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NEAR INVARIANCE AND SYMMETRIC OPERATORS

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Abstract. Let S be a subspace of $L^2(\mathbb{R})$. We show that the operator M of multiplication by the independent variable has a simple symmetric regular restriction to S with deficiency indices $(1,1)$ if and only if $S = uhK^2_\theta$ is a nearly invariant subspace, with θ a meromorphic inner function vanishing at i. Here u is unimodular, h is an isometric multiplier of $K^2_\theta := H^2 \ominus \theta H^2$ into H^2 and H^2 is the Hardy space of the upper half plane. Our proof uses the dilation theory of completely positive maps.

1. Introduction

A closed subspace $S \subset H^2(\mathbb{C}_+)$, where $\mathbb{C}_+$ denotes the complex upper half plane is called nearly invariant [3, Section 12], [12, 6] if the following condition holds:

$$f \in S \text{ and } f(i) = 0 \;\; \Rightarrow \;\; \frac{f(z)}{z-i} \in S. \tag{1.1}$$

In other words the backward shift (the adjoint of the restriction of multiplication by $\frac{z-i}{z+i}$ to H^2) maps the subspace $S' := \{f \in S | \;\; f(i) = 0\} \subset S$ into S. Let θ denote an inner function, *i.e.* θ is analytic in $\mathbb{C}_+$,

$$|\theta(z)| < 1 \qquad z \in \mathbb{C}_+$$

and θ has non-tangential boundary values on the real line $\mathbb{R}$ which exist almost everywhere with respect to Lebesgue measure and satisfy

$$|\theta(x)| = 1$$

almost everywhere with respect to Lebesgue measure on $\mathbb{R}$. Any model subspace $K^2_\theta := H^2 \ominus \theta H^2$ is nearly invariant since it is by definition invariant for the backward shift. Any nearly invariant subspace of $H^2(\mathbb{C}_+)$ can be written as $S = hK^2_\theta$ where θ is inner, $\theta(i) = 0$, h is a certain function such that $\frac{h(z)}{z+i} \in S$, and h is an isometric multiplier of K^2_θ into H^2 (see [3] and Subsection 1.1). A subspace $S \subset L^2(\mathbb{R})$ is said to be nearly invariant if $S = uS'$ where u is a unimodular function and $S' \subset H^2$ is nearly invariant.

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If θ is meromorphic, it is not difficult to show that any nearly invariant subspace $S = uhK^2_\theta \subset L^2(\mathbb{R})$ is a reproducing kernel Hilbert space (RKHS) of functions on $\mathbb{R}$ with a $\mathbb{T}$-parameter family of total orthogonal sets of point evaluation vectors. Here $\mathbb{T}$ denotes the unit circle. This follows, for example, from the results of [9, 10] (these results show that any K^2_θ has these properties for meromorphic inner θ). It also follows that there is a linear manifold (non-closed subspace) $\mathfrak{D}_S \subset S$ such that $M_S := M|_{\mathfrak{D}_S}$ is a closed, regular and simple symmetric linear transformation with deficiency indices $(1,1)$ with domain $\mathrm{Dom}(M_S) = \mathfrak{D}_S$. Here M denotes the self-adjoint operator of multiplication by the independent variable in $L^2(\mathbb{R})$.

Recall that a linear transformation T is simple, symmetric and closed if it is defined on a domain $\mathrm{Dom}(T)$ contained in a separable Hilbert space $\mathscr{H}$ and has the following properties:

$$\langle Tx, y\rangle = \langle x, Ty\rangle, \qquad \forall x,y \in \mathrm{Dom}(T), \qquad \text{T is } \textit{symmetric}; \tag{1.2}$$

$$\bigcap_{z\in\mathbb{C}\setminus\mathbb{R}} \mathrm{Ran}(T-z) = \{0\}, \qquad \text{T is } \textit{simple}; \tag{1.3}$$

$$\{(x,Tx) |\, x \in \mathrm{Dom}(T)\} \quad \text{is a closed subset of } \mathscr{H}\oplus\mathscr{H}, \quad \text{T is } \textit{closed}; \tag{1.4}$$

$$\dim\left(\mathrm{Ran}(T-i)^\perp\right) = 1 = \dim\left(\mathrm{Ran}(T+i)^\perp\right) \qquad \text{T } \textit{has deficiency indices (1,1).} \tag{1.5}$$

The condition (1.3) (T is simple) can be restated equivalently as: T is simple if T has no non-trivial self-adjoint restrictions [1]. Further recall that a point $z \in \mathbb{C}$ is said to be a regular point for a closed linear transformation T if $T - z$ is bounded below. If T is symmetric and Ω_T is its set of regular points, then it follows that

$$\mathbb{C}\setminus\mathbb{R} \subset \Omega_T \subset \mathbb{C},$$

and T is said to be regular if $\Omega_T = \mathbb{C}$.

Note that M_S may not be densely defined, but the co-dimensions of its domain and range are at most one [17, Proposition 2.1]. We will denote the family of all closed, regular, simple symmetric linear transformations with deficiency indices $(1,1)$ on a separable Hilbert space $\mathscr{H}$ by $\mathscr{S}ym^R_1(\mathscr{H})$ for brevity. Here the R stands for regular. Similarly let $\mathscr{S}ym_1(S)$ denote the family of all simple symmetric linear transformations with deficiency indices $(1,1)$ that are defined in S. If $T \in \mathscr{S}ym_1(\mathscr{H})$, one can construct an analytic function θ_T on $\mathbb{C}_+$ which is contractive ($|\theta_T(z)| < 1$), and which is a complete unitary invariant for T [8]. That is, given any $T_1, T_2 \in \mathscr{S}ym_1(\mathscr{H})$, T_1 is unitarily equivalent to T_2 if and only if $\theta_{T_1} = \alpha\theta_{T_2}$ for some α on the unit circle in the complex plane. This function θ_T is called the Livšic characteristic function of T. If $T \in \mathscr{S}ym^R_1(\mathscr{H})$, then its characteristic function θ_T is an inner function which is analytic in a neighbourhood of the real axis $\mathbb{R}$, and has a meromorphic extension to $\mathbb{C}$ (see *e.g.* [9, Theorem 5.0.7]).

The goal of this paper is to show that the two conditions: (i) S is nearly invariant with $S = uhK^2_\theta$ for meromorphic θ (with $\theta(i) = 0$) and (ii) M has a symmetric restriction $M_S \in \mathscr{S}ym^R_1(S)$, are in fact equivalent. This will show in particular that the latter condition implies that S is a RKHS with a $\mathbb{T}$-parameter family of total orthogonal sets of point evaluation vectors. One direction of (i) $\Leftrightarrow$ (ii) follows from known results - it is easy to show that if S is nearly invariant, then M has a symmetric restriction $M_S \in \mathscr{S}ym_1(S)$ (in the next subsection we observe that this follows from *e.g.* [9]). Proving the converse appears to be more difficult, and the goal of this paper is to accomplish this for the special case where $M_S \in \mathscr{S}ym_1^R(\mathscr{H})$. Namely, our main result will be to prove the following:

THEOREM 1. *Let $S \subset L^2(\mathbb{R})$ be a closed subspace. The multiplication operator M has a symmetric restriction $M_S \in \mathscr{S}ym_1^R(S)$ if and only if $S = uhK_\theta^2$ is nearly invariant with meromorphic inner function θ.*

If M has such a restriction M_S, then it follows that the characteristic function of M_S is θ [9]. In the above theorem, u is a unimodular function and h is an isometric multiplier of K_θ^2 into H^2, as described previously. In fact we expect that a more general result holds for arbitrary inner θ. That is, we conjecture that S is nearly invariant if and only if the multiplication operator M has a simple symmetric restriction M_S to a linear manifold in S such that the Livšic characteristic function [8] of M_S is inner (see also [1, Appendix 1, Section 5]). Our approach to proving this result, however, would require the extension of several results in Krein's representation theory of simple symmetric operators to the non-regular case [4]. We will discuss this in more detail in the final section.

Given any symmetric operator $T \in \mathscr{S}ym_1^R(\mathscr{H})$ the results of [15, 16] essentially show how to construct an isometry $V : \mathscr{H} \to L^2(\mathbb{R})$ such that $\mathrm{Ran}(V) = uK_\theta^2$ for a meromorphic inner θ and $VTV^* = M_\theta$ acts as multiplication by the independent variable on its domain. They accomplish this by modifying and extending Krein's original representation theory for regular symmetric operators as presented in [4]. Using this result, the theory of [4], and some dilation theory (Stinespring's dilation theorem for completely positive maps) we show that if M has a symmetric restriction belonging to $\mathscr{S}ym_1^R(S)$ where $S \subset L^2(\mathbb{R})$, that $S = uhK_\theta^2$ must be nearly invariant with meromorphic inner θ such that $\theta(i) = 0$. This provides another connection between the classical theory of representations of symmetric operators as originated by Krein and the theory of model subspaces of Hardy space.

1.1. Nearly invariant subspaces of $H^2(\mathbb{C}_+)$.

Although it will be most convenient to work with the upper half-plane, nearly invariant subspaces of $H^2(\mathbb{D})$ have a more elegant description. Here $\mathbb{D}$ denotes the unit disc. A subspace $S \subset H^2(\mathbb{D})$ is called nearly invariant if the following condition holds:

$$f \in S \text{ and } f(0) = 0 \;\Rightarrow\; f(z)/z \in S. \tag{1.6}$$

If a subspace $S \subset H^2(\mathbb{D})$ is nearly invariant then $S = hK_\varphi^2$ where φ is inner with $\varphi(0) = 0$, multiplication by $h \in S$ is an isometry of K_φ^2 onto S, and h is the unique solution to the extremal problem [6]:

$$\sup\{\mathrm{Re}\,(h(0)) \mid h \in S \text{ and } \|h\| = 1\}. \tag{1.7}$$

Note that $h \in H^2$ since $\varphi(0) = 0$ implies that $k_0^\varphi(z) = 1 \in K_\varphi^2$ is the point evaluation vector at 0. Conversely if h is any isometric multiplier of K_φ^2 into H^2 where $\varphi(0) = 0$, then $S = hK_\varphi^2$ is nearly invariant with extremal function h, and h must have the form [13]:

$$h = \frac{a}{1 - b\varphi}, \tag{1.8}$$

where a, b belong to the unit ball of H^∞ and obey $|a|^2 + |b|^2 = 1$ a.e. on the unit circle $\mathbb{T}$.

Nearly invariant subspaces of $H^2(\mathbb{C}_+)$ have a similar description as follows. Let $\mu(z) := \frac{z-i}{z+i}$, $\mu : \overline{\mathbb{C}_+} \to \overline{\mathbb{D}} \setminus \{1\}$, which has compositional inverse $\mu^{-1}(z) = i\frac{1+z}{1-z}$. Then $\mathscr{U} : H^2(\mathbb{D}) \to H^2(\mathbb{C}_+)$ defined by

$$\mathscr{U} f(z) := \frac{1 - \mu(z)}{\sqrt{\pi}} (f \circ \mu)(z), \tag{1.9}$$

is a unitary transformation which maps $K_\varphi^2 \subset H^2(\mathbb{D})$ onto $K_{\varphi\circ\mu}^2 \subset H^2(\mathbb{C}_+)$. If $S \subset H^2(\mathbb{C}_+)$ is nearly invariant, it follows that $S' := \mathscr{U}^* S$ is nearly invariant and hence $S' = hK_\varphi^2$ for some inner $\varphi \in H^\infty(\mathbb{D})$ such that $\varphi(0) = 0$ and $h \in H^2(\mathbb{D})$. It follows that $S = \mathscr{U} S' = (h \circ \mu) K_{\varphi\circ\mu}^2$ where $\mathscr{U} h = \pi^{-1/2}(1 - \mu) h \circ \mu \in S$, so that $\theta := \varphi \circ \mu$ vanishes at i and $\frac{h\circ\mu}{z+i} \in S \subset H^2(\mathbb{C}_+)$. This shows that if h' is any isometric multiplier of K_θ^2 into $H^2(\mathbb{C}_+)$ (where $\theta(i) = 0$), that $\frac{h'}{z+i} \in H^2$.

Given any inner function $\theta \in H^\infty(\mathbb{C}_+)$, it is well known that M has a restriction $M_\theta \in \mathscr{S}ym_1(K_\theta^2)$ (see *e.g* [9, 10]). Suppose $S := hK_\theta^2$ is nearly invariant ($\theta(i) = 0$) and h is an isometric multiplier of K_θ^2. Since $V :=$multiplication by h commutes with M and is an isometry of K_θ^2 onto S, it is not hard to see that $M_S = P_S V M_\theta V^* P_S$ is a symmetric restriction of M to S with domain $\mathrm{Dom}(M_S) = V\mathrm{Dom}(M_\theta)$. Moreover, since $V\mathrm{Ran}(M_\theta \pm_i) = \mathrm{Ran}(M_S \pm i)$, it follows that $M_S \in \mathscr{S}ym_1(S)$, and that the Livšic characteristic function of M_S is θ (recall here that $\theta(i) = 0$). This shows that any nearly invariant subspace has the property that M has a restriction $M_S \in \mathscr{S}ym_1(S)$. The main goal of this paper is to show the converse (in the special case where θ is meromorphic), namely that if $S \subset L^2(\mathbb{R})$ is such that $M_S \in \mathscr{S}ym_1^R(S)$, that $S = uhK_\theta^2$ is nearly invariant.

2. Representation theory for symmetric operators

Let $\mathscr{H}$ be a separable Hilbert space and let $\mathscr{S}ym_1(\mathscr{H})$ denote the family of all closed simple symmetric linear transformations in $\mathscr{H}$ with deficiency indices $(1,1)$. By a linear transformation we mean a linear map which is not necessarily densely defined, we reserve the term operator for a densely defined linear map. Notice that $\mathscr{S}ym_1(\mathscr{H}) \supset \mathscr{S}ym_1^R(\mathscr{H})$.

Choose $\psi(i) \in \mathrm{Ran}(T + i)^\perp$ ($= \mathrm{Ker}(T^* - i)$ in the case where T is densely defined), and define the vector-valued function

$$\psi(z) := (T' - i)(T' - z)^{-1}\psi(i) = \psi(i) + (z - i)(T' - z)^{-1}\psi(i), \tag{2.1}$$

where T' is any densely defined self-adjoint extension of T within $\mathscr{H}$. If T is regular then T' has purely point spectrum consisting of eigenvalues of multiplicity one with no finite accumulation point, and it follows that $\psi(z)$ is meromorphic in $\mathbb{C}$, with simple poles at each point in $\sigma(T') \subset \mathbb{R}$. Also it can be shown that $0 \neq \psi(z) \in \mathrm{Ran}(T - \overline{z})^{\perp}$ for all $z \in \mathbb{C} \setminus \mathbb{R}$, see *e.g.* [4, Section 1.2, pgs. 8–9].

Choose $0 \neq u \in \mathrm{Ran}(T + i)^{\perp}$. One can establish the following:

LEMMA 1. *If* $T \in \mathscr{S}ym_1^R(\mathscr{H})$ *and* $z \in \overline{\mathbb{C}_+}$, *then for any non-zero* $\psi_z \in \mathrm{Ran}(T - \overline{z})^{\perp}$, $\langle \psi_i, \psi_z \rangle \neq 0$ *(so that* $\langle u, \psi_z \rangle \neq 0$*).*

The above lemma is a consequence of the following considerations:

Recall that $w \in \mathbb{C}$ is called a regular point of T if $T - w$ is bounded below. Let Ω_T^+ denote the intersection of $\overline{\mathbb{C}_+}$ with the set of all regular points of T. Then $\mathbb{C}_+ \subset \Omega_T^+ \subset \overline{\mathbb{C}_+}$ and $\Omega_T^+ = \overline{\mathbb{C}_+}$ if and only if T is regular, *i.e.* if and only if $T \in \mathscr{S}ym_1^R(\mathscr{H})$.

Now for any $w \in \Omega_T^+$, $\mathrm{Ran}(T - \overline{w})^{\perp} = \mathbb{C}\{\phi_w\}$ is one dimensional, spanned by a fixed non-zero vector ϕ_w. For each $w \in \Omega_T^+$, let $\mathfrak{D}_w := \mathrm{Dom}(T) + \mathbb{C}\{\phi_w\}$, and define the linear transformation T_w with domain $\mathfrak{D}_w$ by

$$T_w(\phi + c\phi_w) = T\phi + wc\phi_w, \tag{2.2}$$

for any $\phi \in \mathrm{Dom}(T)$ and $c \in \mathbb{C}$. It is not difficult to verify that T_w is a well-defined and closed linear extension of T. Clearly T_w is densely defined if T is, in which case $T \subset T_w \subset T^*$. A quick calculation verifies that iT_w is dissipative, *i.e.* $\mathrm{Im}\left(\langle T_w\phi, \phi \rangle\right) \geqslant 0$ for all $\phi \in \mathfrak{D}_w$. It follows from this that $T_w - z$ is bounded below for all $z \in \mathbb{C}_-$, so that one can define $(T_w - z)^{-1}$ as a linear transformation from $\mathrm{Ran}(T_w - z)$ onto $\mathrm{Dom}(T_w) = \mathfrak{D}_w$. Observe that ϕ_w is an eigenvector of T_w to eigenvalue w by construction.

REMARK 1. More can be said about the extensions T_w. Since we will not have need of these facts, we will state them here without proof. If T is not densely defined, then one can show that there is exactly one proper closed linear extension T' of T which is not densely defined, and this extension must be self-adjoint. The transformations T_w are self-adjoint if and only if $w \in \mathbb{R}$. (If T_x is the self-adjoint extension of T which is not densely defined, it is self-adjoint in the sense of a linear relation, *i.e.* its graph is self-adjoint as a subspace of $\mathscr{H} \oplus \mathscr{H}$ [5]). One can show that if T_w is densely defined that $\sigma(T_w) \subset \overline{\mathbb{C}_+}$. Since iT_w is dissipative, it follows that the Cayley transform $\mu(T_w)$ is a contractive linear operator which extends the isometric linear transformation $\mu(T)$. One can further show that $w \in \Omega_T^+$ is an eigenvalue of multiplicity one for T_w, and that $w \in \Omega_T^+$ is an eigenvalue for both T_w and T_z if and only if $T_w = T_z$.

Proof of Lemma 1. Choose $w = i \in \mathbb{C}_+$, and recall that $u \in \mathrm{Ran}(T + i)^{\perp}$. Suppose that $z \in \Omega_T^+$ ($= \overline{\mathbb{C}_+}$ since we assume T is regular). Then there is an extension T_z of T for which ψ_z is an eigenvector with eigenvalue z (as described above).

If it were true that $\langle u, \psi_z \rangle = 0$ then we would have that $\psi_z \in \mathrm{Ran}(T + i)$ so that $\psi_z = (T + i)\phi$ for some $\phi \in \mathrm{Dom}(T)$. But then since $T_z - w$ is bounded below for all $w \in \mathbb{C}_-$ it would follow that $(z + i)^{-1}\psi_z = (T_z + i)^{-1}\psi_z = \phi$ so that $\psi_z \in \mathrm{Dom}(T)$. This contradicts the fact that T is simple (it also contradicts the fact that T is symmetric if $z \notin \mathbb{R}$). $\square$

It follows that the function $\langle u, \psi(\bar{z})\rangle$ is meromorphic on $\mathbb{C}$ with zeroes contained strictly in the lower half-plane.

Now we can define the vector-valued function

$$\delta(z) := \frac{\psi(z)}{\langle \psi(z), u\rangle}.$$

By the previous lemma, this is meromorphic in $\mathbb{C}$ with poles contained in the lower half-plane (the poles of $\psi(z)$ on $\mathbb{R}$ cancel out with those of $\langle \psi(z), u\rangle$, see *e.g.* [18]).

Hence one can define a linear map V of $\mathscr{H}$ into a vector space of functions analytic on an open neighbourhood of the closed upper half-plane by

$$(Vf)(z) := \langle f, \delta(\bar{z})\rangle =: \hat{f}(z)$$

for any $f \in \mathscr{H}$. We can endow the range of V, $V\mathscr{H} =: \hat{\mathscr{H}}$ with an inner product which makes it a Hilbert space (and $V : \mathscr{H} \to \hat{\mathscr{H}}$ an isometry) as follows.

Let Q denote any unital $B(\mathscr{H})$-valued POVM (Positive Operator Valued Measure) which diagonalizes T. In this case $Q(\Omega) = P\chi_\Omega(S)P$ where S is a self-adjoint extension of T (to perhaps a larger Hilbert space $\mathscr{K} \supset \mathscr{H}$), and $P : \mathscr{K} \to \mathscr{H}$ is orthogonal projection. Here we assume that $Q(\mathbb{R}) = \mathbb{1}$ so that S is a densely defined linear operator in $\mathscr{K}$ (this is always the case if T is densely defined). Also here, $\Omega \in \mathrm{Bor}(\mathbb{R}) :=$ the Borel sigma algebra of subsets of $\mathbb{R}$. The Borel measure defined by $\sigma(\Omega) := \langle Q(\Omega)u, u\rangle = \langle \chi_\Omega(S)u, u\rangle$ is called a u-spectral measure for T, and we have the following theorem [4, Theorem 2.1.2, pg. 51]:

THEOREM 2. (Krein) *The map $Vf = \hat{f}$ is an isometric map of $\mathscr{H}$ into $L^2(\mathbb{R}, d\sigma)$. It is onto if and only if Q is a projection-valued measure (PVM).*

It is not hard to check that $VTV^* = \hat{T}$ acts as multiplication by the independent variable in $\hat{\mathscr{H}}$.

Silva and Toloza modify this construction slightly as follows [15]. Let $h(z)$ be any entire function whose zero set is equal to $\sigma(T')$ (such an entire function always exists, since the spectrum of $\sigma(T')$ is a discrete set of real eigenvalues of multiplicity one with no finite accumulation point). Then define $\gamma(z) := h(z)\psi(z)$. Then they define the linear map

$$\widetilde{V}f(z) := \widetilde{f}(z) := \langle f, \gamma(\bar{z})\rangle,$$

which maps elements of $\mathscr{H}$ into a vector space $\widetilde{\mathscr{H}}$ of entire functions. If one endows $\widetilde{\mathscr{H}}$ with the inner product $\langle \widetilde{f}, \widetilde{g}\rangle_{\widetilde{\mathscr{H}}} = \langle f, g\rangle$, then $\widetilde{\mathscr{H}}$ is a Hilbert space, $\widetilde{V}$ is an isometry, and one can further verify that $\widetilde{\mathscr{H}}$ is actually an axiomatic de Branges space of entire functions. It follows from results of de Branges that there is an entire de Branges function E (which we can assume has no real zeroes by de Branges [2, Problem 44, pg. 52]) such that $\widetilde{\mathscr{H}}$ with the inner product

$$\langle \widetilde{f}, \widetilde{g}\rangle_E := \int_{-\infty}^{\infty} \widetilde{f}(x)\overline{\widetilde{g}(x)}\frac{1}{|E(x)|^2}dx$$

is a de Branges space of entire functions and $\langle \widetilde{f}, \widetilde{g}\rangle_E = \langle \widetilde{f}, \widetilde{g}\rangle_{\widetilde{\mathscr{H}}}$ for all $\widetilde{f}, \widetilde{g} \in \widetilde{\mathscr{H}} =: \mathscr{H}(E)$.

Now let

$$r(z) := h(z)\hat{u}(z) = h(z)\langle u, \psi(\overline{z})\rangle.$$

By Lemma 1, $\langle u, \psi(x)\rangle \neq 0$ for any $x \in \mathbb{R}$, and it follows that r has no zeroes or poles on $\mathbb{R}$ (the simple zeroes of h on $\mathbb{R}$ coincide with the simple poles of $\hat{u}$). Hence for any $f \in \mathscr{H}$, $\widetilde{f} = r\hat{f}$, so that for any $f, g \in \mathscr{H}$,

$$\langle f, g\rangle = \int_{-\infty}^{\infty} \widetilde{f}(x)\overline{\widetilde{g}(x)} \frac{1}{|E(x)|^2} dx = \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)} \left|\frac{r(x)}{E(x)}\right|^2 dx. \tag{2.3}$$

The following theorem of Krein then implies that this measure σ defined by

$$d\sigma(x) := \left|\frac{r(x)}{E(x)}\right|^2 dx$$

is in fact a u-spectral measure for T [4, Theorem 2.1.1, pg. 49].

THEOREM 3. (Krein) *A Borel measure ν on $\mathbb{R}$ is a $u-$spectral measure if and only if $\langle f, g\rangle = \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)} d\nu(x)$ for all $f, g \in \mathscr{H}$.*

Note that since $E(x)$ has no real zeroes and r has no real zeroes or poles, that σ is in fact equivalent to Lebesgue measure on $\mathbb{R}$, and that $\sigma', \frac{1}{\sigma'}$ are both locally L^∞.

The following theorem on $u-$spectral measures (the form below is valid for $T \in \mathscr{S}ym_1^R(\mathscr{H})$, and for our choice of gauge $u \in \mathrm{Ker}(T^* - i)$) is also due to Krein [4, Corollary 2.1, pg. 16]:

THEOREM 4. (Krein) *Suppose that $T \in \mathscr{S}ym_1^R(\mathscr{H})$, and $0 \neq u \in \mathrm{Ker}(T^* - i)$. Let Q be the POVM obtained by compression of the PVM of some densely defined self-adjoint extension $T' \supset T$ to $\mathscr{H}$, and let $\nu(\cdot) := \langle Q(\cdot)u, u\rangle$ be a $u-$spectral measure of T. Then for any Borel set Ω,*

$$\langle Q(\Omega)f, g\rangle = \int_\Omega \hat{f}(x)\overline{\hat{g}(x)} d\nu(x). \tag{2.4}$$

REMARK 2. Krein's theorems, Theorem 2, Theorem 3 and Theorem 4, were originally stated for densely defined $T \in \mathscr{S}ym_1(\mathscr{H})$ [4]. However, the extended statements above hold for non-densely defined T with essentially no modification of Krein's original proofs.

Now suppose that $S \subset L^2(\mathbb{R})$ and that $T = M_S \in \mathscr{S}ym_1^R(S)$ is a restriction of M. Then M is a self-adjoint extension of M_S, so that we can define the u-spectral measure $\mu(\Omega) := \langle \chi_\Omega(M)u, u\rangle$. Since M is multiplication by x in $L^2(\mathbb{R})$, the measure μ is absolutely continuous with respect to Lebesgue measure so that $d\mu(x) = \mu'(x)dx$. Hence if

$$\langle \hat{f}, \hat{g}\rangle_\mu := \int_{-\infty}^{\infty} \hat{f}(x)\overline{\hat{g}(x)}\mu'(x)dx,$$

then $\langle \hat{f}, \hat{g} \rangle_\mu = \langle f, g \rangle$ by Theorem 3.

Moreover, Theorem 4 implies that for any $f, g \in S$,

$$\langle \chi_\Omega(M) f, g \rangle = \int_\Omega \hat{f}(x) \overline{\hat{g}(x)} \mu'(x) dx \tag{2.5}$$

$$= \int_\Omega \left| \frac{E(x)}{r(x)} \right|^2 \mu'(x) \widetilde{f}(x) \overline{\widetilde{g}(x)} \frac{1}{|E(x)|^2} dx \tag{2.6}$$

$$= \langle R(\widetilde{M}) \chi_\Omega(\widetilde{M}) \widetilde{f}, \widetilde{g} \rangle_E, \tag{2.7}$$

where $R(x) := \left| \frac{E(x)}{r(x)} \right|^2 \mu'(x)$ is locally L^1. Here $\widetilde{M}$ denotes multiplication by the independent variable in $L^2(\mathbb{R}, |E(x)|^{-2} dx) \supset \mathscr{H}(E) = \widetilde{\mathscr{H}}$.

REMARK 3. In fact $\mu'(x) > 0$ a.e.. Otherwise there would be a Borel subset $\Omega \subset \mathbb{R}$ of non-zero Lebesgue measure such that $\langle \chi_\Omega(\hat{M}) \hat{f}, \hat{g} \rangle_\mu = 0$ for all $f, g \in \mathscr{H}$, where $\hat{M}$ denotes multiplication by the independent variable in $L^2(\mathbb{R}, d\mu)$. But this would imply that

$$\left\langle \left| \frac{E(\widetilde{M})}{r(\widetilde{M})} \chi_\Omega(\widetilde{M}) \right|^2 \widetilde{f}, \widetilde{g} \right\rangle_E = 0, \tag{2.8}$$

for all $\widetilde{f}, \widetilde{g} \in \mathscr{H}(E)$, where $\widetilde{M}$ denotes multiplication by the independent variable in $L^2(\mathbb{R}, |E(x)|^{-2} dx)$. Since $E(x)/r(x)$ is non-zero almost everywhere with respect to Lebesgue measure, this would imply that elements of $\mathscr{H}(E)$ vanish almost everywhere on Ω. This is impossible as elements of $\mathscr{H}(E)$ are entire functions. In conclusion $\mu' > 0$ almost everywhere. The fact that $\mu' > 0$ almost everywhere where $\mu(\Omega) = \langle \chi_\Omega(M) u, u \rangle$ also shows that the gauge u is non-zero almost everywhere. This shows that the subspace S contains an element which is non-zero almost everywhere with respect to Lebesgue measure so that S is cyclic (and separating) for the von Neumann algebra generated by bounded functions of M. The fact that $\mu' > 0$ almost everywhere also implies that $R(x) > 0$ a.e.. These facts will be useful later.

Observe that

$$\langle R(\widetilde{M}) \widetilde{f}, \widetilde{g} \rangle_E = \int_{-\infty}^{\infty} \frac{\widetilde{f}(x)}{r(x)} \frac{\overline{\widetilde{g}(x)}}{r(x)} \mu'(x) dx \tag{2.9}$$

$$= \langle \hat{f}, \hat{g} \rangle_\mu = \langle f, g \rangle = \langle \widetilde{f}, \widetilde{g} \rangle_E. \tag{2.10}$$

This calculation shows that $R^{1/2}(\widetilde{M}) P_E$ is a partial isometry in $L^2(\mathbb{R}, |E(x)|^{-2} dx)$ with initial space $\mathscr{H}(E)$.

Now let

$$\theta := \frac{E^*}{E}, \tag{2.11}$$

a meromorphic inner function. Then multiplication by $\frac{1}{E}$ is an isometry of $L^2(\mathbb{R}, |E(x)|^{-2} dx)$ onto $L^2(\mathbb{R})$ that takes $\mathscr{H}(E)$ onto K^2_θ, and which intertwines $\widetilde{M}$ and

M, the operators of multiplication by the independent variable in $L^2(\mathbb{R}, |E(x)|^{-2}dx)$ and $L^2(\mathbb{R})$. Let

$$V : S \to K_\theta^2$$

be the isometry defined by

$$Vf := \frac{\widetilde{f}}{E},$$

and let $V_0 := VP_S$ be the corresponding partial isometry on $L^2(\mathbb{R})$.

REMARK 4. Since V implements a unitary equivalence between $M_S \in \mathscr{S}ym_1^R(S)$ and $M_\theta \in \mathscr{S}ym_1^R(K_\theta^2)$, the symmetric linear transformation of multiplication by z in K_θ^2, it follows that

$$\theta_S := \frac{\theta - \theta(i)}{1 - \overline{\theta(i)}\theta}$$

is the Livšic characteristic function of M_S [9].

It then follows from equation (2.7) that given any Borel set Ω and $f, g \in L^2(\mathbb{R})$,

$$\langle P_S \chi_\Omega(M) P_S f, g\rangle = \langle P_\theta R(M) \chi_\Omega(M) P_\theta V_0 f, V_0 g\rangle. \tag{2.12}$$

Let $\mathrm{vN}(M)$ denote the von Neumann algebra of L^∞ functions of M, and let $R := R(M) \geqslant 0$, which is affiliated with $\mathrm{vN}(M)$. It follows that for any $m \in \mathrm{vN}(M)$.

$$P_S m P_S = V_0^* P_\theta \sqrt{R} m \sqrt{R} P_\theta V_0. \tag{2.13}$$

Given a projector P, we let $\mathscr{P}$ denote the completely positive map $\mathscr{P}(A) = PAP$, and if $B \in B(L^2(\mathbb{R}))$, the Banach space of bounded linear operators on $L^2(\mathbb{R})$, let Ad_B denote the completely positive map $\mathrm{Ad}_B(A) = BAB^*$. The above equation shows that

$$\mathrm{Ad}_{V_0^*} \circ \mathscr{P}_\theta \circ \mathrm{Ad}_{\sqrt{R}}\big|_{\mathrm{vN}(M)} = \mathscr{P}_S\big|_{\mathrm{vN}(M)}. \tag{2.14}$$

Note that since, by equation (2.10), $R^{1/2}P_\theta$ is a partial isometry, that the completely positive map

$$\Phi_1 := \mathscr{P}_\theta \circ \mathrm{Ad}_{\sqrt{R}} : B(L^2(\mathbb{R})) \to B(K_\theta^2)$$

is unital.

In the next section we will use the dilation theory of completely positive maps to show that equation (2.14) implies that the partial isometry $V_0^* : K_\theta^2 \to S$ acts as the restriction of an element affiliated with $\mathrm{vN}(M)$ to S, *i.e.* V_0^* acts as multiplication by a function $\overline{v(x)}$. It will follow easily from this that S is nearly invariant.

3. Application of Dilation Theory

It will be convenient to use a number of acronyms. CP means completely positive, CPU means CP and unital, TP means trace preserving. A CPTPU map is a completely positive unital and trace preserving map, which is also sometimes called a quantum channel. SSD stands for Stinespring dilation.

The following lemma can be proven using Stinespring's theorem.

LEMMA 2. *Let $\mathscr{A}$ be a unital C^* algebra. Suppose that $\phi_1 : \mathscr{A} \to B(\mathscr{H}_1)$ and $\phi_2 : \mathrm{Ran}(\phi_1) \to B(\mathscr{H}_2)$ are CP maps such that $\mathscr{H}_i$ are separable. If π_1 and π_2 are the minimal Stinespring dilations of the $\Phi_1 = \phi_1$ and $\Phi_2 := \phi_2 \circ \phi_1$, then there is a contractive $*$-homomorphism π such that $\pi \circ \pi_1 = \pi_2$.*

One can prove this by inspecting the proof of Stinespring's theorem as presented in [11].

Proof. Begin by constructing the representations π_i as in the proof of Stinespring's theorem. Consider the algebraic tensor products $\mathscr{A} \otimes \mathscr{H}_i =: \mathscr{K}_i'$. Then define inner products on the $\mathscr{K}_i'$ by $(a \otimes x_i, b \otimes y_i)_i = \langle \Phi_i(b^* a)x_i, y_i\rangle_i$ where $a, b \in \mathscr{A}$, $x_i, y_i \in \mathscr{H}_i$. Then as per the usual proof, the Cauchy-Schwarz inequality can be applied to show that $\mathscr{N}_i := \{u \in \mathscr{K}_i' | \ (u,u)_i = 0\}$ is a vector subspace of $\mathscr{K}_i'$. One then defines the Hilbert spaces $\mathscr{K}_i$ to be the completions of $\mathscr{K}_i'/\mathscr{N}_i$ with respect to the inner product $\langle u_j + \mathscr{N}_i, v_j + \mathscr{N}_i\rangle_i := (u_j, v_j)_i$. Now for $a \in \mathscr{A}$ define $\pi_i(a) : \mathscr{K}_i \to \mathscr{K}_i$ by $\pi_i(a) \sum a_k \otimes x_k = \sum a a_k \otimes x_k$. The usual proof of Stinespring's theorem shows that this yields (not necessarily minimal) Stinespring dilations of the CP maps Φ_i.

Now,

$$
\begin{aligned}
\|\pi_1(a)\| &= \sup_{u=\sum a_j \otimes x_j + \mathscr{N}_1 \in \mathscr{K}_1/\mathscr{N}_1 \quad \|u\|_1 = 1} \left(\pi_1(a) \sum a_j \otimes x_j, \pi_1(a) \sum a_j \otimes x_j\right)_1 \\
&= \sup \sum \langle \Phi_1(a_i^* a^* a a_j) x_j, x_i\rangle_{\mathscr{H}_1} \qquad (3.1)
\end{aligned}
$$

It follows that if $\pi_1(a) = 0$ that for any $(a_1, \ldots, a_N) \in \mathscr{A}^{(N)} = \bigoplus_{i=1}^N \mathscr{A}$, and any $\vec{x} = (x_1, \ldots x_N) \in \mathscr{H}_1^{(N)} := \bigoplus_{i=1}^N \mathscr{H}_1$ that $\langle \Phi_1^{(N)}([a_i^* a^* a a_j])\vec{x}, \vec{x}\rangle_{\mathscr{H}_1^{(N)}} = 0$ so that $[a_i^* a^* a a_j] \in \mathrm{Ker}(\Phi_1^{(N)})$. Here, $\Phi_1^{(N)} = \Phi_1 \otimes \mathbb{1}_N$. Hence

$$
[a_i^* a^* a a_j] \in \mathrm{Ker}(\Phi_2^{(N)}) = \mathrm{Ker}(\phi_2^{(N)} \circ \Phi_1^{(N)})
$$

for any $N \in \mathbb{N}$, which in turn shows that $\|\pi_2(a)\| = 0$. Hence $\mathrm{Ker}(\pi_1) \subset \mathrm{Ker}(\pi_2)$.

Define $\pi : \pi_1(\mathscr{A}) \to \pi_2(\mathscr{A})$ by $\pi \circ \pi_1 = \pi_2$. The above calculation shows that π is a well-defined $*-$homomorphism. Also $\pi_1(a) \in \mathrm{Ker}(\pi)$ if and only if $a \in \mathrm{Ker}(\pi_2) \supset \mathrm{Ker}(\pi_1)$. Hence $\mathrm{Ker}(\pi)$ is closed and is isomorphic to $\frac{\mathrm{Ker}(\pi_2)}{\mathrm{Ker}(\pi_1)}$. If we define the map $\hat{\pi} : \pi_1(\mathscr{A})/\mathrm{Ker}(\pi) \to \pi_2(\mathscr{A})$ by $\hat{\pi}(\pi_1(a) + \mathrm{Ker}(\pi)) = \pi(\pi_1(a))$ then this is an isomorphism of C^* algebras and is hence isometric. It follows that π is a contractive $*-$homomorphism. $\square$

This basic fact will now be used to prove the following lemma:

LEMMA 3. *Let $\mathscr{B} \subset \mathscr{A}$ be C^*-algebras. Let Φ_i be CP maps from $\mathscr{A}$ into $B(\mathscr{H}_i)$. Let $\Phi : B(\mathscr{H}_1) \to B(\mathscr{H}_2)$ be a CPU map such that $\Phi \circ \Phi_1|_{\mathscr{B}} = \Phi_2|_{\mathscr{B}}$. Further assume that Φ_i and $\Phi_i|_{\mathscr{B}}$ have the same minimal Stinespring dilations. Let $(\pi_i, V_i, \mathscr{K}_i)$ be the minimal SSD's of the Φ_i, $(\pi', V', \mathscr{K}')$ the minimal SSD of $\Phi \circ \Phi_1$. Then $\mathscr{K}_2 \subset \mathscr{K}'$ is reducing for $\pi'|_{\mathscr{B}}$ and there is an onto $*$-homomorphism $\pi : \pi_1(\mathscr{A}) \to \pi'(\mathscr{A})$ such that $\pi \circ \pi_1|_{\mathscr{B}} = \pi_2|_{\mathscr{B}} = \mathscr{P}_{\mathscr{K}_2} \circ \pi'|_{\mathscr{B}}$.*

Proof. If $(\pi', V', \mathscr{K}')$ is the minimal SSD of $\Phi \circ \Phi_1$, then it is automatically an SSD of $\Phi \circ \Phi_1|_{\mathscr{B}} = \Phi_2|_{\mathscr{B}}$. Since Φ_2 and its restriction to $\mathscr{B}$ have the same minimal SSD $(\pi_2, V_2, \mathscr{K}_2)$ it follows that we can assume $\mathscr{K}_2 \subset \mathscr{K}'$, that $\mathscr{K}_2$ is reducing for $\pi'|_{\mathscr{B}}$ and that $\mathscr{P}_{\mathscr{K}_2} \circ \pi'|_{\mathscr{B}} = \pi_2|_{\mathscr{B}}$. By the previous lemma, there is an onto *-homomorphism $\pi : \pi_1(\mathscr{A}) \to \pi'(\mathscr{A})$ such that $\pi \circ \pi_1 = \pi'$. Hence $\pi \circ \pi_1|_{\mathscr{B}} = \mathscr{P}_{\mathscr{K}_2} \circ \pi'|_{\mathscr{B}} = \pi_2|_{\mathscr{B}}$. $\square$

Define $\Theta := \mathscr{P}_{\mathscr{K}_2} \circ \pi'$. This is a CPU map which is a contractive *-homomorphism when restricted to $\mathscr{B}$.

LEMMA 4. *If $S \subset L^2(\mathbb{R})$ contains a function which is cyclic and separating for* $\mathrm{vN}(M)$, i.e. *a function f which is non-zero almost everywhere with respect to Lebesgue measure, and P is the projection onto S, then the minimal SSD of $\mathscr{P} : \mathrm{vN}(M) \subset B(L^2(\mathbb{R})) \to B(S)$ is the identity map on $B(L^2(\mathbb{R}))$.*

Here, as before $\mathscr{P}(A) = PAP$ for any $A \in B(L^2(\mathbb{R}))$.

Proof. Straightforward: the identity map on $B(L^2(\mathbb{R}))$ is clearly an SSD of $\mathscr{P}|_{\mathrm{vN}(M)}$. To show that it is minimal one just needs to check that $\mathrm{vN}(M)S$ is dense in $L^2(\mathbb{R})$. As S contains an element which is cyclic for M, this is clear. $\square$

Applying this to our specific situation yields:

PROPOSITION 1. *Suppose that $S_i \subset L^2(\mathbb{R})$ are cyclic (and hence separating) for* $\mathrm{vN}(M)$ *with projections P_i, and that there exists a CPU map $\Phi_1 : B(L^2(\mathbb{R})) \to B(S_1)$ with minimal SSD $(\mathrm{id}, V, L^2(\mathbb{R}))$ for some contraction $V : B(S_1) \to B(L^2(\mathbb{R}))$. If there exists a CPU map $\Phi : B(S_1) \to B(S_2)$ such that $\Phi \circ \Phi_1|_{\mathrm{vN}(M)} = \mathscr{P}_2|_{\mathrm{vN}(M)}$, then there is a CPTPU map $\Theta : B(L^2(\mathbb{R})) \to B(L^2(\mathbb{R}))$, such that $\Theta(m) = m$ for all $m \in \mathrm{vN}(M)$ so that the effects of Θ belong to $\mathrm{vN}(M)$ and $\mathscr{P}_2 \circ \Theta = \Phi \circ \Phi_1$.*

Recall here that any completely positive map $\Phi : B(\mathscr{H}) \to B(\mathscr{H})$ can be expressed as $\Phi(A) = \sum_i E_i A E_i^*$ where the E_i are contractions in $B(\mathscr{H})$ and $\sum E_i E_i^* \leqslant \mathbb{1}$. If Φ is unital then it follows that $\sum E_i E_i^* = \mathbb{1}$. These operators are called the effects of Φ, or sometimes the Kraus operators of Φ and we write $\Phi \equiv \{E_i\}$. The set of effects of Φ is not unique, but two different sets of effects for Φ are related as described in Lemma 5 below.

Proof. Let $\mathscr{H} := L^2(\mathbb{R})$. We apply Lemma 3 with $\Phi_2 = \mathscr{P}_2$. By Lemma 4 the minimal SSD of $\mathscr{P}_2$ is $(\mathrm{id}, P_2, L^2(\mathbb{R}))$. By Lemma 3, there is a $*$-isomorphism $\pi : B(L^2(\mathbb{R})) \to B(L^2(\mathbb{R}))$ such that $\pi \circ \mathrm{id} = \pi'$, where π' is the minimal SSD of $\Phi \circ$

Φ_1, and $\pi|_{\mathrm{vN}(M)} = \pi \circ \mathrm{id}|_{\mathrm{vN}(M)} = \mathrm{id}|_{\mathrm{vN}(M)}$. Hence $\Theta := \mathscr{P}_{L^2(\mathbb{R})} \circ \pi' = \mathscr{P}_{L^2(\mathbb{R})} \circ \pi \circ \mathrm{id}$ is a CPU map (π_1 is the identity map) $\Theta : B(L^2(\mathbb{R})) \to B(L^2(\mathbb{R}))$ and we have that $\Theta|_{\mathrm{vN}(M)} = \pi_2|_{\mathrm{vN}(M)} = \mathrm{id}|_{\mathrm{vN}(M)}$.

In other words $\Theta(m) = m$ for all $m \in \mathrm{vN}(M)$ and hence if $\{E_i\}$ are the effects of Θ, then the E_i commute with spectral projections of M and must belong to $\mathrm{vN}(M)$ (this is not hard to show, see [7, pgs. 7–8]). In particular the effects of Θ are normal operators. Such a CP map is called hermitian. Given a completely positive map Φ on $B(\mathscr{H})$, one can define its dual $\Phi^\dagger : T(\mathscr{H}) \to T(\mathscr{H})$, with respect to the canonical trace on $B(\mathscr{H})$ by $\Phi^\dagger(T) \in T(\mathscr{H})$ is the unique trace-class operator obeying $\mathrm{Tr}(T\Phi(A)) = \mathrm{Tr}(\Phi^\dagger(T)A)$ for all $A \in B(\mathscr{H})$. Here $T(\mathscr{H})$ denotes the trace-class operators. It is easy to show that Φ is unital if and only if $\Phi^\dagger$ is trace-preserving, and vice versa. Since Θ is hermitian, it follows that $\Theta^\dagger$ is also unital. It follows that Θ is trace-preserving and unital, hence Θ is a CPTPU map, *i.e.* a quantum channel of $B(L^2(\mathbb{R}))$.

Now

$$\mathscr{P}_2 \circ \Theta = \mathscr{P}_2 \circ \pi \circ \pi_1 = \mathscr{P}_2 \circ \pi' = \Phi \circ \Phi_1, \tag{3.2}$$

and this completes the proof. $\square$

We will need the following fact which relates two different sets of effects which define the same CP map acting on $B(\mathscr{H})$ when $\mathscr{H}$ is separable.

LEMMA 5. *Let $\Phi : B(\mathscr{H}) \to B(\mathscr{H})$ be a normal CPU map and let $(E_l)_{l=1}^k$ and $(F_j)_{j=1}^l$ be two sets of effects for Φ. Then there is an isometry $U : l_k^2(\mathscr{H}) \to l_l^2(\mathscr{H})$ whose entries are scalars multiplied by the identity in $\mathscr{H}$ such that $U(E_l^*) = (F_j^*)$. Here (E_l^*) denotes the column vector with entries E_l^*. In particular the two sets of effects have the same closed linear span.*

Proof. In finite dimensions this is well-known to experts in quantum error correction, and the proof for the separable case is virtually identical. Here we sketch the proof.

Let $(\mathscr{K}, V, \pi)$ denote the minimal SSD of Φ so that $V : \mathscr{H} \to \mathscr{K}$ is an isometry such that $V\pi(A)V^* = \Phi(A)$. Since Φ is normal it follows that π is normal. Also since π is a minimal SSD of Φ, it is an irreducible normal representation of the type I factor $B(\mathscr{H})$.

It follows from the representation theory of factors of type I that we can assume that $\mathscr{K} = l_k^2(\mathscr{H}) \simeq \mathscr{H} \otimes l_k^2$ for some $k \in \mathbb{N} \cup \{\infty\}$ where l_k^2 is the Hilbert space of square summable sequences of length k, and that $\pi(A) = A \otimes \mathbb{1}$. Since $V : \mathscr{H} \to l_k^2(\mathscr{H})$ we can define $E_k^* : \mathscr{H} \to \mathscr{H}$ by choosing $E_k^* h = h_k$ where $Vh = (h_1, h_2, \ldots)$. The $\{E_k\}$ are a set of effects for Φ, *i.e.* $\Phi(A) = \sum_k E_k A E_k^*$, $\|E_k\| \leqslant 1$ and $\sum_k E_k E_k^* = \mathbb{1}$.

Now suppose that $\{F_j\}_{j=1}^n$ are another set of effects for Φ. Then we can construct a SSD of Φ by letting $\pi'(A) = A \otimes \mathbb{1}$ on $l_n^2(\mathscr{H}) =: \mathscr{K}'$ and defining $V' : \mathscr{H} \to \mathscr{K}'$ by $V'h = (F_1^* h, F_2^* h, \ldots)$. Now $(\mathscr{K}', V', \pi')$ contains a minimal SSD $(\mathscr{K}_2, V', \pi_2)$ (when constructing the minimal SSD from an arbitrary SSD, this does not change the isometry V', this can be observed from [11, pg. 46]) such that $\pi'(B(\mathscr{H}))V'\mathscr{H} = \mathscr{K}_2$.

By the uniqueness of the minimal SSD, there is a unitary operator $U : \mathscr{K} = l_j^2(\mathscr{H}) \to \mathscr{K}_2 \subset l_n^2(\mathscr{H})$ such that $\mathrm{Ad}_U \circ \pi = \pi_2$ and $UV = V'$. The first equation implies that if we write U as an $n \times j$ matrix with entries in $B(\mathscr{H})$, then each entry U_{ik} belongs to the commutant of $B(\mathscr{H})$ and hence must be a scalar times the identity. The second equation tells us that this scalar matrix multiplying the column vector $\{E_i^*\}$ equals the column vector $\{F_j^*\}$. In particular the $\{E_i\}$ and $\{F_i\}$ have the same closed linear span. $\square$

To apply the result of the previous proposition to the situation of the previous section, equation (2.14), we will need one final lemma:

LEMMA 6. *Consider* $\Phi_1 := \mathscr{P}_\theta \circ \mathrm{Ad}_{\sqrt{R}} : B(L^2(\mathbb{R})) \to B(K_\theta^2)$. *Then the minimal SSD's of both* Φ_1 *and* $\Phi_1|_{\mathrm{vN}(M)}$ *are both equal to* $(\mathrm{id}, \sqrt{R}P_\theta, L^2(\mathbb{R}))$, *where* id *denotes the identity isomorphism.*

Proof. Recall that $V = \sqrt{R}P_\theta : K_\theta^2 \to L^2(\mathbb{R})$ is an isometry. For any $A \in B(L^2(\mathbb{R}))$, we have that $V^*\mathrm{id}(A)V = \mathscr{P}_\theta \circ \mathrm{Ad}_{\sqrt{R}}(A) = \Phi_1(A)$, this shows that id is a SSD of Φ_1, and hence of $\Phi_1|_{\mathrm{vN}(M)}$. To show that this is minimal we need to show that both $B(L^2(\mathbb{R}))VK_\theta^2$ and $\mathrm{vN}(M)VK_\theta^2$ are dense in $L^2(\mathbb{R})$. Clearly the first set is dense in $L^2(\mathbb{R})$. Now it is not difficult to show that $L^2(\mathbb{R}) = \bigoplus_{k\in\mathbb{Z}} \theta^k K_\theta^2$. Since $\sqrt{R}$ is non-zero almost everywhere with respect to Lebesgue measure, it follows that $\mathrm{vN}(M)VK_\theta^2$ is dense in $L^2(\mathbb{R})$. $\square$

We now have all the necessary tools to provide a proof of the main theorem of this paper:

THEOREM 1. *Let* $S \subset L^2(\mathbb{R})$ *be a closed subspace. The multiplication operator* M *has a symmetric restriction* $M_S \in \mathscr{S}ym_1^R(S)$ *if and only if* $S = uhK_{\theta_S}^2$ *is nearly invariant for some meromorphic inner function* θ_S. *Moreover if* M *has such a restriction* M_S *then* θ_S *is the Livšic characteristic function of* M_S.

In the above u is a unimodular function and h is an isometric multiplier of K_θ^2 onto $\overline{u}S$ (so that $\frac{h}{z+i} \in \overline{u}S$). Recall that if θ is defined as in equation (2.11) of Section 2, that

$$\theta_S = \frac{\theta - \theta(i)}{1 - \overline{\theta(i)}\theta},$$

as discussed in Remark 4. Also recall that as discussed at the end of Subsection 1.1, if $S = uhK_\phi^2$ is nearly invariant for some meromorphic inner ϕ, then it is clear that M has a symmetric restriction $M_S \in \mathscr{S}ym_1^R(S)$. Also since M_S is unitarily equivalent to M_ϕ, the symmetric operator of multiplication by z in K_ϕ^2, it follows as in Remark 4 that the characteristic function of M_S will be ϕ. Hence to complete the proof of the above theorem it suffices to prove that $M_S \in \mathscr{S}ym_1^R(S)$ implies that $S = uhK_{\theta_S}^2$ for some unimodular u, and h an isometric multiplier of $K_{\theta_S}^2$ into H^2.

Proof. Given S, and $M_S \in \mathscr{S}ym_1^R(S)$, let θ be defined as in equation (2.11) of Section 2.

Let $S_1 := K_\theta^2$, $S_2 = S$, with projectors P_i. Let $\Phi_1 := \mathscr{P}_1 \circ \mathrm{Ad}_{\sqrt{R}}$, $\Phi_2 = \mathscr{P}_2$ and $\Phi = \mathrm{Ad}_{V_0^*}$. Then by equation (2.14) of the previous section, the previous lemma, and Remark 3, it follows that the conditions of Proposition 1 are satisfied, so that there is a quantum channel Θ on $B(L^2(\mathbb{R}))$ with effects $\{E_k\} \subset \mathrm{vN}(M)$ and $\mathscr{P}_2 \circ \Theta = \Phi \circ \Phi_1$. Taking adjoints yields $\Theta^\dagger \circ \mathscr{P}_2 = \Phi_1^\dagger \circ \Phi^\dagger$. Hence both $\{E_k^* P_2\}$ and $\{\sqrt{R} P_1 V_0\}$ are sets of effects for the same map, and so by Lemma 5, they must have the same linear span. This shows that for any k, there is an $\alpha_k \in \mathbb{C}$ so that $E_k^* P_2 = \alpha_i \sqrt{R} P_1 V_0 P_2$ (recall $V_0 : S_2 \to S_1$ is a partial isometry). Hence,

$$\left(E_k^* - \frac{\alpha_i}{\alpha_1} E_1^* \right) P_2 = 0, \tag{3.3}$$

and since $S = S_2$ is cyclic and separating for $\mathrm{vN}(M)$, we conclude that $E_k^* = \frac{\alpha_k}{\alpha_1} E_1^*$. Since Θ is unital, we have $1 = \sum |c_k|^2 ||E_1(x)|^2 =: k^2 |E_1(x)|^2$. This shows that $U := kE_1$ is a unimodular function such that $\Theta = \mathrm{Ad}_U$, so that Θ is actually a $*$-isomorphism. Now $\{UP_2\}$ and $\{\sqrt{R} P_1 V_0\}$ have the same linear span, and there is an $\alpha \in \mathbb{C}$ so that

$$\alpha U P_2 = \sqrt{R} P_1 V_0 = \sqrt{R} V_0. \tag{3.4}$$

Hence $V_0 = \frac{\alpha U}{\sqrt{R}} P_2$. Actually, since $\frac{U}{\sqrt{R}} P_2$ and V_0 are both partial isometries, it follows that $|\alpha|^2 = 1$ so we can assume $\alpha = 1$. This shows that multiplication by the function $U/\sqrt{R}$ is an isometry from S onto K_θ^2. Hence multiplication by $\overline{U}\sqrt{R}$ is an isometry from K_θ^2 onto S. Also by known results there is a function q such that multiplication by q is an isometry from $K_{\theta_S}^2$ onto K_θ^2, this mapping is called a Crofoot transform [14, Section 13]. It follows that if $g := q\overline{U}\sqrt{R}$, that multiplication by g is an isometry from $K_{\theta_S}^2$ onto S. Since $\theta_S(i) = 0$, $k_i(z) = \frac{i}{2\pi} \frac{1}{z+i}$ is the point evaluation vector at i in $K_{\theta_S}^2$, it follows that $\frac{g}{z+i} \in L^2(\mathbb{R})$. It follows that $S = gK_{\theta_S}^2$ is nearly invariant, and if uh is the Beurling-Nevanlinna factorization of $\frac{g}{z+i}$, $h \in H^2$, u unimodular, that $S' = \overline{u}S$ is a nearly invariant subspace of H^2 such that $S' = h(z+i)K_{\theta_S}^2$. Since M_S is unitarily equivalent to M_{θ_S}, it follows that the characteristic function of M_S is θ_S. □

COROLLARY 1. *If $R = 1$, then S is seminvariant.*

Here $S \subset L^2(\mathbb{R})$ is called seminvariant if it is seminvariant for the shift (multiplication by $\mu(x) = \frac{x-i}{x+i}$). Recall that a subspace is seminvariant for an operator if it is the direct difference of two invariant subpsaces, one of which contains the other. A subspace is seminvariant for the shift if and only if $S = uK_\theta^2$ where u is unimodular and θ is an inner function. This follows from the Beurling-Lax theorem, see for example the proof of [9, Theorem 5.2.2].

Proof. Suppose that $R = 1$. In this case $UP_2 = V_0$ (we can assume $\alpha = 1$), so that $U^* P_1 = V_0^*$ and $S = S_2 = U^* K_\theta^2$ where $U^* \in \mathrm{vN}(M)$ is unitary. □

It seems possible that the converse to the above corollary is also true.

COROLLARY 2. *If $S \subset L^2(\mathbb{R})$ is such that M has a restriction $M_S \in \mathscr{S}ym_1^R(S)$, then S is a reproducing kernel Hilbert space with a $\mathbb{T}$-parameter family of total orthogonal sets of point evaluation vectors.*

Proof. This follows as S is the image of K_θ^2 under an isometric multiplier and K_θ^2 has these properties when θ is inner and meromorphic. $\square$

4. Outlook

We have proven that a subspace $S \subset L^2(\mathbb{R})$ is nearly invariant with $S = hK_\theta^2$, and θ meromorphic and inner, $\theta(i) = 0$, if and only if the multiplication operator M has a restriction $M_S \in \mathscr{S}ym_1^R(S)$ with meromorphic inner characteristic function θ. We expect a similar result to hold whenever θ is inner and not necessarily meromorphic, and perhaps an analogous result could be established for arbitrary contractive analytic θ. However to generalize the approach presented here would require generalizing Krein's results of Section 2 to the case of more general contractive analytic functions.

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