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ON A CLASS OF BOUNDARY CONTROL PROBLEMS

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Abstract. We discuss a class of linear control problems in a Hilbert space setting, which covers diverse systems such as hyperbolic and parabolic equations with boundary control and boundary observation even including memory terms. We introduce abstract boundary data spaces in which the control and observation equations can be formulated without strong geometric constraints on the underlying domain. The results are applied to a boundary control problem for the equations of visco-elasticity.

1. Introduction

Linear control systems are typically given by a differential equation (∂_0 denotes the derivative with respect to time), linking the state x and the control u

$$\partial_0 x(t) = Ax(t) + Bu(t), \quad t \in \mathbb{R}_{>0},$$

usually completed by an initial condition $x(0+) = x_0$, and an algebraic equation linking state, control and the observation y

$$y(t) = Cx(t) + Du(t) \quad t \in \mathbb{R}_{>0},$$

where A,B,C and D are matrices of appropriate sizes. Following general practice this system may be recorded by the 2×2-block-matrix[1]

$$\begin{pmatrix} A-\partial_0 & B \\ C & D \end{pmatrix}. \tag{1}$$

Rewriting these equations as one system acting on the whole real line $\mathbb{R}$ instead of the positive half-line $\mathbb{R}_{>0}$ we end up with an differential-algebraic system of the form

$$\partial_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -C & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -A & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \delta \otimes \begin{pmatrix} x_0 \\ 0 \end{pmatrix} + \begin{pmatrix} B \\ D \end{pmatrix} u, \tag{2}$$

where the initial condition for the state variable x transforms into an additional Dirac-δ-source term on the right hand side. Systems of this form have been studied in the finite- and infinite-dimensional case in various works. In the infinite-dimensional case

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[1] Usually the time derivative is not mentioned, since it is assumed that the time enters in this fashion, compare Remark 3.2.

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the operators B, C and D, acting on some suitable Banach- or Hilbert-spaces, are usually assumed to be bounded, while the operator A is the generator of a C_0-semigroup. In this case the solution theory is rather straightforward. However, in case of boundary control and observation it turns out that the operators B and C are in general unbounded and hence, a more sophisticated theory is needed. The classical approach is to consider so-called admissible operators B and C as it was done for instance in [14, 15, 2, 3, 7, 8, 21, 22, 24, 4]. We focus on a class of linear control problems, where the operators B and C are unbounded, but in our approach the admissibility for these operators has not to be verified.

The solution theory provided in this article is based on the theory of evolutionary equations as they were considered in [9]. As it was shown in [13, 12], linear control systems (including the case of unbounded operators B and C) are just a subclass of evolutionary equations. In this note we will generalize the solution theory presented in [13] to a broader class of so-called *(linear) material laws*. This generalization allows us to study control problems including delay terms. As in [13] we introduce abstract boundary data spaces, which enable us to formulate boundary control and observation equations without strong smoothness assumptions on the boundary of the underlying domain. Indeed, it will suffice to guarantee a Poincare-type estimate for the involved differential operators.

Section 2 recalls some preliminaries on evolutionary equations, linear material laws and extrapolation spaces (so-called Sobolev chains) and we refer to [9, 11, 6] for the proofs and a deeper study of the related topics.

In Section 3 we introduce the notion of linear control systems, which will be a special case of the broader class of abstract evolutionary equations. In contrast to [13] we will generalize the class of possible control problems to the case of arbitrary material laws (while in [13] just the so-called (P)-degenerate case, cf. [9], was treated). We provide a well-posedness result for this class, which is in essence just an application of [9, Solution theory], and show the causality of the solution operator.

Boundary control problems are introduced in Section 4 and we show that they fit into the abstract class of linear control problems introduced previously. In order to formulate boundary control and observation equations, without imposing strong smoothness constraints on the domain, we introduce abstract traces and recall the notion of abstract boundary data spaces. Finally, we apply our findings to a boundary control problem for the equations of visco-elasticity.

2. Preliminaries

In this section we recall the notion of evolutionary equations. Following [6] we begin to introduce the time derivative ∂_0 as a boundedly invertible operator on an exponentially weighted L_2-space.

DEFINITION 2.1. For $\nu \in \mathbb{R}_{>0}$ we denote by $H_{\nu,0}(\mathbb{R})$ the space of all square-integrable functions[2] with respect to the exponentially weighted Lebesgue-measure

[2]Throughout we identify the equivalence classes induced by the equality almost everywhere with their

$\exp(-2\nu t)\,\mathrm{d}t$, endowed with the inner product given by

$$\langle f|g\rangle_{\nu,0} := \int_{\mathbb{R}} f(t)^* g(t) \exp(-2\nu t)\,\mathrm{d}t \quad (f,g \in H_{\nu,0}(\mathbb{R})).$$

We define the operator $\partial_{0,\nu}$ on $H_{\nu,0}(\mathbb{R})$ as the closure of the derivative

$$\begin{aligned}\partial_{0,\nu}|_{C_c^\infty(\mathbb{R})} : C_c^\infty(\mathbb{R}) \subseteq H_{\nu,0}(\mathbb{R}) &\to H_{\nu,0}(\mathbb{R})\\ \phi &\mapsto \phi'.\end{aligned}$$

This operator is normal with $\mathfrak{Re}\,\partial_{0,\nu} = \nu$ and hence, $0 \in \rho(\partial_{0,\nu})$ with $\|\partial_{0,\nu}^{-1}\| \leqslant \frac{1}{\nu}$. If the choice of ν is clear from the context we will omit the additional index ν.

We can extend the operator ∂_0 to the space of H-valued functions $H_{\nu,0}(\mathbb{R};H)$, where H is a Hilbert space, in a canonical way.

REMARK 2.2.

(a) For $u \in H_{\nu,0}(\mathbb{R})$ the function $\partial_0^{-1}u$ is given by

$$\left(\partial_0^{-1}u\right)(x) = \int_{-\infty}^{x} u(t)\,\mathrm{d}t \quad (x \in \mathbb{R} \text{ a.e.}).$$

This especially yields the causality[3] of ∂_0^{-1}.

(b) Let $\mathscr{F} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ denote the Fourier-transform and define for $\nu > 0$ the operator $e^{-\nu m} : H_{\nu,0}(\mathbb{R}) \to L_2(\mathbb{R})$ by $(e^{-\nu m}f)(x) = e^{-\nu x}f(x)$ for almost every $x \in \mathbb{R}$, which is obviously unitary. Then we define the *Fourier-Laplace-transform* $\mathscr{L}_\nu := \mathscr{F}e^{-\nu m} : H_{\nu,0}(\mathbb{R}) \to L_2(\mathbb{R})$, which gives the following spectral representation of the derivative $\partial_{0,\nu}$:

$$\partial_{0,\nu} = \mathscr{L}_\nu^*(\mathrm{i}m + \nu)\mathscr{L}_\nu,$$

where m denotes the "multiplication-by-the-argument" operator on $L_2(\mathbb{R})$ with maximal domain.

With the spectral representation of ∂_0 we can define so-called linear material laws (cf. [9]).

DEFINITION 2.3. Let $r > 0, H$ an arbitrary Hilbert space and $M : B_{\mathbb{C}}(r,r) \to L(H)$ be bounded and analytic. Then we define the operator $M\left(\frac{1}{\mathrm{i}m+\nu}\right)$ on $L_2(\mathbb{R})$ for $\nu > \frac{1}{2r}$ by

$$\left(M\left(\frac{1}{\mathrm{i}m+\nu}\right)f\right)(x) = M\left(\frac{1}{\mathrm{i}x+\nu}\right)f(x) \quad (x \in \mathbb{R} \text{ a.e.})$$

representatives.

[3]A mapping $F : H_{\nu,0}(\mathbb{R};H) \to H_{\nu,0}(\mathbb{R};H)$, where H is an arbitrary Hilbert space, is called *causal*, if for each $a \in \mathbb{R}$ it holds $\chi_{(-\infty,a]}(m)F\chi_{(-\infty,a]}(m) = \chi_{(-\infty,a]}(m)F$, where by $\chi_{(-\infty,a]}(m)$ we denote the operator on $H_{\nu,0}(\mathbb{R};H)$ mapping a function f to the truncated function $t \mapsto \chi_{(-\infty,a]}(t)f(t)$.

and the *linear material law* $M(\partial_0^{-1}) \in L(H_{\nu,0}(\mathbb{R};H))$ for $\nu > \frac{1}{2r}$ by

$$M\left(\partial_0^{-1}\right) := \mathscr{L}_\nu^* M\left(\frac{1}{\mathrm{i}m+\nu}\right)\mathscr{L}_\nu.$$

REMARK 2.4. Due to the analyticity of M we obtain by a Paley-Wiener result that the operator $M(\partial_0^{-1})$ is causal.

Note that any densely defined closed linear operator A defined in a Hilbert space H gives rise to densely defined closed linear operator in $H_{\nu,0}(\mathbb{R};H)$ defined as the canonical extension of the operator acting as $(Af)(t) := Af(t)$ for all $t \in \mathbb{R}$ and simple functions f taking values in the domain of A. Henceforth, we will identify A with its extension without further notice.

THEOREM 2.5. (Solution theory for evolutionary equations [9, Solution theory]) *Let H be a Hilbert space and $A : D(A) \subseteq H \to H$ be skew-selfadjoint. Furthermore let $r > 0$ and $M : B_{\mathbb{C}}(r,r) \to L(H)$ be analytic, bounded and assume that there exists $c > 0$ such that for all $z \in B_{\mathbb{C}}(r,r)$*

$$\mathfrak{Re}\, z^{-1}M(z) \geqslant c. \tag{3}$$

Then there exists $\nu_0 > 0$ such that for all $\nu \geqslant \nu_0$ the evolutionary equation

$$\left(\overline{\partial_0 M(\partial_0^{-1}) + A}\right) u = f, \tag{4}$$

admits for every $f \in H_{\nu,0}(\mathbb{R};H)$ a unique solution $u \in H_{\nu,0}(\mathbb{R};H)$, which depends continuously on f. More precisely, $0 \in \rho\left(\overline{\partial_0 M(\partial_0^{-1}) + A}\right)$ and the solution operator $\left(\overline{\partial_0 M(\partial_0^{-1}) + A}\right)^{-1}$ *is causal.*

Next we introduce the concept of Sobolev-chains. For the proofs and further details we refer to [11, Chapter 2].

DEFINITION 2.6. Let H be a Hilbert space and $C : D(C) \subseteq H \to H$ be a densely defined, closed linear operator with $0 \in \rho(C)$. For $k \in \mathbb{Z}$ we define $H_k(C)$ as the completion of the domain $D(C^k)$ with respect to the norm $|C^k \cdot|_H$. Then $(H_k(C))_{k\in\mathbb{Z}}$ is a sequence of Hilbert spaces with $H_k(C) \hookrightarrow H_{k-1}(C)$ for $k \in \mathbb{Z}$. The sequence $(H_k(C))_{k\in\mathbb{Z}}$ is called the *Sobolev-chain of* C. For each $k \in \mathbb{Z}$ the operator $C : H_{|k|+1}(C) \subseteq H_{k+1}(C) \to H_k(C)$ possesses a unitary extension to $H_{k+1}(C)$, which will be again denoted by C. Furthermore $H_k(C)^*$ can be identified with $H_{-k}(C^*)$ for each $k \in \mathbb{Z}$ via a unitary operator.

REMARK 2.7.

(a) Let H_0, H_1 be two Hilbert spaces over the same field and $A : D(A) \subseteq H_0 \to H_1$ be densely defined, closed and linear. For each $k \in \mathbb{Z}$ the operator

$$A : H_{|k|+1}(|A|+\mathrm{i}) \subseteq H_{k+1}(|A|+\mathrm{i}) \to H_k(|A^*|+\mathrm{i})$$

has a unique continuous extension to $H_{k+1}(|A|+\mathrm{i})$.

(b) Let $(H_k(C))_{k\in\mathbb{Z}}$ be a Sobolev-chain associated with some operator C and $A : H_1(C) \to H$ be linear and bounded, where H denotes an arbitrary Hilbert-space. Then the operator $A^\diamond : H \to H_{-1}(C^*)$ is defined as the dual operator of A, where we identify the dual of H with H and $H_1(C)^*$ is identified with $H_{-1}(C^*)$.

3. Abstract linear control systems

In this section we introduce the shape of linear control systems and show that they fit into the class of evolutionary equations introduced in the previous section. We consider a densely defined closed linear operator $F : D(F) \subseteq H_0 \to H_1$ for two Hilbert spaces H_0 and H_1. Furthermore let U and Y be Hilbert spaces, which will serve as control and observation space, respectively.

DEFINITION 3.1. Let $M_{1,i2} \in L(Y;H_i)$ and $M_{1,2i} \in L(H_i;Y)$ for $i \in \{0,1\}$ and $M_{1,22} \in L(Y;Y)$. Let

$$M(z) := \begin{pmatrix} K(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 \end{pmatrix} + z \begin{pmatrix} 0 & \begin{pmatrix} M_{1,02} \\ M_{1,12} \end{pmatrix} \\ \begin{pmatrix} M_{1,20} & M_{1,21} \end{pmatrix} & M_{1,22} \end{pmatrix} \quad (z \in B_{\mathbb{C}}(r,r)), \tag{5}$$

where $K : B_{\mathbb{C}}(r,r) \to L(H_0 \oplus H_1)$ is a linear material law. An *abstract linear control system* $\mathscr{C}_{M,F,B}$ is an evolutionary equation of the form

$$\left(\partial_0 M(\partial_0^{-1}) + \begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \begin{pmatrix} x \\ \xi \\ y \end{pmatrix} = f + Bu. \tag{6}$$

Here $f \in H_{\nu,0}(\mathbb{R};H_0 \oplus H_1 \oplus Y)$ is an arbitrary source term, $u \in H_{\nu,0}(\mathbb{R};U)$ is the control and $B \in L(H_{\nu,0}(\mathbb{R};U);H_{\nu,0}(\mathbb{R};(H_0 \oplus H_1 \oplus Y)))$ is the control operator. We call an abstract linear control system *well-posed,* if the operator

$$\partial_0 M(\partial_0^{-1}) + \begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \subseteq H_{\nu,0}(\mathbb{R};H_0 \oplus H_1 \oplus Y) \oplus H_{\nu,0}(\mathbb{R};H_0 \oplus H_1 \oplus Y)$$

possesses a densely defined bounded inverse for sufficiently large ν. The continuation to the space $H_{\nu,0}(\mathbb{R};H_0 \oplus H_1 \oplus Y)$ of the inverse is called *solution operator*.

REMARK 3.2. In the literature, the concept of well-posedness for linear control systems of the form (1) is commonly based on expressing $(A - \partial_0)^{-1}$ by convolution with the fundamental solution, i.e. via semigroup theory. Indeed, it was shown by Salamon ([14, 15]) that a well-posed linear system in the sense of [22, 23] can be represented in the form (1), where A is the generator of a C_0-semigroup and B and C are admissible (possibly unbounded) control and observation operators. In our approach, we reformulate the problem into a system of the form (6), where the unbounded coefficient operators will be incorporated into the operator matrix $\begin{pmatrix} \widetilde{A} & 0 \\ 0 & 0 \end{pmatrix}$ with $\widetilde{A} = \begin{pmatrix} 0 & -F^* \\ F & 0 \end{pmatrix}$, which is indeed a skew-selfadjoint operator (we have assumed skew-selfadjointness as a matter of simplification throughout although the solution theory can be extended to include the case of accretive operators in time and space, see [16, 18]). In the simplest case, where $M\left(\partial_0^{-1}\right) = \begin{pmatrix} M_{0,00} & 0 \\ 0 & 0 \end{pmatrix} + \partial_0^{-1} M_1$, this reformulation suggests, by modifying the state space appropriately, to consider a control system with a different block partition

$$\begin{pmatrix} \widetilde{A} - \partial_0 M_{0,00} & \widetilde{B} \\ \widetilde{C} & \widetilde{D} \end{pmatrix},$$

where now $\widetilde{B}, \widetilde{C}$ and $\widetilde{D}$ are bounded again. However, as a trade-off $M_{0,00}$ – far from being the identity, as in (1) – is now allowed to be non-invertible, i.e. the state equation is itself a differential-algebraic system. Indeed, due to the generality of the material law operator $M(\partial_0^{-1})$, systems of the form (6) cover a broad class of differential-algebraic equations, such as parabolic, hyperbolic and (quasi-static) elliptic state equations, even those of changing type, as well as equations with operators non-local in time or space, such as fractional derivatives (see [10, 19]) or convolution operators (see [17]), where semigroup techniques are not appropriate.

In contrast to the classical concept of well-posedness, where as a consequence of the regularizing effect of the semi-group one obtains continuous solutions, our well-posedness concept hinges on square-integrability in time with respect to an exponentially weighted Lebesgue-measure. Thus, in our setting, if semigroups can be utilized at all, their use is related rather to the regularity of solutions of a well-posed system than to the well-posedness of the system itself (see [12]).

It is clear that an abstract linear control system $\mathscr{C}_{M,F,B}$ is of the form (4) given in Theorem 2.5 with $A = \begin{pmatrix} 0 & -F^* & 0 \\ F & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Hence, the solution theory for evolutionary equations is applicable. It is obvious that if M satisfies the condition (3), then so does K and the operator $\mathfrak{Re}\, M_{1,22}$ is strictly positive definite. However, the latter is not a sufficient condition for the positive definiteness of $\mathfrak{Re}\, z^{-1} M(z)$. We define the operator $J \in L(Y; H_0 \oplus H_1)$ by

$$J := \frac{1}{2} \begin{pmatrix} M_{1,02} + M_{1,20}^* \\ M_{1,12} + M_{1,21}^* \end{pmatrix}.$$

THEOREM 3.3. *Let $\mathscr{C}_{M,F,B}$ be an abstract linear control system. Assume that $\mathfrak{Re}\, z^{-1} K(z) \geqslant c_0 > 0$ and $\mathfrak{Re}\, M_{1,22} \geqslant c_1 > 0$. Assume that there is $\delta > 0$ such that $c_0 - \delta \|J\| > 0$ and $c_1 - \frac{1}{\delta}\|J\| > 0$. Then $\mathscr{C}_{M,F,B}$ is well-posed and the solution operator is causal.*

Proof. Since,

$$\mathfrak{Re} \begin{pmatrix} 0 & \begin{pmatrix} M_{1,02} \\ M_{1,12} \end{pmatrix} \\ \begin{pmatrix} M_{1,20} & M_{1,21} \end{pmatrix} & M_{1,22} \end{pmatrix} = \begin{pmatrix} 0 & J \\ J^* & \mathfrak{Re}\, M_{1,22} \end{pmatrix}$$

we get for $w := (x,\xi,y) \in H_0 \oplus H_1 \oplus Y$

$$
\begin{aligned}
\mathfrak{Re}\langle z^{-1}M(z)w|w\rangle &= \mathfrak{Re}\left\langle z^{-1}K(z)\begin{pmatrix} x \\ \xi \end{pmatrix}\middle|\begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle + \langle \mathfrak{Re}\, M_{1,22}y|y\rangle + 2\,\mathfrak{Re}\left\langle Jy\middle|\begin{pmatrix} x \\ \xi \end{pmatrix}\right\rangle \\
&\geqslant c_0\left|\begin{pmatrix} x \\ \xi \end{pmatrix}\right|^2 + c_1|y|^2 - 2\|J\|\left|\begin{pmatrix} x \\ \xi \end{pmatrix}\right||y| \\
&\geqslant (c_0 - \delta\|J\|)\left|\begin{pmatrix} x \\ \xi \end{pmatrix}\right|^2 + \left(c_1 - \frac{1}{\delta}\|J\|\right)|y|^2.
\end{aligned}
$$

The assertion then follows by Theorem 2.5. □

4. Boundary control systems

This section is devoted to the study of boundary control systems. At first we show how boundary control and observation equations can be handled within the framework presented in the previous sections. This will mainly be done by a particular choice for the unbounded operator F. As it was pointed out in [13, Subsection 5.1], the resulting class of control systems can be interpreted as a generalization of a subclass of port-Hamiltonian systems (cf. [5]) to the higher dimensional case. Moreover, we recall the notion of so-called boundary data spaces, introduced in [13, Subsection 5.2], as well as abstract traces, which enable us to treat boundary values as suitable distributions belonging to some extrapolation space.

First let us fix some notation. For Hilbert spaces $H_0,\ldots,H_n$ we define for $i \in \{0,\ldots,n\}$ the operator

$$\pi_{H_i} : H_0 \oplus \ldots \oplus H_n \to H_i$$

as the orthogonal projection on H_i. Note that then $\pi_{H_i}^*$ is the canonical embedding from H_i to $H_0 \oplus \ldots \oplus H_n$.

We begin this section with an illustrative example.

EXAMPLE 4.1. Let $\Omega \subseteq \mathbb{R}^n$ be an arbitrary domain and define the operators $\mathring{\mathrm{grad}}$ and $\mathring{\mathrm{div}}$ as the closures of

$$
\begin{aligned}
\mathrm{grad}|_{C_c^\infty(\Omega)} : C_c^\infty(\Omega) \subseteq L_2(\Omega) &\to L_2(\Omega)^n \\
\phi &\mapsto (\partial_i\phi)_{i\in\{1,\ldots,n\}}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{div}|_{C_c^\infty(\Omega)^n} : C_c^\infty(\Omega)^n \subseteq L_2(\Omega)^n &\to L_2(\Omega) \\
(\phi_i)_{i\in\{1,\ldots,n\}} &\mapsto \sum_{i=1}^n \partial_i\phi_i,
\end{aligned}
$$

respectively. These operators are formally skew-adjoint, i.e., $\mathring{\mathrm{grad}} \subseteq -\left(\mathring{\mathrm{div}}\right)^* =: \mathrm{grad}$ and $\mathring{\mathrm{div}} \subseteq -\left(\mathring{\mathrm{grad}}\right)^* =: \mathrm{div}$. Then, using the extrapolation spaces of the operators

$|\operatorname{div}|+\mathrm{i}$ and $|\operatorname{grad}|+\mathrm{i}$ we define the *Dirichlet-trace* and the *Neumann-trace* by

$$\begin{aligned}\gamma_{\operatorname{grad}} : H_1(|\operatorname{grad}|+\mathrm{i}) &\to H_{-1}(|\operatorname{div}|+\mathrm{i})\\ u &\mapsto \left(\operatorname{grad}-\mathring{\operatorname{grad}}\right)u\end{aligned}$$

and

$$\begin{aligned}\gamma_{\operatorname{div}} : H_1(|\operatorname{div}|+\mathrm{i}) &\to H_{-1}(|\operatorname{grad}|+\mathrm{i})\\ \zeta &\mapsto \left(\operatorname{div}-\mathring{\operatorname{div}}\right)\zeta,\end{aligned}$$

respectively. Note that in the case of a smooth boundary, the distributions $\gamma_{\operatorname{grad}}u$ and $\gamma_{\operatorname{div}}\zeta$ for $u\in H_1(|\operatorname{grad}|+\mathrm{i})$ and $\zeta\in H_1(|\operatorname{div}|+\mathrm{i})$ are supported on $\partial\Omega$. More precisely with the help of the divergence theorem,

$$\langle\gamma_{\operatorname{grad}}u|\zeta\rangle=\int_{\partial\Omega}u^*\zeta\cdot n\,\mathrm{d}S=\langle u|\gamma_{\operatorname{div}}\zeta\rangle,$$

where n denotes the unit outward normal and S the surface measure on $\partial\Omega$. Note that $\gamma_{\operatorname{grad}}u=0$ if and only if $u\in D(\mathring{\operatorname{grad}})$ and $\gamma_{\operatorname{div}}\zeta=0$ if and only if $\zeta\in D(\mathring{\operatorname{div}})$.

In the rest of this subsection we generalize the concepts illustrated in the example above. For that purpose let H_0 and H_1 be two complex Hilbert spaces, $\mathring{G}\subseteq H_0\oplus H_1$ and $\mathring{D}\subseteq H_1\oplus H_0$ two densely defined closed linear operators, which are formally skew-adjoint. We define $G:=-\left(\mathring{D}\right)^*$ and $D:=-\left(\mathring{G}\right)^*$.

DEFINITION 4.2. (Abstract traces) We define the *abstract traces* γ_G and γ_D by

$$\begin{aligned}\gamma_G : H_1(|G|+\mathrm{i}) &\to H_{-1}(|D|+\mathrm{i})\\ v &\mapsto \left(G-\mathring{G}\right)v\end{aligned}$$

and

$$\begin{aligned}\gamma_D : H_1(|D|+\mathrm{i}) &\to H_{-1}(|G|+\mathrm{i})\\ w &\mapsto \left(D-\mathring{D}\right)w.\end{aligned}$$

Furthermore we define the *abstract trace spaces* as the image spaces of the respective trace operators, i.e.

$$\begin{aligned}\operatorname{TR}(G)&:=\gamma_G[H_1(|G|+\mathrm{i})],\\ \operatorname{TR}(D)&:=\gamma_D[H_1(|D|+\mathrm{i})].\end{aligned}$$

Clearly the kernels of γ_G and γ_D are given by $H_1(|\mathring{G}|+\mathrm{i})$ and $H_1(|\mathring{D}|+\mathrm{i})$ respectively. This leads to the following definition.

DEFINITION 4.3. (Boundary data spaces) We define the *boundary data spaces* as

$$\mathrm{BD}(G) := H_1(|\mathring{G}|+\mathrm{i})^{\perp_{H_1(|G|+\mathrm{i})}}$$

and

$$\mathrm{BD}(D) := H_1(|\mathring{D}|+\mathrm{i})^{\perp_{H_1(|D|+\mathrm{i})}}.$$

LEMMA 4.4. *The boundary data spaces are given by* $\mathrm{BD}(G) = N(1-DG)$ *and* $\mathrm{BD}(D) = N(1-GD)$.

Proof. See [13, Lemma 5.1]. $\square$

THEOREM 4.5. *The operators*

$$\gamma_G|_{\mathrm{BD}(G)} : \mathrm{BD}(G) \to \mathrm{TR}(G)$$

and

$$\gamma_D|_{\mathrm{BD}(D)} : \mathrm{BD}(D) \to \mathrm{TR}(\mathrm{D})$$

are unitary.

Proof. Let $u \in \mathrm{BD}(G)$. Then we get for each $v \in H_1(|D|+\mathrm{i})$ that

$$\begin{aligned}
|\langle \gamma_G u|v\rangle| &= |\langle Gu|v\rangle + \langle u|Dv\rangle| \\
&= |\langle Gu|v\rangle + \langle DGu|Dv\rangle| \\
&= |\langle Gu|v\rangle_{H_1(|D|+\mathrm{i})}| \\
&\leqslant |Gu|_{H_1(|D|+\mathrm{i})}|v|_{H_1(|D|+\mathrm{i})}
\end{aligned}$$

and hence,

$$|\gamma_G u|_{H_{-1}(|D|+\mathrm{i})} \leqslant |Gu|_{H_1(|D|+\mathrm{i})} = \sqrt{|Gu|^2+|DGu|^2} = \sqrt{|Gu|^2+|u|^2} = |u|_{H_1(|G|+\mathrm{i})}.$$

On the other hand we have

$$\langle \gamma_G u|Gu\rangle = \langle Gu|Gu\rangle + \langle u|DGu\rangle = \langle Gu|Gu\rangle + \langle u|u\rangle = |u|^2_{H_1(|G|+\mathrm{i})},$$

which gives $|u|_{H_1(|G|+\mathrm{i})} \leqslant |\gamma_G u|_{H_{-1}(|D|+\mathrm{i})}$. That $\gamma_G|_{\mathrm{BD}(G)}$ is onto, follows by the definition of $\mathrm{BD}(G)$ and $\mathrm{TR}(G)$. The assertion for $\gamma_D|_{\mathrm{BD}(D)}$ follows by interchanging the roles of D and G. $\square$

Since $G[\mathrm{BD}(G)] \subseteq \mathrm{BD}(D)$ and $D[\mathrm{BD}(D)] \subseteq \mathrm{BD}(G)$ we may consider the following restrictions of G and D

$$\begin{aligned}
\overset{\bullet}{D} : \mathrm{BD}(D) &\to \mathrm{BD}(G) \\
\phi &\mapsto D\phi
\end{aligned}$$

and

$$\begin{aligned} \overset{\bullet}{G} : \mathrm{BD}(G) &\to \mathrm{BD}(D) \\ \phi &\mapsto G\phi. \end{aligned}$$

The operators $\overset{\bullet}{D}$ and $\overset{\bullet}{G}$ enjoy the following unexpected property.

THEOREM 4.6. *We have that*

$$\left(\overset{\bullet}{G}\right)^{*} = \overset{\bullet}{D} = \left(\overset{\bullet}{G}\right)^{-1}.$$

In particular, $\overset{\bullet}{G}$ *and* $\overset{\bullet}{D}$ *are unitary.*

Proof. See [13, Theorem 5.2.]. □

REMARK 4.7. The operator $\gamma_D|_{\mathrm{BD}(D)} \overset{\bullet}{G} \left(\gamma_G|_{\mathrm{BD}(G)}\right)^{-1} : \mathrm{TR}(G) \to \mathrm{TR}(D)$ is unitary and can be interpreted as an abstract version of the Dirichlet-to-Neumann operator.

After these preparations we show, how systems with boundary control and boundary observation can be treated within the framework of Section 3. For doing so, let $C \in L(H_1(|G|+\mathrm{i});V)$ for some Hilbert space V and assume that F is given by

$$F := \begin{pmatrix} -G \\ C \end{pmatrix} : H_1(|G|+\mathrm{i}) \subseteq H_0(|G|+\mathrm{i}) \to H_0(|\mathring{D}|+\mathrm{i}) \oplus V. \tag{7}$$

As in the definition of an abstract linear control system $\mathscr{C}_{M,F,B}$ the adjoint of F comes into play. We compute it explicitly in the next theorem for the case, when G is assumed to be boundedly invertible. In applications this requirement can be guaranteed by assuming certain geometric properties of the underlying domain (e.g. segment property, Lipschitz boundary and so on).

THEOREM 4.8. *Let F be given as above and let G be boundedly invertible. Then*

$$\begin{aligned} F^* : D(F^*) \subseteq H_0(|\mathring{D}|+\mathrm{i}) \oplus V &\to H_0(|G|+\mathrm{i}) \\ (\zeta, w) &\mapsto \mathring{D}\zeta + C^{\diamond} w, \end{aligned}$$

where $C^{\diamond}$ is the dual operator of C with respect to the Gelfand-triplet $H_1(|G|+\mathrm{i}) \subseteq H_0(|G|+\mathrm{i}) \subseteq H_{-1}(|G|+\mathrm{i})$ and

$$D(F^*) = \{(\zeta, w) \in H_0(|\mathring{D}|+\mathrm{i}) \oplus V \,|\, \mathring{D}\zeta + C^{\diamond} w \in H_0(|G|+\mathrm{i})\}.$$

Proof. See [13, Theorem 5.4.]. □

REMARK 4.9. Let $\mathscr{C}_{M,F,B}$ be an abstract linear control system, where F is given by (7). We assume that G is boundedly invertible. Note that, as a consequence, $\mathring{D}$ is boundedly invertible as well. An element $(x,(\zeta,w)) \in H_0(|G|) \oplus \left(H_0(|\mathring{D}|) \oplus V\right)$ belongs to the domain of $\begin{pmatrix} 0 & -F^* \\ F & 0 \end{pmatrix}$ if and only if $x \in H_1(|G|)$ and $\mathring{D}\zeta + C^\diamond w \in H_0(|\mathring{D}|)$. The latter is equivalent to

$$\gamma_D(\zeta + (\mathring{D})^{-1} C^\diamond w) = 0. \tag{8}$$

Recall from (5) that M is of the form

$$M(z) = \begin{pmatrix} K(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 0 \; 0 \end{pmatrix} & 0 \end{pmatrix} + z \begin{pmatrix} 0 & \begin{pmatrix} M_{1,02} \\ M_{1,12} \end{pmatrix} \\ \begin{pmatrix} M_{1,20} \; M_{1,21} \end{pmatrix} & M_{1,22} \end{pmatrix}$$

for suitable operators $M_{1,ij}$ and $K : B_{\mathbb{C}}(r,r) \to L(H_0(|G|) \oplus H_0(|\mathring{D}|) \oplus V)$. Note that due to the block structure of F^*, this operator has indeed four lines and columns. With $(x,\xi) = (x,(\zeta,w))$, the third and fourth line of the equation given by $\mathscr{C}_{M,F,B}$ read as

$$\partial_0 \pi_V K(\partial_0^{-1}) \left(\pi^*_{H_0(|G|)} x + \pi^*_{H_0(|\mathring{D}|)} \zeta\right) + \partial_0 \pi_V K(\partial_0^{-1}) \pi_V^* w + \pi_V M_{1,12} y + Cx = \pi_V f + \pi_V Bu$$

and

$$M_{1,20} x + M_{1,21} \pi^*_{H_0(|\mathring{D}|)} \zeta + M_{1,21} \pi_V^* w + M_{1,22} y = \pi_Y f + \pi_Y Bu,$$

respectively. We may rewrite this as

$$\begin{aligned} &\begin{pmatrix} \partial_0 \pi_V K(\partial_0^{-1}) \pi_V^* & \pi_V M_{1,12} \\ M_{1,21} \pi_V^* & M_{1,22} \end{pmatrix} \begin{pmatrix} w \\ y \end{pmatrix} \\ &= \begin{pmatrix} \pi_V f + \pi_V Bu - \partial_0 \pi_V K(\partial_0^{-1})(\pi^*_{H_0(|G|)} x + \pi^*_{H_0(|\mathring{D}|)} \zeta) - Cx \\ \pi_Y f + \pi_Y Bu - M_{1,20} x - M_{1,21} \pi^*_{H_0(|\mathring{D}|)} \zeta \end{pmatrix} \end{aligned} \tag{9}$$

or equivalently as

$$\begin{aligned} &\begin{pmatrix} \partial_0 \pi_V K(\partial_0^{-1}) \pi_V^* & -\pi_V B \\ M_{1,21} \pi_V^* & -\pi_Y B \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix} \\ &= \begin{pmatrix} \pi_V f - \pi_V M_{1,12} y - \partial_0 \pi_V K(\partial_0^{-1})(\pi^*_{H_0(|G|)} x + \pi^*_{H_0(|\mathring{D}|)} \zeta) - Cx \\ \pi_Y f - M_{1,22} y - M_{1,20} x - M_{1,21} \pi^*_{H_0(|\mathring{D}|)} \zeta \end{pmatrix} \end{aligned} \tag{10}$$

If the material law M satisfies the solvability condition (3), then the operator on the left hand side of (9) is boundedly invertible and thus, we can express w in terms of x, ζ, f and u. Plugging this representation into (8) we obtain the boundary control equation. Analogously, by assuming that the operator on the left hand side of Equation (10) is boundedly invertible, we can express w in terms of x, ζ, f and y and hence, Equation (8) yields the boundary observation equation.

5. A boundary control problem in visco-elasticity

In this section we apply our results to a boundary control problem for the equations of visco-elasticity. For this purpose we introduce the required differential operators. Throughout let $\Omega \subseteq \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$, $n \in \mathbb{N}$, $n \geqslant 1$.

DEFINITION 5.1. We denote by $L_2(\Omega)^{n\times n}$ the Hilbert space of $n \times n$-matrices with entries in $L_2(\Omega)$ endowed with the inner product

$$\langle \Phi | \Psi \rangle_{L_2(\Omega)^{n\times n}} := \int_\Omega \operatorname{trace}\left(\Phi(x)^* \Psi(x)\right)\, \mathrm{d}x \quad \left(\Phi, \Psi \in L_2(\Omega)^{n\times n}\right).$$

Moreover let $H_{\mathrm{sym}}(\Omega) \subseteq L_2(\Omega)^{n\times n}$ denote the closed subspace of symmetric $n\times n$ matrices. We define the operator $\mathring{\mathrm{Grad}}$ as the closure of

$$\begin{aligned} \mathrm{Grad}\,|_{C_c^\infty(\Omega)^n} : C_c^\infty(\Omega)^n \subseteq L_2(\Omega)^n &\to H_{\mathrm{sym}}(\Omega) \\ (\phi_i)_{i\in\{1,\ldots,n\}} &\mapsto \left(\frac{1}{2}\left(\partial_j \phi_i + \partial_i \phi_j\right)\right)_{i,j\in\{1,\ldots,n\}} \end{aligned}$$

and $\mathring{\mathrm{Div}}$ as the closure of

$$\begin{aligned} \mathrm{Div}\,|_{C_{c,\mathrm{sym}}^\infty(\Omega)^{n\times n}} : C_{c,\mathrm{sym}}^\infty(\Omega)^{n\times n} \subseteq H_{\mathrm{sym}}(\Omega) &\to L_2(\Omega)^n \\ (\Phi_{ij})_{i,j\in\{1,\ldots,n\}} &\mapsto \left(\sum_{j=1}^n \partial_j \Phi_{ij}\right)_{i\in\{1,\ldots,n\}}, \end{aligned}$$

where we denote by $C_{c,\mathrm{sym}}^\infty(\Omega)^{n\times n}$ the space of symmetric $n\times n$-matrices with entries in $C_c^\infty(\Omega)$.

Furthermore we extend the meaning of $\mathring{\mathrm{grad}}$ by defining it as the closure of

$$\begin{aligned} \mathrm{grad}\,|_{C_c^\infty(\Omega)^n} : C_c^\infty(\Omega)^n \subseteq L_2(\Omega)^n &\to L_2(\Omega)^{n\times n} \\ (\psi_i)_{i\in\{1,\ldots,n\}} &\mapsto (\partial_j \psi_i)_{i,j\in\{1,\ldots,n\}} \end{aligned}$$

and similarly $\mathring{\mathrm{div}}$ as the closure of

$$\begin{aligned} \mathrm{div}\,|_{C_c^\infty(\Omega)^{n\times n}} : C_c^\infty(\Omega)^{n\times n} \subseteq L_2(\Omega)^{n\times n} &\to L_2(\Omega)^n \\ (\Psi_{ij})_{i,j\in\{1,\ldots,n\}} &\mapsto \left(\sum_{j=1}^n \partial_j \Psi_{ij}\right)_{i\in\{1,\ldots,n\}} \end{aligned}$$

leaving it to the context to determine if the scalar or the matrix version of these operations are meant.

An easy computation shows that $\mathring{\mathrm{Grad}}$ and $\mathring{\mathrm{Div}}$ are formally skew-adjoint, likewise the extended operations $\mathring{\mathrm{grad}}$ and $\mathring{\mathrm{div}}$ are formally skew-adjoint. Following the notation introduced in Section 4 we define $\operatorname{Grad} := -(\mathring{\operatorname{Div}})^*$, $\operatorname{Div} := -(\mathring{\operatorname{Grad}})^*$, $\operatorname{grad} := -(\mathring{\operatorname{div}})^*$ and $\operatorname{div} := -(\mathring{\operatorname{grad}})^*$. The equations of visco-elasticity are given by

$$\partial_0^2 \rho x(t) - \operatorname{Div} T(t) = f(t), \tag{11}$$

$$T(t) = M \operatorname{Grad} x(t) - \int_{-\infty}^{t} g(t-s) \operatorname{Grad} x(s) \, \mathrm{d}s. \tag{12}$$

Here $x \in H_{\nu,0}(\mathbb{R}; L_2(\Omega)^n)$ and $T \in H_{\nu,0}\left(\mathbb{R}; H_{\mathrm{sym}}(\Omega)\right)$ are the unknowns, denoting the displacement field and the stress tensor, respectively. The density function $\rho \in L_\infty(\Omega)$ is assumed to be real-valued and uniformly strictly positive, i.e. $\rho \geqslant c_1 > 0$. The tensor $M \in L\left(H_{\mathrm{sym}}(\Omega)\right)$, linking the stress and the strain tensor, is assumed to be selfadjoint and satisfies $M \geqslant c_2 > 0$. The function $g : \mathbb{R}_{\geqslant 0} \to \mathbb{C}$ is assumed to be absolutely continuous[4], i.e. $g(t) = \int_0^t h(s) \, \mathrm{d}s + g_0$ for some $h \in L_1(\mathbb{R}_{\geqslant 0})$ and $g_0 \in \mathbb{C}$. An easy computation shows that the convolution operator $g* : H_{\nu,0}(\mathbb{R}; H_{\mathrm{sym}}(\Omega)) \to H_{\nu,0}(\mathbb{R}; H_{\mathrm{sym}}(\Omega))$ is continuous for each $\nu > 0$ with

$$\|g *\|_{L(H_{\nu,0}(\mathbb{R}; H_{\mathrm{sym}}(\Omega)))} \leqslant \frac{1}{\nu} \left(|h|_{L_1(\mathbb{R}_{\geqslant 0})} + |g_0| \right).$$

Thus, for $\nu > 0$ large enough, the operator $(1 - M^{-1} g*)$ is invertible, and hence we may write (12) as

$$(M - g*)^{-1} T = (1 - M^{-1} g*)^{-1} M^{-1} T = \operatorname{Grad} x. \tag{13}$$

The boundary control and observation equations are given by

$$TN = \partial_0 x + \sqrt{2} u, \qquad TN = \sqrt{2} y - \partial_0 x, \tag{14}$$

on $\partial\Omega$, where we denote by N the outer unit normal vector field.

REMARK 5.2. Since we have to compare Neumann-type traces and Dirichlet-type traces we have to determine a suitable control and observation space. For doing so, let us assume for the moment that $\partial\Omega$ is smooth. We consider the space $L_2(\partial\Omega)^n$. We assume that the outer unit normal vector field N can be extended to Ω such that $N \in L_\infty(\Omega; \mathbb{R})^n$ and $\operatorname{div} N \in L_\infty(\Omega)$. For $f, g \in \mathrm{BD}(\operatorname{grad})$ we formally compute using the divergence theorem[5]

$$\int_{\partial\Omega} f \cdot g \, \mathrm{d}S$$

$$= \int_{\partial\Omega} (f \cdot g)(N \cdot N) \, \mathrm{d}S$$

[4]The equations can also be studied in a more general setting, for instance g can attain values in $L(H_{\mathrm{sym}}(\Omega))$ (cf. [17]).

[5]Note that due to the assumptions on the vector field N, the matrix-valued function $(f_i N_k)_{i,k \in \{1,\ldots,n\}}$ lies in $D(\operatorname{div})$ for each $f \in D(\operatorname{grad})$.

$$
\begin{aligned}
&= \frac{1}{2}\int_{\partial\Omega} \left((N_k f_i)_{k,i} g^*\right)\cdot N \,\mathrm{d}S + \frac{1}{2}\int_{\partial\Omega} \left((N_k g_i^*)_{k,i} f\right)\cdot N \,\mathrm{d}S \\
&= \frac{1}{2}\int_{\Omega} \operatorname{div}\left((N_k f_i^*)_{k,i} g\right) + \frac{1}{2}\int_{\Omega} \operatorname{div}\left((N_k g_i)_{k,i} f^*\right) \\
&= \frac{1}{2}\left(\left\langle \operatorname{div}(f_i N_k)_{i,k} \middle| g\right\rangle_{L_2(\Omega)^n} + \left\langle (f_i N_k)_{i,k} \middle| \operatorname{grad} g\right\rangle_{L_2(\Omega)^{n\times n}}\right) \\
&\quad + \frac{1}{2}\left(\left\langle f \middle| \operatorname{div}(g_i N_k)_{i,k}\right\rangle_{L_2(\Omega)^n} + \left\langle \operatorname{grad} f \middle| (g_i N_k)_{i,k}\right\rangle_{L_2(\Omega)^{n\times n}}\right) \\
&= \frac{1}{2}\left(\left\langle \pi_{\mathrm{BD}(\mathrm{div})} (f_i N_k)_{i,k} \middle| \overset{\bullet}{\operatorname{grad}}\, g\right\rangle_{\mathrm{BD}(\mathrm{div})} + \left\langle \overset{\bullet}{\operatorname{grad}}\, f \middle| \pi_{\mathrm{BD}(\mathrm{div})} (g_i N_k)_{i,k}\right\rangle_{\mathrm{BD}(\mathrm{div})}\right).
\end{aligned}
$$

This leads to the following choice for the control space U.

DEFINITION 5.3. Let $N \in L_\infty(\Omega;\mathbb{R})^n$ be such that $\operatorname{div} N \in L_\infty(\Omega)$. We define the bounded linear operator $\nu : BD(\operatorname{grad}) \to \mathrm{BD}(\operatorname{div})$ by $\nu f := \pi_{\mathrm{BD}(\mathrm{div})} (f_i N_k)_{i,k\in\{1,\ldots,n\}}$. We assume that the operator $\overset{\bullet}{\operatorname{div}}\,\nu + \nu^* \overset{\bullet}{\operatorname{grad}}$ is positive, i.e. for every $f \in \mathrm{BD}(\operatorname{grad}) \setminus \{0\}$ we have

$$
\left\langle \left(\overset{\bullet}{\operatorname{div}}\,\nu + \nu^* \overset{\bullet}{\operatorname{grad}}\right) f \middle| f\right\rangle_{\mathrm{BD}(\operatorname{grad})} > 0.
$$

We define the Hilbert space U as the completion of $\mathrm{BD}(\operatorname{grad})$ with respect to the inner product

$$
\begin{aligned}
&\langle\cdot|\cdot\rangle_U : \mathrm{BD}(\operatorname{grad}) \times \mathrm{BD}(\operatorname{grad}) \to \mathbb{C} \\
&\qquad (f,g) \mapsto \frac{1}{2}\left(\left\langle \nu f \middle| \overset{\bullet}{\operatorname{grad}}\, g\right\rangle_{\mathrm{BD}(\mathrm{div})} + \left\langle \overset{\bullet}{\operatorname{grad}}\, f \middle| \nu g\right\rangle_{\mathrm{BD}(\mathrm{div})}\right).
\end{aligned}
$$

We denote the embedding $\mathrm{BD}(\operatorname{grad}) \hookrightarrow U$ by ι.

In the following we require that Korn's inequality holds, i.e.,

$$
H_1(|\operatorname{Grad}|+\mathrm{i}) \overset{\kappa}{\hookrightarrow} H_1(|\operatorname{grad}|+\mathrm{i})
$$

(for sufficient criteria see [1] and the references therein). We consider the bounded operator $j : \mathrm{BD}(\operatorname{Grad}) \to U$ given by $j = \iota \circ \pi_{\mathrm{BD}(\operatorname{grad})} \circ \kappa \circ \pi^*_{\mathrm{BD}(\operatorname{Grad})}$ and compute

$$
\begin{aligned}
\langle jf|g\rangle_U &= \langle \pi_{\mathrm{BD}(\operatorname{grad})} \kappa \pi^*_{\mathrm{BD}(\operatorname{Grad})} f|g\rangle_U \\
&= \frac{1}{2}\left(\left\langle \nu \pi_{\mathrm{BD}(\operatorname{grad})} \kappa \pi^*_{\mathrm{BD}(\operatorname{Grad})} f \middle| \overset{\bullet}{\operatorname{grad}}\, g\right\rangle_{\mathrm{BD}(\mathrm{div})}\right. \\
&\qquad \left. + \left\langle \overset{\bullet}{\operatorname{grad}}\, \pi_{\mathrm{BD}(\operatorname{grad})} \kappa \pi^*_{\mathrm{BD}(\operatorname{Grad})} f \middle| \nu g\right\rangle_{\mathrm{BD}(\mathrm{div})}\right)
\end{aligned}
$$

$$= \frac{1}{2}\left(\left\langle f \,\middle|\, \pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\nu^*\,\mathring{\mathrm{grad}}\,g\right\rangle_{\mathrm{BD(Grad)}} + \left\langle f \,\middle|\, \pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\mathring{\mathrm{div}}\,\nu g\right\rangle_{\mathrm{BD(Grad)}}\right)$$
$$= \left\langle f \,\middle|\, \frac{1}{2}\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\left(\nu^*\,\mathring{\mathrm{grad}} + \mathring{\mathrm{div}}\,\nu\right)g\right\rangle_{\mathrm{BD(Grad)}}$$

for each $f \in \mathrm{BD(Grad)}$, $g \in \mathrm{BD(grad)}$. This gives

$$j^* = \frac{1}{2}\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\left(\nu^*\,\mathring{\mathrm{grad}} + \mathring{\mathrm{div}}\,\nu\right)$$

and, consequently,

$$\begin{aligned}&\gamma_{\mathrm{Div}}\,\pi^*_{\mathrm{BD(Div)}}\,\mathring{\mathrm{Grad}}\,j^*g \\ &= \frac{1}{2}\gamma_{\mathrm{Div}}\,\pi^*_{\mathrm{BD(Div)}}\,\mathring{\mathrm{Grad}}\,\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\left(\nu^*\,\mathring{\mathrm{grad}} + \mathring{\mathrm{div}}\,\nu\right)g.\end{aligned} \tag{15}$$

REMARK 5.4. We give a possible interpretation of the latter equality. For this purpose we compute formally using the divergence theorem

$$\begin{aligned}
&\int_{\partial\Omega} (\mathrm{Grad}\,j^*g)N\cdot f\,\mathrm{d}S \\
&= \langle \gamma_{\mathrm{Div}}\,\pi^*_{\mathrm{BD(Div)}}\,\mathring{\mathrm{Grad}}\,j^*g|\pi^*_{\mathrm{BD(Grad)}}f\rangle \\
&= \frac{1}{2}\langle \gamma_{\mathrm{Div}}\,\pi^*_{\mathrm{BD(Div)}}\,\mathring{\mathrm{Grad}}\,\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\nu^*\,\mathring{\mathrm{grad}}\,g|\pi^*_{\mathrm{BD(Grad)}}f\rangle \\
&\quad + \frac{1}{2}\langle \gamma_{\mathrm{Div}}\,\pi^*_{\mathrm{BD(Div)}}\,\mathring{\mathrm{Grad}}\,\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\mathring{\mathrm{div}}\,\nu g|\pi^*_{\mathrm{BD(Grad)}}f\rangle \\
&= \frac{1}{2}\langle \pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\nu^*\,\mathring{\mathrm{grad}}\,g|f\rangle \\
&\quad + \frac{1}{2}\langle \mathring{\mathrm{Grad}}\,\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\nu^*\,\mathring{\mathrm{grad}}\,g|\,\mathring{\mathrm{Grad}}\,f\rangle \\
&\quad + \frac{1}{2}\langle \pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\mathring{\mathrm{div}}\,\nu g|f\rangle \\
&\quad + \frac{1}{2}\langle \mathring{\mathrm{Grad}}\,\pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\mathring{\mathrm{div}}\,\nu g|\,\mathring{\mathrm{Grad}}\,f\rangle \\
&= \frac{1}{2}\langle \pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\nu^*\,\mathring{\mathrm{grad}}\,g|f\rangle_{\mathrm{BD(Grad)}} \\
&\quad + \frac{1}{2}\langle \pi_{\mathrm{BD(Grad)}}\,\kappa^*\pi^*_{\mathrm{BD(grad)}}\,\mathring{\mathrm{div}}\,\nu g|f\rangle_{\mathrm{BD(Grad)}} \\
&= \frac{1}{2}\langle \mathring{\mathrm{grad}}\,g|\nu\pi_{\mathrm{BD(grad)}}\,\kappa\pi^*_{\mathrm{BD(Grad)}}f\rangle_{\mathrm{BD(div)}}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\langle \nu g|\overset{\bullet}{\operatorname{grad}}\,\pi_{\mathrm{BD(grad)}}\kappa\pi^*_{\mathrm{BD(Grad)}}f\rangle_{\mathrm{BD(div)}}\\
&=\langle g|\pi_{\mathrm{BD(grad)}}\kappa\pi^*_{\mathrm{BD(Grad)}}f\rangle_U\\
&=\int_{\partial\Omega} g\cdot\pi_{\mathrm{BD(grad)}}\kappa\pi^*_{\mathrm{BD(Grad)}}f\,\mathrm{d}S\\
&=\int_{\partial\Omega} g\cdot\pi_{\mathrm{BD(grad)}}\kappa\pi^*_{\mathrm{BD(Grad)}}f\,\mathrm{d}S+\int_{\partial\Omega} g\cdot(1-\pi_{\mathrm{BD(grad)}})\kappa\pi^*_{\mathrm{BD(Grad)}}f\,\mathrm{d}S\\
&=\int_{\partial\Omega} g\cdot\kappa\pi^*_{\mathrm{BD(Grad)}}f\,\mathrm{d}S\\
&=\int_{\partial\Omega} g\cdot f\,\mathrm{d}S
\end{aligned}
$$

for each $f\in\mathrm{BD(Grad)}, g\in\mathrm{BD(grad)}$. Hence, equality (15) can be seen as a generalization of

$$(\operatorname{Grad} j^*g)N=g \text{ on } \partial\Omega \tag{16}$$

to the case of non-smooth boundaries.

We now want to transform the equations (11), (13) and (14) into a system of the form treated in the previous subsections. In the terminology of Subsection 4 the operator Grad should play the role of G and Div the role of D. Since we have assumed in Theorem 4.8 that G is boundedly invertible, we require that $\operatorname{Grad}[L_2(\Omega)^n]$ is closed in $H_{\mathrm{sym}}(\Omega)$.[6] Then the projection theorem yields the following orthogonal decompositions[7]

$$
\begin{aligned}
L_2(\Omega)^n&=[\{0\}]\operatorname{Grad}\oplus\overline{\overset{\circ}{\operatorname{Div}}\,[H_{\mathrm{sym}}(\Omega)]},\\
H_{\mathrm{sym}}(\Omega)&=[\{0\}]\,\overset{\circ}{\operatorname{Div}}\oplus\operatorname{Grad}[L_2(\Omega)^n].
\end{aligned}
$$

We define the following orthogonal projections $\pi_{\mathrm{Div}}:L_2(\Omega)^n\to\overline{\overset{\circ}{\operatorname{Div}}\,[H_{\mathrm{sym}}(\Omega)]}$ and $\pi_{\mathrm{Grad}}:H_{\mathrm{sym}}(\Omega)\to\operatorname{Grad}[L_2(\Omega)^n]$. Note that due to the closed graph theorem the operator $\widetilde{\operatorname{Grad}}:=\pi_{\mathrm{Grad}}\operatorname{Grad}\pi^*_{\mathrm{Div}}$ is boundedly invertible and so is $\left(\widetilde{\operatorname{Grad}}\right)^*=-\pi_{\mathrm{Div}}\overset{\circ}{\operatorname{Div}}\,\pi^*_{\mathrm{Grad}}$. Furthermore let us denote by ι_{Grad} the canonical embedding

$$H_1\left(\left|\widetilde{\operatorname{Grad}}\right|+\mathrm{i}\right)\hookrightarrow H_1\left(|\operatorname{Grad}|+\mathrm{i}\right).$$

We consider the following evolutionary problem

[6] The closedness of the range $\operatorname{Grad}[L_2(\Omega)^n]$ holds, for instance if $H_1(|\operatorname{Grad}|+\mathrm{i})$ is compactly embedded in $L_2(\Omega)^n$ and we refer to [20] for sufficient conditions on Ω yielding this compact embedding. Note that this compact embedding yields then a Poincare-type estimate, which in turn yields the closedness of the range. Thus, the minimal assumption is the validity of a Poincare-type estimate.

[7] Note that the closedness of the range of Grad also yields the closedness of range of $\overset{\circ}{\operatorname{Div}}$. Since we do not want to give the details of the proof here, we use the closure bar for convenience.

$$\left(\partial_0 \begin{pmatrix} \pi_{\mathrm{Div}}\rho\pi_{\mathrm{Div}}^* & (0\ 0) & 0 \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}\pi_{\mathrm{Grad}}(M-g*)^{-1}\pi_{\mathrm{Grad}}^* & 0\\ 0 & \partial_0^{-1}\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & (0\ 0) & 0\end{pmatrix} + \begin{pmatrix} 0 & (0\ 0) & 0 \\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} \\ 0 & (0\ \sqrt{2}) & 1\end{pmatrix} + \begin{pmatrix} 0 & \left(-\left(\widetilde{\mathrm{Grad}}\right)^* \ -C^\diamond\right) & 0 \\ \begin{pmatrix}-\widetilde{\mathrm{Grad}}\\C\end{pmatrix} & \begin{pmatrix}0&0\\0&0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix}\\ 0 & (0\ 0) & 0\end{pmatrix}\right)\begin{pmatrix}v\\ \begin{pmatrix}T\\w\end{pmatrix}\\ y\end{pmatrix}$$
$$= \begin{pmatrix} f\\ \begin{pmatrix}0\\0\end{pmatrix}\\0\end{pmatrix} + \begin{pmatrix}0\\ \begin{pmatrix}0\\-\sqrt{2}\end{pmatrix}\\ -1\end{pmatrix}u, \quad (17)$$

where $C : H_1\left(\left|\widetilde{\mathrm{Grad}}\right|+\mathrm{i}\right) \to U$ is given by $Cx := j\pi_{\mathrm{BD(Grad)}}\iota_{\mathrm{Grad}}x$. The material law K is given by

$$K(z) = \begin{pmatrix} \pi_{\mathrm{Div}}\rho\pi_{\mathrm{Div}}^* & 0 & 0 \\ 0 & \pi_{\mathrm{Grad}}\left(M-\sqrt{2\pi}\widehat{g}(-\mathrm{i}z^{-1})\right)^{-1}\pi_{\mathrm{Grad}}^* & 0 \\ 0 & 0 & z \end{pmatrix}$$

and satisfies the solvability condition (3). Indeed, using the representation $z^{-1} = \mathrm{i}t+\nu$ for some $\nu > \frac{1}{2r}, t \in \mathbb{R}$ if $z \in B_{\mathbb{C}}(r,r)$ we estimate

$$\mathfrak{Re}\, z^{-1}\pi_{\mathrm{Div}}\rho\pi_{\mathrm{Div}}^* \geqslant \nu c_1$$

and

$$\begin{aligned}
&\mathfrak{Re}\, z^{-1}\pi_{\mathrm{Grad}}\left(M-\sqrt{2\pi}\widehat{g}(-\mathrm{i}z^{-1})\right)^{-1}\pi_{\mathrm{Grad}}^* \\
&= \mathfrak{Re}\, z^{-1}\pi_{\mathrm{Grad}}\left(\sum_{k=0}^{\infty}(2\pi)^{\frac{k}{2}}M^{-k}\widehat{g}(-\mathrm{i}z^{-1})^k\right)M^{-1}\pi_{\mathrm{Grad}}^* \\
&= \nu\pi_{\mathrm{Grad}}M^{-1}\pi_{\mathrm{Grad}}^* \\
&\quad + \mathfrak{Re}\,\pi_{\mathrm{Grad}}z^{-1}\left(\sqrt{2\pi}M^{-1}\widehat{g}(-\mathrm{i}z^{-1})\right)\left(\sum_{k=0}^{\infty}(2\pi)^{\frac{k}{2}}M^{-k}\widehat{g}(-\mathrm{i}z^{-1})^k\right)M^{-1}\pi_{\mathrm{Grad}}^* \\
&= \nu\pi_{\mathrm{Grad}}M^{-1}\pi_{\mathrm{Grad}}^* \\
&\quad + \mathfrak{Re}\,\pi_{\mathrm{Grad}}M^{-1}\left(\sqrt{2\pi}\widehat{h}(-\mathrm{i}z^{-1})+g_0\right)\left(\sum_{k=0}^{\infty}(2\pi)^{\frac{k}{2}}M^{-k}\widehat{g}(-\mathrm{i}z^{-1})^k\right)M^{-1}\pi_{\mathrm{Grad}}^* \\
&\geqslant \nu c_2 - \frac{\|M^{-1}\|^2\left(|h|_{L_1(\mathbb{R}_{\geqslant 0})}+|g_0|\right)}{1-\nu^{-1}\|M^{-1}\|\left(|h|_{L_1(\mathbb{R}_{\geqslant 0})}+|g_0|\right)},
\end{aligned}$$

where we have used that $\widehat{g}(-\mathrm{i}z^{-1}) = z\widehat{h}(-\mathrm{i}z^{-1}) + \frac{z}{\sqrt{2\pi}} g_0$. Summarizing this gives $\mathfrak{Re}\, z^{-1}K(z) \geqslant 1$. The operator J is given by $J = \left(\begin{array}{c} 0 \\ \left(\begin{array}{c} 0 \\ \frac{1}{\sqrt{2}} \end{array}\right) \end{array}\right)$ and thus, $\|J\| = \frac{1}{\sqrt{2}}$. Since $\mathfrak{Re}\, M_{1,22} = 1$, Theorem 3.3 applies and thus, the control system given by (17) is well-posed. Next, we compute $C^\diamond$. For that purpose let $x \in H_1\left(\left|\widetilde{\operatorname{Grad}}\right| + \mathrm{i}\right)$ and $w \in U$. Then

$$
\begin{aligned}
\langle C^\diamond w | x\rangle &= \langle w | Cx\rangle_U \\
&= \langle w | j\pi_{\mathrm{BD}(\operatorname{Grad})}\iota_{\operatorname{Grad}} x\rangle_U \\
&= \langle \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w | \iota_{\operatorname{Grad}} x\rangle_{H_1(|\operatorname{Grad}|+\mathrm{i})} \\
&= \langle \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w | \iota_{\operatorname{Grad}} x\rangle + \langle \operatorname{Grad} \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w | \operatorname{Grad} \iota_{\operatorname{Grad}} x\rangle \\
&= \langle \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w | x\rangle + \left\langle \pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^* w \middle| \widetilde{\operatorname{Grad}} x \right\rangle \\
&= \langle \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w | x\rangle + \left\langle \left(\widetilde{\operatorname{Grad}}\right)^* \pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^* w \middle| x \right\rangle
\end{aligned}
$$

Summarizing, we get that $C^\diamond = \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* + \left(\widetilde{\operatorname{Grad}}\right)^* \pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^*$. According to the definition of the domain of $\left(-\left(\widetilde{\operatorname{Grad}}\right)^* - C^\diamond\right)$ the implicit boundary condition for the system reads as

$$
\left(\widetilde{\operatorname{Grad}}\right)^* T + C^\diamond w \in H_0\left(\left|\widetilde{\operatorname{Grad}}\right| + \mathrm{i}\right).
$$

Hence,

$$
T + \left(\left(\widetilde{\operatorname{Grad}}\right)^*\right)^{-1} C^\diamond w \in H_1\left(\left|\left(\widetilde{\operatorname{Grad}}\right)^*\right| + \mathrm{i}\right) \subseteq H_1\left(\left|\overset{\circ}{\operatorname{Div}}\right| + \mathrm{i}\right) \subseteq H_1\left(|\operatorname{Div}| + \mathrm{i}\right).
$$

From

$$
\pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^* w = \pi_{\operatorname{Grad}} \operatorname{Grad} \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w \in H_1(|\operatorname{Div}| + \mathrm{i})
$$

it thus follows that $T \in H_1\left(|\operatorname{Div}| + \mathrm{i}\right)$ and

$$
\begin{aligned}
\gamma_{\operatorname{Div}} T &= \gamma_{\operatorname{Div}}\left(-\left(\left(\widetilde{\operatorname{Grad}}\right)^*\right)^{-1} C^\diamond w\right) \\
&= \gamma_{\operatorname{Div}}\left(-\left(\left(\widetilde{\operatorname{Grad}}\right)^*\right)^{-1} \pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Grad})}^* j^* w - \pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^* w\right) \\
&= -\gamma_{\operatorname{Div}} \pi_{\operatorname{Grad}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^* w \\
&= -\gamma_{\operatorname{Div}} \pi_{\mathrm{BD}(\operatorname{Div})}^* \overset{\bullet}{\operatorname{Grad}}\, j^* w,
\end{aligned}
$$

where we used that γ_{Div} vanishes on the domain of $\overset{\circ}{\mathrm{Div}}$, which is a superset of the domain of $\left(\widetilde{\mathrm{Grad}}\right)^*$. Using (9) we get

$$\begin{pmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} -\sqrt{2}u - Cv \\ -u \end{pmatrix}$$

and hence

$$w = -\sqrt{2}u - Cv = -\sqrt{2}u - j\pi_{\mathrm{BD(Grad)}}\iota_{\mathrm{Grad}}v.$$

Analogously one obtains, using (10), that

$$\begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix}\begin{pmatrix} w \\ u \end{pmatrix} = \begin{pmatrix} -Cv \\ -y \end{pmatrix}$$

and thus,

$$w = Cv - \sqrt{2}y = j\pi_{\mathrm{BD(Grad)}}\iota_{\mathrm{Grad}}v - \sqrt{2}y.$$

Following the reasoning of Remark 5.4, we may interpret the resulting boundary control and observation equations as

$$TN = v - u = \partial_0 x + \sqrt{2}u,$$
$$TN = \sqrt{2}y - v = \sqrt{2}y - \partial_0 x$$

on $\partial\Omega$.

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