A FUNCTIONAL ANALYTIC PERSPECTIVE
TO DELAY DIFFERENTIAL EQUATIONS

RAINER PICARD, SASCHA TROSTORFF AND MARCUS WAURICK

Abstract. We generalize the solution theory for a class of delay type differential equations developed in a previous paper, dealing with the Hilbert space case, to a Banach space setting. The key idea is to consider differentiation as an operator with the whole real line as the underlying domain as a means to incorporate pre-history data. We focus our attention on the issue of causality of the differential equations as a characterizing feature of evolutionary problems and discuss various examples. The arguments mainly rely on a variant of the contraction mapping theorem and a few well-known facts from functional analysis.

1. Introduction

In this note, we present a unified Banach space perspective to a class of ordinary differential equations with memory and delay effects. This class is often summarized under the umbrella term delay differential equations:

$$\dot{x}(t) = f(t, x_t, \dot{x}_t) \; t \in I, \; I \subseteq \mathbb{R} \text{ interval,}$$

where $\dot{x}$ denotes the (time-)derivative of x and x_t denotes the *history segment of x*, cf. e.g. [7, 11] see also Example 3.16 below. These equations have many applications in engineering or sciences. We refer to [1, 6] and the references therein for an account of various applications.

The class considered here covers ordinary differential equations, differential difference equations, integro-differential equations or even neutral differential equations. Using some basic functional analysis, the main contribution of this note is that the aforementioned equations can be treated in a unified manner. The core idea consists in the treatment of the problem on the whole real axis as time-line. This enables us to conveniently detour the introduction of certain (pre-)history spaces, cf. e.g. [7, 13]. Moreover, we do not need to introduce an extended state space as it can be done for linear theory using semi-group methods, see e.g. [2]. Dealing with possibly non-linear equations, we cannot use the rich theory of Fourier-multipliers for delay equations, see e.g. [16, 17] for a possible account of this technique being applied to partial differential delay equations.

Our perspective also shows that finite and infinite delay do not need different treatment. However, our focus is on existence of solutions and continuous dependence on the data rather than the continuity or differentiability or other qualitative properties of

the solution itself. This yields an *elementary* solution theory at least for L_p-spaces. Though parts of our results are covered by well-known theory, we have more freedom in choosing right-hand sides. In particular, we can solve delay differential equations with certain measures or functions with unbounded variation as right-hand sides, cf. Subsection 4.2.

Our development of a solution theory for delay differential equations starts in Section 2, where we state a variant of the contraction mapping theorem tailored for operator equations in the abstract form

$$Cx = F(x),$$

for x residing in some Banach space X and F being Lipschitz-continuous in a suitable sense and $C : D(C) \subseteq X \to X$ being a closed, densely defined, continuously invertible linear operator. The remaining parts of this paper are essentially applications of the results from Section 2. To this end, we have to establish the (time-)derivative as a *continuously invertible* operator. Thus, both the Sections 3 and 4 start with the definition of the time-derivative as a continuously invertible operator in a L_p-space and in a space of continuous functions. Having our main applications of delay differential equations in mind, we introduce the (time-)derivative on functions defined on the whole real line as time axis. If one wants to recover classical theory, e.g. initial value problems for ordinary differential equations, one has to know that the solution of a differential equation depends only on the past of the right-hand side. The latter is summarized by the notion of causality, see [15] or Definition 3.9 below. Thus, in both the Section 3 and 4, we also show causality of the respective solution operators. These sections are complemented by the discussion of several examples.

It should be noted that many ideas rely on results and strategies used in reference [14], where a Hilbert space perspective is preferred. The idea of introducing the time-derivative as a continuously invertible operator stems from references [19, 20, 21], which in view of the L_∞-considerations in Section 4 shows its kinship to ideas developed back in 1952 by Morgenstern, [18].

2. The general solution theory – a variant of the contraction mapping theorem

Let X be a Banach space and let $C : D(C) \subseteq X \to X$ be a densely defined closed linear operator with $0 \in \rho(C)$. Then $\mathscr{X}_1(C) := (D(C), |C \cdot|_X)$ is a Banach space. Moreover, define $\mathscr{X}_{-1}(C)$ to be the completion $(X, |C^{-1} \cdot|_X)^{\sim}$ of $(X, |C^{-1} \cdot|_X)$ and let $\mathscr{X}_0(C) := X$. If the operator C is clear from the context, we omit the reference to the operator C in the notation of the associated spaces. We have

$$\mathscr{X}_1 \hookrightarrow \mathscr{X}_0 \hookrightarrow \mathscr{X}_{-1}$$

in the sense of continuous and dense embeddings. Furthermore, $C : D(C) \subseteq \mathscr{X}_0 \to \mathscr{X}_{-1}$ is isometric and densely defined with dense range. Hence, C can be extended to an isometric isomorphism. We identify C with its extension. The fundamental solution theory is based on the following variant of the contraction mapping theorem.

THEOREM 2.1. *Let* $F : \mathscr{X}_0 \to \mathscr{X}_{-1}$ *be a strict contraction and let* $|F|_{\mathrm{Lip}}(< 1)$ *be the smallest Lipschitz constant for* F. *Then the equation*

$$Cx = F(x)$$

has a unique fixed point $x \in \mathscr{X}_0$. *If* $y \in \mathscr{X}_0$ *and* $n \in \mathbb{N}$ *then the following estimates hold*

$$\left| C^{-1}F(y) - x \right| \leqq \frac{|F|_{\mathrm{Lip}}}{1 - |F|_{\mathrm{Lip}}} |y - x|,$$

$$\left| \left(C^{-1}F \right)^n (y) - x \right| \leqq \frac{|F|_{\mathrm{Lip}}}{1 - |F|_{\mathrm{Lip}}} \left| \left(C^{-1}F \right)^n (y) - \left(C^{-1}F \right)^{n-1} (y) \right|,$$

$$\left| \left(C^{-1}F \right)^n (y) - x \right| \leqq \frac{|F|_{\mathrm{Lip}}^n}{1 - |F|_{\mathrm{Lip}}} |y - x|.$$

Proof. The operator $C^{-1} : \mathscr{X}_{-1} \to \mathscr{X}_0$ is an isometric isomorphism. Hence, $C^{-1}F(\cdot)$ is a strict contraction in $\mathscr{X}_0$. The contraction mapping theorem yields the assertion. The estimates are well-known. $\square$

COROLLARY 2.2. *Let* $F : \mathscr{X}_1 \to \mathscr{X}_0$ *be a strict contraction. Then*

$$Cx = F(x)$$

has a unique fixed point $x \in \mathscr{X}_1$.

Proof. The mapping $CF(C^{-1}\cdot)$ satisfies the assumptions from Theorem 2.1. Hence, there exists a unique fixed point $\tilde{x} \in \mathscr{X}_0$ such that

$$C\tilde{x} = CF(C^{-1}\tilde{x}).$$

Therefore $x := C^{-1}\tilde{x} \in \mathscr{X}_1$ satisfies

$$Cx = F(x).$$

Now, let $u \in \mathscr{X}_1$ satisfy $Cu = F(u)$. Then Cu satisfies the equation $C(Cu) = CF(u) = CF(C^{-1}(Cu))$ and thus, $Cu = \tilde{x}$, which gives $u = x$. $\square$

THEOREM 2.3. *Let* $F, G : \mathscr{X}_0 \to \mathscr{X}_{-1}$ *be Lipschitz continuous and assume that the respective Lipschitz semi-norms, i.e., the smallest Lipschitz constants,* $|F|_{\mathrm{Lip}}$ *and* $|G|_{\mathrm{Lip}}$ *satisfy*

$$\frac{|F|_{\mathrm{Lip}} + |G|_{\mathrm{Lip}}}{2} < 1.$$

Let $x, y \in \mathscr{X}_0$ *satisfy*

$$Cx = F(x) \text{ and } Cy = G(y).$$

Then

$$|x - y|_{\mathscr{X}_0} \leqq \frac{1}{1 - \frac{|F|_{\mathrm{Lip}} + |G|_{\mathrm{Lip}}}{2}} \sup_{u \in \mathscr{X}_0} |F(u) - G(u)|_{\mathscr{X}_{-1}}.$$

Proof. By assumption, we have

$$x - y = C^{-1}F(x) - C^{-1}G(y)$$
$$= \frac{1}{2}C^{-1}(F(x) - F(y))$$
$$+ \frac{1}{2}C^{-1}(G(x) - G(y)) - \frac{1}{2}C^{-1}(G(x) - F(x)) + \frac{1}{2}C^{-1}(F(y) - G(y)).$$

This yields the assertion. □

3. The reflexive case – delay differential equations in L_p-spaces

For the whole section, let $p \in (1, \infty)$ and denote by q the conjugate exponent such that $\frac{1}{p} + \frac{1}{q} = 1$. Let X be a Banach space.

3.1. Definition of the time-derivative

Denoting by μ_ν the weighted Lebesgue measure on $\mathbb{R}$ with Radon-Nikodym derivative $x \mapsto e^{-\nu p x}$ for $\nu \in \mathbb{R}$, we define

$$W_{p,\nu}^0(\mathbb{R};X) := L_{p,\nu}(\mathbb{R};X) := L_p(\mu_\nu;X).$$

Note that the mapping $e^{-\nu m} : L_{p,\nu}(\mathbb{R};X) \to L_p(\mathbb{R};X), f \mapsto (x \mapsto e^{-\nu x}f(x))$ is isometrically isomorphic[1].

DEFINITION 3.1. We define

$$\partial_\nu : C_c^\infty(\mathbb{R};X) \subseteq L_{p,\nu}(\mathbb{R};X) \to L_{p,\nu}(\mathbb{R};X), f \mapsto f',$$

where

$$C_c^\infty(\mathbb{R};X) := \{\phi; \phi \text{ indefinitely differentiable, } \operatorname{supp}\phi \text{ compact}\}.$$

The operator ∂_ν is clearly closable and its closure coincides with the distributional derivative. Henceforth, we will not distinguish notationally between ∂_ν and its closure. In order to apply the general solution theory to ∂_ν in place of C, we need the following theorem:

THEOREM 3.2. *Assume* $\nu > 0$. *Then we have that the convolution operator* $\chi_{[0,\infty)}* : L_{p,\nu}(\mathbb{R};X) \to L_{p,\nu}(\mathbb{R};X)$ *is continuous with operator norm equal to* $\frac{1}{\nu}$. *Moreover, it holds* $(\chi_{[0,\infty)}*)^{-1} = \partial_\nu$.

---

[1]The m in the expression $e^{-\nu m}$ serves as reminder of multiplication. We will frequently use this notation. For instance, let $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to E$ for some vector space E. Then we define $\phi(m)\psi$ to be the mapping

$$\phi(m)\psi : \mathbb{R} \to E, t \mapsto \phi(t)\psi(t).$$

Proof. Let $\phi \in C_c^\infty(\mathbb{R};X)$. Then we have, invoking Young's inequality, that

$$
\begin{aligned}
\left| \chi_{[0,\infty)} * \phi \right|_{p,\nu} &= \left| e^{-\nu m} \left(\chi_{[0,\infty)} * \phi \right) \right|_{p,0} \\
&= \left| \left(e^{-\nu m} \chi_{[0,\infty)} \right) * \left(e^{-\nu m} \phi \right) \right|_{p,0} \\
&\leq \int_{\mathbb{R}} \left| e^{-\nu t} \chi_{[0,\infty)}(t) \right| \mathrm{d}t \left| e^{-\nu m} \phi \right|_{p,0} \\
&= \frac{1}{\nu} \left| \phi \right|_{p,\nu}.
\end{aligned}
$$

Now, for $n \in \mathbb{N}$ and some $x \in X$ with $|x| = 1$ define $\phi_n(t) := \frac{1}{n^{1/p}} e^{\nu m} \chi_{[0,n]}(t) x$ for all $t \in \mathbb{R}$. For $n \in \mathbb{N}$ let $u_n := \chi_{[0,\infty)} * \phi_n$. It is easy to that $u_n(t) = \frac{1}{\nu n^{1/p}} \left(e^{\nu \min\{t,n\}} - 1 \right) x$ for all $t \in \mathbb{R}$ and that $|\phi_n|_{p,\nu} = 1$ for all $n \in \mathbb{N}$. Moreover, an easy computation shows that $|u_n| \to \frac{1}{\nu}$ as $n \to \infty$, for the details we refer to [22, Proposition 2.2].

The equality $\left(\chi_{[0,\infty)} * \right)^{-1} = \partial_\nu$ follows by differentiation of the convolution integral. $\square$

REMARK 3.3. If $\nu < 0$, then a similar result holds. The respective inverse, however, is now given by $\left(-\chi_{(-\infty,0]} * \right)^{-1}$.

Now, we are in the situation of our general solution theory with $C = \partial_\nu$.
For convenience, we describe the space $\mathscr{X}_{-1}(\partial_\nu)$ in more detail. We let

$$
W_{p,\nu}^1(\mathbb{R};X) := \mathscr{X}_1(\partial_\nu).
$$

THEOREM 3.4. *Assume that X is reflexive[2]. We have*

$$
\left(W_{q,-\nu}^1(\mathbb{R};X) \right)' = \left(L_{p,\nu}(\mathbb{R};X'); \left| \partial_\nu^{-1} \cdot \right|_{p,\nu} \right)^{\sim},
$$

in the sense of the dual pairing

$$
L_{p,\nu}(\mathbb{R};X') \times L_{q,-\nu}(\mathbb{R};X) \ni (\phi,\psi) \mapsto \int_{\mathbb{R}} \langle \phi(t), \psi(t) \rangle_{X',X} \mathrm{d}t =: \langle \phi, \psi \rangle_{0,0}.
$$

Proof. Let $\phi \in L_{p,\nu}(\mathbb{R};X')$ be such that $\left| \partial_\nu^{-1} \phi \right|_{p,\nu} = 1$. Then, for $\psi \in W_{q,-\nu}^1(\mathbb{R};X)$ with $|\partial_{-\nu} \psi|_{q,-\nu} = 1$ we have

$$
\begin{aligned}
\left| \langle \phi, \psi \rangle_{0,0} \right| &= \left| \langle \partial_\nu \partial_\nu^{-1} \phi, \psi \rangle_{0,0} \right| \\
&= \left| - \langle \partial_\nu^{-1} \phi, \partial_{-\nu} \psi \rangle_{0,0} \right| \\
&= \left| \langle e^{-\nu m} \partial_\nu^{-1} \phi, e^{\nu m} \partial_{-\nu} \psi \rangle_{0,0} \right| \\
&\leq \left| e^{-\nu m} \partial_\nu^{-1} \phi \right|_{p,0} \left| e^{\nu m} \partial_{-\nu} \psi \right|_{q,0} = 1.
\end{aligned}
$$

---

[2]Note that, as a consequence, we have for any σ-finite measure space (Ω, μ) the property $L_p(\mu;X)' = L_q(\mu;X')$ (cf. [8, p. 82: Corollary 4 and p. 98: Theorem 1]).

This establishes the continuity of $\iota : L_{p,v}(\mathbb{R};X') \to W^1_{q,-v}(\mathbb{R};X)'$, $f \mapsto f$ with norm less than or equal to 1. Now, since $L_{q,0}(\mathbb{R};X)' = L_{p,0}(\mathbb{R};X')$ by the reflexivity of X, we find $\tilde{\psi}$ in the unit ball of $L_{q,0}(\mathbb{R};X')$ such that $\langle e^{-vm}\partial_v^{-1}\phi, \tilde{\psi}\rangle_{0,0} = 1$. Defining $\psi := -\partial_{-v}^{-1}(e^{vm})^{-1}\tilde{\psi}$, we get that $|\psi|_{q,-v,1} = 1$. Thus, ι is isometric.

It remains to prove that $L_{p,v}(\mathbb{R};X')$ is dense in $\left(W^1_{q,-v}(\mathbb{R};X)\right)'$. Let ϕ be a continuous linear functional on $\left(W^1_{q,-v}(\mathbb{R};X)\right)'$ vanishing on $L_{p,v}(\mathbb{R};X')$. By the reflexivity of X, we deduce that $\phi \in W^1_{q,-v}(\mathbb{R};X)$. Hence, $\phi = 0$. $\square$

For our general solution theory the following corollary will be useful.

COROLLARY 3.5. *Let $\ell \in \{1,0,-1\}$ and $v \in \mathbb{R}\setminus\{0\}$ and assume X to be reflexive. Then, we have*
$$W^{-\ell}_{q,-v}(\mathbb{R};X')' = W^{\ell}_{p,v}(\mathbb{R};X).$$

Proof. The result is clear as a consequence of Theorem 3.4. $\square$

3.2. Solution theory

We restate the basic solution theory in our particular situation. However, we shall restrict ourselves to a particular form of right-hand sides. We will need the following types of additional test function spaces:

$C_c^{\infty,+}(\mathbb{R};X) := \{\phi; \phi \text{ indefinitely differentiable,}$

$$\sup\operatorname{supp}\phi < \infty, \exists n \in \mathbb{N} : \operatorname{supp}\phi^{(n)} \text{ compact}\}$$

and

$$C_c^{\infty,+}(\mathbb{R};X)' := \{u : C_c^{\infty,+}(\mathbb{R};X) \to \mathbb{K};\ u \text{ linear}\}.$$

We note here that we do not assume any specific continuity property of the functionals in $C_c^{\infty,+}(\mathbb{R};X)'$. The particular continuity property will be assumed in the following definition.

DEFINITION 3.6. (eventually (k,ℓ)-contracting) Let $k,\ell \in \{1,0,-1\}$ and let Y be a reflexive Banach space. A mapping $F : C_c^{\infty}(\mathbb{R};X) \to C_c^{\infty,+}(\mathbb{R};Y')'$ is called *eventually (k,ℓ)-Lipschitz continuous* if the following assumptions are satisfied:

- there exists $v_0 > 0$ such that for all $v \geqq v_0$ we have $F(0) \in W^{-\ell}_{q,-v}(\mathbb{R};Y')'$,

- there exists $v_1 > 0$ and $C > 0$ such that for all $v \geqq v_1$, $u,v \in C_c^{\infty}(\mathbb{R};X)$, $\phi \in C_c^{\infty,+}(\mathbb{R};Y')$ we have

$$|F(u)(\phi) - F(v)(\phi)| \leqq C |\phi|_{W^{-\ell}_{q,-v}(\mathbb{R};Y')} |u - v|_{W^k_{p,v}(\mathbb{R};X)}.$$

For an eventually (k,ℓ)-Lipschitz continuous mapping F, we denote by F_v its Lipschitz continuous extension from $W^k_{p,v}(\mathbb{R};X)$ to $W^{\ell}_{p,v}(\mathbb{R};Y)$. Moreover, denote by $|F_v|_{\text{Lip}}$ the infimum over all possible Lipschitz constants for F_v. We call F *eventually (k,ℓ)-contracting* if $\limsup_{v\to\infty}|F_v|_{\text{Lip}} < 1$.

THEOREM 3.7. (L_p-solution theory) *Let X be reflexive and let $F : C_c^\infty(\mathbb{R};X) \rightarrow C_c^{\infty,+}(\mathbb{R};X')'$ be $(0,-1)$-contracting. Then, for v large enough, the equation*

$$\partial_v u = F_v(u)$$

admits a unique solution $u \in L_{p,v}(\mathbb{R};X)$.

Proof. This result is a special case of Theorem 2.1.   □

REMARK 3.8. The analogous result also holds for $(1,0)$-contracting mappings. Moreover, since the norm of ∂_v^{-1} as a mapping from $L_{p,v}$ into itself is bounded by $\frac{1}{v}$, we also get a solution theory for $(0,0)$- and $(1,1)$-Lipschitz mappings, cf. [14, Corollary 3.4].

As causality is a characterizing feature of time-evolution, we are particularly interested in establishing causality of the solution operator.

DEFINITION 3.9. (Causality) Let E,F be vector spaces. A mapping $W : D(W) \subseteq E^{\mathbb{R}} \rightarrow F^{\mathbb{R}}$ is called *causal* if for all $x,y \in D(W)$ and $t \in \mathbb{R}$ we have

$$\chi_{\mathbb{R}_{<t}}(m)(x-y) = 0 \Rightarrow \chi_{\mathbb{R}_{<t}}(m)(W(x) - W(y)) = 0.$$

Similar to [14, Definition 4.3], we have to define a notion of distributional integrals or distributional convolutions.

DEFINITION 3.10. Let $w \in C_c^{\infty,+}(\mathbb{R};X)'$. Then we define

$$\chi_{[0,\infty)} * w : C_c^{\infty,+}(\mathbb{R};X) \rightarrow \mathbb{K}, \phi \mapsto w(\chi_{(-\infty,0]} * \phi).$$

REMARK 3.11. Assume that X is reflexive. For $w \in W_{p,v}^{-1}(\mathbb{R};X)$ we have that $\chi_{[0,\infty)} * w = \partial_v^{-1} w$. Indeed, by Theorem 3.4, we have $w \in W_{q,-v}^1(\mathbb{R};X')'$ and thus, for $\phi \in C_c^{\infty,+}(\mathbb{R};X')$, we get that

$$\chi_{[0,\infty)} * w(\phi) = w(\chi_{(-\infty,0]} * \phi) = w(-\partial_{-v}^{-1}\phi) = \langle w, -\partial_{-v}^{-1}\phi\rangle_{0,0} = \langle \partial_v^{-1} w, \phi\rangle_{0,0} = \partial_v^{-1} w(\phi).$$

THEOREM 3.12. (Causality) *Assume that X is reflexive. Let $F : C_c^\infty(\mathbb{R};X) \rightarrow C_c^{\infty,+}(\mathbb{R};X')'$ be $(0,-1)$ contracting. Then $\partial_v^{-1}F$ is causal.*

Proof. The proof follows essentially along the lines of [14, Theorem 4.5]. Since we are, however, dealing with a Banach space setting here, the arguments are more delicate and thus worth recalling in detail. Let $t \in \mathbb{R}$, v_1 such that $|F_v|_{\text{Lip}} < 1$ for

all $v \geqq v_1$. Let $\phi \in C^\infty(\mathbb{R})$ be bounded. For $v \geqq v_1$ and $w \in C_c^\infty(\mathbb{R};X)$ and $\psi \in C_c^{\infty,+}(\mathbb{R};X')$ with $\sup\operatorname{supp}\psi \leqq t$ we compute

$$
\begin{aligned}
\left|\partial_{v_1}^{-1}F_{v_1}(w)(\psi) - \partial_{v_1}^{-1}F_{v_1}(\phi(m)w)(\psi)\right| &= \left|\chi_{[0,\infty)} * F(w)(\psi) - \chi_{[0,\infty)} * F(\phi(m)w)(\psi)\right| \\
&= \left|\partial_v^{-1}F_v(w)(\psi) - \partial_v^{-1}F_v(\phi(m)w)(\psi)\right| \\
&= \left|F_v(w)(-\partial_{-v}^{-1}\psi) - F_v(\phi(m)w)(-\partial_{-v}^{-1}\psi)\right| \\
&\leqslant \left|-\partial_{-v}^{-1}\psi\right|_{W_{q,-v}^1(\mathbb{R};X')}\left|w - \phi(m)w\right|_{L_{p,v}(\mathbb{R};X)} \cdot \\
&= \left|\psi\right|_{L_{q,-v}(\mathbb{R};X')}\left|w - \phi(m)w\right|_{L_{p,v}(\mathbb{R};X)} \\
&\leqslant \left|\psi\right|_{L_{q,0}(\mathbb{R};X')}e^{vt}\left|w - \phi(m)w\right|_{L_{p,v}(\mathbb{R};X)} \cdot
\end{aligned}
$$

By continuity, we deduce that

$$
\begin{aligned}
\left|\partial_{v_1}^{-1}F_{v_1}(w)(\psi) - \partial_{v_1}^{-1}F_{v_1}(\chi_{\mathbb{R}_{<t}}(m)w)(\psi)\right| &\leqq \left|\psi\right|_{L_{q,0}(\mathbb{R};X')}e^{vt}\left(\int_t^\infty |w(\tau)|^p e^{-pv\tau}d\tau\right)^{\frac{1}{p}} \\
&= \left|\psi\right|_{L_{q,0}(\mathbb{R};X')}\left(\int_0^\infty |w(\tau+t)|^p e^{-pv\tau}d\tau\right)^{\frac{1}{p}}.
\end{aligned}
$$

Hence, letting $v \to \infty$, we get the assertion. $\quad\square$

THEOREM 3.13. (Independence of v) *Assume that X is a reflexive Banach space. Let $F : C_c^\infty(\mathbb{R};X) \to C_c^{\infty,+}(\mathbb{R};X')'$ be $(0,-1)$-contracting. Let $v_1 \in \mathbb{R}_{>0}$ be such that $|F_v|_{\mathrm{Lip}} < 1$ for all $v \geqq v_1$. Let $v_2 \geqq v_1$. Then, if w_{v_1}, w_{v_2} satisfy*

$$\partial_{v_1}w_{v_1} = F_{v_1}(w_{v_1}) \text{ and } \partial_{v_2}w_{v_2} = F_{v_2}(w_{v_2}),$$

we have $w_{v_1} = w_{v_2}$.

Proof. The proof follows the ideas of the proof of [14, Theorem 4.6]: Let $t \in \mathbb{R}$, $v \in \mathbb{R}_{\geqq v_1}$. Denoting by w_v the solution of

$$\partial_v w_v = F_v(w_v) \in W_{p,v}^{-1}(\mathbb{R};X),$$

we recall $w_v \in L_{p,v}(\mathbb{R};X)$. Moreover, we have due to Theorem 3.12

$$
\begin{aligned}
\chi_{\mathbb{R}_{<t}}(m)w_v &= \chi_{\mathbb{R}_{<t}}(m)\partial_v^{-1}F_v(w_v) \\
&= \chi_{\mathbb{R}_{<t}}(m)\partial_v^{-1}F_v(\chi_{\mathbb{R}_{<t}}(m)w_v).
\end{aligned}
$$

Then, as $\partial_{v_1}^{-1}F_{v_1}$ coincides with $\partial_{v_2}^{-1}F_{v_2}$ on $C_c^\infty(\mathbb{R};X)$ and as an approximating sequence of $C_c^\infty(\mathbb{R};X)$-functions for $\chi_{\mathbb{R}_{<t}}(m_0)w_{v_2}$ can be chosen to converge in both $L_{p,v_1}(\mathbb{R};X)$ and $L_{p,v_2}(\mathbb{R};X)$, we arrive at

$$\chi_{\mathbb{R}_{<t}}(m)\partial_{v_2}^{-1}F_{v_2}(\chi_{\mathbb{R}_{<t}}(m)w_{v_2}) = \chi_{\mathbb{R}_{<t}}(m)\partial_{v_1}^{-1}F_{v_1}(\chi_{\mathbb{R}_{<t}}(m)w_{v_2}).$$

Hence,

$$\left|\chi_{\mathbb{R}_{<t}}(m)(w_{v_1} - w_{v_2})\right|_{L_{p,v_1}(\mathbb{R};X)}$$

$$= \left|\chi_{\mathbb{R}_{<t}}(m)(\partial_{v_1}^{-1} F_{v_1}(\chi_{\mathbb{R}_{<t}}(m)w_{v_1}) - \partial_{v_1}^{-1} F_{v_1}(\chi_{\mathbb{R}_{<t}}(m)w_{v_2}))\right|_{L_{p,v_1}(\mathbb{R};X)}$$

$$\leqq \left|\partial_{v_1}^{-1} F_{v_1}(\chi_{\mathbb{R}_{<t}}(m)w_{v_1}) - \partial_{v_1}^{-1} F_{v_1}(\chi_{\mathbb{R}_{<t}}(m)w_{v_2})\right|_{L_{p,v_1}(\mathbb{R};X)}$$

$$\leqq |F_{v_1}|_{\text{Lip}} \left|\chi_{\mathbb{R}_{<t}}(m)(w_{v_1} - w_{v_2})\right|_{L_{p,v_1}(\mathbb{R};X)}.$$

Since $|F_{v_1}|_{\text{Lip}} < 1$ the assertion follows. $\square$

3.3. Examples

Before we illustrate the applicability of our abstract theorems, we introduce the notion of having delay and of being amnesic for mappings from function spaces into function spaces.

DEFINITION 3.14. Let E, F be vector spaces. A mapping $W : D(W) \subseteq E^{\mathbb{R}} \to F^{\mathbb{R}}$ is called *amnesic* if for all $t \in \mathbb{R}$, $x, y \in D(W)$ we have

$$\chi_{\mathbb{R}_{>t}}(m)(x - y) = 0 \Rightarrow \chi_{\mathbb{R}_{>t}}(m)(W(x) - W(y)) = 0$$

W is said to *have delay* if W is not amnesic.

We shall give some examples of mappings having delay.

EXAMPLE 3.15. (Discrete delay) For $\theta \in \mathbb{R}$ we define

$$\tau_\theta : L_{p,v}(\mathbb{R};X) \to L_{p,v}(\mathbb{R};X),$$
$$f \mapsto (t \mapsto f(t + \theta))$$

It is easy to see, that τ_θ is not causal for $\theta > 0$, whereas it is amnesic. For $\theta < 0$, τ_θ is causal and has delay. For convenience, we compute the operator norm of τ_θ. For $f \in C_c^\infty(\mathbb{R};X)$, we have

$$|\tau_\theta f|_{p,v}^p = \int_{\mathbb{R}} |f(t+\theta)|_X^p \, e^{-pvt} \mathrm{d}t = \int_{\mathbb{R}} |f(t+\theta)|_X^p \, e^{-pv(t+\theta)} \mathrm{d}t \, e^{pv\theta} = |f|_{p,v}^p \, e^{pv\theta}.$$

Thus, $\|\tau_\theta\| = e^{v\theta}$.

EXAMPLE 3.16. (Continuous delay) The mapping

$$\Theta : L_{p,v}(\mathbb{R};X) \to L_{p,v}(\mathbb{R};L_p(\mathbb{R}_{<0};X))$$
$$\phi \mapsto \phi_{(\cdot)} := (t \mapsto (\theta \mapsto \phi(t+\theta)))$$

clearly has delay. We compute its operator norm. For $f \in C_c^\infty(\mathbb{R};X)$, we have

$$
\begin{aligned}
|\Theta f|_{p,v}^p &= \int_{\mathbb{R}} \int_{\mathbb{R}_{<0}} |f(t+\theta)|_X^p \, d\theta e^{-pvt} dt \\
&= \int_{\mathbb{R}_{<0}} \int_{\mathbb{R}} |f(t+\theta)|_X^p \, e^{-pv(t+\theta)} dt \, e^{pv\theta} d\theta \\
&= \int_{\mathbb{R}_{<0}} |f|_{p,v}^p \, e^{pv\theta} d\theta \\
&= |f|_{p,v}^p \frac{1}{pv}.
\end{aligned}
$$

Hence, $\|\Theta\| = \frac{1}{\sqrt[p]{pv}}$.

To incorporate initial value problems, it is convenient to have an adapted point trace result.

THEOREM 3.17. (Sobolev embedding) *For $v \in \mathbb{R} \setminus \{0\}$, define*

$$
C_v(\mathbb{R};X) := \big\{ f : \mathbb{R} \to X; f \text{ continuous,}
$$

$$
|f|_{v,\infty} := \sup\{ |e^{-vt} f(t)|_X ; t \in \mathbb{R}\} < \infty, \ e^{-vt} f(t) \to 0 (t \to \pm\infty) \big\}.
$$

We endow $C_v(\mathbb{R};X)$ with the norm $|\cdot|_{v,\infty}$ such that it becomes a Banach space. The mapping

$$
\iota : C_c^\infty(\mathbb{R};X) \subseteqq W_{p,v}^1(\mathbb{R};X) \to C_v(\mathbb{R};X), f \mapsto f
$$

is continuous.

Proof. We shall only prove the case $v > 0$. The case $v < 0$ can be dealt with similarly. Let $f \in C_c^\infty(\mathbb{R};X)$ and $s,t \in \mathbb{R}, s < t$. Then

$$
\begin{aligned}
|f(t) - f(s)| &\leq \int_s^t |\partial_v f(\xi)| \, d\xi \\
&= \int_s^t |\partial_v f(\xi)| e^{-v\xi} e^{v\xi} d\xi \\
&\leq \left(\int_s^t |\partial_v f(\xi)|^p e^{-pv\xi} d\xi \right)^{\frac{1}{p}} \left(\int_s^t e^{qv\xi} d\xi \right)^{\frac{1}{q}} \\
&\leq |f|_{1,v,p} \left(\frac{1}{qv} \left(e^{qvt} - e^{qvs} \right) \right)^{\frac{1}{q}}.
\end{aligned}
$$

Letting $s \to -\infty$ in this inequality, we arrive at

$$
e^{-vt} |f(t)| \leq \frac{1}{\sqrt[q]{qv}} |f|_{1,v,p},
$$

which gives the continuity result. $\square$

By the aforementioned theorem, ι has a unique continuous extension to the space $W_{p,\nu}^1(\mathbb{R};X)$. As the extension of ι is the extension of the identity mapping, we omit ι in the following, and choose, without giving explicit reference to it, the continuous representer of a $W_{p,\nu}^1(\mathbb{R};X)$-function. (It is easy to see that the continuous extension is one-to-one.)

Now, we have all the tools at hand to apply our general solution theory to a number of example cases.

EXAMPLE 3.18. (Initial value problems, cf. [14, Theorem 5.4]) Let $\nu > 0$, $u_0 \in X$. Let F be $(0,0)$-Lipschitz and such that $F(\phi) = 0$ for all $\phi \in C_c^\infty(\mathbb{R};X)$ with $\operatorname{supp}\phi \subseteq (-\infty,0]$. Then the equation[3]

$$\partial_\nu u = F_\nu(u) + \delta u_0$$

admits a unique solution $u \in L_{p,\nu}(\mathbb{R};X)$ and such that $u - \chi_{\mathbb{R}_{>0}}(m)u \in W_{p,\nu}^1(\mathbb{R};X)$ and $u(0+) = u_0$ if ν is chosen sufficiently large.

Unique existence of u follows from our general solution theory. The remaining facts follow from the representation

$$u - \chi_{\mathbb{R}_{>0}}(m)u = u - \partial_\nu^{-1}\delta u_0 = \partial_\nu^{-1}F_\nu(u)$$

and causality of $\partial_\nu^{-1}F_\nu$.

EXAMPLE 3.19. (Finite discrete delay) Let $\theta_1,\ldots,\theta_n \in \mathbb{R}_{\leq 0}$ be distinct, and let $\Phi : C_c^\infty(\mathbb{R};X^n) \to C_c^{\infty,+}(\mathbb{R};X')'$ be $(0,-1)$-contracting. Then, for ν sufficiently large, the equation

$$\partial_\nu u = \Phi_\nu(\tau_{\theta_1}u,\cdots,\tau_{\theta_n}u)$$

admits a unique solution $u \in L_{p,\nu}(\mathbb{R};X)$.

It suffices to observe that the operator norm of

$$\Theta : L_{p,\nu}(\mathbb{R};X) \to L_{p,\nu}(\mathbb{R};X^n), f \mapsto (\tau_{\theta_1}f,\ldots,\tau_{\theta_n}f)$$

can be estimated arbitrarily close to 1, if ν was chosen sufficiently large.

EXAMPLE 3.20. (Continuous delay) Let $\Phi : C_c^\infty(\mathbb{R};L_p(\mathbb{R}_{<0};X)) \to C_c^{\infty,+}(\mathbb{R};X')'$ be $(0,-1)$-Lipschitz. Then, for ν sufficiently large, the equation

$$\partial_\nu u = \Phi_\nu(u_{(.)})$$

admits a unique solution.

The assertion follows from the Example 3.16, where we estimated the operator norm of the mapping $\phi \mapsto \phi_{(.)}$ in the weighted spaces under consideration.

---

[3]By Theorem 3.17, we have that the point evaluation at 0, denoted by δ, is an element of $W_{p,\nu}^{-1}$. For a Banach space element u_0 we write δu_0 for the derivative of the map $t \mapsto \chi_{[0,\infty)}(t)u_0$. Thus, in this sense it holds $\delta u_0 \in W_{p,\nu}^{-1}(\mathbb{R};X)$.

EXAMPLE 3.21. (Neutral differential equations) Let $\Phi : C_c^\infty(\mathbb{R}; L_p(\mathbb{R}_{<0}; X)^2) \to C_c^{\infty,+}(\mathbb{R}; X')'$ be $(0,0)$-Lipschitz. Then the equation

$$\partial_\nu u = \Phi(u_{(\cdot)}, (\partial_\nu u)_{(\cdot)})$$

admits a unique solution $u \in W_{p,\nu}^1(\mathbb{R}; X)$, if ν was chosen large enough.

Consider the mapping

$$\Theta : W_{p,\nu}^1(\mathbb{R}; X) \to L_{p,\nu}(\mathbb{R}; L_p(\mathbb{R}_{<0}; X)^2), f \mapsto (f_{(\cdot)}, (\partial_\nu f)_{(\cdot)}).$$

Note that the operator norm of $W_{p,\nu}^1(\mathbb{R}; X) \to L_{p,\nu}(\mathbb{R}; X)^2, f \mapsto (f, \partial_\nu f)$ is bounded if $\nu \to \infty$ and that the mapping $L_{p,\nu}(\mathbb{R}; X) \ni f \mapsto f_{(\cdot)} \in L_{p,\nu}(\mathbb{R}; L_p(\mathbb{R}_{<0}; X))$ has operator norm tending to 0 if $\nu \to \infty$, by the aforementioned example. We deduce that Θ is eventually $(1,0)$-contracting, with arbitrarily small operator norm and that the map $\Phi \circ \Theta$ is eventually $(1,0)$-contracting. Hence, our general solution theory applies.

In the following, we will treat some more concrete examples from the literature.

EXAMPLE 3.22. The following example has been considered in [3, 4, 12] and the references therein. Let $B \in L_1(\mathbb{R}_{>0}; \mathbb{R}^{n \times n})$, $(A_j)_j \in \ell_1(\mathbb{N}; \mathbb{R}^{n \times n})$, $(t_j)_j \in \mathbb{R}_{\geqslant 0}^{\mathbb{N}}$ and let $f \in L_p(\mathbb{R}; \mathbb{R}^n)$ be such that the support is bounded from below. Consider the problem of finding $x \in L_{p,\nu}(\mathbb{R}; \mathbb{R}^{n \times n})$ such that

$$\partial_\nu x = \sum_{j=0}^\infty A_j \tau_{-t_j} x + B * x + f.$$

The unique existence of x follows by observing that the operator

$$F : L_{p,\nu}(\mathbb{R}; \mathbb{R}^n) \to L_{p,\nu}(\mathbb{R}; \mathbb{R}^n)$$

$$x \mapsto \left(\sum_{j=0}^\infty A_j \tau_{-t_j} x + B * x \right)$$

is Lipschitz continuous. Indeed, Young's inequality ensures $|B * x| \leqslant |B|_{L^1} |x|$ for all $x \in L_{p,\nu}(\mathbb{R}; \mathbb{R}^n)$. The first term we estimate as follows. Let $x \in L_{p,\nu}(\mathbb{R}; \mathbb{R}^n)$. Then

$$\left| \sum_{j=0}^\infty A_j \tau_{-t_j} x \right| \leqslant \sum_{j=0}^\infty |A_j| \, |x| = \left| (A_j)_j \right|_{\ell^1} |x|$$

In [5] the oscillations of possible solutions to the following problem are discussed.

EXAMPLE 3.23. Let $k \in \mathbb{N}, n \in \mathbb{N}_{>0}$ and for $j \in \{0, \ldots, k\}$ let $p_j : \mathbb{R} \to \mathbb{R}$ be continuous and bounded and $\sigma_j \in \mathbb{R}_{>0}$. Let $f \in L_p(\mathbb{R})$ with support bounded from below and consider the following neutral differential equation of n'th order

$$(x - p_0(m) \tau_{-\sigma_0} x)^{(n)} = \sum_{j=1}^k p_j(m) \tau_{-\sigma_j} x + f.$$

For $v \in \mathbb{R}_{>0}$, we may equally discuss

$$\partial_v^n (x - p_0(m)\tau_{-\sigma_0}x) = \sum_{j=1}^{k} p_j(m)\tau_{-\sigma_j}x + f.$$

The latter is the same as

$$x = \partial_v^{-n} \left(\sum_{j=1}^{k} p_j(m)\tau_{-\sigma_j}x \right) + p_0(m)\tau_{-\sigma_0}x + \partial_v^{-n}f.$$

We observe that this is a fixed point problem, which admits a unique solution for v large enough. Indeed, the operator norm of

$$\left(\partial_v^{-n} \left(\sum_{j=1}^{k} p_j(m)\tau_{-\sigma_j} \right) + p_0(m)\tau_{-\sigma_0} \right) : L_{p,v}(\mathbb{R}) \to L_{p,v}(\mathbb{R})$$

can be estimated by

$$\left\| \partial_v^{-n} \left(\sum_{j=1}^{k} p_j(m)\tau_{-\sigma_j} \right) + p_0(m)\tau_{-\sigma_0} \right\| \leqslant \left\| \partial_v^{-n} \left(\sum_{j=1}^{k} p_j(m)\tau_{-\sigma_j} \right) \right\| + \left\| p_0(m)\tau_{-\sigma_0} \right\|$$

$$\leqslant \frac{1}{v^n} \sum_{j=1}^{k} |p_j|_\infty \exp(-\sigma_j v) + |p_0|_\infty \exp(-\sigma_0 v),$$

which is eventually less than 1 if v is large enough. Note that for having a solution theory, the continuity of the p_j's was not needed.

A different class of neutral differential equations, which was considered in [9, 10] is as follows. We shall treat the Hilbert space case here for convenience.

EXAMPLE 3.24. Let H be a Hilbert space and $M, L \in L(L_2(\mathbb{R}_{<0};H);H)$. Consider the following equation

$$(t \mapsto Mx_t)' = (t \mapsto Lx_t) + f, \tag{1}$$

where $f \in L_2(\mathbb{R};H)$ with support bounded from below is given. Our space-time approach prerequisites the consideration of the operator

$$\Theta : C_c^\infty(\mathbb{R};H) \to C_c^\infty(\mathbb{R};L_2(\mathbb{R}_{<0};H)), \phi \mapsto \phi_{(\cdot)}$$

in a slightly different version than before. We note that for $\phi \in C_c^\infty(\mathbb{R};H)$ and $v \in \mathbb{R}_{>0}$, we have

$$\partial_v \Theta \phi = \Theta \partial_v \phi. \tag{2}$$

For any $v \in \mathbb{R}_{>0}$ there is a continuous extension Θ_v as a mapping from $L_{2,v}(\mathbb{R};H)$ to $L_{2,v}(\mathbb{R};L_2(\mathbb{R}_{<0};H))$. Moreover, for all $x \in L_{2,v}(\mathbb{R};H)$ we have

$$\langle \Theta x, \Theta x \rangle = \frac{1}{2v} \langle x, x \rangle.$$

This equality yields the closedness of the range of Θ_ν. Continuous extension of (2) for all $\phi \in W^1_{2,\nu}(\mathbb{R};H)$ yields that ∂_ν leaves $R(\Theta_\nu)$ invariant. Furthermore, we have the same property for ∂_ν^{-1}. Hence, our perspective on (1) is the following. Consider

$$\partial_\nu M(\Theta_\nu x) = L\Theta_\nu x + f.$$

By the continuous invertibility of Θ and the intertwining relation (2), this is the same as to consider

$$\partial_\nu \Theta_\nu M(\Theta_\nu x) = \Theta_\nu L\Theta_\nu x + \Theta_\nu f.$$

Assume $\mathcal{M}_\nu : R(\Theta_\nu) \to R(\Theta_\nu), y \mapsto \Theta_\nu My$ to be continuously invertible. Therefore, we formulate the equation as follows

$$\partial_\nu \Theta_\nu M(y) = \Theta_\nu Ly + \Theta_\nu f.$$

for $y \in R(\Theta_\nu)$. Now, our general solution theory applies to the equation

$$\partial_\nu y = \Theta_\nu \left(L\mathcal{M}_\nu^{-1} y + f \right).$$

This yields a unique solution $y \in R(\Theta_\nu)$. The solution of equation (1) is then given by $x = \Theta_\nu^{-1} \mathcal{M}_\nu^{-1} y$.

4. The non-reflexive case – spaces of continuous functions

In this section, we will describe how to adapt the general solution theory of Section 2 to the non-reflexive setting. The main difficulty to overcome is to give appropriate meaning to "eventually (k,ℓ)-Lipschitz continuous" in order to state a coherent theory. For the whole section, let X be a Banach space. We focus here on the L_∞-norm, we could, however, also treat the case of L_1-functions. As the case of L_1 is a hybrid of distributional derivatives similar to the previous part and the issues resulting from the non-reflexivity of the underlying space as discussed in the following sections, we only consider the L_∞-norm here.

4.1. The time-derivative

The distributional time-derivative as presented in Section 3 cannot be used in the straightforward way by choosing L_∞ as underlying space, since the (distributional) time-derivative would not be densely defined anymore. Thus, we consider the more or less classical way of discussing delay differential equations and use the space of Banach space valued continuous functions $C_\nu(\mathbb{R};X)$, which we have already defined in Theorem 3.17, as the underlying space.

DEFINITION 4.1. For $\nu \in \mathbb{R}$, define $\partial_\nu : C^1_\nu(\mathbb{R};X) \subseteq C_\nu(\mathbb{R};X) \to C_\nu(\mathbb{R};X), f \mapsto f'$, where $C^1_\nu(\mathbb{R};X) := \{f \in C_\nu(\mathbb{R};X); f' \in C_\nu(\mathbb{R};X)\}$.

PROPOSITION 4.2. Let $\nu \in \mathbb{R} \setminus \{0\}$. Then $0 \in \rho(\partial_\nu)$, $\partial_\nu^{-1} f(t) = \int_{-\infty}^t f(\tau) d\tau$ ($t \in \mathbb{R}$, $\nu > 0$) and $\left\| \partial_\nu^{-1} \right\| = \frac{1}{|\nu|}$.

Proof. For $f \in C_c^\infty(\mathbb{R};X)$ and $\nu > 0$, we compute

$$\left| e^{-\nu t} \int_{-\infty}^t f(\tau) d\tau \right| = \left| \int_{-\infty}^t e^{-\nu t + \nu \tau} f(\tau) e^{-\nu \tau} d\tau \right|$$

$$\leq \int_{-\infty}^t e^{-\nu t + \nu \tau} d\tau \, |f|_{C_\nu(\mathbb{R};X)} = \frac{1}{\nu} |f|_{C_\nu(\mathbb{R};X)} \,.$$

In order to see the remaining inequality, for $n \in \mathbb{N}$ take a function $\phi_n \in C_c^\infty(\mathbb{R})$ such that $0 \leq \phi_n \leq 1$ and $\phi_n = 1$ on $[-n,n]$. Let $x \in X$ with $|x| = 1$ and define $f_n(t) := e^{\nu t} \phi_n(t) x$ for $n \in \mathbb{N}, t \in \mathbb{R}$. Note that $|f_n|_{C_\nu(\mathbb{R};X)} \leq 1$ for all $n \in \mathbb{N}$. Moreover, observe that for $n \in \mathbb{N}$ we have

$$\sup\{ |e^{-\nu t}(\partial_\nu^{-1} f_n)(t)| ; t \in \mathbb{R} \} \geq e^{-\nu n} \int_{-n}^n e^{\nu \tau} d\tau$$

$$= \frac{1}{\nu} e^{-\nu n} \left(e^{\nu n} - e^{-\nu n} \right)$$

$$= \frac{1}{\nu} \left(1 - e^{-2\nu n} \right) \to \frac{1}{\nu} \quad (n \to \infty).$$

This yields $\left\| \partial_\nu^{-1} \right\| \geq \frac{1}{\nu}$. The case $\nu < 0$ is similar. $\quad\square$

Hence, ∂_ν is a possible choice for C in the basic solution theory. Before, we state the solution theory also in this case, we define eventually Lipschitz continuous mappings to have a prototype of right-hand sides at hand. We denote $C_\nu^k(\mathbb{R};X) := \mathscr{X}_k(\partial_\nu)$ for $k \in \{1,0,-1\}$. Due to the non-reflexivity of $C_\nu(\mathbb{R};X)$, we cannot define eventual Lipschitz continuity for mappings with values in a space of linear functionals. Instead of characterizing the negative extrapolation spaces as suitable duals, we introduce the space $C_{-\infty}(\mathbb{R};X) := \bigcup_{\nu \in \mathbb{R}_{>0}} C_\nu^{-1}(\mathbb{R};X)$. In order to compare elements of "negative" spaces for different parameters ν, we define the following equality relation between these elements: For $\phi \in C_\nu^{-1}(\mathbb{R};X)$ and $\psi \in C_\mu^{-1}(\mathbb{R};X)$ we define

$$\phi = \psi :\Leftrightarrow \partial_\nu^{-1} \phi = \partial_\mu^{-1} \psi.$$

REMARK 4.3. Let $\phi \in C_\nu^{-1}(\mathbb{R};X), \psi \in C_\mu^{-1}(\mathbb{R};X)$ with $\phi = \psi$. Then there exists a sequence $(\rho_n)_{n \in \mathbb{N}}$ in $C_c^\infty(\mathbb{R};X)$ such that $\rho_n \to \phi$ in $C_\nu^{-1}(\mathbb{R};X)$ and $\rho_n \to \psi$ in $C_\mu^{-1}(\mathbb{R};X)$ as $n \to \infty$. Indeed, let $(\gamma_n)_{n \in \mathbb{N}} \in C_c^\infty(\mathbb{R})^{\mathbb{N}}$ be a mollifier and define $\tilde{\rho}_n := \gamma_n * \partial_\nu^{-1} \phi \in C_\nu(\mathbb{R};X) \cap C_\mu(\mathbb{R};X)$. Then we obtain, due to the continuity of the translation operator

$$[0,1] \ni s \mapsto (f \mapsto f(\cdot + s)) \in L(C_\nu(\mathbb{R};X))$$

for each $\nu \in \mathbb{R}_{>0}$, that $\tilde{\rho}_n \to \partial_\nu^{-1} \phi$ in $C_\nu(\mathbb{R};X)$ and $C_\mu(\mathbb{R};X)$ as $n \to \infty$. For $k \in \mathbb{N}$ let now $\chi_k \in C_c^\infty(\mathbb{R})$ such that $0 \leqslant \chi_k \leqslant 1$ and $\chi_k = 1$ on $(-k,k)$. Then an easy computation shows $\chi_k \tilde{\rho}_n \to \tilde{\rho}_n$ in $C_\nu(\mathbb{R};X)$ and $C_\mu(\mathbb{R};X)$ as $k \to \infty$. Hence, we find a strictly increasing sequence $(k_n)_n$ of integers such that $(\rho_n)_n := ((\chi_{k_n} \tilde{\rho}_n)')_n$ has the desired properties.

DEFINITION 4.4. (eventually (k,ℓ)-contracting) Let $k,\ell \in \{1,0,-1\}$ and let Y be a Banach space. A mapping $F : C_c^\infty(\mathbb{R};X) \to C_{-\infty}(\mathbb{R};Y)$ is called *eventually* (k,ℓ)-*Lipschitz continuous* if the following assumptions are satisfied:

- there exists $v_0 > 0$ such that for all $v \geqq v_0$ we have $F(0) \in C_v^\ell(\mathbb{R};Y)$,

- there exists $v_1 > 0$ and $C > 0$ such that for all $v \geqq v_1$, $u,v \in C_c^\infty(\mathbb{R};X)$ we have

$$|F(u) - F(v)|_{C_v^\ell(\mathbb{R};Y)} \leqq C |u-v|_{C_v^k(\mathbb{R};X)}.$$

For an eventually (k,ℓ)-Lipschitz continuous mapping F, we denote by F_v its Lipschitz continuous extension from $C_v^k(\mathbb{R};X)$ to $C_v^\ell(\mathbb{R};Y)$. Moreover, denote by $|F_v|_{\text{Lip}}$ the infimum over all possible Lipschitz constants for F_v. We call F *eventually* (k,ℓ)-*contracting* if $\limsup_{v\to\infty} |F_v|_{\text{Lip}} < 1$.

REMARK 4.5. Note that for a (k,ℓ)-Lipschitz continuous mapping we have the inclusion $F[C_c^\infty(\mathbb{R};X)] \subseteqq \bigcap_{v \geqq v_0} C_v^{-1}(\mathbb{R};X)$ for some $v_0 > 0$, where the intersection is understood with respect to the equality relation defined above.

THEOREM 4.6. (Solution theory) *Let* $F : C_c^\infty(\mathbb{R};X) \to C_{-\infty}(\mathbb{R};X)$ *be eventually* $(0,-1)$-*contracting. Then there exists a unique solution* $u \in C_v(\mathbb{R};X)$ *of the equation*

$$\partial_v u = F_v(u)$$

if v *is chosen sufficiently large.*

Proof. Clear. □

REMARK 4.7. The latter theorem also extends to the case of $(1,0)$-contracting. Moreover, similar to Section 3 and due to Proposition 4.2, we also have a solution theory for $(0,0)$- or $(1,1)$-Lipschitz continuous mappings, which is the common situation.

THEOREM 4.8. (Causality) *Let* $F : C_c^\infty(\mathbb{R};X) \to C_{-\infty}(\mathbb{R};X)$ *be* $(0,-1)$-*contracting. Then* $\partial_v^{-1} F_v$ *is causal if* v *is chosen sufficiently large.*

Proof. Let $v_0 \in \mathbb{R}_{>0}$ be such that F admits a Lipschitz-continuous extension for all $v \geqq v_0$. Let $\tau \in \mathbb{R}$ and let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leqq \phi \leqq 1$, $\phi(s) = 0$ for $s \geqq \tau$ and $\phi(t) = 1$ for $t \leqq \tau - \varepsilon$ for some $\varepsilon \in \mathbb{R}_{>0}$. We show that $\partial_v^{-1} F_v(v)(t) = \partial_v^{-1} F_v(\phi(m)v)(t)$ for $v \in C_c^\infty(\mathbb{R};X)$ and $t \leqq \tau - \varepsilon$. Let $\psi \in C_c^\infty(\mathbb{R};X')$ be such that $\sup \text{supp}\, \psi \leqq \tau - \varepsilon$. We compute for $v \in C_c^\infty(\mathbb{R};X)$ and $\eta \geqq v$

$$\int_\mathbb{R} |\langle \partial_v^{-1} F_v(v) - \partial_v^{-1} F_v(\phi(m)v), \psi \rangle|$$

$$\leqq \int_\mathbb{R} |\partial_v^{-1} F_v(v) - \partial_v^{-1} F_v(\phi(m)v)| \, |\psi| = \int_\mathbb{R} |\partial_\eta^{-1} F_\eta(v) - \partial_\eta^{-1} F_\eta(\phi(m)v)| \, |\psi|$$

$$\leq \left|\partial_\eta^{-1} F_\eta(v) - \partial_\eta^{-1} F_\eta(\phi(m)v)\right|_{\infty,\eta} \int_{-\infty}^{\tau-\varepsilon} |\psi(t)| e^{\eta t} dt$$

$$\leq |(1 - \phi(m))v|_{\infty,\eta} \int_{-\infty}^{\tau-\varepsilon} |\psi(t)| e^{\eta t} dt$$

$$\leq \sup\{\left|e^{-\eta t} v(t)\right|; \tau - \varepsilon \leq t < \infty\} \int_{-\infty}^{\tau-\varepsilon} |\psi(t)| e^{\eta t} dt$$

$$= \sup\left\{\left|e^{-\eta(t+\tau-\varepsilon)} v(t + \tau - \varepsilon)\right|; 0 \leq t < \infty\right\} \int_{-\infty}^{0} |\psi(t+\tau-\varepsilon)| e^{\eta(t+\tau-\varepsilon)} dt$$

$$= \sup\{\left|e^{-\eta t} v(t + \tau - \varepsilon)\right|; 0 \leq t < \infty\} \int_{-\infty}^{0} |\psi(t+\tau-\varepsilon)| e^{\eta t} dt$$

$$\to 0 \quad (\eta \to \infty),$$

where in the second line we have used the definition of equality of elements in $C_\nu^{-1}(\mathbb{R};X)$ and $C_\eta^{-1}(\mathbb{R};X)$. Thus,

$$\sup\{\left|\partial_\nu^{-1} F_\nu(v)(t) - \partial_\nu^{-1} F_\nu(\phi(m)v)(t)\right|; -\infty < t \leq \tau - \varepsilon\} = 0.$$

This yields causality. $\quad\square$

THEOREM 4.9. (Independence of ν) *Let* $F : C_c^\infty(\mathbb{R};X) \to C_{-\infty}(\mathbb{R};X)$ *be* $(0, -1)$-*contracting and* $\nu_0 > 0$ *such that* $|F_\nu|_{\text{Lip}} < 1$ *for each* $\nu \geq \nu_0$. *Let* $\nu \geq \mu \geq \nu_0$ *and let* $v_\nu \in C_\nu(\mathbb{R};X)$, $v_\mu \in C_\mu(\mathbb{R};X)$ *denote the solutions of the equations*

$$\partial_\nu v_\nu = F_\nu(v_\nu) \text{ and } \partial_\mu v_\mu = F_\mu(v_\mu),$$

respectively. Then $v_\nu = v_\mu$.

Proof. Let $t \in \mathbb{R}$ and let $\phi \in C^\infty(\mathbb{R})$ be such that $0 \leq \phi \leq 1$ and $\phi(s) = 0$ for $s \geq t$ and $\phi(s) = 1$ for $s \leq t - \varepsilon$ for some $\varepsilon > 0$. Note that for $w \in C_\nu(\mathbb{R};X)$ we have $\phi(m)w \in C_\mu(\mathbb{R};X)$ with $|\phi(m)w|_{\mu,\infty} \leq e^{(\nu-\mu)t} |\phi(m)w|_{\nu,\infty}$. Hence, $\partial_\nu^{-1} F_\nu(\phi(m)v_\nu) = \partial_\mu^{-1} F_\mu(\phi(m)v_\nu)$, since we can approximate $\phi(m)v_\nu$ by the same sequence of test functions in both spaces $C_\nu(\mathbb{R};X)$ and $C_\mu(\mathbb{R};X)$. Hence, we obtain by using the causality of $\partial_\nu^{-1} F_\nu$

$$|\chi_{\mathbb{R}_{<t-\varepsilon}}(m)(v_\nu - v_\mu)|_{\infty,\mu} = |\chi_{\mathbb{R}_{<t-\varepsilon}}(m)(\partial_\nu^{-1} F_\nu(\phi(m)v_\nu) - \partial_\mu^{-1} F_\mu(\phi(m)v_\mu))|_{\infty,\mu}$$

$$= |\chi_{\mathbb{R}_{<t-\varepsilon}}(m)(\partial_\mu^{-1} F_\mu(\phi(m)v_\nu) - \partial_\mu^{-1} F_\mu(\phi(m)v_\mu))|_{\infty,\mu}$$

$$\leq |F_\mu|_{\text{Lip}} |\phi(m)v_\nu - \phi(m)v_\mu|_{\infty,\mu}.$$

$$\leq |F_\mu|_{\text{Lip}} |\chi_{\mathbb{R}_{<t}}(m)(v_\nu - v_\mu)|_{\infty,\mu}.$$

Thus, we get $\chi_{\mathbb{R}_{<t}}(m)v_\nu = \chi_{\mathbb{R}_{<t}}(m)v_\mu$ for each $t \in \mathbb{R}$ and hence, $v_\nu = v_\mu$. $\quad\square$

4.2. Examples

Let us describe the space $C_{-\infty}(\mathbb{R};\mathbb{R}) := C_{-\infty}(\mathbb{R})$ in more detail. Let μ be a Borel measure on $\mathbb{R}$ such that for some $\nu > 0$ we have $t \mapsto \mu((-\infty,t]) \in C_\nu(\mathbb{R})$. Then $\mu \in C_\nu^{-1}(\mathbb{R})$. Indeed, let $(g_n)_n$ be a $C_c^\infty(\mathbb{R})$ sequence approximating $\mu((-\infty,\cdot])$ in $C_\nu(\mathbb{R})$. Then g_n' converges to μ in $C_\nu^{-1}(\mathbb{R})$. Moreover, the derivative applied to $\mu((-\infty,\cdot])$ is just the distributional derivative: Let $\phi \in C_c^\infty(\mathbb{R})$ then we have for $n \in \mathbb{N}$

$$\int_{\mathbb{R}} g_n'\phi = -\int_{\mathbb{R}} g_n\phi' \rightarrow -\int_{\mathbb{R}} \int_{-\infty}^t \mathrm{d}\mu(s)\phi'(t)\mathrm{d}t = -\int_{\mathbb{R}} \int_s^\infty \phi'(t)\mathrm{d}t\,\mathrm{d}\mu(s) = \int_{\mathbb{R}} \phi\,\mathrm{d}\mu.$$

A particular instance of such measures are bounded measures with support bounded below and a continuous cumulative distribution function. As a non-trivial example we mention the derivative of the "devil's staircase".

Another example is the derivative of the function $f : x \mapsto \chi_{[0,\infty)}(x)\cos(\frac{\pi}{x})x$. Since f is of unbounded variation, its derivative $(x \mapsto \chi_{[0,\infty)}(x)\cos(\frac{\pi}{x}) + \chi_{[0,\infty)}(x)\frac{\sin(\frac{\pi}{x})}{x}\pi)$ is not a Borel measure.

EXAMPLE 4.10. (IVP for ODE) Let $F : D(F) \subseteq X^{\mathbb{R}} \to C_{-\infty}(\mathbb{R};X)$ with $F(\phi) = 0$ for each $\phi \in C_c^\infty(\mathbb{R};X)(\subseteq D(F))$ with $\mathrm{supp}\,\phi \subseteq (-\infty,0)$. We assume that for every $x \in X$ the mapping

$$G_x : C_c^\infty(\mathbb{R};X) \to C_{-\infty}(\mathbb{R};X)$$
$$\phi \mapsto F(\phi + \chi_{[0,\infty)}x)$$

is $(0,-1)$-contracting. Then, by Theorem 4.6, we find for every $u^{(0)} \in X$ a unique solution $v \in C_\nu(\mathbb{R};X)$ of the equation

$$\partial_\nu v = G_{u^{(0)}}(v).$$

Moreover, due to causality, v is supported on $[0,\infty)$. We define $u := v + \chi_{[0,\infty)}u^{(0)} \in C_\nu^{-1}(\mathbb{R};X)$, which solves the equation $\dot{u} = F(u)$ on the half axis $\mathbb{R}_{>0}$. Furthermore, $v = u - \chi_{[0,\infty)}u^{(0)}$ is continuous and thus, $0 = v(0-) = v(0+) = u(0+) - u^{(0)}$, which gives that u satisfies the initial condition $u(0+) = u^{(0)}$.

REMARK 4.11. (Nemitzkii-operator) The mapping $F : \phi \mapsto (t \mapsto f(\phi(t)))$ for some Lipschitz continuous $f : X \to X$ satisfies the assumptions from the previous example.

EXAMPLE 4.12. Let H be a Hilbert space. Let $T,K \in L(H)$. Consider the initial value problem:

$$\begin{cases} \dot{S}(t) = TS(t) - S(t)T & ,t > 0, \\ S(0) = K. \end{cases}$$

The latter equation has a unique continuous solution $S : \mathbb{R}_{\geq 0} \to L(H)$. Indeed, consider the operator

$$[\cdot,T] : L(H) \to L(H), S \mapsto TS - ST.$$

The latter is a continuous mapping with norm bounded by $2\|T\|$. Considering G_K as above with $F(\phi) := (t \mapsto [\phi(t), T])$ for functions $\phi : \mathbb{R} \to L(H)$, we are in the situation of Example 4.10.

Similarly to Section 3, we are now in the position to discuss several delay type equations. For sake of brevity we shall only list the operators involved and compute their operator norms.

EXAMPLE 4.13. The operator $\tau_\theta : C_\nu(\mathbb{R}; X) \to C_\nu(\mathbb{R}; X), f \mapsto f(\cdot + \theta)$ has operator norm $e^{\nu\theta}$, which can be read off from the following. From

$$e^{-\nu t} f(t + \theta) = e^{\nu\theta} e^{-\nu(t+\theta)} f(t + \theta)$$

for $t \in \mathbb{R}$ and $f \in C_c^\infty(\mathbb{R}; X)$, we see that $\|\tau_\theta\| = e^{\nu\theta}$.

EXAMPLE 4.14. The operator $C_\nu(\mathbb{R}; X) \ni \phi \mapsto \phi_{(\cdot)} \in C_\nu(\mathbb{R}; C_b(\mathbb{R}_{<0}; X))$ has a norm bounded by 1. Indeed, for $t \in \mathbb{R}$ and $\phi \in C_c^\infty(\mathbb{R}; X)$ we compute

$$\left|\phi_{(t)}\right|_{C_b(\mathbb{R}_{<0}; X)} = \sup_{\theta \in \mathbb{R}_{<0}} |\phi(t + \theta)| = \sup_{\theta \in \mathbb{R}_{<0}} \left|\phi(t + \theta) e^{-\nu(t+\theta)}\right| e^{\nu(t+\theta)} \leqq |\phi|_{\nu,\infty} e^{\nu t}.$$

REMARK 4.15. Note that $\phi \mapsto \phi_{(\cdot)}$ is not a strict contraction for ν large as it has been in the L_p-case. Hence, in order to solve equations of the form

$$\partial_\nu u = F_\nu(u_{(\cdot)})$$

F_ν needs to be $(0, -1)$-contracting. So, Lipschitz-continuity does not suffice to establish a well-posedness theorem, at least in the continuous case. Moreover, note that this perspective also effects neutral differential equations of the form $\partial_\nu u = F_\nu(u_{(\cdot)}, (\partial_\nu u)_{(\cdot)})$ for suitable F. In that case one has to assume that F is $(0, 0)$-contracting and not only $(0, 0)$-Lipschitz.

REFERENCES

[1] B. BALACHANDRAN, D. E. GILSINN, T. KALMÁR-NAGY, *Delay Differential Equations*, Springer Verlag, Berlin 2009.

[2] A. BÁTKAI, S. PIAZZERA, *Semigroups for delay equations*, Research Notes in Mathematics 10. Wellesley: A K Peters, 2005.

[3] C. CORDUNEANU, *Some differential equations with delay*, Proc. equadiff III, 3rd Czech. Conf. Differ. Equat. Appl., Brno 1972, 105–114, 1973.

[4] C. CORDUNEANU AND N. LUCA, *The stability of some feedback systems with delay*, J. Math. Anal. Appl., **51**: 377–393, 1975.

[5] P. DAS, B. B. MISHRA, AND C. R. DASH, *Oscillation theorems for neutral delay differential equations of odd order*, Bull. Inst. Math., Acad. Sin. (N.S.), **1** (4): 557–568, 2006.

[6] O. DIEKMANN, M. GYLLENBERG, *Abstract delay equations inspired by population dynamics*, Functional analysis and evolution equations, pp. 187–200, Birkhäuser, Basel, 2008.

[7] O. DIEKMANN, S. A. VAN GILS, S. M. VERDUYN LUNEL, H.-O. WALTHER, *Delay Equations*, Springer-Verlag, New York, 1995.

[8] J. DIESTEL AND J. J. UHL JUN., *Vector measures*, Providence, 1977.

[9] M. FRASSON, *Large Time Behaviour of Neutral Delay Systems*, PhD thesis, Universiteit Leiden, 2005.

[10] J. K. Hale, *A class of neutral equations with the fixed point property*, Proc. Natl. Acad. Sci. USA, **67**: 136–137, 1970.

[11] J. K. Hale, S. M. Verduyn Lunel, *Introduction to functional differential equations*, Applied Mathematical Sciences. 99. Springer-Verlag, New York, 1993.

[12] T. Havarneanu, *Semigroups and differential equations with infinite delay*, Nonlinear Anal., Theory Methods Appl., **5**: 737–756, 1981.

[13] Y. Hino, S. Murakami, and T. Naito, *Functional differential equations with infinite delay*, Lecture Notes in Mathematics, 1473. Berlin, Springer, 1991.

[14] A. Kalauch, R. Picard, S. Siegmund, S. Trostorff, and M. Waurick, *A Hilbert Space Perspective on Ordinary Differential Equations with Memory Term*, J. Dynam. Differential Equations, to appear, `arXiv:1204.2924`.

[15] V. Lakshmikantham, S. Leela, Z. Drici, F. A. McRae, *Theory of Causal Differential Equations*, Atlantis Studies in Mathematics for Engineering and Science – Vol. 5, 2010.

[16] C. Lizama, *Fourier multipliers and periodic solutions of delay equations in Banach spaces*, J. Math. Anal. Appl., **324** (2): 921–933, 2006.

[17] C. Lizama, and V. Poblete, *Periodic solutions of fractional differential equations with delay*, J. Evol. Equ., **11** (1): 57–70, 2011.

[18] D. Morgenstern, *Beiträge zur nichtlinearen Funktionalanalysis*, PhD thesis, TU Berlin, 1952.

[19] R. Picard, *Hilbert Space Approach to Some Classical Transforms*, Pitman Research Notes in Mathematics Series. 196. Harlow: Longman Scientific & Technical, New York: John Wiley & Sons, Inc. 203 p., 1989.

[20] R. Picard, *A structural observation for linear material laws in classical mathematical physics*, Math. Methods Appl. Sci., **32** (14): 1768–1803, 2009.

[21] R. Picard, D. McGhee, *Partial Differential Equations: A Unified Hilbert Space Approach*, De Gruyter Expositions in Mathematics 55, 2011.

[22] S. Trostorff, *Well-posedness and causality for evolutionary inclusions*, PhD Thesis, TU Dresden, 2011, `http://www.qucosa.de/fileadmin/data/qucosa/documents/7832/phd-thesis_trostorff.pdf`

Rainer Picard
Zellescher Weg 12-14, 01062 Dresden
Institut für Analysis, Fachrichtung Mathematik
Technische Universität Dresden, Germany
e-mail: `rainer.picard@tu-dresden.de`

Sascha Trostorff
Zellescher Weg 12-14, 01062 Dresden
Institut für Analysis, Fachrichtung Mathematik
Technische Universität Dresden, Germany
e-mail: `sascha.trostorff@tu-dresden.de`

Marcus Waurick
Zellescher Weg 12-14, 01062 Dresden
Institut für Analysis, Fachrichtung Mathematik
Technische Universität Dresden, Germany
e-mail: `marcus.waurick@tu-dresden.de`