Since $\lim_{n\to\infty} S_n x_k$ exists, there exist N_0 in $\mathbb{N}$, $N_0 \geqslant N$, such that $\|S_m x_k - S_n x_k\|_E < \frac{\varepsilon}{3}$ for all $m,n \geqslant N_0$. Thus $\|S_m x - S_n x\|_E \leqslant \|S_m x - S_m x_k\|_E + \|S_m x_k - S_n x_k\|_E + \|S_n x_k - S_n x\|_E < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$ for all $m,n \geqslant N_0$.

It follows that $(S_n x)_{n\in\mathbb{N}}$ is a Cauchy sequence in E. Hence $\sum_{i=0}^{\infty} B_i x = \lim_{n\to\infty} S_n x$ exists in E. Hence $x \in D(B)$. As consequence, there exists $k_0 \in \mathbb{N}$ such that for all $m,k \geqslant k_0$,

$$\|x_m - x_k\|_L \leqslant \frac{\varepsilon}{2} \text{ and } \sup_{n\in\mathbb{N}} \|S_n x_m - S_n x_k\|_E \leqslant \frac{\varepsilon}{2}, \ \varepsilon > 0$$

Thus, for all $m,k \geqslant k_0$,

$$\begin{aligned} \|x_m - x\|_B &= \|x_m - x\|_L + \sup_{n\in\mathbb{N}} \lim_{k\to\infty} \|S_n x_m - S_n x_k\|_E \\ &\leqslant \|x_m - x\|_L + \|x_m - x_k\|_B \leqslant \varepsilon \end{aligned}$$

Hence, F is a Banach space and $F \hookrightarrow E$. On the other hand, $F \hookrightarrow (D(S_N), \|.\|_{S_N})$ for all $N \in \mathbb{N}$. Obviously, $Ax = \lim_{N\to\infty} S_N x$ and

$$D(B) = \left\{ x \in \left(\bigcap_{n\in\mathbb{N}} D(S_n) \right) \cap L \colon \lim_{N\to\infty} S_N x \text{ exists} \right\}$$

Then, according to Theorem 12 we conclude that B is an almost closed extension of A. $\square$

THEOREM 14. *Let J be a (Lebesgue) measurable subset of $\mathbb{R}$ and $A_t \in ACl(E)$ with an extension $B_t \in AC(E)$ and the associated auxiliary Banach space $E_t = (D(B_t), \|.\|_{B_t})$ for all $t \in J$. Let $Ax = \int_J A_t x dt$ with domain*

$$D(A) = \left\{ x \in \left(\bigcap_{t \in J} D(A_t) \right) \cap L : A_t x \in L^1(J,E) \right\}$$

where L is a Banach space such that $L \hookrightarrow E_t$ for all $t \in J$. Then, $A \in ACl(E)$ with the almost closed extension $Bx = \int_J B_t x dt$ defined on E of domain

$$E_B = \left\{ x \in \left(\bigcap_{t \in J} D(B_t) \right) \cap L : B_t x \in L^1(J,E) \right\} \tag{4.3}$$

equipped with the norm $\|x\|_{E_B} = \|x\|_L + \int_J \|B_t x\|_E \, dt$.

Proof. Let $(x_n)_{n\in\mathbb{N}}$ a Cauchy sequence in E_B. Then, $(x_n)_{n\in\mathbb{N}}$ and $(B_t x_n)_{n\in\mathbb{N}}$ are Cauchy sequences respectively in E and $L^1(J,E)$. Hence $x_n \to x$ in L and there exists a function $y_t \in L^1(J,E)$ such that $B_t x_n \to y_t$ in $L^1(J,E)$. So, since $\|B_t x_n - y_t\|_E \to 0$ in $L^1(J,\mathbb{R})$, there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ for which $\left\|B_t x_{n_k} - y_t\right\|_E \to 0$ converges pointwise almost everywhere in $L^1(J,\mathbb{R})$, that is, $B_t x_n \to y_t$ converges pointwise almost everywhere in E. The boundedness of B_t from E_t into E implies that $x \in D(B_t)$ and $B_t x = y_t$ for almost all $t \in J$. Consequently, $B_t x \in L^1(J,E)$ and therefore $x \in E_B$. Furthermore, $\|x_n - x\|_{E_B} = \|x_n - x\|_L + \int_J \|B_t x_n - y_t\|_E dt \to 0$ as $n \to \infty$. Thus, E_B is a Banach space and $E_B \hookrightarrow E$. Since

$$\begin{aligned} \|Bx - Bx_n\|_E &= \left\| \int_J (B_t x - B_t x_n) dt \right\|_E \\ &\leqslant \int_J \|B_t x - B_t x_n\|_E \, dt \leqslant \|x - x_n\|_{E_B} \end{aligned}$$

it follows that B is a bounded extension of A from E_B into E. $\square$

5. Topology in the class of almost closable operators

We introduce in this section a topology in the class $ACl(E)$ of all almost closable linear operators on E. We use here the constructions made by Caradus in [5]. Let $A \in ACl(E)$ with the almost closed extension B on the auxilliary Banach space E_B and suppose α denotes a canonical decomposition $A = A_0 C$ for A on $D(A)$ according to

4) of Theorem 10, we use the notation $A = A_0C[E_B]$ to represent this decomposition. It is clear that, given A, the space E_B is unique up to isomorphism. For $\varepsilon > 0$, we define an ε-neighborhood of an almost closable operator $A = A_0C[E_B]$ by

$$V(A;\alpha,\varepsilon) = \{S \in ACl(E)\colon D(S) = D(A),\ S \text{ has a canonical decomposition } S = \widetilde{A_0}C[E_B] \text{ and } \left\|A_0 - \widetilde{A_0}\right\|_{B(E_B,E)} < \varepsilon\} \tag{5.1}$$

The family $\{V(A;\alpha,\varepsilon)\colon A \in ACl(E) \text{ and } \varepsilon > 0\}$ constitute a subbasis of $ACl(E)$ since $\bigcup\{V(A;\alpha,\varepsilon)\colon A \in ACl(E) \text{ and } \varepsilon > 0\} = ACl(E)$. The topology τ_α generated by the subbasis $\{V(A;\alpha,\varepsilon)\}_{\varepsilon>0}$ is the collection of all unions of finite intersections of elements of $\{V(A;\alpha,\varepsilon)\}_{\varepsilon>0}$, that is τ_α is the smallest topology on $ACl(E)$ in which the elements $V(A;\alpha,\varepsilon)$ are open. This definition of ε-neighborhoods is not restrictive knowing that in the mathematical literature several interesting studies concern some classes of operators all defined on the same domain, one can consult for example the recent work of Polakovič and Riečanová [24].

REMARK 4. As any two Banach space norms $\|.\|_1$ and $\|.\|_2$ on $D(B)$ which make $D(B)$ complete are equivalent, then if we take another representation β instead of α the topologies τ_α and τ_β coincide on $ACl(E)$, for this reason we denote τ_α by τ.

If $T \in B(E)$, the representation α of T is given by $T = TI[E]$, it resuls then that τ coincide with the uniform operator topology on $B(E)$. τ coincide on the set $C(E)$, of all closed densely defined operators on E, with the topology introduced by Caradus in [5]. Thus, τ is stronger than that induced by the gap metric g on $C(E)$; that is, if $A_n \to A$ in τ implies $g(A_n,A) \to 0$ as $n \to \infty$ for $A_n, A \in C(E)$.

The topology τ in $ACl(E)$ possesses the following fundamental properties.

THEOREM 15. *τ is a locally convex Hausdorff topology on $ACl(E)$.*

Proof. We have to show at first that each $V(A;\alpha,\varepsilon)$ is convex. Let $A^{(1)}, A^{(2)} \in V(A;\alpha,\varepsilon)$ where $A = A_0C[E_B]$. Then, $A^{(1)} = A_0^{(1)}C[E_B]$ and $A^{(2)} = A_0^{(2)}C[E_B]$ with $\left\|A_0 - A_0^{(1)}\right\|_{B(E_B,E)} < \varepsilon$ and $\left\|A_0 - A_0^{(2)}\right\|_{B(E_B,E)} < \varepsilon$.
Since $\lambda A^{(1)} + (1-\lambda)A^{(2)} = (\lambda A_0^{(1)} + (1-\lambda)A_0^{(2)})C[E_B]$ for $0 \leqslant \lambda \leqslant 1$, then

$$\begin{aligned}
&\left\|A_0 - (\lambda A_0^{(1)} + (1-\lambda)A_0^{(2)})\right\|_{B(E_B,E)} \\
&= \left\|\lambda(A_0 - A_0^{(1)}) + (1-\lambda)(A_0 - A_0^{(2)})\right\|_{B(E_B,E)} \\
&< \lambda\varepsilon + (1-\lambda)\varepsilon = \varepsilon
\end{aligned}$$

Hence, $\lambda A^{(1)} + (1-\lambda)A^{(2)} \in V(A;\alpha,\varepsilon)$. On the other hand, to establish the Hausdorff separability of τ we take $A^{(1)}, A^{(2)} \in ACl(E)$ such that $A^{(1)} \neq A^{(2)}$. If $D(A^{(1)}) \neq D(A^{(2)})$ then $V(A^{(1)};\alpha,\varepsilon) \cap V(A^{(2)};\alpha,\varepsilon) = \emptyset$ for any $\varepsilon > 0$. If $D(A^{(1)}) = D(A^{(2)})$, by virtue of the previous remark, we can consider $A^{(1)}$ and $A^{(2)}$ with the same representation α and the same auxiliary space E_B, $A^{(1)} = A_0^{(1)}C_1[E_B]$ and $A^{(2)} = A_0^{(2)}C_2[E_B]$. Suppose that $S \in V(A^{(1)};\alpha,\varepsilon) \cap V(A^{(2)};\alpha,\varepsilon)$. So we have $D(S) = D(A^{(1)}) = D(A^{(2)})$, $S = S_1C_1[E_B] = S_2C_2[E_B]$, $\left\|S_1 - A_0^{(1)}\right\|_{B(E_B,E)} < \varepsilon$ and $\left\|S_2 - A_0^{(2)}\right\|_{B(E_B,E)} < \varepsilon$. Thus, for any $x \in D(A^{(1)}) = D(A^{(2)})$, we obtain

$$\begin{aligned}\left\|A^{(1)}x - A^{(2)}x\right\|_E &\leqslant \left\|A_0^{(1)}C_1x - S_1C_1x\right\|_E + \left\|A_0^{(2)}C_2x - S_2C_2x\right\|_E \\ &< \varepsilon(\|C_1x\|_{E_B} + \|C_2x\|_{E_B})\end{aligned}$$

Then, $A^{(1)} = A^{(2)}$, which contradicts the hypothesis. Consequently, $V(A^{(1)};\alpha,\varepsilon) \cap V(A^{(2)};\alpha,\varepsilon) = \emptyset$. $\square$

THEOREM 16. *The mappings* $s\colon\ (A^{(1)},A^{(2)}) \to A^{(1)} + A^{(2)}$ *and* $p:\ (A^{(1)},A^{(2)}) \to A^{(1)}A^{(2)}$ *are continuous in* $(ACl(E),\tau)$.

Proof. Let $A^{(1)},A^{(2)} \in ACl(E)$. $A^{(1)} = A_0^{(1)}C_1[E_{B_1}]$ and $A^{(2)} = A_0^{(2)}C_2[E_{B_2}]$ are respectively the canonical decompostions α_1 and α_2 of $A^{(1)}$ and $A^{(2)}$. From the definition of ε-neighborhoods, we see that

$$V(A^{(1)};\alpha_1,\varepsilon_1) + V(A^{(2)};\alpha_2,\varepsilon_2) \subseteq V(A^{(1)} + A^{(2)};\alpha_1 + \alpha_2,\varepsilon) \tag{5.2}$$

$$V(A^{(1)};\alpha_1,\varepsilon_1)V(A^{(2)};\alpha_2,\varepsilon_2) \subseteq V(A^{(1)}A^{(2)};\alpha_1\alpha_2,\varepsilon)$$

with, in each case, $M(\varepsilon_1 + \varepsilon_2) < \varepsilon$, where M is a constant depending on $A^{(1)}$ and $A^{(2)}$. $\alpha_1 + \alpha_2$ and $\alpha_1\alpha_2$ are obtained from α_1 and α_2 by the constructions $4)$ of Theorem 10. Hence, it follows from (5.2) the continuity of s and p. $\square$

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REFERENCES

[1] S. AGMON AND L. NIRENBERG, *Properties of solutions of ordinary differential equations in Banach space*, Comm. Pure Appl. Math. **16**, (1963), 121–239.

[2] W. ARENDT, *Vector valued Laplace transforms and Cauchy problems*, Israel J. Math. **59**, (1987), 327–352.

[3] A. AZZOUZ, B. MESSIRDI AND G. DJELLOULI, *New results on the closedness of the product and sum of closed linear operators*, Bull. Math. Anal. Appl. **3**, 2 (2011), 151–158.

[4] B. BAUMER AND F. NEUBRANDER, *Laplace transform methods for evolution equations*, Conferenze del Seminario di Matematica dell' Universita di Bari **259**, (1995), 27–60.

[5] S. R. CARADUS, *Semiclosed operators*, Pacific J. Math. **44**, 1 (1973), 75–79.

[6] J. DIXMIER, *L'adjoint du produit de deux opérateurs fermés*, Ann. Fac. Sci. Toulouse Math. 4ème série, **11**, (1974), 101–106.

[7] G. DJELLOULI, S. MESSIRDI AND B. MESSIRDI, *Some stronger topologies for closed operators in Hilbert space*, Int. J. Contemp. Math. Sciences **5**, 25 (2010), 1223–1232.

[8] R. ENGEL AND K. J. NAGEL, *One-parameter semigroups for linear evolution equations*, Graduate Texts in Mathematics. Springer Verlag, 2000.

[9] M. FERNANDEZ MIRANDA AND J. PH. LABROUSSE, *On the closure of the product and sum of linear relations*, Complex Anal. Oper. Theory **6**, 3 (2012), 613–624.

[10] P. A. FILLMORE AND J. P. WILLIAMS, *On operator ranges*, Adv. Math. **7**, (1971), 254–281.

[11] C. FOIAS, *Invariant semiclosed subspaces*, Preprint, Institute of Mathematics, Bucarest, Romania, 1970.

[12] E. HILLE AND R. S. PHILLIPS, *Functional Analysis and Semi-Groups*, Amer. Math. Soc. Providence, Rhode Island, 1957.

[13] R. V. KADILON AND J. R. RINGROSE, *Fundamentals of the theory of operator algebras (I)*, Acad. Press, 1983.

[14] T. KATO, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[15] W. E. KAUFMAN, *Representing a closed operator as a quotients of continuous operators*, Proc. Amer. Math. Soc. **72**, (1978), 531–534.

[16] W. E. KAUFMAN, *Semiclosed operators in Hilbert space*, Proc. Amer. Math. Soc. **76**, (1979), 67–73.

[17] J. PH. LABROUSSE, *Les opérateurs quasi Fredholm: Une généralisation des opérateurs semi Fredholm*, Rend. Circ. Mat. Palermo (2) Suppl. Tomo **XXIX**, (1980), 161–258.

[18] J. PH. LABROUSSE AND B. MERCIER, *Equivalences compactes entre deux opérateurs fermés sur un espace de Hilbert*, Math. Nachr. **133**, (1987), 91–105.

[19] J. S. MAC NEARNEY, *Investigation concerning positive definite continued fractions*, Duke Math. J. **26**, (1959), 663–678.

[20] I. V. MELNIKOVA AND A. FILINIKOV, *Abstract Cauchy problems: Three approaches*, Chapman and Hall/CRC Monographs ans Surveys in Pure and Applied Mathematics. Boca, Raton, London, New York, Washington, D.C., 2001.

[21] B. MESSIRDI, M. H. MORTAD, A. AZZOUZ AND G. DJELLOULI, *A Topological Characterization of the Product of Two Closed Operators*, Colloq. Math. **112**, 2 (2008), 269–278.

[22] B. MESSIRDI AND M. H. MORTAD, *On Different Products of Closed Operators*, Banach J. Math. Anal. **2**, 1 (2008), 40–47.

[23] S. MESSIRDI, M. DJAA AND B. MESSIRDI, *Stability of Almost Closed Operators on a Hilbert Space*, Sarajevo J. Math. **5**, 17 (2009), 133–141.

[24] M. POLAKOVIC AND Z. RIECANOVA, *Generalized effects algebras of positive operators densely defined on Hilbert spaces*, Int. J. Theor. Phys. **50**, (2011), 1167–1174.

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