A RIEMANN SURFACE APPROACH FOR
DIFFRACTION FROM RATIONAL WEDGES

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Abstract. This paper aims at the explicit analytical representation of acoustic, electromagnetic or elastic, time-harmonic waves diffracted from wedges in $\mathbb{R}^3$ in a correct setting of Sobolev spaces. Various problems are modelled by Dirichlet or Neumann boundary value problems for the $2D$ Helmholtz equation with complex wave number. They have been analyzed before by several methods such as the Malinzhinets method using Sommerfeld integrals, the method of boundary integral equations from potential theory or Mellin transformation techniques. These approaches lead to results which are particularly useful for asymptotic and numerical treatment. Here we develop new representation formulas of the solutions which are based upon the solutions to Sommerfeld diffraction problems. We make use of symmetry properties, which require a generalization of these formulas to Riemann surfaces in order to cover arbitrary rational angles of the wedge. The approach allows us to prove well-posedness in suitable Sobolev spaces and to obtain explicit solutions in a new, perhaps surprising, form provided the angle is rational, i.e., $\alpha = \pi m/n$ where $m, n \in \mathbb{N}$.

1. Introduction

The explicit representation of waves diffracted from non-rectangular wedges belongs to a famous class of open problems in diffraction theory [13, 20, 25]. These problems are often modelled by boundary value problems for the $2D$ Helmholtz equation in a cone Ω with an angle $\alpha \in]\pi, 2\pi[$ (so-called *exterior problems*) and Dirichlet, Neumann or other boundary conditions.

Besides of proving well-posedness in suitable Sobolev spaces, the aim of this paper is to establish *explicit formulas* of the solution if the angle of the cone is rational, i.e., $\alpha = \pi m/n$ with $m, n \in \mathbb{N}$. We present a new approach that is *not* based on the well-known work by Malyuzhinets [20], on Sommerfeld integrals [11], and others (which work for non-rational angles, as well), but on a recently developed potential method which leads to systems of Wiener-Hopf-Hankel equations in Sobolev spaces. The main interest is directed towards explicit formulas in closed analytic form, or, if not available, in terms of series expansions which present the unique solution with the help of a bounded linear operator acting from the space of the boundary data into a subspace of $H^1(\Omega)$ of finite energy solutions [15], or into other appropriate Sobolev

spaces $H^{1+\varepsilon}(\Omega)$. The circumstance that we restrict to rational angles is intimately related to the nature of the representations of the solutions. As a rule one could say that the representation becomes more complicated if n and m become larger.

In this introduction we will define basic notation and recall basic representation formulas for the solutions, which are necessary for the development of the paper. We will also present a short version of the main results. The detailed results will be established in later section, and diagrams summarizing them will be given in the last section.

For $\alpha \in]0, 2\pi[$, let

$$\Omega_{0,\alpha} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : 0 < \arg(x_1 + ix_2) < \alpha \right\} \tag{1}$$

denote the cone in the plane with angle α bordered by the half-lines

$$\begin{aligned}
\Gamma_1 &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 = 0\}, \\
\Gamma_2 &= \{(x_1, x_2) \in \mathbb{R}^2 : \arg(x_1 + ix_2) = \alpha\},
\end{aligned} \tag{2}$$

and the origin. For a regularity parameter $\varepsilon \in [0, 1/2[$, we are looking for the weak solution $u \in H^{1+\varepsilon}(\Omega_{0,\alpha})$ of the *Helmholtz equation (HE)*

$$(\Delta + k^2)u = 0 \text{ in } \Omega_{0,\alpha} \tag{3}$$

satisfying Dirichlet and/or Neumann boundary conditions,

$$u|_{\Gamma_j} = g_j \quad \text{or} \quad \left. \frac{\partial u}{\partial n} \right|_{\Gamma_j} = g_j, \qquad j = 1, 2,$$

on the half-lines Γ_1 and Γ_2 with Dirichlet or Neumann data g_1, g_2 taken from the appropriate Sobolev spaces $H^{1/2+\varepsilon}(\mathbb{R}_+)$ or $H^{-1/2+\varepsilon}(\mathbb{R}_+)$. Here, and throughout the paper, the half-lines Γ_j are identified with the positive real numbers $\mathbb{R}_+ =]0, \infty[$. We consider three types of boundary value problems, namely pure Dirichlet (DD), pure Neumann (NN), and mixed (DN) problems. The space of solutions of the HE (without regard to the boundary condition) will be denoted by

$$\mathscr{H}^{1+\varepsilon}(\Omega_{0,\alpha}) = \left\{ u \in H^{1+\varepsilon}(\Omega_{\beta,\gamma}) : (\Delta + k^2)u = 0 \right\}. \tag{4}$$

Notice that results for the rotated cone $\Omega_{\beta,\gamma} = \{(x_1, x_2) \in \mathbb{R}^2 : \beta < \arg(x_1 + ix_2) < \gamma\}$ with $\gamma - \beta = \alpha$ can be simply obtained by rotation from the solution in $\Omega_{0,\alpha}$. The wave number k is always complex with $\mathrm{Im}(k) > 0$. A discussion of real wave numbers and the limiting absorption principle is beyond this work, although physically most interesting [20, 25] and mathematically challenging. We consider this as an important open problem (see Section 13).

The focus of [7] was rational angles $\alpha = 2\pi/n$, $n = 2, 3, \ldots$, as well as the case of the slit-plane $\Omega_{0,2\pi}$, $\alpha = 2\pi$ (also denoted as Sommerfeld problems [15]). In the present paper we want to consider cones with angles $\alpha = \pi m/n$ for arbitrary $m, n \in \mathbb{N}$. As it will turn out as part of our approach, even if we would restrict to angles $\alpha \in]0, 2\pi[$,

it is necessary to consider "auxiliary" solutions to the HE on cones with angles $\alpha = \pi m$, $m \in \mathbb{N}$. Such cones $\Omega_{0,\alpha}$ are to be regarded as subsets of Riemann surfaces, and we will call them *conical Riemann surfaces (CRS)*. Moreover, though unproblematic, some care is necessary to define the corresponding Sobolev space $H^{1+\varepsilon}(\Omega)$ and the corresponding solution spaces $\mathscr{H}^{1+\varepsilon}(\Omega)$. This will be done in Section 2.

Another remark concerns the range of the *regularity parameter* ε, usually chosen to be $[0, 1/2[$. In [7] it was possible to extend the results in some cases (such as DD) to ranges $]-1/2, 1/2[$. In other cases (such as DN), the range of ε had to be restricted, and arguments were given that such restrictions are necessary to guarantee the well-posedness of the problem. We will encounter the same issue in this paper and thus have to carefully monitor this parameter. However, we will refrain from discussing whether our conditions on ε are inherent to the problem or tied to our approach. An overview of the various possible choices of ε is presented in two tables in Section 13.

In order to describe the spaces for the boundary data we recall the definition of the usual Sobolev spaces $H^s = H^s(\mathbb{R})$ and of the Sobolev spaces $H^s(\mathbb{R}_\pm)$, as well, where $\mathbb{R}_+ =]0, \infty[$ and $\mathbb{R}_- =]-\infty, 0[$ (see, e.g., [8]). The restriction operator which restricts a function or distribution on $\mathbb{R}$ to $\mathbb{R}_\pm$ will be denoted by $r_\pm$. Thus $H^s(\mathbb{R}_\pm) = r_\pm(H^s)$, and the norm in $H^s(\mathbb{R}_\pm)$ can be defined by

$$\|f\|_{H^s(\mathbb{R}_\pm)} = \inf_\ell \|\ell f\|_{H^s}$$

where ℓf stands for any extension of f to a distribution in H^s. An equivalent norm can be defined via the Sobolev-Slobodetski norm for $s > 0$ and via a duality for $s < 0$. Furthermore, we denote by $H^s_\pm = H^s_\pm(\mathbb{R})$ the (closed) subspace of H^s which consists of all distributions with support in the closure of $\mathbb{R}_\pm$. By $\widetilde{H}^s(\mathbb{R}_\pm)$ we denote the space of all distributions which are the restrictions of distributions in $H^s_\pm$, i.e., $\widetilde{H}^s(\mathbb{R}_\pm) = r_\pm(H^s_\pm)$. A norm is defined by

$$\|f\|_{\widetilde{H}^s(\mathbb{R}_\pm)} = \inf_{\ell_0} \|\ell_0 f\|_{H^s}$$

where $\ell_0 f$ stands for any extension of f to a distribution in $H^s_\pm$ (which is not unique for $s < -1/2$). In fact, the map $r_\pm : H^s_\pm \to \widetilde{H}^s(\mathbb{R}_\pm)$ is injective if and only if $s \geq -1/2$, and in this case one can define the extension-by-zero operator $\ell_0 : \widetilde{H}^s(\mathbb{R}_\pm) \to H^s_\pm$. Clearly, in this case $\|f\|_{\widetilde{H}^s(\mathbb{R}_\pm)} = \|\ell_0 f\|_{H^s}$. Notice that while $\widetilde{H}^s(\mathbb{R}_\pm)$ is always continuously embedded in $H^s(\mathbb{R}_\pm)$, these two spaces coincide for $s \in]-1/2, 1/2[$.

Two other spaces are needed. Define, for functions (and appropriately for distributions) in $H^s(\mathbb{R})$ the flip operator

$$(\mathscr{J} f)(x) = f(-x), \qquad x \in \mathbb{R}. \tag{5}$$

The spaces of even/odd distributions on $\mathbb{R}$ are

$$H^s_e(\mathbb{R}) = \left\{ f \in H^s : f = \mathscr{J} f \right\}, \quad H^s_o(\mathbb{R}) = \left\{ f \in H^s : f = -\mathscr{J} f \right\}.$$

It is well known [5] that the even and odd extension operators are well-defined, linear and bounded between the following spaces

$$\ell^e : H^s(\mathbb{R}_\pm) \to H^s_e(\mathbb{R}), \quad \ell^o : H^s(\mathbb{R}_\pm) \to H^s_o(\mathbb{R}) \tag{6}$$

for $s \in]-1/2, 3/2[$ (even) and $s \in]-3/2, 1/2[$ (odd), respectively. In fact, for these values of s, the even or odd extensions are bijective and the inverse maps are given by $r_{\pm}$.

Let us now introduce the boundary operators. Given $\Omega = \Omega_{0,\alpha}$, $\varepsilon \in]-1/2, 1/2[$, and a part of the boundary $\Gamma \subset \partial\Omega$, one can consider the usual trace operator

$$T_{0,\Gamma} : u \in H^{1+\varepsilon}(\Omega) \mapsto u|_{\Gamma} \in H^{1/2+\varepsilon}(\Gamma),$$

which is linear and bounded. Slightly more delicate is the case of the operator $T_{1,\Gamma} = T_{0,\Gamma}\frac{\partial}{\partial n}$, where n stands for the *normal vector* on $\partial\Omega$ *directed towards the interior* of Ω. Here one has to restrict the operator to solutions of the Helmholtz equation

$$T_{1,\Gamma} : u \in \mathscr{H}^{1+\varepsilon}(\Omega) \mapsto \left.\frac{\partial u}{\partial n}\right|_{\Gamma} \in H^{-1/2+\varepsilon}(\Gamma).$$

This last definition as it stands makes sense for u sufficiently smooth on the boundary. The operator then extends via continuity. A precise definition, in which $H^{-1/2+\varepsilon}(\Gamma)$ is identified as the dual of another Sobolev space, was given in Section 4 of [7].

We will consider the operators $T_{0,\Gamma}$ and $T_{1,\Gamma}$ with Γ equal to Γ_1 or Γ_2 as defined in (2). Consequently, we are going to consider the Helmholtz equation with one of the following three types of boundary conditions,

$$
\begin{array}{llll}
\text{(DD)} & T_{0,\Gamma_1}u = g_1, & T_{0,\Gamma_2}u = g_2, \\
\text{(NN)} & T_{1,\Gamma_1}u = g_1, & T_{1,\Gamma_2}u = g_2, & (7) \\
\text{(DN)} & T_{0,\Gamma_1}u = g_1, & T_{1,\Gamma_2}u = g_2.
\end{array}
$$

The boundary data belongs to the space $H^{1/2+\varepsilon}(\mathbb{R}_+)$ in the Dirichlet case, and to the space $H^{-1/2+\varepsilon}(\mathbb{R}_+)$ in the Neumann case.

The Dirichlet problems (DD) are well-posed in the corresponding spaces only if the Dirichlet data satisfies the compatibility condition

$$g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+), \tag{8}$$

i.e., this function is extendible by zero onto the full line $\mathbb{R}$ such that the zero extension $\ell_0(g_1 - g_2)$ belongs to $H^{1/2+\varepsilon}(\mathbb{R})$. The Dirichlet compatibility condition is redundant for $\varepsilon \in]-1/2, 0[$. The Neumann problems (NN) need a compatibility condition

$$g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+), \tag{9}$$

and we need to impose the condition that $\varepsilon \in [0, 1/2[$ (see again Section 4 in [7]). For $\varepsilon \in]0, 1/2[$ the compatibility condition is redundant, hence we only need it for $\varepsilon = 0$. The redundancy stems from the fact that $\widetilde{H}^s(\mathbb{R}_+) = H^s(\mathbb{R}_+)$ for $s \in]-1/2, 1/2[$. In case of the mixed problems (DN) it is not necessary to add compatibility conditions on the data in view of their well-posedness.

For $\varepsilon \in [-1/2, 1/2[$, the spaces of distributions (g_1, g_2) which satisfy the compatibility conditions (8) and (9) are denoted by

$$H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim} = \left\{ (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2 \; : g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) \right\}, \qquad (10)$$

$$H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim} = \left\{ (g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2 : g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+) \right\}, \qquad (11)$$

and can be equipped with the norm

$$\|(g_1, g_2)\|_{H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}} = \|g_1 + g_2\|_{H^{1/2+\varepsilon}(\mathbb{R}_+)} + \|g_1 - g_2\|_{\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)},$$

$$\|(g_1, g_2)\|_{H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}} = \|g_1 - g_2\|_{H^{-1/2+\varepsilon}(\mathbb{R}_+)} + \|g_1 + g_2\|_{\widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)}.$$

In [7], the resolvent operators to the HE,

$$(g_1, g_2) \in H^{\pm 1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim} \mapsto u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,\alpha})$$

for the DD and NN problems (see (7)) were seen to be linear homeomorphisms, if $\alpha = 2\pi/n$, $n = 1, 2, 3, \ldots$, and u was expressed explicitly in terms of (g_1, g_2). Similarly, for the DN problem,

$$(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \mapsto u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,\alpha})$$

were proved to be linear homeomorphisms and explicit formulas were given. Certain restrictions on ε had to be made.

In particular cases, the solution to the HE can be obtained immediately, e.g., for $\alpha = \pi$, $\Omega^+ = \Omega_{0,\pi}$ being the upper half-plane, cases DD and NN, respectively. Therein we have the *double and simple layer potentials* [10] in its simplest form

$$u(x_1, x_2) = (\mathscr{K}_{D,\Omega^+} \iota g)(x_1, x_2) = \mathscr{F}^{-1}_{\xi \mapsto x_1} e^{-t(\xi)x_2} \widehat{\iota g}(\xi)$$
$$\tag{12}$$
$$u(x_1, x_2) = (\mathscr{K}_{N,\Omega^+} \iota g)(x_1, x_2) = -\mathscr{F}^{-1}_{\xi \mapsto x_1} e^{-t(\xi)x_2} t^{-1}(\xi) \widehat{\iota g}(\xi)$$

which are called *line potentials (LIPs)* with density $\iota g \in H^s(\mathbb{R})$. Similarly, the solution of the DD and NN problems in the lower-half plane $\Omega^- = \Omega_{\pi, 2\pi}$ are given, respectively, by

$$u(x_1, x_2) = (\mathscr{K}_{D,\Omega^-} \iota g)(x_1, x_2) = \mathscr{F}^{-1}_{\xi \mapsto x_1} e^{t(\xi)x_2} \widehat{\iota g}(\xi)$$
$$\tag{13}$$
$$u(x_1, x_2) = (\mathscr{K}_{N,\Omega^-} \iota g)(x_1, x_2) = -\mathscr{F}^{-1}_{\xi \mapsto x_1} e^{t(\xi)x_2} t^{-1}(\xi) \widehat{\iota g}(\xi).$$

Here $\mathscr{F}$ denotes the Fourier transformation, $\mathscr{F}_{x \to \xi} f(x) = \int_{-\infty}^{\infty} f(x) e^{i\xi x} \, dx$, and we use the notation $\widehat{f} = \mathscr{F}f$. Furthermore, we need the function

$$t(\xi) = (\xi^2 - k^2)^{1/2}, \qquad \xi \in \mathbb{R}, \qquad (14)$$

with vertical branch cut from k to $-k$ via ∞, not crossing the real line, and with choice of the square-root such that $t(\pm\infty) = +\infty$. Considering $g = (g_1, g_2) \in H^s(\mathbb{R}_+)^2_\sim$, ιg is the "natural composition" of the two boundary data

$$\iota g(x) = \begin{cases} g_1(x) & , x > 0 \\ g_2(-x) & , x < 0. \end{cases} \tag{15}$$

Depending on the value of s and whether we have the Dirichlet or Neumann case, ιg is a function or a distribution, which belongs to $H^s(\mathbb{R})$ if $s \geqslant -1/2$. The precise characterization of $f = \iota g$ is that $g_1 = r_+ f$ and $g_2 = \mathscr{J} r_- f = r_+ \mathscr{J} f$, where $\mathscr{J}$ is the flip operator (5) (see [7, Sections 3 and 4] for details).

The solution of the mixed DN problem for the upper half-plane Ω^+ needs already more sophisticated methods such as the Wiener-Hopf technique, but it can be represented explicitly as well by an analytic formula

$$\begin{aligned} u(x_1, x_2) &= \left(\mathscr{K}_{DN,\Omega^+}(g_1, g_2) \right)(x_1, x_2) \\ &= \mathscr{F}^{-1}_{\xi \mapsto x_1} e^{-t(\xi)x_2} t_-^{-1/2}(\xi) \left\{ \widehat{P_+} t_-^{1/2} \widehat{\ell g_1} - \widehat{P_-} t_+^{-1/2} \widehat{\mathscr{J} \ell g_2} \right\}(\xi) \\ &= \mathscr{K}_{D,\Omega^+} A_{t_-^{-1/2}} \left\{ P_+ A_{t_-^{1/2}} \ell g_1 - P_- A_{t_+^{-1/2}} \mathscr{J} \ell g_2 \right\}(x_1, x_2) \end{aligned} \tag{16}$$

where $t_\pm^s$ $(s = \pm 1/2)$ are the functions

$$t_\pm^s(\xi) = (\xi \pm k)^s, \qquad s \in \mathbb{R}, \tag{17}$$

with the vertical branch cut from $\pm k$ to ∞ not crossing the real line, also stipulating that $t_\pm^s(+\infty) = +\infty$. Therein

$$A_\phi = \mathscr{F}^{-1} \phi \cdot \mathscr{F} \tag{18}$$

is referred to as a (distributional) convolution operator acting between appropriate Sobolev spaces $H^s(\mathbb{R})$. The operators $P_\pm = \ell_0 r_\pm$ are bounded projections in $H^\varepsilon(\mathbb{R})$ $(|\varepsilon| < 1/2)$, $\widehat{P_\pm} = \mathscr{F} \ell_0 r_\pm \mathscr{F}^{-1}$, and $\ell g_1, \ell g_2$ denote any extensions from $\mathbb{R}_+$ to $\mathbb{R}$ such that $\ell g_1 \in H^{1/2+\varepsilon}(\mathbb{R})$ and $\ell g_2 \in H^{-1/2+\varepsilon}(\mathbb{R})$; the operator does not depend on the particular choice of the extension.

Another particular case is $\alpha = \pi/2$. Here the solutions are amazingly simple, and simplest in the DN case, expressed by so-called *half-line potentials (HLPs)*:

$$u(x_1, x_2) = \mathscr{F}^{-1}_{\xi \mapsto x_1} e^{-t(\xi)x_2} \widehat{\ell^e g_1}(\xi) - \mathscr{F}^{-1}_{\xi \mapsto x_2} e^{-t(\xi)x_1} t^{-1}(\xi) \widehat{\ell^o g_2}(\xi) \tag{19}$$

where ℓ^e and ℓ^o denote even and odd extension, respectively. For more details and the DD and NN problems see [5, 14].

The case $\alpha = 3\pi/2$, so-called *exterior rectangular wedge diffraction problems*, is much more complicated and leads to Wiener-Hopf-Hankel equations. It was also explicitly solved and analyzed in [2, 14]. We shall present here a completely different and new representation formula, including series expansion, see Example 10.6 in Section 10.

What is also important for our purposes are the *Sommerfeld potentials (SOPs)* introduced in [7] for the angle $\alpha = 2\pi$. These are the solutions of the Sommerfeld diffraction problem in the slit domain

$$\Omega_{0,2\pi} = \mathbb{R}^2 \setminus \overline{\Sigma}, \tag{20}$$

where boundary data g_1, g_2 are given on the upper and lower banks $\Sigma^\pm$ of $\Sigma = \mathbb{R}_+ \times \{0\}$ (we write $\Gamma_1 = \Sigma^+$ and $\Gamma_2 = \Sigma^-$ here, identified with $\mathbb{R}_+$ again). In the DD case, under condition (8), the solution reads

$$u = \mathcal{K}_{D,\Omega_{0,2\pi}}(g_1, g_2) = \begin{cases} \mathcal{K}_{D,\Omega^+} u_0^+ & \text{in} \quad \Omega^+ \\ \mathcal{K}_{D,\Omega^-} u_0^- & \text{in} \quad \Omega^- \end{cases} \tag{21}$$

with

$$\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix} = \Upsilon_D^{-1} \begin{pmatrix} I & 0 \\ 0 & \Pi_{1/2}^+ \end{pmatrix} \begin{pmatrix} \ell_0 & 0 \\ 0 & \ell \end{pmatrix} \Upsilon_D \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \qquad \Upsilon_D = \begin{pmatrix} I & -I \\ I & I \end{pmatrix}$$

where we use

$$\Pi_s^+ = A_{t_-^{-s}} \ell_0 r_+ A_{t_-^s} : H^{s+\varepsilon}(\mathbb{R}) \to H^{s+\varepsilon}(\mathbb{R}), \qquad s \in \mathbb{R}, |\varepsilon| < 1/2. \tag{22}$$

Note that the solution space consists of all $H^{1+\varepsilon}$ functions which satisfy the HE in any proper sub-cone of $\Omega_{0,2\pi}$ and can be written as

$$\mathcal{H}^{1+\varepsilon}(\Omega_{0,2\pi}) = \Big\{ u \in L^2(\mathbb{R}^2) : u|_{\Omega^\pm} \in H^{1+\varepsilon}(\Omega^\pm), (\Delta + k^2)u = 0 \text{ in } \Omega^+ \cup \Omega^-,$$
$$u_0^+ - u_0^- \in H_+^{1/2+\varepsilon}, \ u_1^+ - u_1^- \in H_+^{-1/2+\varepsilon} \Big\}. \tag{23}$$

The two differences in the last line of the formula denote the jumps of the traces $u_0^+ - u_0^- = u(x_1, 0+0) - u(x_1, 0-0)$ or of the x_2-derivatives of u, namely $u_1^+ - u_1^- = \frac{\partial u}{\partial x_2}(x_1, 0+0) - \frac{\partial u}{\partial x_2}(x_1, 0-0)$, respectively, across the line $x_2 = 0$. Note that the explicit representation of $u_0^\pm$ on the full line in terms of $g_1 \pm g_2$ on the half line in (21) is equivalent to the so-called jump relations, cf. formulas (62) later on.

The Neumann problem was uniquely solved by

$$u = \mathcal{K}_{N,\Omega_{0,2\pi}}(g_1, g_2) = \begin{cases} \mathcal{K}_{N,\Omega^+} u_1^+ & \text{in} \quad \Omega^+ \\ \mathcal{K}_{N,\Omega^-} u_1^- & \text{in} \quad \Omega^- \end{cases} \tag{24}$$

with

$$\begin{pmatrix} u_1^+ \\ u_1^- \end{pmatrix} = \Upsilon_N^{-1} \begin{pmatrix} \Pi_{-1/2}^+ & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & \ell_0 \end{pmatrix} \Upsilon_N \begin{pmatrix} g_1 \\ -g_2 \end{pmatrix}, \qquad \Upsilon_N = \begin{pmatrix} I & I \\ I & -I \end{pmatrix},$$

provided (9) is satisfied (in case $\varepsilon = 0$ only, superfluous for $\varepsilon \in]0, 1/2[$). Note that in some publications (such as [15]) the normal derivative was taken in the positive x_2-direction in both banks $\Sigma^\pm$ of the screen $\Sigma = \partial\Omega$ (i.e., the interior derivative g_2 on the lower bank has here to be replaced by $-g_2$ in the cited formulas, such that $g_1 + g_2 = r_+(u_1^+ - u_1^-)$ satisfies (9) with identification of Σ and $\mathbb{R}_+$).

Finally, the solution of the DN problem (considered by Meister in 1977 [12]) was given by a celebrated matrix factorization due to Rawlins in 1981 [21], see also [9, 13, 22], and its operator theoretical interpretation [15]:

$$u = \mathcal{K}_{DN,\Omega_{0,2\pi}}(g_1,g_2) = \mathcal{K}_{D,\Omega_{0,2\pi}}B_-^{-1}W_{DN}^{-1}\begin{pmatrix} g_1 \\ -g_2 \end{pmatrix} \tag{25}$$

$$B_-^{-1} = \begin{pmatrix} I & -I \\ -A_t & -A_t \end{pmatrix}^{-1} = \tfrac{1}{2}\begin{pmatrix} I & I \\ -I & I \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -A_{t^{-1}} \end{pmatrix}$$

$$W_{DN}^{-1} = \mathcal{A}_+^{-1}\ell_0 r_+ \mathcal{A}_-^{-1}\ell$$

$$\mathcal{A} = \mathcal{F}^{-1}\begin{pmatrix} 1 & -t^{-1} \\ -t & -1 \end{pmatrix}\mathcal{F} = \mathcal{A}_- \mathcal{A}_+$$

$$= -\tfrac{1}{\sqrt{4k}}\,\mathcal{F}^{-1}\begin{pmatrix} -t_{+-} & t^{-1}t_{--} \\ -t_{--} & -t_{+-} \end{pmatrix}\begin{pmatrix} t_{++} & -t^{-1}t_{-+} \\ t_{-+} & t_{++} \end{pmatrix}\mathcal{F},$$

where $t_{\pm\pm}(\xi) = \left(\sqrt{2k} \pm \sqrt{k \pm \xi}\right)^{1/2}$ and the first/second index corresponds to the first/second sign, respectively. In some papers the factors are written in terms of $\sqrt{\xi - k}$ instead of $\sqrt{k-\xi}$ and one has to substitute $\sqrt{k-\xi} = i\sqrt{\xi-k}$, $\sqrt{k^2 - \xi^2} = i\sqrt{\xi^2 - k^2}$, due to the vertical branch cut from k to ∞ in the upper half-plane and from ∞ to $-k$ in the lower half-plane. In Prop. 5.7 of [7] it was shown that the DN problem is solvable in $\Omega_{0,2\pi}$ provided that $\varepsilon \in]-1/4, 1/4[$.

All the previous formulas (for $\alpha = \pi$ and $\alpha = 2\pi$) were used in [7] to solve the BVPs for $\alpha = 2\pi/n$, namely by symmetry arguments. Let us explain this symmetry idea in two examples.
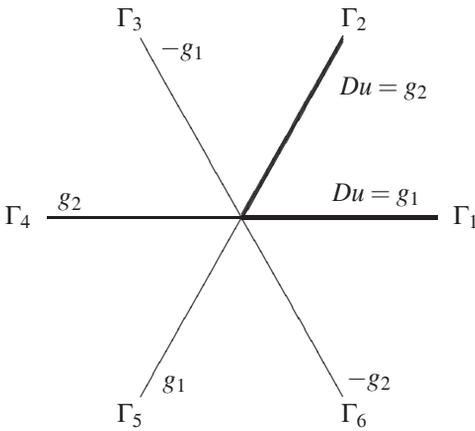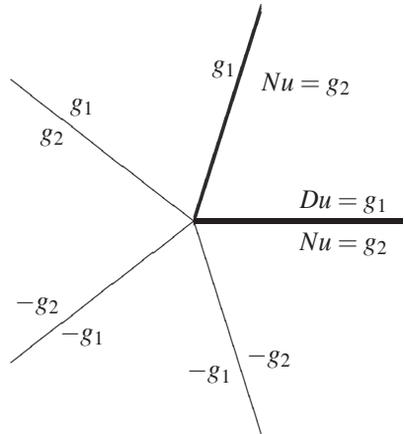

Figure 1: $\alpha = \pi/3$, *case DD*   Figure 2: $\alpha = 2\pi/5$, *case DN*

In the first case, consider the DD problem for $n = 6$ outlined in Fig. 1. We add

five $60°$ sectors to the scene, impose Dirichlet boundary conditions on the banks of the half-lines as indicated and define u by taking the sum of three potentials in the three half-planes overlapping $\Omega_{0,\alpha}$ constructed by formula (12) and rotation. One can prove that, e.g., the contributions from $\Gamma_1 \cup \Gamma_3$ to the trace of u on Γ_2, as well as those from $\Gamma_4 \cup \Gamma_6$, annihilate. In this way one obtains a solution of the Dirichlet problem in $\Omega_{0,\alpha}$. Uniqueness is clear from Green's formula arguments.

In the second example, the DN problem for $n = 5$, we define u by taking the sum of five Sommerfeld potentials for the five slit planes with mixed boundary conditions, hence using formula (25) and rotation.

It has been proved in [7] that the method works for all DD, NN, DN problems and $\alpha = 2\pi/n$ but hardly for $\alpha = 2\pi m/n$, $m \geqslant 2$. Up to now, only a few more problems have been solved in a similar way (using the potentials mentioned before), e.g., for the case $\alpha = 3\pi/2$ which includes the study of Hankel operators [2, 14]. Remarkably in this way an exterior problem could be solved in explicit analytical form.

In contrast to these very special cases, the main results of this paper are given in Section 12 by Theorem 12.1, Theorem 12.2, and Theorem 12.3. As a brief version of these results, let us state the following theorem, in which for sake of simplicity and illustration the assumptions are stricter than in the afore-mentioned theorems.

For the wave number $k = k_x + ik_y$, $k_y > 0$, underlying the HE (3), it is necessary to define the constant

$$c(k) = \left(\frac{|k| + |k_x|}{|k| - |k_x|} \right)^{1/4}. \tag{26}$$

Note that always $c(k) \geqslant 1$.

THEOREM 1.1. *Let* $\alpha = m\pi/n$ *with* $m, n \in \{1, 2, ...\}$ *and* $\varepsilon \in]-1/2, 1/2[$.

(i) *If, in addition,*

$$\varepsilon \in]-1/m, 1/m[\quad and \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)},$$

then for each $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, *the DD problem for the Helmholtz equation in* $\Omega_{0,\alpha}$ *admits a unique solution in* $H^{1+\varepsilon}(\Omega_{0,\alpha})$.

(ii) *If, in addition,*

$$\varepsilon \in]0, 1/m[\quad and \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)},$$

then for each $g \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, *the NN problem for the Helmholtz equation in* $\Omega_{0,\alpha}$ *admits a unique solution in* $H^{1+\varepsilon}(\Omega_{0,\alpha})$.

(iii) If, in addition,

$$\varepsilon \in\,]-1/2m, 1/2m[\quad \text{and} \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/2m)} \quad \text{in case } n \text{ is odd,}$$

or

$$\varepsilon \in\,]0, 1/m[\quad \text{and} \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)} \quad \text{in case } n \text{ is even,}$$

then for each $g \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$, the DN problem for the Helmholtz equation in $\Omega_{0,\alpha}$ admits a unique solution in $H^{1+\varepsilon}(\Omega_{0,\alpha})$.

The resolvent operator can be written in closed analytical form (by a finite number of algebraic and analytic operations) or by series expansion, in terms of LIPs, HLPs or/and SOPs.

The particular formulas will be presented later (see the diagrams in the last section as a guide for more details). Moreover, a generalization to BVPs *in special (conical) Riemann surfaces will be provided.*

2. BVPs in conical Riemann surfaces: compatibility conditions and uniqueness

Let $\mathscr{R}^2$ stand for the universal covering surface of $\mathbb{R}^2 \setminus \{0\}$ and let $\tau : \mathscr{R}^2 \to \mathbb{R}^2 \setminus \{0\}$ stand for the covering projection. Points in $\mathscr{R}^2$ will be written as x and their corresponding projections as $(x_1, x_2) = \tau(x)$. One can define the angle or argument of $x \in \mathscr{R}^2$ as $\arg(x) = \arg(x_1 + ix_2) + 2\pi\kappa(x)$ with appropriate $\kappa(x) \in \mathbb{Z}$. It is illustrative to recall the situation of the covering surface of $\mathbb{C} \setminus \{0\}$, the natural domain of definition of the complex logarithm.

For $\beta < \gamma$, we will consider *conical Riemann surfaces* (CRS)

$$\Omega_{\beta,\gamma} = \{x \in \mathscr{R}^2 : \beta < \arg(x) < \gamma\} \tag{27}$$

and the two half-lines

$$\Gamma_1 = \{x \in \mathscr{R}^2 : \arg(x) = \beta\}, \quad \Gamma_2 = \{x \in \mathscr{R}^2 : \arg(x) = \gamma\},$$

which together with the origin form the boundary of $\Omega_{\beta,\gamma}$. We will always identify Γ_1 and Γ_2 with $\mathbb{R}_+$.

We are going to look for weak solutions $u \in H^{1+\varepsilon}(\Omega_{\beta,\gamma})$ of the HE

$$(\Delta + k^2)u = 0 \text{ in } \Omega_{\beta,\gamma}. \tag{28}$$

The space of solutions will be denoted by

$$\mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma}) = \left\{ u \in H^{1+\varepsilon}(\Omega_{\beta,\gamma}) \,:\, (\Delta + k^2)u = 0 \right\}, \tag{29}$$

where we assume $\varepsilon \in]-1/2,1/2[$ or make even further restrictions on the regularity parameter ε. We will consider three types of boundary value problems (BVPs), according to the boundary conditions

$$(DD) \quad T_{0,\Gamma_1}u = g_1, \quad T_{0,\Gamma_2}u = g_2, \tag{30}$$

$$(NN) \quad T_{1,\Gamma_1}u = g_1, \quad T_{1,\Gamma_2}u = g_2, \tag{31}$$

$$(DN) \quad T_{0,\Gamma_1}u = g_1, \quad T_{1,\Gamma_2}u = g_2, \tag{32}$$

where the boundary data g_j are given in the corresponding Sobolev spaces $H^{1/2+\varepsilon}(\mathbb{R}_+)$ or $H^{-1/2+\varepsilon}(\mathbb{R}_+)$. For a detailed description of the situation in which $\varepsilon < 0$ see [7].

A few clarifications are in order. There are several equivalent definitions of the Sobolev space $H^{1+\varepsilon}(\Omega_{\beta,\gamma})$. For instance, one can stipulate that $H^{1+\varepsilon}(\Omega_{\beta,\gamma})$ consists of all functions u defined on $\Omega_{\beta,\gamma}$ such that $u \circ \tau_\Omega^{-1} \in H^{1+\varepsilon}(\widehat{\Omega})$ for each proper cone $\widehat{\Omega} \subset \Omega_{\beta,\gamma}$ of angle less than 2π, where the bijective map $\tau_\Omega : \Omega \to \widehat{\Omega} \subset \mathbb{R}^2 \setminus \{0\}$ is the restriction of the covering map τ onto Ω. It is possible to restrict oneself in this definition to finitely many such Ω's which cover $\Omega_{\beta,\gamma}$. By the same idea, the HE (28) on the Riemann surface $\Omega_{\beta,\gamma}$ can be understood locally by restricting the "global" solution u to "local" solutions $u \circ \tau_\Omega^{-1}$ defined on a cone in $\mathbb{R}^2 \setminus \{0\}$. The interpretation of the boundary operators T_{0,Γ_j} and T_{1,Γ_j} is also easy to give (see the remarks made in the introduction).

Let us make the trivial observation that the HE in $\Omega_{\beta,\gamma}$ can be reduced via rotation to the HE in $\Omega_{0,\alpha}$ where $\alpha = \gamma - \beta$. More specifically, if $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,\alpha})$, then $v \in \mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ where

$$v(x) = u(\mathscr{R}_\beta^{-1}x), \qquad \mathscr{R}_\beta \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{33}$$

Strictly speaking this formula holds in $\mathbb{R}^2$, but the interpretation for the Riemann surface $\mathscr{R}^2$ should also be clear. Therefore it is no loss of generality to restrict our considerations to cones $\Omega_{0,\alpha}$.

In this connection and for later constructions, it is useful to introduce the different *leaves* of the Riemann surface $\mathscr{R}^2$,

$$\Lambda_j = \{x \in \mathscr{R}^2 : 2\pi(j-1) < \arg(x) < 2\pi j\}, \quad j \in \mathbb{Z}, \tag{34}$$

which also constitute the leaves of the CRS $\Omega_{0,\alpha}$, except possibly for the last one $\Omega_{0,\alpha} \cap \Lambda_k$, $k = \lceil \frac{\alpha}{2\pi} \rceil$.

Our first result concerns the compatibility conditions which the solutions of the problems (30)–(32) have to satisfy. We encounter compatibility conditions in the DD case and the NN case (see (8) and (9)), whereas we do not expect compatibility conditions in the DN case. That no compatibility conditions are necessary in the DN case was already observed in [7] in the case $\alpha = \gamma - \beta = 2\pi/n$. In the DD case for $\varepsilon \in]-1/2,0[$, the compatibility conditions are automatically fulfilled since $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) = H^{1/2+\varepsilon}(\mathbb{R}_+)$ as they are in the NN case for $\varepsilon \in]0,1/2[$ for the same reason.

PROPOSITION 2.1. *Let $\varepsilon \in [0, 1/2[$. A solution $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ of the DD prob-lem satisfies the compatibility condition $g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$. A solution of the NN problem satisfies the compatibility condition $g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$, which is only necessary for $\varepsilon = 0$ and redundant for $\varepsilon \in]0, 1/2[$.*

Proof. Without loss of generality we can restrict ourselves to cones $\Omega_{0,\alpha}$. The result is known for BVPs in proper cones ($0 < \alpha < 2\pi$) [7] and also for Sommerfeld problems ($\alpha = 2\pi$) [7, 15, 24], see the introduction. A solution of the DD problem satisfies, by restriction, certain DD problems on leaves $\Lambda_1, \Lambda_2, \ldots$. Thus the condition (8) is found by iteration. A solution u of the NN problem has restriction, say, u_j on Λ_j, $j = 1, 2, \ldots, k$, and traces of the normal derivatives which coincide on the common boundaries of the leaves, if taken always in the same direction, say of the inner normal n of Γ_1 (anti-clockwise), are given by

$$g_1 = \frac{\partial u_1}{\partial n}\Big|_{\Gamma_1}, h_1 = \frac{\partial u_1}{\partial n}\Big|_{\overline{\Lambda}_1 \cap \overline{\Lambda}_2} = \frac{\partial u_2}{\partial n}\Big|_{\overline{\Lambda}_1 \cap \overline{\Lambda}_2}, h_2 = \frac{\partial u_2}{\partial n}\Big|_{\overline{\Lambda}_2 \cap \overline{\Lambda}_3} = \frac{\partial u_3}{\partial n}\Big|_{\overline{\Lambda}_2 \cap \overline{\Lambda}_3}, \ldots.$$

From the NN problem on the last leaf we have

$$\frac{\partial u_k}{\partial n}\Big|_{\overline{\Lambda}_{k-1} \cap \overline{\Lambda}_k} + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$$

due to the result for proper cones. So we obtain iteratively $g_1 - h_1, h_1 - h_2, \ldots, h_{k-1} - h_k, h_k + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$ and $g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$. □

PROPOSITION 2.2. *Let $\varepsilon \geqslant 0$. Any solution $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,\alpha})$ of the DD, NN or DN problem is unique.*

Proof. (Sketch) It is sufficient to consider the case $\varepsilon = 0$, which is known for proper cones, see, e.g., [2, 10]. The proof is based upon Green's identity and easily generalized to CRSs by splitting $\Omega_{\beta,\gamma}$ into proper cones. □

Uniqueness does not always hold in the case of $\varepsilon < 0$. For instance, it is known (see, e.g., Prop. 5.6 of [7]) that the DN problem in the slit-plane $\Omega_{0,2\pi}$ is not unique for $\varepsilon \in]-1/2, -1/4[$. Modifying this example one can show that for $\alpha > \pi$ the DN problem is not unique in $\Omega_{0,\alpha}$ for $\varepsilon \in]-1/2, -\pi/2\alpha[$. The DD problem is not unique in $\Omega_{0,\alpha}$ whenever $\alpha > 2\pi$ and $\varepsilon \in]-1/2, -\pi/\alpha[$.

However, sometimes uniqueness also holds for negative ε.

PROPOSITION 2.3. *For $\varepsilon \in]-1/2, 0[$, any solution to the DD problem in $\Omega_{0,\pi} = \Omega^+$ is unique.*

Proof. Let $u \in \mathscr{H}^{1+\varepsilon}(\Omega^+)$ be a solution with zero DD data, $T_{0,\mathbb{R}}u = 0$. Define

$$v(x_1, x_2) = \begin{cases} u(x_1, x_2) & \text{if } x_2 > 0 \\ -u(x_1, -x_2) & \text{if } x_2 < 0 \end{cases}$$

and $v(x_1,0) = 0$. Then $v \in H^{1+\varepsilon}(\mathbb{R}^2)$ and v satisfies the HE equation (in a distributional sense) on $\mathbb{R}^2$ except, perhaps, on the line $\mathbb{R} \times \{0\}$. Since Dirichlet and Neumann jump conditions hold on $\mathbb{R}$,

$$v(x_1,x_2)\Big|_{x_2 \to +0} = v(x_1,x_2)\Big|_{x_2 \to -0}, \qquad \frac{\partial v}{\partial x_2}\Big|_{x_2 \to +0} = \frac{\partial v}{\partial x_2}\Big|_{x_2 \to -0},$$

v satisfies the HE on all of $\mathbb{R}^2$. Taking the 2D Fourier transform $\hat{v}(\xi_1,\xi_2)$ of v, the HE becomes $(k^2 + \xi_1^2 + \xi_2^2)\hat{v}(\xi_1,\xi_2) = 0$, and this implies $\hat{v} = 0$ and thus $v = 0$ and $u = 0$. $\square$

The uniqueness results have a simple, but important consequence regarding the representation of the solution of the HE in the half-plane. The result was known in the case $\varepsilon \geqslant 0$.

COROLLARY 2.4. *Let $\varepsilon > -1/2$. Then each $u \in \mathscr{H}^{1+\varepsilon}(\Omega^+)$ can be represented as $u = \mathscr{K}_{D,\Omega^+} f$ with unique $f \in H^{1/2+\varepsilon}(\mathbb{R})$.*

Proof. Clearly, if $u = \mathscr{K}_{D,\Omega^+} f$, then $f = T_{0,\mathbb{R}} u$ and this implies the uniqueness of f. Furthermore, each f gives rise to a solution of the HE equation in the corresponding Sobolev space. It remains to show existence. Given $u \in \mathscr{H}^{1+\varepsilon}(\Omega^+)$ put $f = T_{0,\mathbb{R}} u \in H^{1/2+\varepsilon}(\mathbb{R})$ and $v = \mathscr{K}_{D,\Omega^+} f$. Then $u - v \in \mathscr{H}^{1+\varepsilon}(\Omega^+)$ and $T_{0,\mathbb{R}}(u - v) = 0$. The uniqueness of the DD problem implies that $u = v$ and thus $u = \mathscr{K}_{D,\Omega^+} f$ as desired. $\square$

A similar result can be obtained for the lower half-plane Ω^- as well. Regarding the NN problem and the simple layer potential, the corresponding result is known and holds for $\varepsilon \geqslant 0$.

The significance of the previous results should be clear. They allow us to represent Helmholtz solutions via double or simple layer potentials. We will make use of this in the next section. It is perhaps interesting to remark that if the DD or NN problem in the half-plane would not be unique, then there would exist solutions of the HE which could not be represented as double or simple layer potentials (12) and (13).

Uniqueness results in the DD and DN case of negative $\varepsilon < 0$ in different cones $\Omega_{0,\alpha}$ will also be obtained in what follows, although we do not settle the issue completely.

3. BVPs in a CRS for $\alpha = m\pi$ via LIPs

Now we begin studying the questions of existence and representation of the solution and start with Dirichlet problems in the CRS $\Omega_{0,m\pi}$ where $m = 1,2,3,\ldots$, using LIPs (line potentials) as a first approach. The leaves of $\Omega_{0,m\pi}$ take the form

$$\Lambda_j = \{(x_1,x_2) : 2\pi(j-1) < \arg(x_1 + ix_2) < 2\pi j\}, \qquad j = 1,\ldots,q$$

($q = [\frac{m+1}{2}]$ denoting the integer part of $\frac{m+1}{2}$) and are split into the corresponding upper and lower half-planes Λ_j^+ and Λ_j^-. Moreover, let

$$\Sigma_j = \{ x \in \mathscr{R}^2 : \arg(x) = j\pi \}, \qquad j = 0,1,2,\ldots$$

be half-lines on the Riemann surface starting at the origin.

PROPOSITION 3.1. *Let* $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}$ *with* $\varepsilon \in]-1/2, 1/2[$ *be given. A function u belongs to $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ and satisfies the Dirichlet conditions* (30) *iff*

$$u = \begin{cases} \mathscr{K}_{D,\Omega^+} \, u_{0j}^+ & in \quad \Lambda_j^+, \quad j = 1, ..., [\frac{m+1}{2}] \\ \mathscr{K}_{D,\Omega^-} \, u_{0j}^- & in \quad \Lambda_j^-, \quad j = 1, ..., [\frac{m}{2}] \end{cases} \tag{35}$$

where (ι being defined in (15) *and* $q = [\frac{m+1}{2}]$ *)*

$$u_{0j}^+ = \begin{cases} \iota(f_0, f_1) & , \; j = 1 \\ \iota(f_2, f_3) & , \; j = 2 \\ \quad \vdots \\ \iota(f_{m-1}, f_m) & , \; j = q, \, m = 2q-1 \\ \iota(f_{m-2}, f_{m-1}) & , \; j = q, \, m = 2q \end{cases}$$

$$\tag{36}$$

$$u_{0j}^- = \begin{cases} \iota(f_2, f_1) & , \; j = 1 \\ \iota(f_4, f_3) & , \; j = 2 \\ \quad \vdots \\ \iota(f_{m-1}, f_{m-2}) & , \; j = q-1, \, m = 2q-1 \\ \iota(f_m, f_{m-1}) & , \; j = q, \, m = 2q \end{cases}$$

and $f_0, f_1, \ldots, f_{m-1}, f_m \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ *with* $f_0 = g_1$ *and* $f_m = g_2$ *satisfy the compatibility conditions*

$$f_0 - f_1, f_1 - f_2, \ldots, f_{m-1} - f_m \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) \tag{37}$$

and the Wiener-Hopf-Hankel equations

$$r_+ A_t \begin{pmatrix} 2I & \mathscr{J} & 0 & 0 & \ldots & 0 \\ \mathscr{J} & 2I & \mathscr{J} & 0 & \ldots & 0 \\ 0 & \mathscr{J} & 2I & \mathscr{J} & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathscr{J} & 2I & \mathscr{J} \\ 0 & \ldots & 0 & 0 & \mathscr{J} & 2I \end{pmatrix} \ell_0 \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{m-2} \\ f_{m-1} \end{pmatrix} = -r_+ A_t \, \mathscr{J} \, \ell_0 \begin{pmatrix} f_0 \\ 0 \\ \vdots \\ 0 \\ f_m \end{pmatrix}. \tag{38}$$

Proof. Denote the traces of u on the x_1-half-lines $\Sigma_0 = \Gamma_1, \Sigma_1, \ldots, \Sigma_m = \Gamma_2$, that we meet on $\Omega_{0,m\pi}$ surrounding the origin anti-clockwise, by

$$f_0, f_1, f_2, \ldots, f_{m-1}, f_m \in H^{1/2+\varepsilon}(\mathbb{R}_+) \tag{39}$$

noting that $f_0 = g_1$ and $f_m = g_2$. We know that the HE is satisfied near points of these half-lines iff the jumps of the normal derivatives are zero when representing u by the
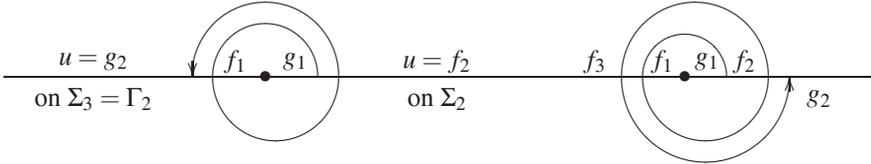
Figure 3: $m = 3$ *(odd)*   Figure 4: $m = 4$ *(even)*

formulas (12) locally, i.e. in upper/lower half-planes $\Lambda_j^{\pm}$ of the leaves Λ_j of $\Omega_{0,m\pi}$, $j = 1, \ldots, [\frac{m+1}{2}]$. Hence a representation of u is given by (35) and (36) with unknowns $f_j, j = 1, \ldots, m-1$.

The interface conditions can be written in the following form (first for smooth functions, and by a density argument as relations in $H^{-1/2+\varepsilon}(\mathbb{R}_{\pm})$):

$$
\begin{aligned}
T_{1,\Sigma_1}\left(u|_{\Lambda_1^+} - u|_{\Lambda_1^-}\right) &= 0, \\
T_{1,\Sigma_2}\left(u|_{\Lambda_1^-} - u|_{\Lambda_2^+}\right) &= 0, \\
&\ \ \vdots \\
T_{1,\Sigma_{m-1}}\left(u|_{\Lambda_{q-1}^-} - u|_{\Lambda_q^+}\right) &= 0 \ \ \text{if } m = 2q-1, \\
T_{1,\Sigma_{m-1}}\left(u|_{\Lambda_q^+} - u|_{\Lambda_q^-}\right) &= 0 \ \ \text{if } m = 2q.
\end{aligned}
\tag{40}
$$

Using (12), (35) and (36), we obtain the $m-1$ identities

$$
\begin{aligned}
r_- A_t\left(\imath(f_0,f_1) + \imath(f_2,f_1)\right) &= 0, \\
r_+ A_t\left(\imath(f_2,f_1) + \imath(f_2,f_3)\right) &= 0, \\
&\ \ \vdots \\
r_+ A_t\left(\imath(f_{m-1},f_{m-2}) + \imath(f_{m-1},f_m)\right) &= 0 \ \ \text{if } m = 2q-1, \\
r_- A_t\left(\imath(f_{m-2},f_{m-1}) + \imath(f_m,f_{m-1})\right) &= 0 \ \ \text{if } m = 2q.
\end{aligned}
\tag{41}
$$

Hence for j even we have

$$
r_+ A_t\left(2\ell_0 f_j + \mathscr{J}\ell_0 f_{j-1} + \mathscr{J}\ell_0 f_{j+1}\right) = 0,
\tag{42}
$$

while for j odd we have

$$
r_- A_t\left(2\mathscr{J}\ell_0 f_j + \ell_0 f_{j-1} + \ell_0 f_{j+1}\right) = 0.
$$

Applying the reflection operator $\mathscr{J}$ to the last equation we obtain the first equation because $\mathscr{J} r_- A_t = r_+ A_t \mathscr{J}$ noting that $t(-\xi) = t(\xi)$. Hence we derive (42) for all $j = 1, \ldots, m-1$. These equations can be written in the form of the Wiener-Hopf-Hankel system.

Conversely, assume $f_0, f_1, \ldots, f_{m-1}, f_m$ with $f_0 = g_1$ and $f_m = g_2$ is a solution of (38) satisfying the compatibility conditions (37). The compatibility conditions imply that $u_{0j}^{\pm} \in H^{1/2+\varepsilon}(\mathbb{R})$ and hence u is well defined by (35) and (36). The function u belongs to $H^{1+\varepsilon}$ and satisfies the HE on each $\Lambda_j^{\pm}$. Reversing the above arguments it follows that the appropriate interface conditions are satisfied on Λ_j. Hence $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ and the Dirichlet conditions are fulfilled as well. $\quad\square$

REMARK 3.2. As expected from Proposition 2.1, the compatibility conditions (37) yield $g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$. Notice that g_1 and g_2 themselves do not need to be in $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, and neither the functions f_j. This however means that the extension-by-zero operator appearing in the Wiener-Hopf system (38) need not map into $H^{1/2+\varepsilon}(\mathbb{R})$. Hence the system needs to be properly interpreted. One way of doing this is by rewriting (42) as

$$
\begin{aligned}
0 &= r_+ A_t \Big(2(I + \mathscr{J})\ell_0 f_j + \mathscr{J}\ell_0(f_{j-1} - f_j) + \mathscr{J}\ell_0(f_{j+1} - f_j) \Big) \\
&= r_+ A_t \Big(2\ell^e f_j + \mathscr{J}\ell_0(f_{j-1} - f_j) + \mathscr{J}\ell_0(f_{j+1} - f_j) \Big),
\end{aligned} \tag{43}
$$

which features the even extension $\ell^e : H^{1/2+\varepsilon}(\mathbb{R}_+) \to H^{1/2+\varepsilon}(\mathbb{R})$ and $\ell_0 : \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) \to H^{1/2+\varepsilon}(\mathbb{R})$ both now well-defined and bounded (see (6)). Notice that A_t maps $H^{1/2+\varepsilon}(\mathbb{R})$ boundedly into $H^{-1/2+\varepsilon}(\mathbb{R})$.

There is an analogue of Proposition 3.1 for the Neumann problem, exchanging the roles of Dirichlet and Neumann data, as follows:

PROPOSITION 3.3. *Let* $(g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}$ *with* $\varepsilon \in [0, 1/2[$ *be given. A function* u *belongs to* $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ *and satisfies the Neumann conditions* (31) *iff*

$$
u = \begin{cases} \mathscr{K}_{N,\Omega^+}\, u_{1j}^+ & in \quad \Lambda_j^+, \quad j = 1, ..., [\frac{m+1}{2}] \\ \mathscr{K}_{N,\Omega^-}\, u_{1j}^- & in \quad \Lambda_j^-, \quad j = 1, ..., [\frac{m}{2}] \end{cases} \tag{44}
$$

where (ι being defined in (15) *and* $q = [\frac{m+1}{2}]$)

$$
u_{1j}^+ = \begin{cases} \iota(f_0, f_1) & , \ j = 1 \\ \iota(f_2, f_3) & , \ j = 2 \\ \quad\vdots & \\ \iota(f_{m-1}, f_m) & , \ j = q, \ m = 2q - 1 \\ \iota(f_{m-2}, f_{m-1}) & , \ j = q, \ m = 2q \end{cases}
$$

$$
\tag{45}
$$

$$
u_{1j}^- = \begin{cases} -\iota(f_2, f_1) & , \ j = 1 \\ -\iota(f_4, f_3) & , \ j = 2 \\ \quad\vdots & \\ -\iota(f_{m-1}, f_{m-2}) & , \ j = q - 1, \ m = 2q - 1 \\ -\iota(f_m, f_{m-1}) & , \ j = q, \ m = 2q \end{cases}
$$

and $f_0, f_1, \ldots, f_{m-1}, f_m \in H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *with* $f_0 = g_1$ *and* $f_m = (-1)^{m+1} g_2$ *satisfy the compatibility conditions*

$$
f_0 + f_1, f_1 + f_2, \ldots, f_{m-1} + f_m \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+) \tag{46}
$$

and the system of Wiener-Hopf-Hankel equations

$$
r_+ A_{t^{-1}}
\begin{pmatrix}
2I & \mathscr{J} & 0 & 0 & \ldots & 0 \\
\mathscr{J} & 2I & \mathscr{J} & 0 & \ldots & 0 \\
0 & \mathscr{J} & 2I & \mathscr{J} & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & \mathscr{J} & 2I & \mathscr{J} \\
0 & \ldots & 0 & 0 & \mathscr{J} & 2I
\end{pmatrix}
\ell_0
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_{m-2} \\
f_{m-1}
\end{pmatrix}
= -r_+ A_{t^{-1}} \mathscr{J} \ell_0
\begin{pmatrix}
f_0 \\
0 \\
\vdots \\
0 \\
f_m
\end{pmatrix}.
\qquad (47)
$$

Proof. The proof is analogous to that of Proposition 3.1, so we remark only the differences. We are going to define f_j as the traces of the $\frac{\partial}{\partial x_2}$-derivatives on Σ_j, and then Σ_j being identified with $\mathbb{R}_+$,

$$
f_j = \mathscr{J}^j \left. \frac{\partial u}{\partial x_2} \right|_{\Sigma_j}.
$$

Clearly, $f_0 = g_1$ and $f_m = (-1)^{m+1} g_2$ taking into account the direction of the normal derivatives. Notice that the minus sign in (45) is due to the opposite direction of the derivative defining f_j and the Neumann data occurring in the definition of (13).

As interface conditions we obtain

$$
\begin{aligned}
T_{0,\Sigma_j}(u|_{\Lambda_k^+} - u|_{\Lambda_k^-}) = 0, && j = 2k-1, \\
T_{0,\Sigma_j}(u|_{\Lambda_k^-} - u|_{\Lambda_{k+1}^+}) = 0, && j = 2k,
\end{aligned}
$$

which amount to

$$
\begin{aligned}
r_- A_{t^{-1}}(\iota(f_{j-1}, f_j) + \iota(f_{j+1}, f_j)) = 0, && j \text{ odd}, \\
r_+ A_{t^{-1}}(\iota(f_j, f_{j-1}) + \iota(f_j, f_{j+1})) = 0, && j \text{ even}.
\end{aligned}
$$

For all $j = 1, \ldots, m-1$, this is equivalent to

$$
r_+ A_{t^{-1}}(2\ell_0 f_j + \mathscr{J} \ell_0 f_{j-1} + \mathscr{J} \ell_0 f_{j+1}) = 0, \qquad (48)
$$

which gives rise to the Wiener-Hopf-Hankel system.

Conversely, assume that the Wiener-Hopf-Hankel system and the compatibility conditions are satisfied. The compatibility conditions imply that $u_{1j}^\pm \in H^{-1/2}(\mathbb{R})$ and thus we can define u by (44). The Wiener-Hopf-Hankel system implies the interface condition. Everything together then yields $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ with the (NN) boundary conditions. $\square$

REMARK 3.4. The compatibility conditions (46) imply the compatibility condition $g_1 + g_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$ (see Proposition 2.1). Notice that this condition is relevant only for $\varepsilon = 0$, but redundant for $\varepsilon \in {]0, 1/2[}$. In the case $\varepsilon = 0$, the distributions g_1, g_2

and f_j need not belong to $\widetilde{H}^{-1/2}(\mathbb{R}_+)$. The proper interpretation of (47) is to rewrite (48) as

$$
\begin{aligned}
0 &= r_+ A_{t^{-1}} \Big(2(I - \mathscr{J})\ell_0 f_j + \mathscr{J}\ell_0(f_{j-1} + f_j) + \mathscr{J}\ell_0(f_{j+1} + f_j) \Big) \\
&= r_+ A_{t^{-1}} \Big(2\ell^o f_j + \mathscr{J}\ell_0(f_{j-1} + f_j) + \mathscr{J}\ell_0(f_{j+1} + f_j) \Big).
\end{aligned}
\tag{49}
$$

Therein, the odd extension $\ell^o : H^{-1/2}(\mathbb{R}_+) \to H^{-1/2}(\mathbb{R})$ is bounded (see (6)), along with ℓ_0 acting on $\widetilde{H}^{-1/2}(\mathbb{R}_+)$. Note that $A_{t^{-1}}$ maps $H^{-1/2+\varepsilon}(\mathbb{R})$ to $H^{1/2+\varepsilon}(\mathbb{R})$.

Let us finally look at the DN system. The first question is what kind of ansatz to choose. We decide to work with the Dirichlet ansatz (35)–(36). One could also work with the Neumann ansatz or perhaps mixed ansatzes. This might lead to different systems, which however, in the end should all be equivalent since the DD, NN, and DN data on $\partial\Lambda_j^{\pm}$ are all in one-to-one correspondence.

PROPOSITION 3.5. *Let $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ with $\varepsilon \in\,]-1/2, 1/2[$ be given. A function u belongs to $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ and satisfies the DN conditions* (32) *iff u is represented by* (35)–(36), *where $f_0, f_1, \ldots, f_{m-1}, f_m \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ with $f_0 = g_1$ satisfying the compatibility conditions* (37) *and the Wiener-Hopf-Hankel equations*

$$
r_+ A_t
\begin{pmatrix}
2I & \mathscr{J} & 0 & 0 & \cdots & 0 \\
\mathscr{J} & 2I & \mathscr{J} & 0 & \cdots & 0 \\
0 & \mathscr{J} & 2I & \mathscr{J} & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \mathscr{J} & 2I & \mathscr{J} \\
0 & \cdots & 0 & 0 & \mathscr{J} & I
\end{pmatrix}
\ell_0
\begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_{m-2} \\
f_{m-1} \\
f_m
\end{pmatrix}
= -
\begin{pmatrix}
r_+ A_t \mathscr{J}\ell_0 g_1 \\
0 \\
\vdots \\
0 \\
0 \\
g_2
\end{pmatrix}.
\tag{50}
$$

REMARK 3.6. Notice that the system (50) has a larger size than the system (38). Moreover, there is no factor "2" in the lower right entry. No compatibility condition is required between g_1 and g_2. However, compatibility conditions occur in the ansatz between the f_j's including $f_0 = g_1$.

Proof. The arguments are as in the (DD) case. The only difference is the connection with g_2. Let us assume m is even. The odd case gives the same results without any change of sign. Then $g_2 = T_{1,\Sigma_2} u|_{\Lambda_q^-}$, $2m = q$, gives

$$
g_2 = r_+ u_{1q}^- = -r_+ A_t u_{0q}^- = -r_+ A_t (\ell_0 f_m + \mathscr{J}\ell_0 f_{m-1}).
$$

This is precisely what the last row in the system (50) means. $\square$

REMARK 3.7. Since we do not assume $g_1 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$ we need a proper interpretation in the cases where we merely have $g_1 \in H^{1/2+\varepsilon}(\mathbb{R}_+)$, $\varepsilon \in [0, 1/2[$. The first

$m-1$ rows of (50) can be understood as in (42). The last row can be rewritten as

$$0 = r_+ A_t \left(2\ell^e f_m + \mathscr{J}\ell_0(f_{m-1} - f_m) \right). \tag{51}$$

The same remarks as for (43) apply here.

4. Equivalent systems of the basic BVPs in $\Omega_{0,m\pi}$ via SOPs

In the following (second) approach we establish another (equivalent) characterization of the solutions $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ of the three basic BVPs (DD, NN, DN) in the special CRS $\Omega_{0,m\pi}$. We will transform the Wiener-Hopf-Hankel systems obtained in the previous section into other types of systems which involve projection operators. The LIP's (line potentials) will be replaced by SOP's (Sommerfeld potentials); see (21) and (24). Though the approach is equivalent it seems more transparent and more suitable for later analysis.

The following notation will be useful: For $s, \varepsilon \in \mathbb{R}$ and $|\varepsilon| < 1/2$ let

$$\begin{aligned}
P_s^+ &= A_{t_+^{-s}} \ell_0 r_+ A_{t_+^s} : H^{s+\varepsilon} \to H^{s+\varepsilon}, \\
P_s^- &= A_{t_-^{-s}} \ell_0 r_- A_{t_-^s} : H^{s+\varepsilon} \to H^{s+\varepsilon},
\end{aligned} \tag{52}$$

which are bounded projections in $H^{s+\varepsilon}$ acting onto $H_+^{s+\varepsilon}$ and $H_-^{s+\varepsilon}$, respectively. The latter implies that

$$P_s^\pm f_\pm = f_\pm \tag{53}$$

whenever $f_\pm \in H_\pm^{s+\varepsilon}$. The complementary projections $\Pi_s^+ = I - P_s^-$ appeared already in the context of (21), (22), and (24). To simplify notation, we define for $j \in \mathbb{N}$:

$$P_s^{(j)} = \begin{cases} P_s^+ & \text{if } j \text{ is even} \\ P_s^- & \text{if } j \text{ is odd} \end{cases} \tag{54}$$

The transformed systems will no longer involve $f_1, ..., f_{m-1}$ (and f_m in the DN case), but instead

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \vdots \\ \varphi_{m-1} \end{pmatrix} = \mathscr{J}_\# \ell_0 \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_{m-1} \end{pmatrix} = \begin{pmatrix} \mathscr{J}\ell_0 f_1 \\ \mathscr{J}^2 \ell_0 f_2 \\ \mathscr{J}^3 \ell_0 f_3 \\ \vdots \\ \mathscr{J}^{m-1} \ell_0 f_{m-1} \end{pmatrix}. \tag{55}$$

In the DN case the vectors have one more component. Still, the compatibility conditions (37) and (46) will be expressed in terms of the f_j's.

4.1. The DD case

First we present the result concerning the DD problem.

PROPOSITION 4.1. *Let* $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)_{\sim}^2$ *with* $\varepsilon \in]-1/2, 1/2[$ *be given. A function* u *belongs to* $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ *and satisfies the DD conditions* (30) *iff* u *is represented by* (35)–(36) *such that* $f_0, \ldots, f_m \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ *satisfy the compatibility condition* (37), *and* $f_0 = g_1, f_m = g_2$, *and* φ *defined by* (55) *satisfies the system*

$$M_{DD,m}\, \varphi = \begin{pmatrix} I & \frac{1}{2}P_{1/2}^- & 0 & \cdots & & 0 \\ \frac{1}{2}P_{1/2}^+ & I & \frac{1}{2}P_{1/2}^+ & & & \vdots \\ 0 & \frac{1}{2}P_{1/2}^- & I & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \frac{1}{2}P_{1/2}^{(m-2)} \\ 0 & \cdots & & 0 & \frac{1}{2}P_{1/2}^{(m-1)} & I \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{m-2} \\ \varphi_{m-1} \end{pmatrix} = \begin{pmatrix} g_1^* \\ 0 \\ \vdots \\ 0 \\ g_2^* \end{pmatrix} \qquad (56)$$

with

$$\begin{aligned} g_1^* &= -\tfrac{1}{2} P_{1/2}^- \ell_0 g_1, \\ g_2^* &= -\tfrac{1}{2} P_{1/2}^{(m-1)} \mathscr{J}^m \ell_0 g_2. \end{aligned} \qquad (57)$$

REMARK 4.2. *For* $\varepsilon \in]-1/2, 1/2[$, *the operator*

$$M_{DD,m} : \mathscr{J}_{\#} H_+^{1/2+\varepsilon}(\mathbb{R})^{m-1} \to \mathscr{J}_{\#} H_+^{1/2+\varepsilon}(\mathbb{R})^{m-1} \qquad (58)$$

is always well-defined and bounded. If $\varepsilon \in]-1/2, 0[$, then φ and the right hand side in (56) belong to theses spaces, and thus the system (56) can be understood as it is.

However, if $\varepsilon \in [0, 1/2[$, then the functions $\varphi_j = \mathscr{J}^j \ell_0 f_j$ do not necessarily belong to $H^{1/2+\varepsilon}(\mathbb{R})$. Hence the system must be properly interpreted.

One possibility will be pointed out in the proof. Another one consists in noticing that $\varphi_j \in H^{1/2-\delta}(\mathbb{R})$ for (any) $\delta \in]0, 1/2[$, i.e., one considers $M_{DD,m}$ as being defined on a larger space and then restricts it. A third possibility (in case $\varepsilon = 0$) is to consider $M_{DD,m}$ as being defined on the dense subspace

$$\mathscr{J}_{\#} \ell_0 \widetilde{H}^{1/2}(\mathbb{R}_+)^{m-1} = \mathscr{J}_{\#} H_+^{1/2}(\mathbb{R})^{m-1} \subset \mathscr{J}_{\#} \ell_0 H^{1/2}(\mathbb{R}_+)^{m-1}.$$

and then look for an (unbounded) extension of this operator. (Notice that for $\varepsilon \in]0, 1/2[$, $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$ is a closed subspace of $H^{1/2+\varepsilon}(\mathbb{R}_+)$ of codimension one.)

Proof. The proof does follow in large parts (in particular, in view of the compatibility conditions) the proof of Proposition 3.1. Therein we arrive at equation (42),

$$r_+ A_t (2\ell_0 f_j + \mathscr{J} \ell_0 f_{j-1} + \mathscr{J} \ell_0 f_{j+1}) = 0$$

for $j = 1, \ldots, m-1$. Our goal is now to show that this is equivalent to the system (56), which reads

$$\varphi_j + \tfrac{1}{2} P^{(j)}_{1/2} (\varphi_{j-1} + \varphi_{j+1}) = 0 \tag{59}$$

with $\varphi_j = \mathscr{J}^j f_j$. This is nothing but

$$\ell_0 f_j + \tfrac{1}{2} A_{t_+^{-1/2}} \ell_0 r_+ A_{t_+^{1/2}} (\mathscr{J} \ell_0 f_{j-1} + \mathscr{J} \ell_0 f_{j-1}) = 0$$

noting that $A_{t_+^{-1/2}} \ell_0 r_+ A_{t_+^{1/2}} = \mathscr{J} A_{t_-^{-1/2}} \ell_0 r_- A_{t_-^{1/2}} \mathscr{J}$. We get from the last equation to the first equation by applying the Wiener-Hopf operator

$$W_t = r_+ A_t = r_+ A_{t_-^{1/2}} A_{t_+^{1/2}} : H_+^{1/2+\varepsilon}(\mathbb{R}) \to H^{-1/2+\varepsilon}(\mathbb{R}_+).$$

Vice versa, we need to apply its inverse

$$W_t^{-1} = A_{t_+^{-1/2}} \ell_0 r_+ A_{t_-^{-1/2}} \ell : H^{-1/2+\varepsilon}(\mathbb{R}_+) \to H_+^{1/2+\varepsilon}(\mathbb{R}).$$

In the case $\varepsilon \in [0, 1/2[$ some more care is necessary. Here one should not start with (42) but with (43). This implies not directly (59) but

$$P^{(j)}_{1/2}(I + \mathscr{J})\varphi_j + \tfrac{1}{2} P^{(j)}_{1/2}(\varphi_{j-1} + \varphi_{j+1} - 2\varphi_j) = 0. \tag{60}$$

Notice that $\varphi_{j-1} + \varphi_{j+1} - 2\varphi_j \in H^{1/2+\varepsilon}(\mathbb{R})$ due to the compatibility condition, and likewise $(I + \mathscr{J})\varphi_j = \ell^e f_j \in H^{1/2+\varepsilon}(\mathbb{R})$. In the case $\varphi_j \in \mathscr{J}^j H_+^{1/2+\varepsilon}(\mathbb{R})$, equations (59) and (60) coincide. This completes the proof. $\square$

REMARK 4.3. We notice that (59) can also be obtained by direct verification. Consider, e.g., the case $j = 1$. Then

$$\begin{aligned} u_0^+ &= \varphi_0 + \varphi_1 = \ell_0 f_0 + \mathscr{J} \ell_0 f_1 \\ u_0^- &= \varphi_2 + \varphi_1 = \ell_0 f_2 + \mathscr{J} \ell_0 f_1. \end{aligned} \tag{61}$$

and the jump relations on Σ_j ($j = 0, 1, 2$) read as

$$u_0^\pm = \tfrac{1}{2}\left[\pm \ell_0(f_0 - f_2) + \Pi_{1/2}^+ \ell(f_0 + f_2) \right]. \tag{62}$$

Substitution yields

$$2\varphi_1 + \varphi_0 + \varphi_2 = \Pi_{1/2}^+(\varphi_0 + \varphi_2). \tag{63}$$

Using $\Pi_{1/2}^+ + P_{1/2}^- = I$ we obtain (59). The argumentation can also be reversed.

REMARK 4.4. There are at least three possibilities to solve (56) in the above-mentioned spaces ($\varepsilon \in [0, 1/2[$).

1. Invert the extended operator (58) acting on the space $\mathscr{J}_\# H_+^{1/2-\delta}(\mathbb{R})^{m-1}$, $\delta \in {}]0, 1/2[$, and show that data $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, $\varepsilon \in [0, 1/2[$, yield solutions $f_1, \ldots, f_{m-1} \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ satisfying the compatibility conditions.

2. For $\varepsilon = 0$, invert (58) for data in a dense subspace, $(g_1, g_2) \in \widetilde{H}^{1/2}(\mathbb{R}_+)^2$, and show that the (restricted) inverse has a bounded extension to the spaces we want.

3. Reduce the DD problem equivalently to a semi-homogeneous BVP with $g_1 = 0$ and $g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$ (or vice-versa), invert (58) for $\varepsilon \in [0, 1/2[$ and apply the inverse to $(0, \ldots, 0, \widetilde{g}_2^*)$ where $\widetilde{g}_2^* \in H^{1/2+\varepsilon}$. This is possible if we find some $v \in \mathcal{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ with $T_{0,\Gamma_1} v = g_1$ and substitute $u = v + w$. An example in case of $\alpha = \pi/4$ was given in [1].

We will follow yet another way. We are going to reduce the system (56) via a suitable linear substitution to a system of the same kind, but where the operator $M_{DD,m}$ can be considered as a bounded linear operator on $\mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R})^{m-1}$. (Of course, this extra step needs only be done in the case $\varepsilon \in [0, 1/2[$. It is redundant if $g_1, g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$.) Notice that this reduction is carried out by one-to-one mappings, taking into account the compatibility conditions, which means that we have homeomorphisms between the corresponding spaces or linear manifolds.

PROPOSITION 4.5. *Let $\varepsilon \in] - 1/2, 1/2[$. Assume that the operator*

$$M_{DD,m} : \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R})^{m-1} \to \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R})^{m-1}$$

is invertible. Then the DD problem has for each $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ a unique solution, which can be obtained as follows:
Choose an arbitrary $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ such that $g_1 - g, g_2 - g \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, and solve the system $M_{DD,m}\varphi' = \psi'$ with

$$\psi' = - \begin{pmatrix} P_{1/2}^- \ell^e g \\ P_{1/2}^+ \ell^e g \\ \vdots \\ P_{1/2}^{(m-2)} \ell^e g \\ P_{1/2}^{(m-1)} \ell^e g \end{pmatrix} - \frac{1}{2} \begin{pmatrix} P_{1/2}^- \ell_0 (g_1 - g) \\ 0 \\ \vdots \\ 0 \\ P_{1/2}^{(m-1)} \mathscr{J}^m \ell_0 (g_2 - g) \end{pmatrix} \in \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R})^{m-1}.$$

Then $\varphi = \varphi' + \mathscr{J}_\# \ell_0 (g, \ldots, g)^T$ yields the solution of (56).

Proof. We can rewrite the system (56) as follows,

$$\begin{pmatrix} P_{1/2}^- & 2I & P_{1/2}^- & 0 & \ldots & 0 & 0 \\ 0 & P_{1/2}^+ & 2I & P_{1/2}^+ & & \vdots & \vdots \\ 0 & 0 & P_{1/2}^- & 2I & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & P_{1/2}^{(m-2)} & 0 \\ 0 & 0 & \ldots & 0 & P_{1/2}^{(m-1)} & 2I & P_{1/2}^{(m-1)} \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{m-1} \\ \varphi_m \end{pmatrix} = 0 \qquad (64)$$

with $\varphi_0 = \ell_0 g_1$ and $\varphi_m = \mathscr{J}^m \ell_0 g_2$ being given (see (57)) and $\varphi_1, ..., \varphi_{m-1}$ to be determined. The idea is to make a simple substitution, replacing each φ_j by φ'_j via

$$\varphi_j = \varphi'_j + \mathscr{J}^j \ell_0 g, \qquad j = 0, ..., m,$$

where $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ satisfies the stated compatibility condition. The compatibility conditions that the functions $\varphi_j = \mathscr{J}^m \ell_0 f_j$ have to satisfy (see (37)) amount to requiring that $\varphi'_j \in \mathscr{J}^j H_+^{1/2+\varepsilon}(\mathbb{R})$. The substitution now turns the homogeneous system (64) into one with the right hand side given by a vector $\psi = (\psi_j)_{j=1}^{m-1}$, where

$$\psi_j = -2(\mathscr{J}^j + P_{1/2}^{(j)} \mathscr{J}^{j-1}) \ell_0 g = -2P_{1/2}^{(j)}(I + \mathscr{J}) \ell_0 g = -2P_{1/2}^{(j)} \ell^e g.$$

Here we used that $P_{1/2}^{(j)} \mathscr{J}^j \ell_0 g = \mathscr{J}^j \ell_0 g$; see (53). We divide the new system by 2 and make it into a square system by moving $\varphi'_0 = \ell_0(g_1 - g)$ and $\varphi'_m = \mathscr{J}^m \ell_0(g_2 - g)$ to the right hand side. Then we obtain $M_{DD,m} \varphi' = \psi'$ as desired. $\square$

If $g_1, g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, then we can choose $g = 0$ and the new system $M_{DD,m} \varphi' = \psi'$ is identical with the original system (56). In general, if $g_1, g_2 \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ and $g_1 - g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, the "canonical" choice for g might be $g = (g_1 + g_2)/2$. Other possibilities might be $g = g_1$ or $g = g_2$. In the end, they should all give the same answer.

4.2. The NN case

Let us now proceed with discussing the NN problem.

PROPOSITION 4.6. *Let* $(g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ *with* $\varepsilon \in [0, 1/2[$ *be given. A function* u *belongs to* $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ *and satisfies the NN conditions* (31) *iff* u *is represented by* (44)–(45) *such that* $f_0, ..., f_m \in H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *satisfy the compatibility conditions* (46), *and* $f_0 = g_1$, $f_m = (-1)^{m+1} g_2$, *and* φ *defined by* (55) *satisfies the system*

$$M_{NN,m}\varphi = \begin{pmatrix} I & \frac{1}{2}P_{-1/2}^- & 0 & \cdots & & 0 \\ \frac{1}{2}P_{-1/2}^+ & I & \frac{1}{2}P_{-1/2}^+ & & & \vdots \\ 0 & \frac{1}{2}P_{-1/2}^- & I & \ddots & & 0 \\ \vdots & & \ddots & \ddots & \frac{1}{2}P_{-1/2}^{(m-2)} & \\ 0 & \cdots & 0 & \frac{1}{2}P_{-1/2}^{(m-1)} & I \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{m-2} \\ \varphi_{m-1} \end{pmatrix} = \begin{pmatrix} g_1^* \\ 0 \\ \vdots \\ 0 \\ g_2^* \end{pmatrix} \quad (65)$$

with

$$\begin{aligned} g_1^* &= -\tfrac{1}{2}P_{-1/2}^- \ell_0 g_1, \\ g_2^* &= \tfrac{1}{2}(-1)^m P_{-1/2}^{(m-1)} \mathscr{J}^m \ell_0 g_2. \end{aligned} \quad (66)$$

REMARK 4.7. For $\varepsilon \in]-1/2, 1/2[$, the operator

$$M_{NN,m} : \mathscr{J}_{\#} H_+^{-1/2+\varepsilon}(\mathbb{R})^{m-1} \to \mathscr{J}_{\#} H_+^{-1/2+\varepsilon}(\mathbb{R})^{m-1}$$

is always well-defined and bounded. In the case $\varepsilon \in]0, 1/2[$ we have

$$\varphi \in \mathscr{J}_{\#} \ell_0 H^{-1/2+\varepsilon}(\mathbb{R}_+)^{m-1} = \mathscr{J}_{\#} \ell_0 \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)^{m-1} = \mathscr{J}_{\#} H_+^{-1/2+\varepsilon}(\mathbb{R})^{m-1},$$

and as before, the system (65) can be directly understood in the corresponding spaces.

For $\varepsilon = 0$, the operator in (65) must be considered as an unbounded operator, densely defined on $\mathscr{J}_{\#} \ell_0 \widetilde{H}^{-1/2}(\mathbb{R}_+)^{m-1} = \mathscr{J}_{\#} H_+^{-1/2}(\mathbb{R}) \subset \mathscr{J}_{\#} \ell_0 H^{-1/2}(\mathbb{R}_+)^{m-1}$.

Proof. The reasoning is quite similar to that in the proof of Proposition 4.1, but one should pay attention to two differences caused by the sign of g_2 and the spaces $H^{-1/2+\varepsilon}$, $\varepsilon \in [0, 1/2[$. As in the proof of Proposition 3.3, we denote the derivatives of u on $\Gamma_1 = \Sigma_0, \Sigma_1, \ldots, \Sigma_m = \Gamma_2$ in positive x_2-direction by $g_1 = f_0, f_1, f_2, \ldots, f_m = (-1)^{m+1} g_2$. We derive the compatibility conditions (46) and we obtain the equations (48),

$$r_+ A_{t^{-1}} (2\ell_0 f_j + \mathscr{J} \ell_0 f_{j-1} + \mathscr{J} \ell_0 f_{j+1}) = 0$$

for $j = 1, \ldots, m-1$. We have to show that this is equivalent to system (65), which reads

$$\varphi_j + \tfrac{1}{2} P_{-1/2}^{(j)} (\varphi_{j-1} + \varphi_{j+1}) = 0 \tag{67}$$

with $\varphi_j = \mathscr{J}^j \ell_0 f_j$. These last equations can be rewritten as

$$\ell_0 f_j + \tfrac{1}{2} A_{t_+^{1/2}} \ell_0 r_+ A_{t_+^{-1/2}} \left(\mathscr{J} \ell_0 f_{j-1} + \mathscr{J} \ell_0 f_{j-1} \right) = 0.$$

We can now show the equivalence between these two equations by applying the Wiener-Hopf operator

$$W_{t^{-1}} = r_+ A_{t^{-1}} = r_+ A_{t_-^{-1/2}} A_{t_+^{-1/2}} : H_+^{-1/2+\varepsilon}(\mathbb{R}) \to H^{1/2+\varepsilon}(\mathbb{R}_+)$$

and its inverse

$$W_{t^{-1}}^{-1} = A_{t_+^{1/2}} \ell_0 r_+ A_{t_-^{1/2}} \ell : H^{1/2+\varepsilon}(\mathbb{R}_+) \to H_+^{-1/2+\varepsilon}(\mathbb{R}),$$

respectively. This completes the proof. $\square$

REMARK 4.8. As in the DD case, also in the NN case, the equations (67) can be obtained by direct verification. For $j = 1$, we have

$$\begin{aligned} u_1^+ &= \varphi_0 + \varphi_1 = \ell_0 f_0 + \mathscr{J} \ell_0 f_1 \\ u_1^- &= \varphi_2 + \varphi_1 = \ell_0 f_2 + \mathscr{J} \ell_0 f_1. \end{aligned} \tag{68}$$

The jump conditions on Σ_j, $j = 0, 1, 2$, become

$$u_1^\pm = \frac{1}{2} \left[\pm \ell_0 (f_0 - f_2) + \Pi_{-1/2}^+ \ell(f_0 + f_2) \right] \tag{69}$$

and this turns into

$$2\varphi_1 + \varphi_0 + \varphi_2 = \Pi_{-1/2}^+ (\varphi_0 + \varphi_2). \tag{70}$$

Let us now reduce the system (65) via a linear substitution to a system of the same kind, where the operator $M_{DD,m}$ is considered a bounded linear operator on $\mathscr{J}_{\#}H_+^{-1/2+\varepsilon}(\mathbb{R})^{m-1}$. Notice that this is only necessary in case $\varepsilon = 0$ and when g_1, g_2 do not belong to $\tilde{H}^{-1/2+\varepsilon}(\mathbb{R}+)$. Even though the following proposition is needed only for $\varepsilon = 0$, we state it for $\varepsilon \geqslant 0$.

PROPOSITION 4.9. *Let* $\varepsilon \in [0, 1/2[$. *Assume that the operator*

$$M_{NN,m} : \mathscr{J}_{\#}H_+^{-1/2+\varepsilon}(\mathbb{R}_+)^{m-1} \to \mathscr{J}_{\#}H_+^{-1/2+\varepsilon}(\mathbb{R}_+)^{m-1}$$

is invertible. Then the NN problem has for each right hand side $(g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}$
a unique solution, which can be obtained as follows:
Choose an arbitrary $g \in H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *such that* $g_1 - g, g_2 + g \in \tilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$, *and*
solve the system $M_{NN,m}\varphi' = \psi'$ *with*

$$\psi' = - \begin{pmatrix} P_{-1/2}^- \ell^o g \\ P_{-1/2}^+ \ell^o g \\ \vdots \\ P_{-1/2}^{(m-2)} \ell^o g \\ P_{-1/2}^{(m-1)} \ell^o g \end{pmatrix} - \frac{1}{2} \begin{pmatrix} P_{-1/2}^- \ell_0(g_1 - g) \\ 0 \\ \vdots \\ 0 \\ (-1)^{m-1} P_{-1/2}^{(m-1)} \mathscr{J}^m \ell_0(g_2 + g) \end{pmatrix} \in \mathscr{J}_{\#}H_+^{-1/2+\varepsilon}(\mathbb{R})^{m-1}.$$

Then $\varphi = \varphi' + \mathscr{J}_{\#}\ell_0(g, -g, g, ..., (-1)^m g)^T$ *yields the solution of* (65).

Proof. The proof is analogous to the proof of Proposition 4.5. We rewrite the system (65) as follows,

$$\begin{pmatrix} P_{-1/2}^- & 2I & P_{-1/2}^- & 0 & \cdots & 0 & 0 \\ 0 & P_{-1/2}^+ & 2I & P_{-1/2}^+ & & \vdots & \vdots \\ 0 & 0 & P_{-1/2}^- & 2I & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & P_{-1/2}^{(m-2)} & 0 \\ 0 & 0 & \cdots & 0 & P_{-1/2}^{(m-1)} & 2I & P_{-1/2}^{(m-1)} \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{m-1} \\ \varphi_m \end{pmatrix} = 0 \qquad (71)$$

with $\varphi_0 = \ell_0 g_1$ and $\varphi_m = (-1)^{m+1} \mathscr{J}^m \ell_0 g_2$ being given (see (66)) and $\varphi_j = \mathscr{J}^j \ell_0 f_j$ ($j = 1, ..., m-1$) to be determined. We replace each φ_j by φ'_j by stipulating

$$\varphi_j = \varphi'_j + (-1)^j \mathscr{J}^j \ell_0 g, \qquad j = 0, ..., m.$$

The compatibility conditions on $\varphi_j = \mathscr{J}^j \ell_0 f_j$ (see (46)) amount to requiring that $\varphi'_j \in \mathscr{J}^j H_+^{1/2+\varepsilon}(\mathbb{R})$. Indeed, we have $f_j - (-1)^j g \in \tilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$. The substitution now

turns the homogeneous system (71) into one with the right hand side given by a vector $\psi = (\psi_j)_{j=1}^{m-1}$, where

$$\psi_j = -2((-1)^j \mathscr{J}^j + (-1)^{j-1} P_{-1/2}^{(j)} \mathscr{J}^{j-1}) \ell_0 g$$

$$= -2(-1)^j P_{-1/2}^{(j)} (\mathscr{J}^j - \mathscr{J}^{j-1}) \ell_0 g = -2 P_{1/2}^{(j)} \ell^o g.$$

Here we used that $P_{-1/2}^{(j)} \mathscr{J}^j \ell_0 g = \mathscr{J}^j \ell_0 g$; see (53). We divide the new system by 2 and make it into a square system by moving $\varphi_0' = \ell_0(g_1 - g)$ and $\varphi_m' = (-1)^{m+1} \mathscr{J}^m \ell_0 (g_2 + g)$ to the right hand side. Then we obtain $M_{NN,m} \varphi' = \psi'$ as desired. $\square$

If $g_1, g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, then we can choose $g = 0$ and the new system $M_{NN,m} \varphi' = \psi'$ is identical with the original system (65). In general, for $\varepsilon = 0$, if $g_1, g_2 \in H^{1/2}(\mathbb{R}_+)$ and $g_1 + g_2 \in \widetilde{H}^{1/2}(\mathbb{R}_+)$, the "canonical" choice in the NN case might be $g = (g_1 - g_2)/2$. Other possibilities are $g = g_1$ or $g = -g_2$.

4.3. The DN case

Let us finally state the corresponding results for the DN case. The proof is very similar to the DD case (due to a similar ansatz) and is omitted.

PROPOSITION 4.10. *Let $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ with $\varepsilon \in]-1/2, 1/2[$ be given. A function u belongs to $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ and solves the DN problem (32) iff u is represented by (35)-(36) such that $f_0, ..., f_m \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ satisfy the compatibility conditions (37), and $f_0 = g_1$, and φ defined by (55) satisfies the system*

$$M_{DN,m} \varphi = \begin{pmatrix} I & \frac{1}{2}P_{1/2}^- & 0 & 0 & \cdots & 0 \\ \frac{1}{2}P_{1/2}^+ & I & \frac{1}{2}P_{1/2}^+ & 0 & & \vdots \\ 0 & \frac{1}{2}P_{1/2}^- & I & \frac{1}{2}P_{1/2}^- & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & 0 & \frac{1}{2}P_{1/2}^{(m-1)} & I & \frac{1}{2}P_{1/2}^{(m-1)} \\ 0 & \cdots & 0 & 0 & P_{1/2}^{(m)} & I \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_{m-2} \\ \varphi_{m-1} \\ \varphi_m \end{pmatrix} = \begin{pmatrix} g_1^* \\ 0 \\ \vdots \\ 0 \\ 0 \\ g_2^* \end{pmatrix} \quad (72)$$

with

$$g_1^* = -\frac{1}{2} P_{1/2}^- \ell_0 g_1 = -\frac{1}{2} A_{t_-^{-1/2}} \ell_0 r_- A_{t_-^{1/2}} \ell_0 g_1,$$

$$g_2^* = -\mathscr{J}^m W_t^{-1} \ell g_2 = \mathscr{J}^m A_{t_+^{-1/2}} \ell_0 r_+ A_{t_-^{-1/2}} \ell g_2. \quad (73)$$

REMARK 4.11. The system (72) looks almost like the system (56) obtained in the DD case. The difference is in the last line where the factor $\frac{1}{2}$ in front of $P_{1/2}^{(m)}$ does not occur. Also the size of the system is $m \times m$.

There is an important observation about the formulas in Proposition 4.10. Looking at (73) we see that

$$g_2^* \in \mathscr{J}^m H_+^{1/2+\varepsilon}(\mathbb{R}) \subset H^{1/2+\varepsilon}, \tag{74}$$

whilst g_1^* is not necessarily in $H^{1/2+\varepsilon}$. (Indeed, $g_1 \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ but not necessarily $g_1 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$).

Looking at (73), $g_2 \mapsto g_2^* \ (H^{-1/2+\varepsilon}(\mathbb{R}_+) \to H_{\mp}^{1/2+\varepsilon})$ is just a substitution, evidently invertible. The other formula $g_1 \mapsto g_1^*$ contains essentially a Hankel operator, which is not boundedly invertible.

REMARK 4.12. Certainly there are various alternative ways to obtain an equivalent system such as (72). Instead of using SOPs of type DD, DD, ..., DD, DN on the leaves $\Lambda_1, \Lambda_2, \ldots$, one can use SOPs of type NN, NN, ... NN, ND or DN, ND, DN, ... or just use the Rawlins factorization (see (25)) in the very last case. However, the three basic BVPs (DD, NN, DN) in CRSs $\Omega_{0,m\pi}$ are as different as the problems in the slit plane $\Omega_{0,2\pi}$, here reflected by the term $P_{1/2}^{(m)}$ in (72).

PROPOSITION 4.13. *Let $\varepsilon \in]-1/2, 1/2[$ and assume that the operator*

$$M_{DN,m} : \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R})^m \to \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R})^m$$

is invertible. Then the DN problem has for each $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ a unique solution, which can be obtained as follows:
Choose an arbitrary $g \in H^{1/2+\varepsilon}(\mathbb{R}_+)$ such that $g_1 - g \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, and solve the system $M_{DN,m}\varphi' = \psi'$ with

$$\psi' = -\begin{pmatrix} P_{1/2}^- \ell^e g \\ P_{1/2}^+ \ell^e g \\ \vdots \\ P_{1/2}^{(m-1)} \ell^e g \\ P_{1/2}^{(m)} \ell^e g \end{pmatrix} + \begin{pmatrix} -\frac{1}{2} P_{1/2}^- \ell_0 (g_1 - g) \\ 0 \\ \vdots \\ 0 \\ g_2^*. \end{pmatrix}$$

Then $\varphi = \varphi' + \mathscr{J}_\# \ell_0 (g, ..., g)^T$ yields the solution of (72).

Proof. We can rewrite the system as follows,

$$\begin{pmatrix} P_{1/2}^- & 2I & P_{1/2}^- & 0 & \cdots & 0 & 0 \\ 0 & P_{1/2}^+ & 2I & P_{1/2}^+ & & \vdots & \vdots \\ 0 & 0 & P_{1/2}^- & 2I & \ddots & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & P_{1/2}^{(m-2)} & 0 \\ 0 & 0 & \cdots & 0 & P_{1/2}^{(m-1)} & 2I & P_{1/2}^{(m-1)} \\ 0 & 0 & \cdots & 0 & 0 & 2P_{1/2}^{(m)} & 2I \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \vdots \\ \varphi_{m-2} \\ \varphi_{m-1} \\ \varphi_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 2g_2^* \end{pmatrix} \tag{75}$$

with $\varphi_0 = \ell_0 g_1$ and g_2^* being given and $\varphi_1, ..., \varphi_m$ to be determined. We make the same substitution as in the DD case,

$$\varphi_j = \varphi_j' + \mathscr{J}^j \ell_0 g, \qquad j = 0, ..., m.$$

The compatibility conditions now amount to require that $\varphi_j' \in \mathscr{J}^j H_+^{1/2+\varepsilon}(\mathbb{R}_+)$. Since $g - g_1 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$ we have $\varphi_j' \in \mathscr{J}^j H_+^{1/2+\varepsilon}(\mathbb{R})$. The substitution makes the system (75) to one where to the right hand side the vector $\psi = (\psi_j)_{j=1}^m$ with $\psi_j = -2P_{1/2}^{(j)} \ell^e g$ is added. The computation is the same as in the DD case and also the last row give the corresponding entry. Now divide by $1/2$ and move $\phi_0' = \ell_0(g_1 - g)$ to the right hand side. $\square$

The canonical choice for g in the previous proposition is certainly $g = g_1$. The certain "asymmetry" with respect to the Dirichlet and Neumann data, which appears in the formulas (72) and (75) is tied to the ansatz which we chose. As pointed out above, we could have proceeded differently, and obtained different formulas, which of course in the end should give the same results.

4.4. Alternative approaches

In view of the possible representation of $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ by SOPs on the leaves $\Lambda_1, \Lambda_2, ...$ one can think (third approach) of representing u only in terms of data on $\Sigma_0, \Sigma_2, \Sigma_4, ...$ to achieve a shorter system than (56), (65) or (72), respectively. Essentially, this will be done in the proof of Theorem 6.4 in Section 6 for the DD problem, and similarly, for the NN problem in Section 7, and for the DN problem in Section 8.

5. On the norm of the projections $P_s^\pm$

In order to prepare the following sections, in which we want to solve the systems

$$M_{DD,m}\varphi = \psi, \qquad M_{NN,m}\varphi = \psi, \qquad M_{DN,m}\varphi = \psi,$$

respectively, or more precisely, the systems where φ is replaced by φ' and ψ is replaced by ψ' (see Propositions 4.5, 4.9 and 4.13), we need to estimate the norm of the projections $P_s^\pm : H^{s+\varepsilon} \to H^{s+\varepsilon}$. It turns out that we need these estimates only in the cases $s = \pm 1/2$.

We start with the following basic result.

LEMMA 5.1. *Let $|s| < 1/2$. Then the operators $\ell_0 r_+$ and $\ell_0 r_-$ are well defined and bounded projection operators acting on H^s. Their operator norm is at most $1/\cos(\pi s)$.*

In this lemma we consider the Sobolev spaces $H^s = H^s(\mathbb{R})$ with the usual inner product

$$\langle \phi, \psi \rangle_{H^s} = \int_{\mathbb{R}} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} (1 + \xi^2)^s \, d\xi$$

and the corresponding norm.

Proof. Without loss of generality let us focus on $\ell_0 r_+$. We can deduce the result for $\ell_0 r_-$ by applying the operator $\mathscr{J}$ from both sides. Consider the functions $t_\pm^s(\xi) = (\xi \pm i)^s$ defined in (17) with the concrete wave number $k = i$. Then the convolution operators $A_{t_\pm^s}$ are isometries between suitable pairs of Sobolev spaces. In particular, the boundedness of $\ell_0 r_+$ on H^s is equivalent to the boundedness of

$$B = A_{t_+^s} \ell_0 r_+ A_{t_-^{-s}} : L^2(\mathbb{R}) \to L^2(\mathbb{R}),$$

and the norms are the same. Observe that B is (the extension of) the inverse of a Wiener-Hopf operator,

$$B = \ell_0 (W_\phi)^{-1} r_+ : L^2(\mathbb{R}) \to L^2(\mathbb{R})$$

where $W_\phi = r_+ A_{t_+^s t_-^{-s}} \ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, provided that W_ϕ is (boundedly) invertible. Indeed, in this case standard Wiener-Hopf theory tells us that $(W_\phi)^{-1} = r_+ A_{t_+^s} \ell_0 r_+ A_{t_-^{-s}} \ell_0$. The symbol of the Wiener-Hopf operator is the function $\phi(\xi) = ((\xi - i)/(\xi + i))^s$, $\xi \in \mathbb{R}$, whose image is a subarc of the unit circle connecting 1 and $e^{-2\pi i s}$. Thus, after a rotation, we can decompose

$$e^{\pi i s} W_\phi = W_1 + i W_2$$

where W_1, W_2 are self-adjoint bounded linear Wiener-Hopf operators, and $W_1 \geqslant \cos(\pi s) I$. This implies that W_ϕ is invertible and that $\|W^{-1}\| \leqslant 1/\cos(\pi s)$. Thus $\|\ell_0 r_+\|_{\mathscr{L}(H^s)} = \|B\|_{\mathscr{L}(L^2)} \leqslant 1/\cos(\pi s)$. $\quad\square$

Now we are going to estimate the norms of the operators

$$P_s^+ = A_{t_+^{-s}} \ell_0 r_+ A_{t_+^s} : H^{s+\varepsilon} \to H^{s+\varepsilon}$$
$$P_s^- = A_{t_-^{-s}} \ell_0 r_- A_{t_-^s} : H^{s+\varepsilon} \to H^{s+\varepsilon}$$

where $t_\pm^s(\xi) = (\xi \pm k)^s$, $\xi \in \mathbb{R}$ (see (17)), depend on the underlying wave number k ($\mathrm{Im}(k) > 0$). We will not estimate the norm of these operators with respect to the standard norm in $H^{s+\varepsilon}$ but with respect to a convenient equivalent norm.

LEMMA 5.2. *Let $k = k_x + i k_y$, $k_y > 0$, be the wave number, let $|\varepsilon| < 1/2$ and $s \in \mathbb{R}$. Introduce in $H^{s+\varepsilon}$ the inner product*

$$\langle \phi, \psi \rangle_{k,s,\varepsilon} = \int_{\mathbb{R}} \hat{\phi}(\xi) \overline{\hat{\psi}(\xi)} (1 + \xi^2)^\varepsilon (|k|^2 + \xi^2)^s \, d\xi$$

and the corresponding norm. Then, with respect to this norm, the operator norm of $P_s^\pm$ is bounded by

$$\|P_s^\pm\|_{\mathscr{L}(H^{s+\varepsilon})} \leqslant \frac{1}{\cos(\pi\varepsilon)} \left(\frac{|k| + |k_x|}{|k| - |k_x|} \right)^{|s|/2}. \tag{76}$$

Proof. The operators $P_s^{\pm}$ are the product of three operators, where the factors in the middle $\ell_0 r_+$ and $\ell_0 r_-$ will be considered on H^ε equipped with the usual Sobolev norm. By Lemma 5.1 (with ε taking the place of s) we obtain $1/\cos(\pi\varepsilon)$ as bound of the operator norm of $\ell_0 r_\pm : H^\varepsilon \to H^\varepsilon$. It remains to verify that

$$\|A_{t_\pm^s}\|_{\mathscr{L}(H^{s+\varepsilon}, H^\varepsilon)} \cdot \|A_{t_\pm^{-s}}\|_{\mathscr{L}(H^\varepsilon, H^{s+\varepsilon})} \leqslant \left(\frac{|k| + |k_x|}{|k| - |k_x|} \right)^{|s|/2}.$$

This is rather straightforward. Indeed, for $\phi \in H^{s+\varepsilon}$ we get

$$\begin{aligned}
\|A_{t_\pm^s}\phi\|_{H^\varepsilon}^2 &= \int_{\mathbb{R}} |\xi \pm k|^{2s} |\hat{\phi}(\xi)|^2 (1 + \xi^2)^\varepsilon \, d\xi \\
&\leqslant \sup_{\xi \in \mathbb{R}} \frac{|\xi \pm k|^{2s}}{(|k|^2 + \xi^2)^s} \cdot \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 (1 + \xi^2)^\varepsilon (|k|^2 + \xi^2)^s \, d\xi \\
&= \max \left\{ \left(1 + \frac{|k_x|}{|k|} \right)^s, \left(1 - \frac{|k_x|}{|k|} \right)^s \right\} \cdot \|\phi\|_{H^{s+\varepsilon}}^2
\end{aligned}$$

and

$$\begin{aligned}
\|A_{t_\pm^{-s}}\phi\|_{H^{s+\varepsilon}}^2 &= \int_{\mathbb{R}} |\xi \pm k|^{-2s} |\hat{\phi}(\xi)|^2 (1 + \xi^2)^\varepsilon (|k|^2 + \xi^2)^s \, d\xi \\
&\leqslant \sup_{\xi \in \mathbb{R}} \frac{(|k|^2 + \xi^2)^s}{|\xi \pm k|^{2s}} \cdot \int_{\mathbb{R}} |\hat{\phi}(\xi)|^2 (1 + \xi^2)^\varepsilon \, d\xi \\
&= \max \left\{ \left(1 + \frac{|k_x|}{|k|} \right)^{-s}, \left(1 - \frac{|k_x|}{|k|} \right)^{-s} \right\} \cdot \|\phi\|_{H^\varepsilon}^2.
\end{aligned}$$

We omit the details of the computation of the suprema. Distinguishing between $s \geqslant 0$ and $s \leqslant 0$, we obtain (76). $\quad\square$

The spectral radius $\rho(T)$ of an operator T defined on Banach space X is the maximum of $|\lambda|$, where $\lambda \in \mathbb{C}$ is taken from the spectrum of T, i.e., the set of all λ for which $\lambda I - T$ is not (boundedly) invertible on X. The spectral radius of a bounded linear operator does not exceed the operator norm of T,

$$\rho(T) \leqslant \|T\|_{\mathscr{L}(X)}. \tag{77}$$

If we change the norm in X to an equivalent norm, then also the operator norm may change. Estimate (77) holds for any (equivalent) norm, while by definition the spectral radius does not depend on the norm. Notice also that if $\rho(T) < 1$, then $I - T$ is invertible and

$$(I - T)^{-1} = I + \sum_{\kappa=1}^{\infty} T^\kappa \tag{78}$$

converges in (any) operator norm.

For a given wave number k of the HE, recall the definition of the constant $c(k)$ given in (26). Notice that always $c(k) \geqslant 1$ and that equality holds for k being purely imaginary.

From the norm estimate of the previous lemma we immediately obtain the following result. Notice that the above remarks about equivalent norms show that we have to pass to equivalent norms only temporarily in the proof.

PROPOSITION 5.3. *For $s = \pm 1/2$, wave number k, and $|\varepsilon| < 1/2$, the spectral radii of $P_s^{\pm} \mathscr{J}$ and $P_s^{\pm} P_s^{\mp}$, acting on $H^{s+\varepsilon}$, are bounded as follows:*

$$\rho(P_s^{\pm} \mathscr{J}) \leqslant \frac{c(k)}{\cos(\pi\varepsilon)}, \qquad \rho(P_s^{\pm} P_s^{\mp}) \leqslant \frac{c(k)^2}{\cos^2(\pi\varepsilon)}.$$

In view of (78) we want to have the estimates of $P_{\pm 1/2}^{\pm}$ as small as possible. Therefore the "optimal" case $c(k) = 1$ is obtained only for purely imaginary wave numbers (i.e., $k_x = 0$). All other wave numbers yield more restrictive results. We think that this is due to our approach.

Let us also remark that for $\varepsilon = 0$ and for purely imaginary wave numbers k, the previous lemma shows that $P_s^{\pm}$ have norm one. Since they are projections, it follows that they are orthogonal projections.

6. Solution of the DD problem in $\Omega_{0,m\pi}$ by series expansion

We are going to solve the DD problem in $\Omega_{0,m\pi}$. The solution can be represented via a series expansion involving the operators $P^{\pm} = P_{1/2}^{\pm}$. We first consider the special cases $m \leqslant 4$ in detail and later establish the general result.

For $m = 1$ nothing needs to be done (see (12) and also Prop. 3.1). In the case $m = 2$ we know that the solution is given by the SOPs (see (22)). Based on the approach of the previous sections we obtain the same result, which we present for sake of illustration. In what follows we will frequently use

$$g^e = \tfrac{1}{2}(g_1 + g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+), \qquad g^o = \tfrac{1}{2}(g_1 - g_2) \in \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+), \qquad (79)$$

where $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_{\sim}$ are the given Dirichlet data.

PROPOSITION 6.1. *Let $\varepsilon \in]-1/2, 1/2[$. The DD problem in $\Omega_{0,2\pi}$ admits a unique solution given by*

$$\varphi_1 = \mathscr{J} \ell_0 g^e - P^- \ell^e g^e, \qquad (80)$$

(in connection with (35), (36), (55)) in terms of the LIPs given by

$$u_0^+ = \varphi_0 + \varphi_1 = (I - P^-)\ell^e g^e + \ell_0 g^o, \qquad (81)$$

$$u_0^- = \varphi_2 + \varphi_1 = (I - P^-)\ell^e g^e - \ell_0 g^o. \qquad (82)$$

Proof. In the case $m = 2$ the system (56) reduces to $M_{DD,2} = I$ and thus $\varphi_0 = \ell_0 g_1$, $\varphi_2 = \ell_0 g_2$, and

$$\varphi_1 = g_1^* + g_2^* = -\tfrac{1}{2} P^- (\ell_0 g_1 + \ell_0 g_2),$$

or, if we use Proposition 4.5 with $g = g^e$,

$$\varphi_1 = \mathscr{J} \ell_0 g^e - P^- \ell^e g^e - \tfrac{1}{2} P^- (\ell_0(g_1 - g^e) - \ell_0(g_2 - g^e))) = \mathscr{J} \ell_0 g^e - P^- \ell^e g^e.$$

Translating this into LIPs we obtain, after some computation, (81) and (82). The uniqueness follows from Prop. 4.5 since $M_{DD,2} = I$. □

The formulas (81) and (82) coincide with those of the Sommerfeld DD potentials (21). Notice that one can replace the even extension operator ℓ^e therein by an arbitrary extension (see (53)). We do not have any restriction on $\varepsilon \in]-1/2, 1/2[$ or on the wave number.

In what follows we are going to state the results for $m = 3$ and $m = 4$ as an illustration. Here use Prop. 5.3 in order to show that under appropriate conditions we have the following convergent series:

$$\left(I - \mu \mathscr{J} P^{\pm}\right)^{-1} = \sum_{\kappa=0}^{\infty} \left(\mu \mathscr{J} P^{\pm}\right)^{\kappa}$$

and

$$\left(I - \mu^2 P^{\pm} P^{\mp}\right)^{-1} = \sum_{\kappa=0}^{\infty} \left(\mu^2 P^{\pm} P^{\mp}\right)^{\kappa}.$$

Proposition 6.2. *Let* $m = 3$, $k \in \mathbb{C}$, $\text{Im}(k) > 0$, *and* $\varepsilon \in]-1/3, 1/3[$ *be such that*

$$\frac{c(k)}{\cos(\pi\varepsilon)} < 2. \tag{83}$$

Then the system (56) *is uniquely solvable by the following formulas*

$$\varphi_1 = \mathscr{J} \ell_0 g^e - P^- \left(I + \tfrac{1}{2} \mathscr{J} P^-\right)^{-1} \ell^e g^e - \tfrac{1}{2} P^- \left(I - \tfrac{1}{2} \mathscr{J} P^-\right)^{-1} \ell_0 g^o,$$
$$\varphi_2 = \ell_0 g^e - P^+ \left(I + \tfrac{1}{2} \mathscr{J} P^+\right)^{-1} \ell^e g^e + \tfrac{1}{2} P^+ \left(I - \tfrac{1}{2} \mathscr{J} P^+\right)^{-1} \ell_0 g^o, \tag{84}$$

(in connection with (35), (36), (55)) *where* g^e, g^o *are given by* (79). *In case* $g_1, g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$, *the solutions are also given by*

$$\varphi_1 = \left(I - \tfrac{1}{4} P^- P^+\right)^{-1} \left(\tfrac{1}{4} P^- P^+ \mathscr{J} \ell_0 g_2 - \tfrac{1}{2} P^- \ell_0 g_1\right),$$
$$\varphi_2 = \left(I - \tfrac{1}{4} P^+ P^-\right)^{-1} \left(\tfrac{1}{4} P^+ P^- \ell_0 g_1 - \tfrac{1}{2} P^+ \mathscr{J} \ell_0 g_2\right). \tag{85}$$

Proof. For $m = 3$ the system (56) reads as

$$\begin{pmatrix} I & \tfrac{1}{2} P^- \\ \tfrac{1}{2} P^+ & I \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} g_1^* \\ g_2^* \end{pmatrix} = - \begin{pmatrix} \tfrac{1}{2} P^- \ell_0 g_1 \\ \tfrac{1}{2} P^+ \mathscr{J} \ell_0 g_2 \end{pmatrix}. \tag{86}$$

Elimination of φ_1, φ_2 yields (85) in case $g_1, g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R})$. In general, we can apply Proposition 4.5 with $g = g^e$ and obtain

$$\begin{pmatrix} I & \tfrac{1}{2} P^- \\ \tfrac{1}{2} P^+ & I \end{pmatrix} \begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} = \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} -P^- \ell^e g^e - \tfrac{1}{2} P^- \ell_0 g^o \\ -P^+ \ell^e g^e + \tfrac{1}{2} P^+ \mathscr{J} \ell_0 g^o \end{pmatrix},$$

with $\varphi_j = \varphi'_j + \mathscr{J}^j \ell_0 g^e$. Elimination of φ'_1, φ'_2 yields

$$(I - \tfrac{1}{4}P^- P^+)\varphi'_1 = \psi'_1 - \tfrac{1}{2}P^- \psi'_2 = -P^-(I - \tfrac{1}{2}\mathscr{J}P^-)\ell^e g^e - \tfrac{1}{2}P^-(I + \tfrac{1}{2}\mathscr{J}P^-)\ell_0 g^o,$$

$$(I - \tfrac{1}{4}P^+ P^-)\varphi'_2 = \psi'_2 - \tfrac{1}{2}P^+ \psi'_1 = -P^+(I - \tfrac{1}{2}\mathscr{J}P^+)\ell^e g^e + \tfrac{1}{2}P^+(I + \tfrac{1}{2}\mathscr{J}P^+)\ell_0 g^o.$$

As $P^- = \mathscr{J}P^+ \mathscr{J}$ we obtain formulas such as

$$\left(I - \tfrac{1}{4}P^- P^+\right)^{-1} = \left(I - \tfrac{1}{4}P^- \mathscr{J}P^- \mathscr{J}\right)^{-1} = \left(I - \tfrac{1}{2}P^- \mathscr{J}\right)^{-1}\left(I + \tfrac{1}{2}P^- \mathscr{J}\right)^{-1},$$

where the inverses can be written as a convergent series because of Prop. 5.3 and the assumption (83). In this way we arrive at (84). $\square$

We remark that the LIPs in the case $m = 3$ read as follows:

$$u^+_{00} = \varphi_0 + \varphi_1 = \ell^e g^e + \ell_0 g^o - P^-(I + \tfrac{1}{2}\mathscr{J}P^-)^{-1}\ell^e g^e - \tfrac{1}{2}P^-(I - \tfrac{1}{2}\mathscr{J}P^-)^{-1}\ell_0 g^o,$$

$$u^-_{00} = \varphi_1 + \varphi_2 = \ell^e g^e - (I + \mathscr{J})(I + \tfrac{1}{2}\mathscr{J}P^-)^{-1}\ell^e g^e$$
$$\qquad - \tfrac{1}{2}P^-(I - \tfrac{1}{2}\mathscr{J}P^-)^{-1}\ell_0 g^o + \tfrac{1}{2}P^+(I - \tfrac{1}{2}\mathscr{J}P^+)^{-1}\ell_0 g^o,$$

$$u^+_{01} = \varphi_2 + \varphi_3 = \ell^e g^e - \ell_0 g^o - P^+(I + \tfrac{1}{2}\mathscr{J}P^+)^{-1}\ell^e g^e - \tfrac{1}{2}P^+(I - \mathscr{J}\tfrac{1}{2}P^+)^{-1}\ell_0 g^o.$$

For the case $m = 4$, the corresponding result is as follows.

PROPOSITION 6.3. *Let* $m = 4$, $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$, *and* $\varepsilon \in\,]-1/2, 1/2[$ *be such that*

$$\frac{c(k)}{\cos(\pi\varepsilon)} < \sqrt{2}. \tag{87}$$

Then the system (56) *is uniquely solvable by the following formulas*

$$\varphi_1 = -\ell_0 g^e + \left(I - \tfrac{1}{2}P^- P^+\right)^{-1}(I - P^-)\ell^e g^e - \tfrac{1}{2}P^- \ell_0 g^o,$$
$$\varphi_2 = \ell_0 g^e - \left(I - \tfrac{1}{2}P^+ P^-\right)^{-1}P^+(I - P^-)\ell^e g^e, \tag{88}$$
$$\varphi_3 = -\ell_0 g^e + \left(I - \tfrac{1}{2}P^- P^+\right)^{-1}(I - P^-)\ell^e g^e + \tfrac{1}{2}P^- \ell_0 g^o$$

(in connection with (35)*,* (36)*,* (55)*). If* $g_1, g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$ *the solution is also given by*

$$\varphi_1 = -\tfrac{1}{2}P^- \ell_0 g_1 - \tfrac{1}{4}P^- \sum_{\kappa=1}^{\infty} \left(\tfrac{1}{2}P^+ P^-\right)^{\kappa} \ell_0(g_1 + g_2)$$

$$\varphi_2 = \tfrac{1}{2}\sum_{\kappa=1}^{\infty} \left(\tfrac{1}{2}P^+ P^-\right)^{\kappa} \ell_0(g_1 + g_2) \tag{89}$$

$$\varphi_3 = -\tfrac{1}{2}P^- \ell_0 g_2 - \tfrac{1}{4}P^- \sum_{\kappa=1}^{\infty} \left(\tfrac{1}{2}P^+ P^-\right)^{\kappa} \ell_0(g_1 + g_2).$$

Proof. The system (56) for $m = 4$ reads

$$\begin{pmatrix} I & \frac{1}{2}P^- & 0 \\ \frac{1}{2}P^+ & I & \frac{1}{2}P^+ \\ 0 & \frac{1}{2}P^- & I \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} g_1^* \\ 0 \\ g_2^* \end{pmatrix} = -\begin{pmatrix} \frac{1}{2}P^-\ell_0 g_1 \\ 0 \\ \frac{1}{2}P^-\ell_0 g_2 \end{pmatrix}. \tag{90}$$

Substituting φ_1 from the first and φ_3 from the third equation into the second one we get

$$\left(I - \tfrac{1}{4}P^+P^-\right)\varphi_2 = -\tfrac{1}{2}P^+(g_1^* + g_2^*) = \tfrac{1}{4}P^+P^-\ell_0(g_1 + g_2) \tag{91}$$

which gives the second line of (89). The other two result easily from the first and third line of (90), respectively.

In the general situation we have to use Proposition 4.5 and obtain a similar system with $\varphi_j = \varphi_j' + \mathscr{J}^j \ell_0 g^e$ and the right hand side

$$\psi_1' = -P^-\ell^e g^e - \tfrac{1}{2}P^-\ell_0 g^o,$$
$$\psi_2' = -P^+\ell^e g^e,$$
$$\psi_3' = -P^-\ell^e g^e + \tfrac{1}{2}P^-\ell_0 g^o.$$

Solving for φ_j' yields

$$\varphi_2' = \left(I - \tfrac{1}{4}P^+P^-\right)^{-1}\left(\psi_2' - \tfrac{1}{2}P^+(\psi_1' + \psi_3')\right)$$

and

$$\varphi_1' = \psi_1' - \tfrac{1}{2}P^- \varphi_2', \qquad \varphi_3' = \psi_3' - \tfrac{1}{2}P^- \varphi_2',$$

and thus, after simplification, (88). $\square$

The corresponding LIPs in case $m = 4$ can be evaluated as follows:

$$u_{00}^+ = \varphi_0 + \varphi_1 = \left(I - \tfrac{1}{4}P^-P^+\right)^{-1}(I - P^-)\ell^e g^e + (I - \tfrac{1}{2}P^-)\ell_0 g^o,$$
$$u_{00}^- = \varphi_1 + \varphi_2 = (I - P^+)\left(I - \tfrac{1}{4}P^-P^+\right)^{-1}(I - P^-)\ell^e g^e - \tfrac{1}{2}P^-\ell_0 g^o,$$
$$u_{01}^+ = \varphi_2 + \varphi_3 = (I - P^+)\left(I - \tfrac{1}{4}P^-P^+\right)^{-1}(I - P^-)\ell^e g^e + \tfrac{1}{2}P^-\ell_0 g^o,$$
$$u_{01}^- = \varphi_3 + \varphi_4 = \left(I - \tfrac{1}{4}P^-P^+\right)^{-1}(I - P^-)\ell^e g^e - (I - \tfrac{1}{2}P^-)\ell_0 g^o.$$

We now consider the general case and prove that the DD problem has always a solution, which can be obtained by a series representation.

THEOREM 6.4. *Let* $m \geqslant 3$, $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$, *and* $\varepsilon \in\,]-1/m, 1/m[$ *be such that*

$$\frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)}. \tag{92}$$

Then the operator

$$M_{DD,m} : \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R}_+)^{m-1} \to \mathscr{J}_\# H_+^{1/2+\varepsilon}(\mathbb{R}_+)^{m-1}$$

is invertible. Consequently, the DD problem in $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ *admits a unique solution in terms of a series representation.*

Proof. We identify the spaces $\ell_0 \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) = H_+^{1/2+\varepsilon}$ and $\mathscr{J} \ell_0 \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) = H_-^{1/2+\varepsilon}$. Consequently,

$$\mathscr{J}^{\#} \ell_0 \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^{m-1} = H_-^{1/2+\varepsilon} \dot{+} H_+^{1/2+\varepsilon} \dot{+} \ldots \dot{+} H_{\pm}^{1/2+\varepsilon},$$

which we abbreviate by X_{m-1}. The system $M_{DD,m} \varphi = \psi$ ($\varphi, \psi \in X_m$) can be written as

$$
\begin{pmatrix}
I & \frac{1}{2}P^- & 0 & \cdots & & 0 \\
\frac{1}{2}P^+ & I & \frac{1}{2}P^+ & & & \vdots \\
0 & \frac{1}{2}P^- & I & \ddots & & 0 \\
\vdots & & \ddots & \ddots & \frac{1}{2}P^{(m-2)} & \\
0 & \cdots & 0 & \frac{1}{2}P^{(m-1)} & I &
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\vdots \\
\varphi_{m-2} \\
\varphi_{m-1}
\end{pmatrix}
=
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\vdots \\
\psi_{m-2} \\
\psi_{m-1}
\end{pmatrix}.
$$

Looking at the odd rows we can eliminate $\varphi_1, \varphi_3, \varphi_5, \ldots$, and express them in terms of $\varphi_2, \varphi_4, \ldots$. This yields a system of size k where $k = (m-2)/2$ if m is even and $k = (m-1)/2$ if m is odd. The systems has the form

$$
\begin{pmatrix}
I-T & -\frac{1}{2}T & 0 & 0 & \ldots & 0 \\
-\frac{1}{2}T & I-T & -\frac{1}{2}T & 0 & \ldots & 0 \\
0 & -\frac{1}{2}T & I-T & -\frac{1}{2}T & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -\frac{1}{2}T & I-T & -\frac{1}{2}T \\
0 & \ldots & 0 & 0 & -\frac{1}{2}T & I-T
\end{pmatrix}
\begin{pmatrix}
\varphi_2 \\
\varphi_4 \\
\vdots \\
\varphi_{2k-2} \\
\varphi_{2k}
\end{pmatrix}
=
\begin{pmatrix}
\hat{\psi}_2 \\
\hat{\psi}_4 \\
\vdots \\
\hat{\psi}_{2k-2} \\
\hat{\psi}_{2k}
\end{pmatrix}
$$

in case m is even, and

$$
\begin{pmatrix}
I-T & -\frac{1}{2}T & 0 & 0 & \ldots & 0 \\
-\frac{1}{2}T & I-T & -\frac{1}{2}T & 0 & \ldots & 0 \\
0 & -\frac{1}{2}T & I-T & -\frac{1}{2}T & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & 0 & -\frac{1}{2}T & I-T & -\frac{1}{2}T \\
0 & \ldots & 0 & 0 & -\frac{1}{2}T & I-\frac{1}{2}T
\end{pmatrix}
\begin{pmatrix}
\varphi_2 \\
\varphi_4 \\
\vdots \\
\varphi_{2k-2} \\
\varphi_{2k}
\end{pmatrix}
=
\begin{pmatrix}
\hat{\psi}_2 \\
\hat{\psi}_4 \\
\vdots \\
\hat{\psi}_{2k-2} \\
\hat{\psi}_{2k}
\end{pmatrix}
$$

in case m is odd, where $T = \frac{1}{2}P^+ P^-$ and with a certain modified right hand side. Notice the difference in the lower-right entry of the block matrices in the even and odd case. The unique solvability of these systems is equivalent to the unique solvability of the original system. The block operator acts on the space $(H_+^{1/2+\varepsilon})^k$, the k-fold direct

sum of $H_+^{1/2+\varepsilon}$, and can be written as $I - T \otimes B_k$, where

$$B_k = \begin{pmatrix} 1 & \frac{1}{2} & 0 & \ldots & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \ldots & 0 & \frac{1}{2} & \sigma \end{pmatrix}$$

is a (scalar) $k \times k$ matrix and $\sigma = 1$ (m even) or $\sigma = \frac{1}{2}$ (m odd). The tensor product $T \otimes B_k$ is identified with a block operator in the usual way.

With the help of a standard ansatz, namely, $v = [(\xi_1)^j - (\xi_2)^j]_{j=1}^k$ for the eigenvectors, the eigenvalues of the symmetric matrix B_k can be determined. The eigenvalues are $1 + \cos(\frac{\pi j}{k+1})$ (m even) and $1 + \cos(\frac{2\pi j}{2k+1})$ (m odd) with $j = 1, ..., k$. The largest eigenvalue is attained for $j = 1$. Because B_k is symmetric, we obtain for the operator norm

$$\|B_k\| = 1 + \cos(\tfrac{2\pi}{m}) = 2\cos^2(\tfrac{\pi}{m})$$

in both cases (even/odd). It follows that the operator $I - T \otimes B_k$ is invertible and that the series

$$(I - T \otimes B_k)^{-1} = I + \sum_{j=1}^{\infty} T^j \otimes (B_k)^j$$

converges in operator norm if the spectral radius of T is less that $1/\|B_k\|$. As computed above, the spectral radius of T does not exceed $\frac{c(k)^2}{2\cos^2(\pi\varepsilon)}$. Because of (92) this is the case. $\square$

We remark that condition (92) amounts to the condition $|\varepsilon| < 1/m$ in the case when k is purely imaginary (i.e., $c(k) = 1$), whereas otherwise the condition is more restrictive. In the case $m = 2$ we already know that $|\varepsilon| < 1/2$ is a sufficient condition for the unique solvability of the DD problem irrespective of the wave number k. Therefore, it is reasonable to suspect that also for $m \geqslant 3$ only $|\varepsilon| < 1/m$ is necessary in order to guarantee the unique solvability of the DD problem.

7. The NN problem in $\Omega_{0,m\pi}$

The NN problem is similar to the DD problem, except that we make the assumption $\varepsilon \in [0, 1/2[$. The system (65) has the same structure as the system (56). The only difference is that the underlying spaces are different and that $P_{1/2}^{\pm}$ is replaced by $P_{-1/2}^{\pm}$. However, the estimates of the norms of these projections are the same. We notice that the system (65) makes also sense for $\varepsilon \in]-1/2, 1/2[$ (see Remark 4.7). The restriction to nonnegative ε is only due to the connection with the HE.

Due to the similarities with the DD case, we omit to discuss the cases of small m separately. We establish only the main result, whose proof is analogous.

THEOREM 7.1. *Let $m \geqslant 3$, $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$, and $\varepsilon \in]-1/m, 1/m[$ be such that*

$$\frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)}. \tag{93}$$

Then the operator

$$M_{NN,m} : \mathscr{J}_{\#} H_+^{-1/2+\varepsilon}(\mathbb{R}_+)^{m-1} \to \mathscr{J}_{\#} H_+^{-1/2+\varepsilon}(\mathbb{R}_+)^{m-1}$$

is invertible. Consequently, if in addition $\varepsilon \geqslant 0$, the NN problem in $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ admits a unique solution in terms of a series representation.

8. The DN problem in $\Omega_{0,m\pi}$

The DN problem in $\Omega_{0,m\pi}$ can be tackled, as well, resulting in a system which contains the same feature that is known from the last line of (72), namely the appearance of a coefficient 2. It allows similar conclusions by successive elimination and inversion of an operator via series expansion. However, we will also consider a different method for the DN problem as more adequate later in Section 10 (see Remark 10.3 and Example 10.7).

In this section we are going to treat the DN problem with the ideas of Proposition 6.2 and Proposition 6.3. The method via series expansion works for any $m \geqslant 2$, where as before, certain restrictions on ε and k may be necessary. Omitting the case $m = 2$, let us first look at the example $m = 3$. For sake of simplicity let us assume that g_1 belongs to the tilde-space which can be achieved by substitution as shown in Subsection 4.3.

PROPOSITION 8.1. *Let $m = 3$, $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$, $\varepsilon \in]-1/6, 1/6[$ and $\frac{c(k)}{\cos(\pi\varepsilon)} < \frac{2}{\sqrt{3}}$. Then the system (72) with $(g_1, g_2) \in \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R})$ is uniquely solvable by the following formulas*

$$\varphi_1 = g_1^* + \frac{1}{4} P^- \sum_{\kappa=0}^{\infty} \left(\tfrac{3}{4} P^+ P^-\right)^{\kappa} P^+ (g_1^* + g_2^*)$$

$$\varphi_2 = -\frac{1}{2} \sum_{\kappa=0}^{\infty} \left(\tfrac{3}{4} P^+ P^-\right)^{\kappa} P^+ (g_1^* + g_2^*) \tag{94}$$

$$\varphi_3 = g_2^* + \frac{1}{2} P^- \sum_{\kappa=0}^{\infty} \left(\tfrac{3}{4} P^+ P^-\right)^{\kappa} P^+ (g_1^* + g_2^*).$$

where g_1^*, g_2^* are given by (73) with $m = 3$.

Proof. The system (72) for $m = 3$ reads

$$\begin{pmatrix} I & \frac{1}{2}P^- & 0 \\ \frac{1}{2}P^+ & I & \frac{1}{2}P^+ \\ 0 & P^- & I \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \end{pmatrix} = \begin{pmatrix} g_1^* \\ 0 \\ g_2^* \end{pmatrix}. \tag{95}$$

The rest of the proof is similar to what was done in the proof of Proposition 6.2 solving (instead of (91))

$$(I - \tfrac{3}{4}P^+P^-)\varphi_2 = -\tfrac{1}{2}P^+(g_1^* + g_2^*) \qquad (96)$$

by a Neumann series. $\quad \square$

THEOREM 8.2. *Let $m \geqslant 2$, $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$, and $\varepsilon \in]-1/2m, 1/2m[$ be such that*

$$\frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\frac{\pi}{2m})}.$$

Then the operator

$$M_{DN} : \mathscr{J}_{\#}\ell_0 \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^m \to \mathscr{J}_{\#}\ell_0 \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^m$$

is invertible. Consequently, the DN problem in $\mathscr{H}^{1+\varepsilon}(\Omega_{0,m\pi})$ admits a unique solution in terms of a series representation.

Proof. We proceed as in the proof of Theorem 6.4, and eliminate $\varphi_1, \varphi_3, \dots$. We obtain a system of size $k \times k$, where $m = 2k$ if m is even and $m = 2k+1$ if m is odd. The underlying block operator looks like

$$\begin{pmatrix} I-T & -\frac{1}{2}T & 0 & 0 & \dots & 0 \\ -\frac{1}{2}T & I-T & -\frac{1}{2}T & 0 & \dots & 0 \\ 0 & -\frac{1}{2}T & I-T & -\frac{1}{2}T & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -\frac{1}{2}T & I-T & -\frac{1}{2}T \\ 0 & \dots & 0 & 0 & A & B \end{pmatrix}$$

with $T = \frac{1}{2}P^+P^-$ and the last row given by

$$(A,B) = \begin{cases} (-\frac{1}{2}T, I - \frac{3}{2}T) & \text{if } m \text{ even} \\ (-T, I-T) & \text{if } m \text{ odd.} \end{cases}$$

As before, T acts on $H_+^{1/2+\varepsilon}$. We can write this block operator in the form $I - T \otimes B_k$, where the $k \times k$ matrix is given by

$$B_k = \begin{pmatrix} 1 & \frac{1}{2} & 0 & \dots & 0 \\ \frac{1}{2} & 1 & \frac{1}{2} & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & \dots & 0 & a & b \end{pmatrix}$$

with $(a,b) = (\frac{1}{2}, \frac{3}{2})$ for m even and $(a,b) = (1,1)$ for m odd. The eigenvalues of B_k are $1 + \cos(\frac{\pi(2j+1)}{2k+1})$ (m even) and $1 + \cos(\frac{\pi(2j+1)}{2k})$ (m odd) with $j = 0, 1, \dots, k-1$.

Hence in both cases (even/odd) the largest eigenvalue is $1 + \cos(\frac{\pi}{m})$. While in the even case, B_k is symmetric and thus the operator norm equals the largest eigenvalue, a slight modification is necessary in the odd case. Here we have to multiply B_k with the diagonal matrix $\mathrm{diag}(1, ..., 1, \sqrt{2})$ from the left and with the inverse diagonal matrix from the right to obtain a symmetric matrix $\hat{B}_k$. This similarity transform does not affect invertibility and series expansion. In any case we obtain that the norm of B_k or $\hat{B}_k$ is bounded by $1 + \cos(\frac{\pi}{m}) = 2\cos^2(\frac{\pi}{2m})$. Now we can finish the proof in the same way as in the DD case. $\square$

Note that the case $m = 4$ does not anymore reduce to a scalar equation as in Proposition 6.3. But certainly the case $m = 2$ can be considered, as well, resulting in a solution by series expansion instead of the famous Rawlins solution in closed analytical form (25) that was achieved by matrix factorization.

9. Periodic solutions

One can ask for (non-trivial) *periodic* solutions of the HE in $\mathscr{R}^2$ (the universal covering surface of $\mathbb{R}^2 \setminus \{0\}$) with angle of periodicity α. In other words we can look for solutions $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,\alpha})$ satisfying the *periodic boundary conditions*

$$T_{0,\Gamma_1}u = T_{0,\Gamma_2}u, \qquad T_{1,\Gamma_1}u = -T_{1,\Gamma_2}u. \tag{97}$$

Notice the convention on the orientation of the normal derivatives at Γ_1 and Γ_2. Indeed, any solution satisfying (97) can be extended across Γ_1 and Γ_2 onto all of $\mathscr{R}^2$ such that the solution is periodic in the angle $\arg(x)$.

With the help of Green's formula it is easy to see that nonzero solutions cannot exist for any $\alpha > 0$ whenever $\varepsilon \geqslant 0$ (see also Prop. 2.2 in Section 2).

On the other hand, for $\varepsilon < 0$ nonzero solution do exist for each $\alpha > 0$. Indeed, the radially symmetric functions

$$u(x_1, x_2) = H_0^{(1)}(kr), \qquad r = \sqrt{x_1^2 + x_2^2}, \tag{98}$$

are $H^{1+\varepsilon}$-solutions of the HE for each $\varepsilon < 0$. Here $H_0^{(1)}(z)$ denotes the Hankel function (or Bessel function of the third kind). Further linearly independent solutions can be constructed for $\alpha > 4\pi$ and $\varepsilon \in\ ]-1/2, -2\pi/\alpha[$. (This can be done by modifying the example given in Prop. 5.6 in [7].)

For $\alpha = 4\pi$ non-trivial periodic solutions can be regarded as "double space solutions" in the sense of Arnold Sommerfeld (see [18], [23]) who used such functions for the solution of the half-plane problem by series expansion.

THEOREM 9.1. *For $\alpha = 4\pi$ and $\varepsilon \in\ ]-1/2, 0[$, the only solutions to the periodic HE equation are multiples of (98).*

Proof. For $\varepsilon \in\ ]-1/2, 0[$ notice that $\widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+) = H^{1/2+\varepsilon}(\mathbb{R}_+)$ which means that we do not have to worry about the compatibility conditions.

If $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,4\pi})$ satisfies (97), it can be extended across Σ_4 to a function $u^* \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,6\pi})$ by taking the same values on Λ_3 as on Λ_1. Because of periodicity $\varphi_4 = \varphi_0$, $\varphi_5 = \varphi_1$, $\varphi_6 = \varphi_2$. The equations obtained in Prop. 4.1 then become

$$0 = \varphi_0 + \tfrac{1}{2}P^+(\varphi_1 + \varphi_3) = \varphi_2 + \tfrac{1}{2}P^+(\varphi_1 + \varphi_3),$$

$$0 = \varphi_1 + \tfrac{1}{2}P^-(\varphi_0 + \varphi_2) = \varphi_3 + \tfrac{1}{2}P^-(\varphi_0 + \varphi_2),$$

where $P^\pm = P^\pm_{1/2}$. This implies $\varphi_0 = \varphi_2$ and $\varphi_1 = \varphi_3$ and the equations

$$0 = \varphi_0 + P^+\varphi_1, \qquad 0 = \varphi_1 + P^-\varphi_0.$$

Thus $\varphi_0 = P^+P^-\varphi$. Using $\Pi^+ = I - P^-$ and multiplying with the projection P^+ yields

$$P^+\Pi^+\varphi_0 = 0.$$

Note that $\varphi \in H^{1/2+\varepsilon}_+(\mathbb{R})$. By definition,

$$0 = A_{t_+^{-1/2}} \ell_0 r_+ A_{t_+^{1/2}} A_{t_-^{-1/2}} \ell_0 r_+ A_{t_-^{1/2}} \varphi_0,$$

where we can drop the first term and substitute $\psi_0 = \ell_0 r_+ A_{t_-^{1/2}} \varphi_0 \in H^\varepsilon_+(\mathbb{R})$ to obtain

$$0 = \ell_0 r_+ A_{t_+^{1/2} t_-^{-1/2}} \psi_0.$$

This is a Wiener-Hopf equation, however, on $H^\varepsilon_+(\mathbb{R})$. It can be transformed to a Wiener-Hopf equation on $H^0_+(\mathbb{R}) \cong L^2(\mathbb{R}_+)$ with symbol

$$t_+^{1/2-\varepsilon}(\xi) t_-^{-1/2+\varepsilon}(\xi) = \left(\frac{\xi - k}{\xi + k}\right)^{-1/2+\varepsilon}.$$

This Wiener-Hopf operator has a one-dimensional kernel for $\varepsilon \in]-1,0[$, which can be determined via Fourier transformation. (For $\varepsilon = 0$, it is not normally solvable in $L^2(\mathbb{R}_+)$ but injective; for $\varepsilon \in]0,1[$ it is invertible, see [17].) The solutions are given by $\psi_0 = \lambda \cdot \widehat{t_+^{-1/2}} \in H^\varepsilon_+(\mathbb{R})$, $\lambda \in \mathbb{C}$. Hence we arrive at

$$\ell_0 r_+ A_{t_-^{1/2}} \varphi_0 = \lambda \cdot \widehat{t_+^{-1/2}}, \qquad \varphi_0 \in H^{1/2+\varepsilon}_+(\mathbb{R}),$$

and thus, by applying $\ell_0 r_+ A_{t_-^{-1/2}}$,

$$\varphi_0 = \lambda \cdot \ell_0 r_+ \left(\widehat{t_-^{-1/2} t_+^{-1/2}}\right).$$

This shows that the space of solutions to the problem is (at most) one-dimensional. Since we know that multiples of (98) are solutions, we obtain the assertion. $\square$

It is clear that the previous result also holds for angles $\alpha = 4\pi/n$, $n \in \mathbb{N}$, since non-trivial solutions for $\alpha = 4\pi/n$ give rise by periodic continuation to a solution for $\alpha = 4\pi$.

10. The doubling method

We show that the solution of a DD or NN problem for $\Omega_{\beta,\gamma}$ is equivalent to the solution of two BVPs with symmetry in the same $\Omega_{\beta,\gamma}$. Moreover, they are also equivalent to the solution of two BVPs for cones of half the size, e.g., in $\Omega_{\beta,(\beta+\gamma)/2}$.

THEOREM 10.1. *Let* $\beta < \gamma \in \mathbb{R}, \varepsilon \in]-1/2, 1/2[$ *and* $(g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$. *Then the following assertions are equivalent:*

(I) $u \in \mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ *solves the DD problem* (30),

$$T_{0,\Gamma_1} u = g_1, \qquad T_{0,\Gamma_2} u = g_2.$$

(II) $u = u^e + u^o, \qquad u^{e,o} \in \mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma}) \quad$ *where*

u^e *is symmetric with respect to* $\Gamma = \left\{ (x_1, x_2) : \arg(x_1 + ix_2) = \frac{1}{2}(\beta + \gamma) \right\}$,

u^o *is anti-symmetric with respect to* Γ, *and*

$$\begin{aligned} T_{0,\Gamma_1} u^e = g^e = \tfrac{1}{2}(g_1 + g_2), \qquad T_{1,\Gamma} u^e = 0, \\ T_{0,\Gamma_1} u^o = g^o = \tfrac{1}{2}(g_1 - g_2), \qquad T_{0,\Gamma} u^o = 0. \end{aligned} \qquad (99)$$

(III) $u = u^e + u^o, \qquad u^{e,o} \in \mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma}) \quad$ *and*

$$\begin{aligned} T_{0,\Gamma_1} u^e = \; T_{0,\Gamma_2} u^e = g^e = \tfrac{1}{2}(g_1 + g_2), \\ T_{0,\Gamma_1} u^o = -T_{0,\Gamma_2} u^o = g^o = \tfrac{1}{2}(g_1 - g_2). \end{aligned} \qquad (100)$$

Proof. If (I) holds, we decompose u into two parts that are symmetric and anti-symmetric with respect to Γ. These functions, denoted by u^e and u^o, obviously satisfy the conditions in (II). Furthermore, if $u^{e,o}$ are as in (II), then corresponding Dirichlet conditions at Γ_2 hold because of symmetry and anti-symmetry. Thus (III) holds. The implication (III)$\Rightarrow$(I) is trivial. $\square$

To make the connection with "half-cones", consider $\Omega = \Omega_{\beta,(\beta+\gamma)/2}$. If $u^{e,o}$ are given by (II), then their restrictions to Ω, denoted by $\tilde{u}^{e,o}$, belong to $\mathscr{H}^{1+\varepsilon}(\Omega)$ and satisfy the corresponding boundary conditions

$$\begin{aligned} T_{0,\Gamma_1} \tilde{u}^e = g^e, \qquad T_{1,\Gamma} \tilde{u}^e = 0, \\ T_{0,\Gamma_1} \tilde{u}^o = g^o, \qquad T_{0,\Gamma} \tilde{u}^o = 0. \end{aligned} \qquad (101)$$

Conversely, if we are given $\tilde{u}^{e,o} \in \mathscr{H}^{1+\varepsilon}(\Omega)$ satisfying (101), then we can extend $\tilde{u}^{e,o}$ across Γ by symmetry or anti-symmetry to obtain symmetric and anti-symmetric functions $u^{e,o}$ defined on $\Omega_{\beta,\gamma}$. Furthermore, it follows that $u^{e,o} \in \mathscr{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ because the jumps of both $u^{e,o}$ and $\partial u^{e,o}/\partial n$ across Γ are zero, i.e., the functions u^e and u^o

satisfy the HE throughout Γ. The Dirichlet conditions on Γ_1 and Γ_2 are obviously satisfied.

To summarize, this shows that the DD problem in $\Omega_{\beta,\gamma}$ can be equivalently reduced to a DN and a DD problem in $\Omega_{\beta,(\beta+\gamma)/2}$.

Notice that the reduced DD problem for $\tilde{u}^o$ has the boundary data $(g^o, 0)$ with $g^o \in \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$. Thus the compatibility condition is automatically satisfied. The DN problem for $\tilde{u}^e$ with data $(g^e, 0)$, $g^e \in H^{1/2+\varepsilon}(\mathbb{R}_+)$, does not involve a compatibility condition.

THEOREM 10.2. *Let $\beta < \gamma \in \mathbb{R}, \varepsilon \in [0, 1/2[$ and $(g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ (the tilde being relevant only for $\varepsilon = 0$). Then the following assertions are equivalent:*

(I) $u \in \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ solves the NN problem (31),

$$T_{1,\Gamma_1} u = g_1, \qquad T_{1,\Gamma_2} u = g_2.$$

(II) $u = u^e + u^o$, $u^{e,o} \in \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ where

u^e *is symmetric with respect to* $\Gamma = \{(x_1, x_2) : \arg(x_1 + ix_2) = \frac{1}{2}(\beta + \gamma)\}$,

u^o *is anti-symmetric with respect to* Γ, *and*

$$
\begin{aligned}
T_{1,\Gamma_1} u^e = g^e = \tfrac{1}{2}(g_1 + g_2), \qquad & T_{1,\Gamma} u^e = 0, \\
T_{1,\Gamma_1} u^o = g^o = \tfrac{1}{2}(g_1 - g_2), \qquad & T_{0,\Gamma} u^o = 0.
\end{aligned}
\tag{102}
$$

(III) $u = u^e + u^o$, $u^{e,o} \in \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\gamma})$ and

$$
\begin{aligned}
T_{1,\Gamma_1} u^e = \ T_{1,\Gamma_2} u^e = g^e = \tfrac{1}{2}(g_1 + g_2), \\
T_{1,\Gamma_1} u^o = -T_{1,\Gamma_2} u^o = g^o = \tfrac{1}{2}(g_1 - g_2).
\end{aligned}
\tag{103}
$$

Proof. The proof is similar to the previous one. □

As in the DD case, the NN problem in $\Omega_{\beta,\gamma}$ can be reduced equivalently into two NN and ND problem in $\Omega_{\beta,(\beta+\gamma)/2}$. The boundary conditions for the restrictions $\tilde{u}^{e,o}$ are

$$
\begin{aligned}
T_{1,\Gamma_1} \tilde{u}^e = g^e, \qquad & T_{1,\Gamma} \tilde{u}^e = 0, \\
T_{1,\Gamma_1} \tilde{u}^o = g^o, \qquad & T_{0,\Gamma} \tilde{u}^o = 0.
\end{aligned}
\tag{104}
$$

The boundary data for the reduced NN problem is $(g^e, 0)$, $g^e \in \tilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$, and thus satisfies automatically the compatibility condition. No compatibility condition is necessary for the ND problem with $(g^o, 0)$, $g^o \in H^{-1/2+\varepsilon}(\mathbb{R}_+)$.

REMARK 10.3. The idea does not work for reducing the DN problem to problems in half cones.

However, one can reverse the above idea to reduce a DN problem to a DD and a NN problem in cones of twice the size. First reduce the DN problem into two semi-homogeneous DN problems. Then,

- the semi-homogeneous DN problem with zero Neumann data can be reduced to an anti-symmetric DD problem on a "double cone". More specifically, use (99) and (101) with $g^e = 0$.

- the semi-homogeneous DN problem with zero Dirichlet data can be reduced to a symmetric NN problem on a "double cone". Here use (102) and (104) with $g^o = 0$.

THEOREM 10.4. *The DD and NN problems for the HE in $\Omega_{\beta,\gamma}$, $\gamma - \beta = 4\pi/n$, $n \in \mathbb{N}$, are uniquely solvable in closed analytical form, for $\varepsilon \in [0, 1/4[$ if n is odd, and for $\varepsilon \in [0, 1/2[$ if n is even.*

Proof. The reduction to half cones indicated after Theorems 10.1 and 10.2 leads to reduced problems (of DN and DD type or NN and ND type, respectively) in cones of angle $\alpha = (\gamma - \beta)/2 = 2\pi/n$, $n \in \mathbb{N}$. For such angles solutions were obtained in closed analytical form in [7]. The restrictions on the regularity parameter ε were as follows (see [7], Section 6): in the DD case $\varepsilon \in]-1/2, 1/2[$, in the NN case $\varepsilon \in [0, 1/2[$, in the DN case $\varepsilon \in [0, 1/2[$, or $\varepsilon \in]-1/2, 1/2[$, or $\varepsilon \in]-1/4, 1/4[$, depending on whether $n \equiv 0 \mod 8$ or ($n \equiv 4 \mod 8$ or $n \equiv 2 \mod 4$) or $n \equiv 1 \mod 2$, respectively. From the reduction procedure in the NN case we obtain the restriction $\varepsilon \geqslant 0$. Uniqueness also holds for $\varepsilon \geqslant 0$ (see Proposition 2.2).   □

EXAMPLE 10.5. Consider the DD problem in $\Omega_{0,4\pi/3}$. In [7] we found the solution of the DD, NN and DN problems in $\Omega_{0,2\pi/3}$, which can now be used to represent u^e and u^o in closed analytical form, see Theorem 3.9 and Theorem 5.10 in [7]. We used the (local) notation for the resolvent operator $\mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_\alpha}$ of the DD problem in the slit domain $\mathbb{R}^2\setminus\overline{\Sigma}_\alpha = \{(x_1, x_2) \in \mathbb{R}^2 : \arg(x_1 + ix_2) \in]\alpha, \alpha + 2\pi[\}$ and similar notation for the DN problem etc., see [7]. For the DD problem in $\Omega_{0,4\pi/3}$ we have to split the solution u into the two parts $u = u^e + u^o$, symmetric with respect to $\overline{\Sigma}_{2\pi/3}$, and represent those in $\Omega_{0,2\pi/3}$ by the following formulas:

$$u^o = \mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_0}(g^o, 0) + \mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_{2\pi/3}}(g^o, 0) + \mathscr{K}_{D,\mathbb{R}^2\setminus\overline{\Sigma}_{4\pi/3}}(0, -g^o)$$

$$u^e = \mathscr{K}_{DN,\mathbb{R}^2\setminus\overline{\Sigma}_0}(g^e, 0) + \mathscr{K}_{DN,\mathbb{R}^2\setminus\overline{\Sigma}_{2\pi/3}}(-g^e, 0) + \mathscr{K}_{ND,\mathbb{R}^2\setminus\overline{\Sigma}_{4\pi/3}}(0, g^e).$$

It was shown that the three contributions in each formula add up to a solution of the BVP in $\Omega_{0,2\pi/3}$ as illustrated in Figures 5 and 6. Thus the complete solution is given by $u = u^e + u^o$ in $\Omega_{0,2\pi/3}$ and $u = \mathscr{J}_{2\pi/3}\mathscr{J}^{(2)}\mathscr{J}_{-2\pi/3}(u^e - u^o)$ in $\Omega_{2\pi/3,4\pi/3}$ where $\mathscr{J}_\alpha$ is an α-rotation around zero and $\mathscr{J}^{(2)}$ a reflection in x_2-direction. More details are presented in paper [19].
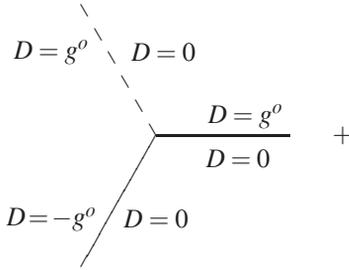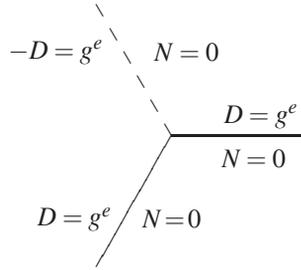
Figure 5: *DD problem for u^o*

Figure 6: *DN problem for u^e*

EXAMPLE 10.6. One can apply Theorem 10.1 also in the opposite direction, e.g., for the reduction of a BVP in $\Omega_{0,\alpha}$ to another in $\Omega_{0,2\alpha}$. For instance let us tackle the DD problem in $\Omega_{0,3\pi/2}$ characterized by $u \in \mathscr{H}^1(\Omega_{0,3\pi/2})$ with

$$T_{0,\Sigma_0}u \;=\; g_1 \,,\; T_{0,\Sigma_{3\pi/2}}u \;=\; g_2 \tag{105}$$

where $g_1, g_2 \in H^{1/2}(\mathbb{R}_+)^2_\sim$ is given. First solve the DD problem in the slit domain $\mathbb{R}^2 \backslash \overline{\Sigma}_{3\pi/2}$ with trace g_2 on the two banks of $\Sigma_{3\pi/2}$ by

$$v \;=\; \mathscr{K}_{D,\mathbb{R}^2 \backslash \overline{\Sigma}_{3\pi/2}}(g_2, g_2) \tag{106}$$

by a rotation in formula (21). The difference $w = u - v$ must satisfy the semi-homogeneous problem where

$$T_{0,\Sigma_0}w \;=\; g_1 - T_{0,\Sigma_0}v \;=\; \widetilde{g}_1 \;\in\; \widetilde{H}^{1/2}(\mathbb{R}_+) \,,\; T_{0,\Sigma_{3\pi/2}}w \;=\; 0. \tag{107}$$

Conversely, it is clear that solving (107) gives the solution $u = w + v$ of (105).

Suppose w is such a solution. Then the odd extension of w across $\Sigma_{3\pi/2}$ onto $\Omega_{0,3\pi}$, denoted by w^o, belongs to $\mathscr{H}^1(\Omega_{0,3\pi})$ and satisfies

$$T_{0,\Sigma_0}w^o \;=\; \widetilde{g}_1 \,,\; T_{0,\Sigma_{3\pi}}w^o \;=\; -\widetilde{g}_1. \tag{108}$$

Thus w_0 must be a solution which can be determined by Proposition 6.2. Conversely, if w_0 is a solution of (108), then its restriction onto the half-cone will be a solution w of (107).

To summarize, we can obtain a solution u of (105) on the cone $\Omega_{0,3\pi/2}$ by first computing v and $\tilde{g}_1$, by solving (108) using Proposition 6.2 (which involves a series expansion), by restricting its solution w^o onto $\Omega_{0,3\pi/2}$ to obtain w, and by finally putting $u = w + v$.

EXAMPLE 10.7. It is possible to reduce the DN problem in $\Omega_{0,\pi/n}$ to a DD problem (or NN problem) in $\Omega_{0,2\pi/n}$. For instance, to solve the DN problem in $\Omega_{0,\pi/n}$ characterized by $u \in \mathscr{H}^1(\Omega_{0,\pi/n})$ with

$$T_{0,\Sigma_0}u \;=\; g_1, \qquad T_{1,\Sigma_{\pi/n}}u \;=\; g_2, \tag{109}$$

where $g_1 \in H^{1/2}(\mathbb{R}_+)$ and $g_2 \in H^{-1/2}(\mathbb{R}_+)$, first obtain an (explicit analytical) solution w of the NN problem in the slit domain $\mathbb{R}^2 \backslash \overline{\Sigma}_{\pi/n}$, i.e.,

$$T_{1,\Sigma_{\pi/n}^{\pm}} w = g_2. \tag{110}$$

Then, via the substitution $u = v + w$, the problem is equivalently reduced to a semi-homogeneous DN problem in $\Omega_{0,\pi/n}$,

$$T_{0,\Sigma_0} v = \tilde{g}_1 := g_1 - T_{0,\Sigma_0} w, \qquad T_{1,\Sigma_{\pi/n}} v = 0. \tag{111}$$

The solution v of this problem can be obtained as the restriction of v^e onto $\Omega_{0,\pi/n}$, where $v^e \in \mathscr{H}^1(\Omega_{0,2\pi/n})$ is the solution of the (symmetric) DD problem

$$T_{0,\Sigma_0} v^e = \tilde{g}_1, \qquad T_{0,\Sigma_{2\pi/n}} v^e = \tilde{g}_1.$$

Notice that, conversely, v^e can be obtained from v by extending it symmetrically across $\Sigma_{\pi/n}$ (see Theorem 10.1).

A modification of this procedure can be developed in which the DN in $\Omega_{0,\pi/n}$ is reduced to an antisymmetric NN problem in $\Omega_{-\pi/n,\pi/n}$.

With this idea we obtain also the explicit solution (in terms of a series) of the DN problem in $\Omega_{0,3\pi/2}$ using Proposition 6.2, which was not yet achieved in previous publications [2, 14]. However, it works for any rational angle $\alpha = 2\pi m/n$ and will therefore be exposed as a general method in Section 12.3.

11. Analytical solution of the DD and NN problems in $\Omega_{0,4\pi}$

So far we have seen that the basic BVPs (DD, NN, DN for appropriate parameters k and ε) in $\Omega_{0,m\pi}, m \in \mathbb{N}$, are solvable with the help of series expansion. The doubling method now gives us the possibility of obtaining solutions in closed analytical form in the special cases DD and NN for $m = 4$. For this purpose recall the resolvent operators defined in (21), (24), and (25).

THEOREM 11.1. *For $\varepsilon \in]-1/4, 1/4[$ the DD problem (30) in $\Omega_{0,4\pi}$ is uniquely solved by*

$$u = u^e + u^o \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,4\pi}) \tag{112}$$

$$= \begin{cases} \mathscr{K}_{DN,\mathbb{R}^2\backslash\overline{\Sigma}_0}(g^e,0) + \mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}_0}(g^o,0) & \text{in } \Lambda_1 \\ \mathscr{K}_{ND,\mathbb{R}^2\backslash\overline{\Sigma}_2}(0,g^e) + \mathscr{K}_{D,\mathbb{R}^2\backslash\overline{\Sigma}_2}(0,-g^o) & \text{in } \Lambda_2. \end{cases} \tag{113}$$

Herein we put $g^e = \frac{1}{2}(g_1 + g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+), g^o = \frac{1}{2}(g_1 - g_2) \in \tilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)$ which are given.

Proof. Theorem 10.1 shows the equivalence with two BVPs (DD and DN) in $\Lambda_1 = \Omega_{0,2\pi}$ which can be solved by use of the formulas (21) and (25). The even/odd

extensions to Λ_2 yields the last line of the formula (112). Notice that the DN problem in $\Omega_{0,2\pi}$ requires the restriction $\varepsilon \in]-1/4, 1/4[$.

The uniqueness is clear if $\varepsilon \geqslant 0$. For $\varepsilon < 0$, assume that a nonzero solution u of the DD problem in $\Omega_{0,4\pi}$ exists. We can make an even/odd decomposition $u = u^e + u^o$ with u^e or u^o being nonzero. This would yield non-trivial solutions of the DD or the DN problem in $\Omega_{0,2\pi}$. However, in $\Omega_{0,2\pi}$ the DD problem is unique for $\varepsilon > -1/2$ (see Prop. 6.1) and the DN problem for $\varepsilon > -1/4$ (see Prop. 5.7 in [7]). $\quad\square$

THEOREM 11.2. *For* $\varepsilon \in [0, 1/4[$, *the NN problem* (31) *in* $\Omega_{0,4\pi}$ *is uniquely solved by*

$$u = u^e + u^o \in \mathcal{H}^{1+\varepsilon}(\Omega_{0,4\pi}) \tag{114}$$
$$= \begin{cases} \mathcal{K}_{N,\mathbb{R}^2\setminus\overline{\Sigma}_0}(g^e, 0) + \mathcal{K}_{ND,\mathbb{R}^2\setminus\overline{\Sigma}_0}(g^o, 0) & \text{in } \Lambda_1 \\ \mathcal{K}_{N,\mathbb{R}^2\setminus\overline{\Sigma}_2}(0, g^e) + \mathcal{K}_{DN,\mathbb{R}^2\setminus\overline{\Sigma}_2}(0, -g^o) & \text{in } \Lambda_2. \end{cases}$$

Herein we put $g^e = \frac{1}{2}(g_1 + g_2) \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+)$, $g^o = \frac{1}{2}(g_1 - g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)$.

Proof. The proof is analogous to the previous one. $\quad\square$

12. Explicit solution of the basic BVPs in $\Omega_{0,\pi m/n}$

We recall that, for $m = 1$ and $m = 2$, all these problems (DD, NN and DN case) were solved in [7] in closed analytical form (for $\mathrm{Im}(k) > 0, \varepsilon \in]-1/2, 1/2[$ or $\varepsilon \in [0, 1/2[$ or $\varepsilon \in]-1/4, 1/4[$). The proofs of the corresponding formulas were based upon symmetry arguments, roughly explained in the introduction of the present paper. These arguments practically hold for $m \in \mathbb{N}$, as well, by using Helmholtz solutions in $\Omega_{0,m\pi}$ instead of $\Omega_{0,\pi} = \Omega^+$ or $\Omega_{0,2\pi} = \mathbb{R}^2\setminus\overline{\Sigma}_0$, respectively.

The new presentation formulas for the HE in $\Omega_{\beta,\gamma}$ for $\gamma - \beta = \pi m/n \in \pi\mathbb{Q}$ are based on the knowledge of the resolvents of solutions in CRS $\Omega_{0,m\pi}$ presented before in closed analytical form *or* by series expansion. Moreover, we need corresponding resolvent operators in rotated cones $\Omega_{\beta,\beta+m\pi}$. Let us record the corresponding notation:

$$\mathcal{K}_{D,\Omega_{\beta,\beta+m\pi}} : H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\beta+m\pi})$$
$$\mathcal{K}_{N,\Omega_{\beta,\beta+m\pi}} : H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\beta+m\pi}) \tag{115}$$
$$\mathcal{K}_{DN,\Omega_{\beta,\beta+m\pi}} : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\beta+m\pi}).$$

Clearly, we have

$$(\mathcal{K}_{B,\Omega_{\beta,\beta+m\pi}} f)(x) = (\mathcal{K}_{B,\Omega_{0,m\pi}} f)(R_\beta^{-1} x)$$

where R_β stands for the rotation by β on the Riemann surface $\mathcal{R}^2$, which is (formally) defined by (33). B stands for D, N, or DN. The explicit form of the potential operators

$\mathcal{K}_{B,\Omega_{0,m\pi}}$ can be taken from the previous sections. It is also easy to obtain resolvent operators for the ND problem,

$$\mathcal{K}_{ND,\Omega_{\beta,\beta+m\pi}} : H^{-1/2+\varepsilon}(\mathbb{R}_+) \times H^{1/2+\varepsilon}(\mathbb{R}_+) \to \mathcal{H}^{1+\varepsilon}(\Omega_{\beta,\beta+m\pi}).$$

We omit the simple details.

To make the connection with [7] we remark that therein we used LIP operators

$$\mathcal{K}_{B,\Omega_\beta^\pm},$$

which gave solutions in $\Omega_\beta^+ = \Omega_{\beta,\beta+\pi}$ and $\Omega_\beta^- = \Omega_{\beta-\pi,\beta}$. In addition, we use SOP operators

$$\mathcal{K}_{B,\mathbb{R}^2 \setminus \overline{\Sigma}_\beta}.$$

Those produced solutions in $\mathbb{R}^2 \setminus \overline{\Sigma}_\beta = \Omega_{\beta,\beta+2\pi}$. In summary, the resolvent operators used in [7] are special cases of the resolvent operators that are being used here.

In order to present our results, let us introduce the involution

$$f^\# = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}^\# = \begin{pmatrix} f_2 \\ f_1 \end{pmatrix} \tag{116}$$

where $f = (f_1, f_2)$ belongs to $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, $H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$, or $H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ depending on the BVP under consideration. For convenience let

$$f^{\#j} = \begin{cases} f^\# & \text{if } j \text{ is odd} \\ f & \text{if } j \text{ is even.} \end{cases}$$

12.1. The Dirichlet problem in $\Omega_{0,\alpha}, \alpha = \pi m/n$

For the Dirichlet problem in $\Omega_{0,\alpha}$ with $\alpha = \pi m/n$ and wave number k, the result is as follows. The Dirichlet problem is reduced to Dirichlet problems in $\Omega_{0,m\pi}$ and in rotated domains $\Omega_{\beta,\beta+2\pi}$. Notice that in one case, the resolvent operator $\mathcal{K}^\varepsilon$ makes sense directly, whereas in the other case it is defined first on a dense subset and can then be extended by continuity.

THEOREM 12.1. *Let $\alpha = \pi m/n$, and assume*

$$\begin{aligned}
&\varepsilon \in]-1/2, 1/2[\quad \text{if } m = 1 \text{ or } m = 2, \\
&\varepsilon \in]-1/4, 1/4[\quad \text{if } m = 4, \\
&\varepsilon \in]-1/m, 1/m[\quad \text{and} \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)} \quad \text{otherwise.}
\end{aligned} \tag{117}$$

Then the DD problem of the HE in $\Omega = \Omega_{0,\alpha}$ with Dirichlet data $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ is uniquely solved by $u = \mathcal{K}^\varepsilon g$, where

(i) *case $n \equiv 1 \mod 2$*

$$\mathcal{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^n (-1)^{j+1} \mathcal{K}_{D,\Omega_{j\alpha-\pi m, j\alpha}} g^{\#(j+1)} \right), \tag{118}$$

(ii) *case* $n \equiv 0 \mod 2$

$\mathscr{K}^\varepsilon$ *is defined in terms of the operator* $\widetilde{\mathscr{K}^\varepsilon} : \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2 \to \mathscr{H}^{1+\varepsilon}(\Omega)$,

$$\widetilde{\mathscr{K}^\varepsilon}g = r_\Omega \left(\sum_{j=1}^{\frac{n}{2}} \left(\mathscr{K}_{D,\Omega_{2(1-j)\alpha,2(1-j)\alpha+m\pi}} \ell^o g_1 - \mathscr{K}_{D,\Omega_{(2j-1)\alpha-m\pi,(2j-1)\alpha}} \ell^o g_2 \right) \right),$$

(119)

$g = (g_1, g_2) \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+)^2$, *as follows:*

    (a) $\varepsilon < 0$: $\mathscr{K}^\varepsilon = \widetilde{\mathscr{K}^\varepsilon}$;
    (b) $\varepsilon = 0$: $\mathscr{K}^0$ *is the continuous extension of* $\widetilde{\mathscr{K}^0}$ *onto* $H^{1/2}(\mathbb{R}_+)^2_\sim$;
    (c) $\varepsilon > 0$: $\mathscr{K}^\varepsilon$ *is the restriction of* $\mathscr{K}^0$ *onto* $H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$.

Moreover, in all cases, the operators

$$\mathscr{K}^\varepsilon : H^{1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

are linear homeomorphisms.

Proof. For case (i), see Theorem 3.9 in [7], using the operator $\mathscr{K}_{D,\Omega_{0,\pi m}}$ instead of $\mathscr{K}_{D,\Omega_{0,2\pi}}$. For case (ii), see Theorem 3.5 in [7], where here the operator $\mathscr{K}_{D,\Omega_{0,m\pi}}$ instead of $\mathscr{K}_{D,\Omega_{0,\pi}}$ occurs. Notice also the change of n to $n/2$.

The uniqueness is clear for $\varepsilon \geqslant 0$ and also in the case of $\alpha = \pi m$. Suppose u is a non-trivial solution of the homogeneous DD problem for $\varepsilon < 0$ in $\Omega_{0,\alpha}$ with $\alpha = m\pi/n$. Then we can continue this solution to onto $\Omega_{0,\pi}$ by antisymmetry with respect to the rays $\{x : \arg(x) = \alpha j\}$, $j = 1, ..., n-1$. The D and N jump conditions are zero along these rays, and therefore we obtain a nonzero $u_{\text{ext}} \in \mathscr{H}^{1+\varepsilon}(\Omega_{0,\pi m})$ satisfying zero DD conditions. This is a contradiction. For details, see also the proof of Prop. 2.3 and Theorem 11.1. $\square$

In [7], the case $\alpha = \pi/n$ with n odd was dealt with in another way (see Theorem 3.3 therein). Careful inspection shows that this case is now contained in case (i) and that the formulas amount to the same.

12.2. The Neumann problem in $\Omega_{0,\alpha}, \alpha = \pi m/n$

The result for the NN problem is analogous to the result for the DD problem. The only difference is that we assume $\varepsilon \geqslant 0$ because we do not formulate the NN problem for $\varepsilon < 0$.

THEOREM 12.2. *Let* $\alpha = \pi m/n$, *and assume*

$$\begin{aligned}
&\varepsilon \in [0, 1/2[&& \text{if } m = 1 \text{ or } m = 2, \\
&\varepsilon \in [0, 1/4[&& \text{if } m = 4, \\
&\varepsilon \in [0, 1/m[&& \text{and} \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)} && \text{otherwise.}
\end{aligned}$$

(120)

Then the NN problem for the HE in $\Omega = \Omega_{0,\alpha}$ with Neumann data $g = (g_1, g_2) \in H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim$ is uniquely solved by:

(i) *case $n \equiv 1 \mod 2$*

$$u = \mathcal{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^{n} \mathcal{K}_{N,\Omega_{j\alpha-\pi m, j\alpha}} g^{\#(j+1)} \right)$$

(ii) *case $n \equiv 0 \mod 2$*

$$u = \mathcal{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^{\frac{n}{2}} \left(\mathcal{K}_{N,\Omega_{2(1-j)\alpha,2(1-j)\alpha+m\pi}} \ell^e g_1 + \mathcal{K}_{N,\Omega_{(2j-1)\alpha-m\pi,(2j-1)\alpha}} \ell^e g_2 \right) \right)$$

Here, for $\varepsilon = 0$, the operator is defined on the dense subspace $\widetilde{H}^{-1/2}(\mathbb{R}_+)^2$ and can be extended by continuity to all of $H^{-1/2}(\mathbb{R}_+)^2_\sim$.

Moreover, in all cases, the operators

$$\mathcal{K}^\varepsilon : H^{-1/2+\varepsilon}(\mathbb{R}_+)^2_\sim \to \mathcal{H}^{1+\varepsilon}(\Omega)$$

are linear homeomorphisms.

Proof. The proof is analogous to the proof of Theorem 4.5 in [7]. The resolvent operators $K_{N,\Omega_{0,m\pi}}$ (and their rotated versions) are well defined and bounded because of Theorem 7.1. For the special cases $m = 1, 2, 4$ see (12), (24), and Theorem 10.4. $\square$

12.3. The DN problem in $\Omega_{0,\alpha}, \alpha = \pi m/n$

We would like to point out two different methods in this context. The first one is based on the idea given in the first paragraph of Section 10 and we present the formulas subsequently, also for comparison with the previous cases. The other method is based on the doubling method, after reduction of the DN problem in $\Omega_{0,2\pi m/n}$ to a DD (or NN) problem with symmetry in $\Omega_{0,4\pi m/n}$ as described in Example 10.7. This will be presented in brief at the end of this subsection.

THEOREM 12.3. *Let $\alpha = \pi m/n$, and assume*

(i) *case $n \equiv 1 \mod 2$*

$$\begin{aligned} \varepsilon &\in [-1/2, 1/2[\quad \textit{if } m = 1, \\ \varepsilon &\in [-1/4, 1/4[\quad \textit{if } m = 2, \\ \varepsilon &\in [-1/2m, 1/2m[\quad \textit{and} \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/2m)} \quad \textit{otherwise,} \end{aligned} \tag{121}$$

(ii/iii) <u>case $n \equiv 0 \mod 2$</u>

$$\begin{array}{ll}
\varepsilon \in [0, 1/2[& \text{if } m = 1 \text{ or } m = 2, \\
\varepsilon \in [0, 1/4[& \text{if } m = 4, \\
\varepsilon \in [0, 1/m[\quad \text{and} \quad \frac{c(k)}{\cos(\pi\varepsilon)} < \frac{1}{\cos(\pi/m)} & \text{otherwise}.
\end{array} \qquad (122)$$

Then the DN problem for the HE in $\Omega = \Omega_{0,\alpha}$ for given data $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ is uniquely solved by $u = K^\varepsilon g$, where

(i) <u>case $n \equiv 1 \mod 2$</u>

$$\mathscr{K}^\varepsilon g = r_\Omega \left(\sum_{j=0}^{\frac{n-1}{2}} (-1)^j \mathscr{K}_{DN, \Omega_{-2j\alpha, m\pi - 2j\alpha}} h - \sum_{j=1}^{\frac{n-1}{2}} (-1)^j \mathscr{K}_{ND, \Omega_{2j\alpha - m\pi, 2j\alpha}} h^\# \right)$$

with $h = (g_1, g_2)$ *if* $n \equiv 1 \mod 4$ *and* $h = (g_1, -g_2)$ *if* $n \equiv 3 \mod 4$,

(ii) <u>case $n \equiv 0 \mod 4$</u>

$$\mathscr{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^{\frac{n}{2}} (-1)^{j+1} \left(\mathscr{K}_{D, \Omega_{2(1-j)\alpha, 2(1-j)\alpha + m\pi}} \ell^o g_1 + \mathscr{K}_{N, \Omega_{(2j-1)\alpha - m\pi, (2j-1)\alpha}} \ell^e g_2 \right) \right),$$

(iii) <u>case $n \equiv 2 \mod 4$</u>

$$\mathscr{K}^\varepsilon g = r_\Omega \left(\sum_{j=1}^{\frac{n}{2}} (-1)^{j+1} \left(\mathscr{K}_{D, \Omega_{2(1-j)\alpha, 2(1-j)\alpha + m\pi}} \ell^e g_1 + \mathscr{K}_{N, \Omega_{(2j-1)\alpha - m\pi, (2j-1)\alpha}} \ell^o g_2 \right) \right).$$

Moreover, in all cases the operators

$$\mathscr{K}^\varepsilon : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathscr{H}^{1+\varepsilon}(\Omega)$$

are linear homeomorphisms.

Proof. See Theorems 5.3, 5.5, and 5.10 in [7], opening the angles by a factor m. In case (ii) the definition of $\mathscr{K}^\varepsilon$ has to be understood similarly as before by continuous extension from $\widetilde{H}^{1/2}(\mathbb{R}_+) \times \widetilde{H}^{-1/2}(\mathbb{R}_+)$ to $H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)$. In case (iii) the direct definition works because the even and odd extension operators ℓ^e and ℓ^o are bounded on the appropriate spaces. The restriction to $\varepsilon \geqslant 0$ in cases (ii) and (iii) is due to the appearance of resolvent operators for the NN problem, for which we imposed this restriction. $\square$

If it was possible to define properly the operators $\mathscr{K}_{N, \Omega_{0,m\pi}}$ for $\varepsilon < 0$, then it is certainly possible to extend the results of cases (ii) and (iii) to negative ε. We refrain from discussing this issue further.

Now we turn to the above-mentioned alternative method.

THEOREM 12.4. *Let* $\alpha = \pi m/n$, *assume*

$$\varepsilon \in [0, 1/2m[\quad and \quad \frac{c(k)}{\cos(\varepsilon\pi)} < \frac{1}{\cos(\pi/2m)} \quad in\ case\ n\ is\ odd;$$

$$\varepsilon \in [0, 1/m[\quad and \quad \frac{c(k)}{\cos(\varepsilon\pi)} < \frac{1}{\cos(\pi/m)} \quad in\ case\ n\ is\ even.$$

Then the DN problem for the HE in $\Omega_{0,\alpha}$ *with given data* $g = (g_1, g_2) \in H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$ *is well-posed and uniquely solved by* $u = (v + w)|_{\Omega_{0,\alpha}}$ *where* w *is the solution of a Neumann problem in* $\Omega_{0,\alpha}$ *with*

$$T_{1,\Gamma_\alpha}w = -T_{1,\Gamma_0}w = g_2$$

and v *is the solution of the Dirichlet problem in* $\Omega_{0,2\alpha}$ *with*

$$T_{0,\Gamma_0}v = T_{0,\Gamma_{2\alpha}}v = \tilde{g}_1 := g_1 - T_{0,\Gamma_0}w.$$

Moreover, the restriction of the resolvent operator

$$\mathcal{K}^\varepsilon : H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+) \to \mathcal{H}^{1+\varepsilon}(\Omega_{0,\alpha})$$

is a linear homeomorphism.

Proof. The idea of Example 10.7 leads to the decomposition of u into a pure NN problem in the given CRS, which can be solved with the help of Theorem 12.2 in Subsection 12.2, and a pure DD problem with symmetry in a cone with double angle, which can be solved with the help of Theorem 12.1 in Subsection 12.1. Uniqueness of the solution results from this decomposition, as well. Notice that the NN and DD problems satisfy the compatibility conditions automatically. $\square$

13. Final remarks and open problems

We observed that wedge diffraction problems with Dirichlet and Neumann conditions (in the sense of this research and by the present methods) can be generally solved in closed analytical form if the angle is $\alpha = \gamma - \beta = 2\pi/n, n \in \mathbb{N}$, or, in case of DD or NN conditions, if $\alpha = 4\pi/n$, as well. The problems where $\alpha = 2\pi m/n, m = 3, 4, ...$ are solvable by series expansion (for suitable k and ε), a solution in closed analytical form is not (yet) available in general. In any case, the explicit solution of DD, NN and DN diffraction problems is possible for arbitrary rational angles (including exterior problems), although sometimes in a rather complicated form.

The following diagram shows the parameter domains for which an explicit solution of the DD problem in a conical Riemann surface $\Omega_{\beta,\gamma}$ is given by the present method and correctness is proved at least for small values of $|\varepsilon|$.

Herein HLP stands for *half-line potential*, a composition of a line potential (LIP) or Sommerfeld potential (SOP) with an operator of even or odd extension from the half-line to the full line. In combination with trace operators they generate Wiener-Hopf plus/minus Hankel operators [14].

$\alpha = \beta - \gamma$	ε	add. condition	form	references
π	$]-1/2,1/2[$	$-$	LIP	(12)
2π	$]-1/2,1/2[$	$-$	SOP	(21), [24]
4π	$]-1/4,1/4[$	$-$	closed	Thm. 11.1
$m\pi$	$]-1/m,1/m[$	$\dfrac{c(k)}{\cos(\pi\varepsilon)} < \dfrac{1}{\cos(\pi/m)}$	series	Thm. 6.4 Prop. 6.2-6.3 ($m = 3,4$)
$2\pi/n$	$]-1/2,1/2[$	$-$	closed	[7]
$4\pi/n$	$]-1/4,1/4[$	$-$	closed	Thm. 11.1 and 12.1
$\pi m/n$	$]-1/m,1/m[$	$\dfrac{c(k)}{\cos(\pi\varepsilon)} < \dfrac{1}{\cos(\pi/m)}$	series	Thm. 6.4, Thm. 12.1
$\pi/2$	$]-1/2,1/2[$	$-$	HLP	[14]
$3\pi/2$	$[0,1/2[$	$-$	closed	[2], [14]
$4\pi/3$	$]-1/4,1/4[$	$-$	closed	Example 10.5, [19]

The special case of $\alpha = 3\pi/2$ is interesting for various reasons, particularly because of the occurrence of Hankel operators. A solution in closed analytical form was given in [14] for $\varepsilon = 0$. Later, in [2], well-posedness was proved for $\varepsilon \in [0,1/2[$. Hence the solution formula of [14] holds for $\varepsilon \in [0,1/2[$, as well.

The situation for the NN problems is similar provided $\varepsilon \geqslant 0$, see Section 7 for details. It is known that a different approach via Sommerfeld integrals allows a closed analytical solution, particularly for the Neumann problem in an arbitrary convex cone, i.e., with angle $\alpha \in]0,\pi[$, see [11]. However, the realization of the solution as an element of $\mathscr{H}^1(\Omega)$ [11] and the proof of $H^{1+\varepsilon}$ regularity for $\varepsilon \in [0,1/2[$ is difficult [16].

For the DN problems we have the following situation.

$\alpha = \beta - \gamma$	ε	add. condition	form	references
π	$]-1/2,1/2[$	$-$	LIP	(16)
2π	$]-1/4,1/4[$	$-$	SOP	(21), (25)
3π	$]-1/6,1/6[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{2}{\sqrt{3}}$	series	Prop. 8.1
$m\pi$	$]-1/2m,1/2m[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{1}{\cos(\pi/2m)}$	series	Thm. 12.3, 12.4
π/n	$[0,1/2[$	$-$	closed	[7]
π/n	$]-1/2,1/2[$	n odd	closed	[7]
$2\pi/n$	$]-1/4,1/4[$	n odd	closed	[7]
$\pi m/n$	$]-1/2m,1/2m[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{1}{\cos(\pi/2m)}$, n odd	series	Thm. 12.3
$\pi m/n$	$[0,1/m[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{1}{\cos(\pi/m)}$, n even	series	Thm. 12.3
$\pi m/n$	$[0,1/2m[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{1}{\cos(\pi/2m)}$, n odd	series	Thm. 12.4
$\pi m/n$	$[0,1/m[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{1}{\cos(\pi/m)}$, n even	series	Thm. 12.4
$\pi/2$	$[0,1/2[$	$-$	HLP	[14]
$3\pi/2$	$[0,1/2[$	$-$	closed	[2], [14]
$4\pi/3$	$[0,1/8[$	$\dfrac{c(k)}{\cos(\varepsilon\pi)} < \dfrac{1}{\cos(\pi/8)}$	series	Thm. 12.4

Finally we like to mention some open problems, which we found interesting.

PROBLEM 1. Find a convenient method for the solution in *closed analytical form* of wedge diffraction problems with Dirichlet and Neumann conditions that works for *arbitrary rational angles*, particularly $\alpha > \pi$.

Connected with the present results, there occur some operator theoretic questions which are of general interest in our opinion. We mention only a few of them.

PROBLEM 2. Explicit inversion of the pure (supposed to be injective) Hankel operators $H_{\zeta \pm 1/2} = r_+ A_{\zeta \pm 1/2} \mathscr{J} \ell_0 : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ and connected questions such as H^s regularity, exact description of the image. We conjecture that the inverse can be presented by series expansion.

PROBLEM 3. The generalization of the previous question to Hankel operators $H_{\zeta \omega} = r_+ A_{\zeta \omega} \mathscr{J} \ell_0$ where $\omega \in \mathbb{C} \setminus \mathbb{Z}$; moreover to Hankel operators $H_\Phi = r_+ A_\Phi \mathscr{J} \ell_0$ where $\Phi \in \mathscr{G} C^\mu(\overline{\overline{\mathbb{R}}}), \mu \in]0,1[, \overline{\overline{\mathbb{R}}} = [-\infty, +\infty]$.

PROBLEM 4. The construction of extension operators from a half-line into CRS:

$$E : H^{1/2}(\mathbb{R}_+) \to \mathscr{H}^1(\Omega_{0,\alpha})$$
$$T_{0,\Sigma_0} E = I_{H^{1/2}(\mathbb{R}_+)}$$

where $\alpha = m\pi, m > 4$. Such a construction would help, e.g., to reduce BVPs in cones or CRS to semi-homogeneous BVPs, cf. Remark 4.4.

PROBLEM 5. Variants of the previous question such as

(a) extension of Neumann or other data into spaces of Helmholtz solutions, (b) extension of pairs of data $T_{0,\Sigma_0} u = g_0, T_{1,\Sigma_0} u = g_1$ (which are correlated), (c) two-sided extension into $\Omega_{\beta,\gamma}$ where $\beta < 0 < \gamma$ from an interior half-line or (d) from a cone.

PROBLEM 6. Determination of mapping properties and invertibility criteria for further operators which appeared in the study of wedge diffraction problems such as the so-called "around the corner operators" in [14] or further Wiener-Hopf plus Hankel operators [2, 14].

PROBLEM 7. Development of a limiting absorption principle for real wave numbers in the sense of the present analysis, i.e., for the setting of Sobolev spaces.

REFERENCES

[1] L. P. CASTRO AND D. KAPANADZE, *Wave diffraction by a 45 degree wedge sector with Dirichlet and Neumann boundary conditions*, Mathematical and Computer Modelling, **48** (2008), 114–121.

[2] L. P. CASTRO AND D. KAPANADZE, *Exterior wedge diffraction problems with Dirichlet, Neumann and impedance boundary conditions*, Acta Appl. Math., **110** (2010), 289–311.

[3] L. P. CASTRO AND F.-O. SPECK, *On the inversion of higher order Wiener-Hopf operators*, J. Integral Equations Appl. **8** (1996), 269–285.

[4] L. P. CASTRO AND F.-O. SPECK, *Regularity properties and generalized inverses of delta-related operators*, Z. Anal. Anwend. **17** (1998), 577–598.

[5] L. P. CASTRO, F.-O. SPECK AND F. S. TEIXEIRA, *On a class of wedge diffraction problems posted by Erhard Meister*, Oper. Theory Adv. Appl. **147** (2004), 211–238.

[6] L. P. CASTRO, F.-O. SPECK AND F. S. TEIXEIRA, *A direct approach to convolution operators with symmetry*, Math. Nachr. **269–270** (2004), 73–85.

[7] T. EHRHARDT, A. P. NOLASCO AND F.-O. SPECK, *Boundary integral methods for wedge diffraction problems: the angle $2\pi/n$, Dirichlet and Neumann conditions*, Operators and Matrices 5 (2011), 1–40.

[8] G. I. ÈSKIN, *Boundary Value Problems for Elliptic Pseudodifferential Equations*, Translations of Mathematical Monographs **52**, AMS, Providence, R. I., 1981.

[9] A. E. HEINS, *The Sommerfeld half-plane problem revisited, II, the factoring of a matrix of analytic functions*, Math. Meth. Appl. Sci. **5** (1983), 14–21.

[10] G. C. HSIAO AND W. L. WENDLAND, *Boundary Integral Equations*, Applied Mathematical Sciences Series **164**, Springer-Verlag, Berlin 2008.

[11] A. I. KOMECH, N. J. MAUSER AND A. E. MERZON, *On Sommerfeld representation and uniqueness in scattering by wedges*, Math. Meth. Appl. Sci. **28** (2005), 147–183.

[12] E. MEISTER, *Ein Überblick über analytische Methoden zur Lösung singulärer Integralgleichungen*, in: *Proceedings of the GAMM Conference in Graz in 1976*, Z. Angew. Math. Mech. **57** (1977), T23–T35.

[13] E. MEISTER, *Some multiple-part Wiener-Hopf problems in Mathematical Physics*, in: *Mathematical Models and Methods in Mechanics*, Banach Center Publications **15**, PWN – Polish Scientific Publishers, Warsaw (1985) 359–407.

[14] E. MEISTER, F. PENZEL, F.-O. SPECK AND F. S. TEIXEIRA, *Some interior and exterior boundary-value problems for the Helmholtz equation in a quadrant*, Proc. R. Soc. Edinb., Sect. A **123** (1993), 193–237.

[15] E. MEISTER AND F.-O. SPECK, *Modern Wiener-Hopf methods in diffraction theory*, Pitman Res. Notes Math. Ser. **216** (1989), 130–171.

[16] A. MERZON, F.-O. SPECK AND T. VILLALBA-VEGA, *On the weak solution of the Neumann problem for the 2D Helmholtz equation in a convex cone and H^s regularity*, Math. Meth. Appl. Sci. 34 (2011), 24–43.

[17] A. MOURA SANTOS, F.-O. SPECK AND F. S. TEIXEIRA, *Minimal normalization of Wiener-Hopf operators in spaces of Bessel potentials*, J. Math. Anal. Appl. **225** (1998), 501–531.

[18] R. J. NAGEM, M. ZAMPOLLI AND G. SANDRI, *Arnold Sommerfeld: Mathematical Theory of Diffraction*, Birkhäuser, Basel, 2004.

[19] A. P. NOLASCO AND F.-O. SPECK, *On some boundary value problems for the Helmholtz equation in a cone of* $240°$, In: Spectral Theory, Mathematical System Theory, Evolution Equations, Differential and Difference Equations, Proceedings of the IWOTA 2010 at Berlin (Eds. W. Arendt, J. A. Ball, K.-H. Förster, V. Mehrmann, C. Trunk), Birkhäuser OTAA series, Vol. 221, Springer, Basel 2012, 509–525.

[20] A. V. OSIPOV AND A. N. NORRIS, *The Malyuzhinets theory for scattering from wedge boundaries: a review*, Wave Motion **29** (1999), 313–340.

[21] A. D. RAWLINS, *The explicit Wiener-Hopf factorization of a special matrix*, Z. Angew. Math. Mech. **61** (1981), 527–528.

[22] A. D. RAWLINS, *The solution of a mixed boundary value problem in the theory of diffraction*, J. Eng. Math. **18** (1984), 37–62.

[23] S. G. SOMMERFELD, *Mathematische Theorie der Diffraction*, Math. Annalen **47** (1896), 317–374.

[24] F.-O. SPECK, *Mixed boundary value problems of the type of Sommerfeld's half-plane problem*, Proc. Royal Soc. Edinburgh **104 A** (1986), 261–277.

[25] P. YA. UFIMTSEV, *Theory of Edge Diffraction in Electromagnetics*, Tech Science Press, Encino, California, 2003.

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