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ESSENTIAL SPECTRA OF SOME MATRIX OPERATORS BY MEANS OF MEASURES OF WEAK NONCOMPACTNESS

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Abstract. In this paper, we give some results concerning stability in the Fredholm theory via the concept of measures of weak noncompactness. These results are exploited to investigate the essential spectra of some matrix operators on Banach spaces. This work contains some results which extend some well known ones in the literature.

1. Introduction

This paper is devoted to study the essential spectra of a general class of operators defined by a 2×2 block operator matrix acting in a product of Banach spaces $X\times X$

$$L_0=\begin{pmatrix} A & B \\ C & D\end{pmatrix},$$

where the operators occurring in the representation of L_0 are unbounded. A acts on the Banach space X and has the domain $\mathscr{D}(A)$, D is defined on $\mathscr{D}(D)$ and acts on X. The intertwining operators B, C are defined respectively on $\mathscr{D}(B)$, $\mathscr{D}(C)$ and act on X. Below, we shall assume that $\mathscr{D}(A)\subset\mathscr{D}(C)$ and $\mathscr{D}(B)\subset\mathscr{D}(D)$. Then the matrix L_0 defines a linear operator in X with domain $\mathscr{D}(A)\times\mathscr{D}(B)$.

One of the problems in the study of such operators is that in general L_0 is not closed or even closable, even if its entries are closed. In [5], the authors give some sufficient conditions under which L_0 is closable and describe its closure which we shall denote L.

The study of the essential spectra of operators defined by a 2×2 block operator matrix has been arround for many years. Among the works in this subject we can quote, for example, [5, 14, 19]. The authors in [5] used the compactness condition for the operator $(\lambda-A)^{-1}$ to describe, under certain additional assumptions, the Fredholm essential spectrum. Whereas in the paper of [19], Shkalikov has assumed that, for some (and hence for all) $\nu\in\rho(A)$, $F(\nu):=C(\nu-A)^{-1}$ and $G(\nu):=\overline{(\lambda-A)^{-1}B}$ are compacts (see Theorem 2) and has proved that

$$\sigma_{eF}(L)=\sigma_{eF}(A)\cup\sigma_{eF}(\overline{D-C(\nu-A)^{-1}B}).$$

Similarly, the authors in [14] extend the obtained results in [5, 19] into a large class of operators and investigate the essential spectra of the matrix operator L. In this paper,

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we prove some localization results on the essential spectra of the matrix operator L with the help of the measures of weak-noncompactness.

The purpose of this work is to pursue the analysis started in [5, 14, 19]. First, we establish some stability results on Fredholm theory (see Theorems 2 and 3). Second, we study the essential spectra of a general class of operators defined by a 2×2 block operator matrix (see Corollaries 2 and 3).

2. Notations and Definitions

For X and Y be two infinite-dimensional Banach spaces, we denote by $\mathscr{C}(X,Y)$ (resp., $\mathscr{L}(X,Y)$) the set of all closed densely defined linear operators (resp., the space of all bounded linear operators) acting from X into Y. The subspace of all compact (resp., weakly compact) operators of $\mathscr{L}(X,Y)$ is designed by $\mathscr{K}(X,Y)$ (resp., $\mathscr{W}(X,Y)$). For $T\in\mathscr{C}(X,Y)$ we use the following notations: $\mathscr{D}(T)$ is the domain, $\mathscr{N}(T)$ is the kernel and $\mathscr{R}(T)$ is the range of T. The nullity, $\alpha(T)$, of T is defined as the dimension of $\mathscr{N}(T)$ and the deficiency, $\beta(T)$, of T is defined as the codimension of $\mathscr{R}(T)$ in Y. We use $\sigma(T)$ and $\rho(T)$ to denote the spectrum and the resolvent set of T. Recall that an operator $T\in\mathscr{C}(X,Y)$ is upper semi-Fredholm if its range is closed and its kernel is finite dimensional; it is lower semi-Fredholm if its range is finite codimension, hence closed; and it is Fredholm if it is upper semi-Fredholm and lower semi-Fredholm. We denote by $\Phi_+(X,Y)$, $\Phi_-(X,Y)$ and $\Phi(X,Y)$ the classes of upper semi-Fredholm, lower semi-Fredholm and Fredholm operators, resp., If $X=Y$, the sets $\mathscr{L}(X,X)$, $\mathscr{C}(X,X)$, $\mathscr{K}(X,X)$, $\mathscr{W}(X,Y)$, $\Phi_+(X,X)$, $\Phi_-(X,X)$, $\Phi(X,X)$, are replaced, resp., $\mathscr{L}(X)$, $\mathscr{C}(X)$, $\mathscr{K}(X)$, $\mathscr{W}(X)$, $\Phi_+(X)$, $\Phi_-(X)$, $\Phi(X)$. If $T\in\Phi_+(X)\cup\Phi_-(X)$, the number $i(T):=\alpha(T)-\beta(T)$ is called the index of T.

Recall that, for $T\in\mathscr{C}(X)$, $X_T:=\mathscr{D}(T)$ (the domain of T) endowed with the graph norm $\|.\|_T$ (i.e., $\|x\|_T=\|x\|+\|Tx\|$) is a Banach space and we have $T\in\mathscr{L}(X_T,X)$. We denote by $\widehat{T}$ the restriction of T to $\mathscr{D}(T)$. Let J be a linear operator on X such that $X_T\subset\mathscr{D}(J)$. We say that J is T-bounded if its restriction to X_T, $\widehat{J}$ belongs to $\mathscr{L}(X_T,X)$.

Notice that if $T\in\mathscr{C}(X)$ and J a T-bounded, then we get the obvious relations

$$\begin{cases}\alpha(\widehat{T})=\alpha(T),\ \beta(\widehat{T})=\beta(T),\ \mathscr{R}(\widehat{T})=\mathscr{R}(T),\\ \alpha(\widehat{T}+\widehat{J})=\alpha(T+J),\\ \beta(\widehat{T}+\widehat{J})=\beta(T+J),\ \mathscr{R}(\widehat{T}+\widehat{J})=\mathscr{R}(T+J).\end{cases}\tag{1}$$

Hence, $T\in\Phi(X)$ (resp., $\Phi_+(X)$) if and only if $\widehat{T}\in\Phi(X_T,X)$ (resp., $\Phi_+(X_T,X)$).

DEFINITION 1. Let X and Y be two Banach spaces.

1. Let $T\in\mathscr{L}(X,Y)$.

(i) T is said to have a left Fredholm inverse (resp., left weak-Fredholm inverse) if there exists $T_l\in\mathscr{L}(Y,X)$ and $K\in\mathscr{K}(X)$ (resp., $T_l^w\in\mathscr{L}(Y,X)$ and $W\in\mathscr{W}(X)$) such that $T_lT=I_X-K$ (resp., $T_l^wT=I_X-W$). The operator T_l (resp., T_l^w) is called left Fredholm inverse of T (resp., left weak-Fredholm inverse of T).

(ii) T is said to have a right Fredholm inverse (resp., right weak-Fredholm inverse) if there exists $T_r \in \mathscr{L}(Y,X)$ (resp., $T_r^w \in \mathscr{L}(Y,X)$) such that $I_Y - TT_r \in \mathscr{K}(Y)$ (resp., $I_Y - TT_r^w \in \mathscr{W}(Y)$). The operator T_r (resp., T_r^w) is called right Fredholm inverse of T (resp., right weak-Fredholm inverse of T).

(iii) T is said to have a Fredholm inverse (resp., weak-Fredholm inverse) if there exists a map which is both a left and a right Fredholm inverse of T (resp., a left and a right weak-Fredholm inverse of T).

2. Let $T \in \mathscr{C}(X)$. T is said to have a left Fredholm inverse (resp., right Fredholm inverse, Fredholm inverse, left weak-Fredholm inverse, right weak-Fredholm inverse, weak-Fredholm inverse) if $\widehat{T}$ has a left Fredholm inverse (resp., right Fredholm inverse, Fredholm inverse, left weak-Fredholm inverse, right weak-Fredholm inverse, weak-Fredholm inverse).

The sets of left, right, left weakly and right weak-Fredholm inverses are resp., defined by:

$$\begin{aligned}
&\Phi_l(X) := \{T \in \mathscr{C}(X) \;\; T \text{ has a left Fredholm inverse}\},\\
&\Phi_r(X) := \{T \in \mathscr{C}(X) \;\; T \text{ has a right Fredholm inverse}\},\\
&\Phi_l^w(X) := \{T \in \mathscr{C}(X) \;\; T \text{ has a left weak-Fredholm inverse}\},\\
&\Phi_r^w(X) := \{T \in \mathscr{C}(X) \;\; T \text{ has a right weak-Fredholm inverse}\}.
\end{aligned}$$

The class of weak-Fredholm operators is $\Phi^w(X) := \Phi_l^w(X) \cap \Phi_r^w(X)$. It is easy to see that $\Phi_l(X) \subset \Phi_l^w(X)$, $\Phi_r(X) \subset \Phi_r^w(X)$ and $\Phi_r(X) \cap \Phi_l(X) = \Phi(X) \subset \Phi^w(X)$.

A complex number λ is in Φ_{lT}, Φ_{rT}, Φ_T, Φ_{lT}^w, Φ_{rT}^w or Φ_T^w if $\lambda - T$ is in $\Phi_l(X)$, $\Phi_r(X)$, $\Phi(X)$, $\Phi_l^w(X)$, $\Phi_r^w(X)$ or $\Phi^w(X)$, resp.,

There are many ways to define the essential spectrum of a closed densely defined linear operator on a Banach space. In this paper, we are concerned, for $T \in \mathscr{C}(X)$, with the following essential spectra:

$$\begin{aligned}
\sigma_{eF}(T) &= \{\lambda \in \mathbb{C} \;\; \lambda - T \notin \Phi(X)\},\\
\sigma_{eW}(T) &= \mathbb{C} \backslash \rho_{eW}(T),\\
\sigma_{eB}(T) &= \mathbb{C} \backslash \rho_{eB}(T),
\end{aligned}$$

where $\rho_{eW}(T) := \{\lambda \in \mathbb{C} \;\; \lambda - T \in \Phi(X) \text{ and } i(\lambda - T) = 0\}$ and $\rho_{eB}(T)$ denotes the set of those $\lambda \in \rho_{eW}(T)$ such that all scalars near λ are in $\rho(T)$.

$$\begin{aligned}
\sigma_{le}(T) &:= \{\lambda \in \mathbb{C} \;\; \lambda - T \notin \Phi_l(X)\},\\
\sigma_{re}(T) &:= \{\lambda \in \mathbb{C} \;\; \lambda - T \notin \Phi_r(X)\}.
\end{aligned}$$

$\sigma_{eF}(.)$ is the Fredholm essential spectrum [20]. $\sigma_{eW}(.)$ is the Wolf essential spectrum [17]. $\sigma_{eB}(.)$ is the Browder essential spectrum [16] and $\sigma_{le}(.)$ (resp., $\sigma_{re}(.)$) is the left (resp., right) essential spectrum [12].

Note that, in general, we have $\sigma_{le}(T) \cup \sigma_{re}(T) = \sigma_{eF}(T) \subset \sigma_{eW}(T) \subset \sigma_{eB}(T)$.

Recall that a Banach space X is said to have the Dunford-Pettis property (for short DP property) if for each Banach space Y every weakly compact operator $T \;\; X \longrightarrow Y$ takes weakly compact sets in X into norm compact sets of Y. It is well known that any L_1-space has the property DP [8]. Also if Ω is a compact Hausdorff space, $C(\Omega)$ has the DP property. For more information we refer to the papers [7, 8, 9, 10].

3. On the Fredholm theory associated with measures of weak noncompactness

The purpose of this section is to establish some results concerning stability in the class of Fredholm operators. Among the works in this direction we can quote, for example, [1, 2, 3, 4]. First, we adopt the following definitions:

DEFINITION 2. ([6]) Let X be a Banach space and let $\mathscr{M}_X$ be the family of all nonempty and bounded subsets of X. A function $\mu : \mathscr{M}_X \longrightarrow [0,+\infty[$ is said to be a measure of weak noncompactness in X if, for all $A,B \in \mathscr{M}_X$, it satisfies the following conditions:

(i) $\mu(A) = 0$ if and only if A is relatively weakly compact.
(ii) $A \subset B \Longrightarrow \mu(A) \leqslant \mu(B)$.
(iii) $\mu(\overline{\text{conv}\ (A)}) = \mu(A)$, where $\text{conv}(A)$ denotes the convex hull of A.
(vi) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$.
(v) $\mu(A+B) \leqslant \mu(A) + \mu(B)$.
(vi) $\mu(\lambda A) = |\lambda|\mu(A)$, $\lambda \in \mathbb{R}$.

DEFINITION 3. Let X and Y be two Banach spaces, $\overline{B}_X$ be the closure of the unit ball of X and let μ be a measure of weak noncompactness in Y. We define the function

$$\begin{aligned} \Psi_\mu : \mathscr{L}(X,Y) &\longrightarrow [0,+\infty[\\ T &\longrightarrow \Psi_\mu(T) = \mu(T(\overline{B}_X)). \end{aligned}$$

(i) Ψ_μ is called a measure of weak noncompactness of operators associated to μ.
(ii) Ψ_μ is said to be algebraic semi-multiplicative if, for all $S \in \mathscr{L}(X)$ and $D \in \mathscr{M}_X$, we have

$$\mu(S(D)) \leqslant \Psi_\mu(S)\mu(D).$$

REMARK 1. Notice that if Ψ_μ has the algebraic semi-multiplicative property, then,

$$\forall S,T \in \mathscr{L}(X),\ \Psi_\mu(ST) \leqslant \Psi_\mu(S)\Psi_\mu(T).$$

As an example of measure of weak noncompactness of operators we have $\Theta_\omega(T) = \omega(T(\overline{B}_X))$, where ω is the measure of weak noncompactness of De Blasi defined by:

$$\omega(A) = \inf\{t > 0 :\ \exists\, C \in \mathscr{W}_X \text{ such that } A \subset C + t\overline{B}_X\},\ \forall A \in \mathscr{M}_X.$$

The function $\Theta_\omega(.)$ possesses several useful properties. For example $\Theta_\omega(.)$ has the algebraic semi-multiplicative property. For further facts concerning measures of weak noncompactness and its properties we refer to [6].

Recall that an operator $T \in \mathscr{L}(X,Y)$ is said to be a Dunford-Pettis operator (for short property DP operator) if T maps weakly compact sets into compact sets.

We will need the next Theorem, which was proved in [1]. We give a proof for the convenience of the reader.

THEOREM 1. [1, Theorem 2.1] *Let X be a complex Banach space. Suppose that, for $T \in \mathscr{L}(X)$,*

$$\begin{cases} (H_1)\ \mu(T(A)) \leqslant \Psi_\mu(T)\mu(A), \text{ for every } A \in \mathscr{M}_X, \\ (H_2)\ \lim_{n\to+\infty} \left(\Psi_\mu(T^n)\right)^{\frac{1}{n}} = 0, \\ (H_3)\ \text{there exists } m \in \mathbb{N}^* \text{ such that } T^m \text{ is DP operator.} \end{cases}$$

Then

$$I - T \in \Phi(X) \text{ and } i(I-T) = 0.$$

Proof. First we show that $I - T \in \Phi_+(X)$. By [15, Theorem 18, p. 161], it suffice to prove that, for any $K \in \mathscr{K}(X)$, $\alpha(I-T-K) < \infty$. To do so, it suffice to establish that the set $A := \mathscr{N}(I-T-K) \cap \overline{B}_X$ is compact, where $\overline{B}_X$ is the closure of the unit ball of X. Since $\lim_{n\to+\infty} \left(\Psi_\mu(T^n)\right)^{\frac{1}{n}} = 0$, then there exists $n_0 \geqslant m$ such that for all $n \geqslant n_0$, $\Psi_\mu(T^n) < 1$. On the other hand, for $n \geqslant n_0$, $I - T^n = R(T)(I-T)$, where $R(T) := I + T + ... + T^{n-1}$. Consider $x \in A$, then $R(T)(I-T)(x) = R(T)K(x)$. Hence, $x = T^n(x) + R(T)K(x)$. Obviously $A \subset T^n(A) + R(T)K(A)$. Applying $\mu(.)$ and taking account the hypothesis (H_1), we infer that

$$\mu(A) \leqslant \mu(T^n(A)) \leqslant \Psi_\mu(T^n)\mu(A).$$

Since $\Psi_\mu(T^n) < 1$, then $\mu(A) = 0$ and therefore A is relatively weakly compact including in $T^n(A) + R(T)K(A)$. Obviously T^{n_0} is DP operator, then $T^{n_0}(A)$ is compact. Hence, A is compact and therefore $I - T \in \Phi_+(X)$. Next, note that for $t \in [0,1]$, we have $(\Psi_\mu(tT))^{n_0} < 1$ and $(tT)^m$ is DP operator. Then, from the above, $(I - tT) \in \Phi_+(X)$. Now, by the continuity of the index on $\Phi_+(X)$, we get $i(I-T) = i(I-tT) = i(I) = 0$. Thus, $I - T \in \Phi(X)$, which completes the proof. □

In what follows, consider μ a measure of weak noncompactness in X and Ψ_μ a measure of weak noncompactness of operators associated to μ.

Now, we are ready to prove the following:

THEOREM 2. *Let X be a Banach space which possess the DP property, $T \in \mathscr{C}(X)$ and let S be a T-bounded operator on X. Assume that, for T_r^w a right weak-Fredholm inverse of T,*

$$\begin{cases} (i)\ \mu(ST_r^w(A)) \leqslant \Psi_\mu(ST_r^w)\mu(A),\ \forall A \in \mathscr{M}_X, \\ (ii)\ \text{there exists } m \in \mathbb{N}^* \text{ such that } (\widehat{S}T_r^w)^m \text{ is DP in } \mathscr{L}(X) \text{ and } \Psi_\mu((\widehat{S}T_r^w)^m) < 1. \end{cases}$$

Then the following statements hold.

$T + S \in \Phi_r(X)$ and $i(T+S) = i(T)$. If moreover $T \in \Phi_+(X)$, then $T + S \in \Phi(X)$.

Proof. (i) Keeping in mind Definition 1, there exists $W \in \mathscr{W}(X)$ such that $(\widehat{T} + \widehat{S})T_r^w = (I_X + \widehat{S}T_r^w) - W$. Thus, Theorem 1 together with [13, Proposition 3.1] imply that $(\widehat{T} + \widehat{S})T_r^w \in \Phi(X)$ and $i((\widehat{T} + \widehat{S})T_r^w) = 0$. Hence, $\widehat{T} + \widehat{S} \in \Phi_r(X_T, X)$. Moreover,

$i((\widehat{T}+\widehat{S})T_r^w) = i(T_r^w)+i(\widehat{T}+\widehat{S}) = 0$ and therefore $i(\widehat{T}+\widehat{S}) = -i(T_r^w) = i(\widehat{T})$. From [13, Proposition 3.1], $\widehat{T}T_r^w \in \Phi(X)$. Taking into account that $T \in \Phi_+(X)$, and by [18, Theorem 2.7 p. 171], we infer that $T_r^w \in \Phi(X, X_T)$. Moreover, the fact that $(\widehat{T}+\widehat{S})T_r^w \in \Phi(X)$, then, according to [18, Theorem 2.5 p. 169], $\widehat{T}+\widehat{S} \in \Phi(X_T, X)$. Finally the result follows from (1). $\square$

In order to give a similar results to Theorem 2, consider μ_T a measure of weak noncompactness in X_T and Ψ_{μ_T} the measure of weak noncompactness of operators associated to μ_T. Now, arguing as in the proof of Theorem 2, we can prove the following:

THEOREM 3. *Let X be a Banach space which possess the DP property, $T \in \mathscr{C}(X)$ and let S be a T-bounded operator on X. Suppose that, for T_l^w a left weak-Fredholm inverse of T,*

$$\begin{cases} (i)\ \mu_T(T_l^w\widehat{S}(A)) \leqslant \Psi_{\mu_T}(T_l^w\widehat{S})\mu_T(A),\ \forall A \in \mathscr{M}_{X_T}, \\ (ii)\ \text{there exists } m \in \mathbb{N}^* \text{ such that } (T_l^w\widehat{S})^m \text{ is DP in } \mathscr{L}(X_T) \text{ and } \Psi_{\mu_T}(T_l^w\widehat{S})^m < 1. \end{cases}$$

Then the following statements hold.

$T+S \in \Phi_l(X)$ and $i(T+S) = i(T)$. If moreover $T \in \Phi_-(X)$, then $T+S \in \Phi(X)$.

Finally, we close this section with the following corollary which extends many known perturbation results in the literature.

COROLLARY 1. *Let X be a Banach space which possess the DP property, $T \in \mathscr{C}(X)$ and let S be a T-bounded operator on X.*

(i) Suppose that, for T_r^w a right weak-Fredholm inverse of T, for some $m \in \mathbb{N}^*$, $(\widehat{S}T_r^w)^m$ is weakly compact operator on X. Then

$T+S \in \Phi_r(X)$ and $i(T+S) = i(T)$. If moreover $T \in \Phi_+(X)$, then $T+S \in \Phi(X)$.

(ii) Suppose that, for T_l^w a left weak-Fredholm inverse of T, $(T_l^w\widehat{S})^m$ is weakly compact operator on X_T, for some $m \in \mathbb{N}^*$. Then $T+S \in \Phi_l(X)$ and $i(T+S) = i(T)$. If moreover $T \in \Phi_-(X)$, then $T+S \in \Phi(X)$.

4. Essential spectra of some matrix operators

The purpose of this section, is to describe the essential spectra of a general class operators defined by a 2×2 block operator matrix.

In the product space $X \times X$, we consider the matrix operators of the form

$$L_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where the operator A acts on X and has domain $\mathscr{D}(A)$, D is defined on $\mathscr{D}(D)$ and acts on X, and the intertwining operator B (resp., C) is defined on the domain $\mathscr{D}(B)$ (resp., $\mathscr{D}(C)$) and acts on X. In what follows, we will assume that the entries of this matrix satisfy the following conditions introduced in [5]:

(H_1) A is densely defined linear operator on X with nonempty resolvent set $\rho(A)$.

(H_2) B and C are densely defined closable operators and $\mathscr{D}(A) \subset \mathscr{D}(C)$.

(H_3) For some (and hence for all) $\nu \in \rho(A)$, $G(\nu) := \overline{(A-\nu)^{-1}B}$ is bounded on its domain $\mathscr{D}(B)$.

(H_4) $\mathscr{D}(B) \subset \mathscr{D}(D)$.

(H_5) For some (and hence for all) $\nu \in \rho(A)$, the operator $D - C(A-\nu)^{-1}B$ is closable. We will denote by $S(\nu)$ its closure.

REMARK 2. (i) The fact that $\mathscr{D}(A) \subset \mathscr{D}(C)$ and from the closed graph theorem we infer that, for each $\nu \in \rho(A)$, $F(\nu) := C(A-\nu)^{-1}$ is defined on X and is bounded.

(ii) If the conditions $(H_1)-(H_5)$ are satisfied, then by [5, Theorem 1.1] the operator L_0 is closable. Moreover, for $\lambda \in \mathbb{C}$ and $\nu \in \rho(A)$, $\lambda - L$ can be written as follows:

$$\begin{aligned}\lambda - L &= \begin{pmatrix} I & 0 \\ F(\nu) & I \end{pmatrix} \begin{pmatrix} A_\lambda & 0 \\ 0 & S_\lambda(\nu) \end{pmatrix} \begin{pmatrix} I & G(\nu) \\ 0 & I \end{pmatrix} - (\lambda-\nu)M(\nu) \\ &:= UV(\lambda)W - (\lambda-\nu)M(\nu),\end{aligned}$$

where $A_\lambda = \lambda - A$, $S_\lambda(\nu) = \lambda - S(\nu)$ and $M(\nu) = \begin{pmatrix} 0 & G(\nu) \\ F(\nu) & F(\nu)G(\nu) \end{pmatrix}$.

In the rest of this section, consider μ a measure of weak-noncompactness in X and Ψ_μ a measure of weak-noncompactness of operators associated to μ. Suppose that Ψ_μ is semi-multiplicative. Unless otherwise stated in all follows, we suppose that, for some $\nu \in \rho(A)$, $F(\nu)$ and $G(\nu)$ satisfy the condition:

$$(H) : \begin{cases} (i)\ F(\nu) \text{ and } G(\nu) \text{ are DP operators,} \\ (ii)\ \max\left(\Psi_\mu(G(\nu)),\ \Psi_\mu(F(\nu))\right) < \frac{1}{2}. \end{cases}$$

For $T \in \mathscr{C}(X)$ and $\lambda \in \mathbb{C}$, we will denote $T_\lambda := \lambda - T$.

THEOREM 4. *Let X be a Banach space which possess the DP property and let $\lambda \in \mathbb{D}(\nu,1)$.*

(i) *Suppose that there exists $A^w_{\lambda l}$ a left weak-Fredholm inverse of $A_\lambda l$ and $S^w_{\lambda l}(\nu)$ a left weak-Fredholm inverse of $S_\lambda(\nu)$ satisfying:*

$$\Psi_\mu(A^w_{\lambda l}G(\nu)) < \frac{1}{2} \text{ and } \Psi_\mu(S^w_{\lambda l}(\nu)F(\nu)) < 1.$$

Then

$$\lambda - L \in \Phi_l(X \times X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$

(ii) *Suppose that there exists $A^w_{\lambda r}$ a right weak-Fredholm inverse of A_λ and $S^w_{\lambda r}(\nu)$ a right weak-Fredholm inverse of $S_\lambda(\nu)$ satisfying:*

$$\Psi_\mu(G(\nu)S^w_{\lambda r}) < \frac{1}{2},\ \Psi_\mu(F(\nu)A^w_{\lambda r}) < 1 \text{ and } \Psi_\mu(S^w_{\lambda r}(\nu)F(\nu)) < 1.$$

Then

$$\lambda - L \in \Phi_r(X \times X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$

(iii) *Suppose that the hypotheses of* (i) *and* (ii) *hold true. Then*

$$\lambda - L \in \Phi(X \times X) and\ i(\lambda - L) = i(V(\lambda)).$$

To prove Theorem 4, we shall need to the following lemma:

LEMMA 1. *For all bounded operator* $T := \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ *on* $X \times X$*, we consider*

$$\varphi_\mu(T) = \max(\Psi_\mu(T_1) + \Psi_\mu(T_2), \Psi_\mu(T_3) + \Psi_\mu(T_4)).$$

Then φ_μ *defines a measure of weak noncompactness on the space* $X \times X$.

Proof. It is easy to verify that φ_μ is a semi-norm on $X \times X$ and satisfying $\varphi_\mu(T) = 0$ if and only if, $\forall i \in \{1, ..., 4\}$, T_i is weakly compact on X. Thus, φ_μ is vanishes on the closed ideal $\mathscr{W}(X \times X)$. □

Proof of Theorem 4.

(i) Let $T_\lambda = UV(\lambda)W$ and $V^w_{\lambda l} = \begin{pmatrix} A^w_{\lambda l} & W_1 \\ W_2 & S^w_{\lambda l}(\nu) \end{pmatrix}$ such that W_1 and W_2 are weakly compact operators. It is easy to see that $V^w_{\lambda l}$ is a left weak-Fredholm inverse of $V(\lambda)$. Thus, $T^w_{\lambda l} = W^{-1} V^w_{\lambda l} U^{-1}$ is a left weak-Fredholm inverse of T_λ. On the other hand, we have:

$$T^w_{\lambda l} M(\nu) = \begin{pmatrix} W_1 F(\nu) - G(\nu) S^w_{\lambda l}(\nu) F(\nu) & A^w_{\lambda l} G(\nu) - G(\nu) W_2 G(\nu) \\ S^w_{\lambda l}(\nu) F(\nu) & W_2 G(\nu) \end{pmatrix}.$$

Now, since the measure of weak noncompactness Ψ_μ is semi-multiplicative, then

$$\varphi_\mu(T^w_{\lambda l} M(\nu)) \leqslant \max[\Psi_\mu(G(\nu))\Psi_\mu(S^w_{\lambda l} F(\nu)) + \Psi_\mu(A^w_{\lambda l} G(\nu)), \Psi_\mu(S^w_{\lambda l} F(\nu))] < 1.$$

Hence, the fact that $|\lambda - \nu| < 1$, we deduce $\varphi_\mu((\lambda - \nu) T^w_{\lambda l} M(\nu)) < 1$. Moreover, according to hypothesis $(H)(i)$, $T^w_{\lambda l} M(\nu)$ is DP operator. Finally, the results follow from Theorem 3.

(ii) Let $V^w_{\lambda r} = \begin{pmatrix} A^w_{\lambda r} & W'_1 \\ W'_2 & S^w_{\lambda r}(\nu) \end{pmatrix}$ be such that W'_1 and W'_2 are weakly compact operators. In the same way one checks that $T^w_{\lambda r} = W^{-1} V^w_{\lambda r} U^{-1}$ is a right weak-Fredholm inverse of T_λ. On the other hand, we have

$$M(\nu) T^w_{\lambda r} = \begin{pmatrix} G(\nu) W'_2 - G(\nu) S^w_{\lambda r}(\nu) F(\nu) & G(\nu) S^w_{\lambda r}(\nu) \\ F(\nu) A^w_{\lambda r} - F(\nu) W'_1 F(\nu) & F(\nu) W'_1 \end{pmatrix}.$$

Thus,

$$\begin{aligned} \varphi_\mu((\lambda - \nu) M(\nu) T^w_{\lambda r}) &\leqslant \max[\Psi_\mu(G(\nu))\Psi_\mu(S^w_{\lambda r} F(\nu)) + \Psi_\mu(G(\nu) S^w_{\lambda r}), \Psi_\mu(F(\nu) A^w_{\lambda r})] \\ &< 1. \end{aligned}$$

Moreover, since $F(\nu)$ and $G(\nu)$ are DP operators, then $M(\nu) T^{wr}_\lambda$ is also. Finally, the results follow from Theorem 2.

(iii) Is a deduction from (i) and (ii). □

Without maintaining to the hypothesis (H), if we suppose that $A^w_{\lambda l}G(\nu)$ and $S^w_{\lambda l}(\nu)F(\nu)$ are weakly compact operators then the results of Theorem 4 remand valid. So, if we translate Theorem 4 in terms of essential spectra we get

COROLLARY 2. *Let X be a Banach space which possess the DP property.*

(i) *Suppose that, for each $\lambda \in \Phi^w_{lA} \cap \Phi^w_{lS(\nu)}$, $A^w_{\lambda l}G(\nu)$ and $S^w_{\lambda l}(\nu)F(\nu)$ are weakly compact operators. Then*

$$\sigma_{le}(L) \subset \sigma_{le}(A) \cup \sigma_{le}(S(\nu)).$$

(ii) *Suppose that, for each $\lambda \in \Phi^w_{rA} \cap \Phi^w_{rS(\nu)}$, $G(\nu)S^w_{\lambda r}$, $F(\nu)A^w_{\lambda r}$ and $S^w_{\lambda r}(\nu)F(\nu)$ are weakly compact operators. Then*

$$\sigma_{re}(L) \subset \sigma_{re}(A) \cup \sigma_{re}(S(\nu)).$$

(iii) *Suppose that, for some $\lambda \in \Phi^w_A \cap \Phi^w_{S(\nu)}$, $A^w_{\lambda l}G(\nu)$, $A^w_{\lambda r}G(\nu)$, $S^w_{\lambda l}(\nu)F(\nu)$ and $S^w_{\lambda r}(\nu)F(\nu)$ are weakly compact operators. Then*

$$\sigma_{eF}(L) \subset \sigma_{eF}(A) \cup \sigma_{eF}(S(\nu)) \text{ and } \sigma_{eW}(L) \subset \sigma_{eW}(A) \cup \sigma_{eW}(S(\nu)).$$

Proof. The statements (i) and (ii) are direct consequence of Theorem 4 (i)-(ii).

(iii) It suffice to prove that if the hypotheses of statements (i)-(ii) hold true for some $\lambda \in \Phi^w_A \cap \Phi^w_{S(\nu)}$, then they also hold true for all $\alpha \in \Phi^w_A \cap \Phi^w_{S(\nu)}$. Suppose, for example, that $A^w_{\lambda l}G(\nu)$ is weakly compact operator, for $\lambda \in \Phi^w_A \cap \Phi^w_{S(\nu)}$. Then there exists A'_λ and W_1, W_2 two weakly compact operators satisfying: $A'_\lambda A_\lambda = I + W_1$ on $\mathcal{D}(A_\lambda)$ and $A_\lambda A'_\lambda = I + W_2$ on X. Let $\alpha \in \Phi^w_A \cap \Phi^w_{S(\nu)}$. There exists A'_α and W_3, W_4 two weakly compact operators satisfying: $A'_\alpha A_\alpha = I + W_3$ on $\mathcal{D}(A_\alpha)$ and $A_\alpha A'_\alpha = I + W_4$ on X. Hence, $A'_\lambda A_\alpha A'_\alpha G(\nu)$ is weakly compact. Now, since $A'_\lambda A_\alpha \in \Phi^w(X)$, then $A'_\alpha G(\nu)$ is weakly compact operator. $\square$

In the rest of this paper, we will study the inverse inclusion given in Corollary 2. For this, we consider $H_\lambda(\nu) := S_\lambda(\nu) + CG(\nu)$, $\nu \in \rho(A)$.

THEOREM 5. *Let X be a Banach space which possess the DP property and let $\lambda \in \mathbb{D}(\nu, 1)$.*

(i) *Suppose that there exists $A^w_{\lambda l}$ (resp., $H^w_{\lambda l}(\nu)$) a left weak-Fredholm inverse of A_λ (resp., a left weak-Fredholm inverse of $H_\lambda(\nu)$) satisfying:*

$$\begin{cases} \bullet\ A^w_{\lambda l}(\nu - A)G(\nu) \text{ and } H^w_{\lambda l}(\nu)C \text{ are weakly compact operators.} \\ \bullet\ \Psi_\mu(A^w_{\lambda l}G(\nu)) < 1. \end{cases}$$

Then

$$V(\lambda) \in \Phi_l(X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$

(ii) *Suppose that there exists $A^w_{\lambda r}$ (resp., $H^w_{\lambda r}(\nu)$) a right weak-Fredholm inverse of A_λ (resp., a right weak-Fredholm inverse of $H_\lambda(\nu)$) satisfying:*

$$\begin{cases} \bullet\ (\nu - A)G(\nu)H^w_{\lambda r}(\nu) \text{ and } CA^w_{\lambda r} \text{ are weakly compact operators.} \\ \bullet\ \Psi_\mu(G(\nu)H^w_{\lambda r}(\nu)) < 1 \text{ and } \Psi_\mu(F(\nu)A^w_{\lambda r}) < 1. \end{cases}$$

Then

$$V(\lambda) \in \Phi_r(X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$

(iii) *Suppose that the hypotheses of* (i) *and* (ii) *hold true. Then*

$$V(\lambda) \in \Phi(X) \text{ and } i(\lambda - L) = i(V(\lambda)).$$

Proof. Consider $L^w_{\lambda l} = \begin{pmatrix} A^w_{\lambda l} & W_1 \\ W_2 & H^w_{\lambda l}(\nu) \end{pmatrix}$ such that W_1 and W_2 are weakly compact operators. Then $L^w_{\lambda l}$ is a left weak-Fredholm inverse of L_λ. On the other hand, we have

$$L^w_{\lambda l} M(\nu) = \begin{pmatrix} W_1 F(\nu) & A^w_{\lambda l} G(\nu) + W_1 F(\nu) G(\nu) \\ H^w_{\lambda l}(\nu) F(\nu) & W_2 G(\nu) + H^w_{\lambda l}(\nu) F(\nu) G(\nu) \end{pmatrix}.$$

Since $H^w_{\lambda l}(\nu)C$ is weakly compact, then $H^w_{\lambda l}(\nu)F(\nu)$ is also. The fact that $|\lambda - \mu| < 1$ and according to Lemma 1, we deduce that $\varphi_\mu((\lambda - \nu)L^w_{\lambda l} M(\mu)) = \Psi_\mu(A^w_{\lambda l} G(\nu)) < 1$. Finally, the results follow from Theorem 3.

(ii) Let $L^w_{\lambda r} = \begin{pmatrix} A^w_{\lambda r} & W'_1 \\ W'_2 & H^w_{\lambda r}(\nu) \end{pmatrix}$ be such that W'_1 and W'_2 are weakly compact operators. It is easy to verify that $L^w_{\lambda r}$ is a right weak-Fredholm inverse of L_λ. On the other hand, we have

$$M(\nu) L^w_{\lambda r} = \begin{pmatrix} G(\nu) W'_2 & G(\nu) H^w_{\lambda r}(\nu) \\ F(\nu) A^w_{\lambda r} + F(\nu) G(\nu) W'_2 & F(\nu) W'_1 + F(\nu) G(\nu) H^w_{\lambda r}(\nu) \end{pmatrix}.$$

According to the hypotheses we infer that

$$\begin{aligned} \varphi_\mu(M(\nu) L^w_{\lambda r}) &\leqslant \max[\Psi_\mu(G(\nu) H^w_{\lambda r}(\nu)), \Psi_\mu(F(\nu) A^w_{\lambda r}) + \Psi_\mu(F(\nu)) \Psi_\mu(G(\nu) H^w_{\lambda r}(\nu))] \\ &< 1. \end{aligned}$$

Finally, the results follow from Theorem 2.

(iii) Is a deduction from (i) and (ii). □

REMARK 3. Remark that if $F(\nu)$, $G(\nu) \in \mathcal{W}(X)$, for some $\nu \in \rho(A)$, then for all $\nu \in \rho(A)$, $F(\nu)$, $G(\nu) \in \mathcal{W}(X)$. Indeed, let $\nu_0 \in \rho(A)$ such that $F(\nu_0)$ and $G(\nu_0)$ are weakly compacts. Then, for all $\nu \in \rho(A)$, we have: $F(\nu) = F(\nu_0)[I + (\nu - \nu_0)(\nu_0 - A)^{-1}]^{-1}$ and $G(\nu) = [I + (\nu - \nu_0)(\nu_0 - A)^{-1}]^{-1}(\nu_0 - A)^{-1}B$. This implies that $F(\nu)$, $G(\nu) \in \mathcal{W}(X)$.

Without maintaining to the hypothesis (H), we can deduce the following:

COROLLARY 3. *Let X be a Banach space which possess the DP property.*

(i) *Suppose that, for each $\lambda \in \Phi^w_{lA} \cap \Phi^w_{lS(\nu)} \cap \Phi^w_{lH(\nu)}$, $S^w_{\lambda l}(\nu)C$ and $G(\nu)$ are weakly compact operators. Then*

$$\sigma_{le}(L) = \sigma_{le}(A) \cup \sigma_{le}(S(\nu)).$$

(ii) *Suppose that, for each* $\lambda \in \Phi^{w}_{rA} \cap \Phi^{w}_{rS(\nu)} \cap \Phi^{w}_{rH(\nu)}$, $G(\nu)H^{w}_{\lambda r}(\nu)$ *and* $F(\nu)$ *are weakly compact operators. Then*

$$\sigma_{re}(L) = \sigma_{re}(A) \cup \sigma_{re}(S(\nu)).$$

(iii) *Suppose that, for some (and hence for all)* $\nu \in \rho(A)$, $F(\nu)$ *and* $G(\nu)$ *are weakly compact operators. Then*

$$\sigma_{eF}(L) = \sigma_{eF}(A) \cup \sigma_{eF}(S(\nu)) \text{ and } \sigma_{eW}(L) = \sigma_{eW}(A) \cup \sigma_{eW}(S(\nu)).$$

If in addition, $\mathbb{C}\backslash\sigma_{eW}(L)$, $\mathbb{C}\backslash\sigma_{eW}(A)$ *and* $\mathbb{C}\backslash\sigma_{eW}(S(\nu))$ *are connected,* $\rho(L) \neq \emptyset$ *and* $\rho(S(\nu)) \neq \emptyset$*, then*

$$\sigma_{eB}(L) = \sigma_{eB}(A) \cup \sigma_{eB}(S(\nu)).$$

Proof. (i) Remark that $A^{w}_{\lambda l}(\nu - A)G(\nu)$ and $A^{w}_{\lambda l}G(\nu)$ are weakly compacts if and only if $G(\nu) \in \mathscr{W}(X)$. Moreover, $S^{w}_{\lambda l}(\nu)$ is a left weak-Fredholm inverse of $H_{\lambda}(\nu)$. Hence, the result follows from Corollary 2 and Theorem 5.

(ii) Remark that $CA^{w}_{\lambda l} = F(\nu)(\nu - A)A^{w}_{\lambda l}$. Thus, $CA^{w}_{\lambda r}$ and $F(\nu)A^{w}_{\lambda r}$ are weakly compacts if and only if $F(\nu) \in \mathscr{W}(X)$. Hence, the result follows from Corollary 2 and Theorem 5.

(iii) The fact that $S^{w}_{\lambda l}(\nu)C = S^{w}_{\lambda l}(\nu)F(\nu)(\nu - A)$, then the results follow from (i) and (ii). To describe the Browder essential spectrum of L, we have $\sigma_{eW}(L) \subset \sigma(L)$. Thus, since $\alpha(\lambda - L)$ and $\beta(\lambda - L)$ are constant on any component of Φ_L except possibly on a discrete set of points at which they have large values (see, for example [18]), then $\sigma_{eB}(L) \subset \sigma_{eW}(L)$ and therefore $\sigma_{eW}(L) = \sigma_{eB}(L)$. Using the same reasoning as before, we show that $\sigma_{eW}(A) = \sigma_{eB}(A)$ and $\sigma_{eW}(L) = \sigma_{eB}(S(\nu))$. $\square$

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