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BISHOP'S PROPERTY (β), HYPERCYCLICITY AND HYPERINVARIANT SUBSPACES

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Abstract. The question whether every operator on H has an hyperinvariant subspace is one of the most difficult problems in operator theory. The purpose of this paper is to make a beginning on the hyperinvariant subspace problems for another class of operators closely related to the normal operators namely, the class of k-quasi-class A operators. A necessary and sufficient condition for the hypercyclicity of the adjoint of a quasi-class A operator is also presented.

1. Introduction and Preliminaries

Let $B(H)$ be the algebra of all bounded linear operators acting on infinite dimensional separable complex Hilbert space H. Let T be an operator in $B(H)$. An operator is said to be positive (denoted $T \geqslant 0$) if $(Tx,x) \geqslant 0$ for all $x \in H$. The operator T is said to be a $p-$ hyponormal operator if and only if $(T^*T)^p \geqslant (TT^*)^p$ for a positive number p. In [30] is defined the class of log-hyponormal operators as follows: T is a log-hyponormal operator if it is invertible and satisfies the following relation $\log T^*T \geqslant \log TT^*$. Class of p-hyponormal operators and class of log-hyponormal operators were defined as extension class of hyponormal operators, i.e, $T^*T \geqslant TT^*$. It is well known that every p-hyponormal operator is a q- hyponormal operator for $p \geqslant q > 0$, by the Löwner-Heinz theorem "$A \geqslant B \geqslant 0$ ensures $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0,1]$", and every invertible p-hyponormal operator is a log-hyponormal operator since $\log$ is an operator monotone function. An operator T is paranormal if

$$||Tx||^2 \leqslant ||T^2x||||x||$$

for all $x \in H$. Let T be an operator whose polar decomposition $T = U|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is a partial isometry with $\ker U = \ker(|T|) = \ker T$. Associated with T is a related operator $|T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ called the Aluthge transform of T denoted by $\widehat{T}$ [2]. For every $T \in B(H)$ the sequence $\{\widehat{T}^{(n)}\}$ of Aluthge iterates of T is defined

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by $\widehat{T}^0 = T$ and $\widehat{T}^{n+1} = \widehat{\widehat{T}^{(n)}}$ for every nonnegative integer n. Aluthge an Wang [3] introduced ω-hyponormal operators defined as follows: An operator T is said to be ω-hyponormal if $|\widehat{T}| \geqslant |T| \geqslant |\widehat{T^*}|$. An operator T such that $|T^2| \leqslant |T|^2$ is called of class A. In [9] authors, Furuta, Ito, Yamazaki introduced the class A operators which includes the class of log-hyponormal operators (see Theorem 2, in [9]) and is included in the class of paranormal operators (see Theorem 1 in [9]). I. Jean and I. Kim [18] introduced quasi-class A operators which includes class A operators. An operator T is said to be quasi-class A if

$$T^*(|T^2| - |T|^2)T \geqslant 0.$$

As a further generalization of both class A operators and quasi-class A operators F. Gao and X. Fang [14] introduced the notion of k-quasi-class A operators. An operator T is called k-quasi-class A if

$$T^{*k}(|T^2| - |T|^2)T^k \geqslant 0$$

where k is a natural number. It is clear that

$$\begin{aligned} &\text{hyponormal} \subseteq p-\text{hyponormal} \subseteq \text{class}\,A\,\text{operators} \\ &\subseteq \text{quasi-class}\,A\,\text{operators} \subseteq k-\text{quasi-class}\,A\,\text{operators}. \end{aligned}$$

EXAMPLE 1.1. Given a bounded sequence of positive numbers $\{\alpha_i\}_{i=0}^{\infty}$, and let T be the unilateral weighted shift operator on l^2 with the canonical orthonormal basis $\{e_n\}_{i=0}^{\infty}$ defined by $Te_n = \alpha_n e_{n+1}$ for all $n \geqslant 0$, that is,

$$T = \begin{pmatrix} 0 & & & & \\ & 0 & & & \\ & \alpha_0 & 0 & & \\ & & \alpha_1 & 0 & \\ & & & \alpha_2 & \ddots \\ & & & & \ddots \\ & & & & \end{pmatrix}$$

Straightforward calculations show that T is a k-quasi-class A operator if and only if $\alpha_k \leqslant \alpha_{k+1} \leqslant \alpha_{k+2} \cdots$. So if $\alpha_{k+1} \leqslant \alpha_{k+2} \leqslant \alpha_{k+3} \cdots$ and $\alpha_k > \alpha_{k+1}$, then T is a $(k+1)$-quasi-class A operator, but is not a k-quasi-class A operator. Thus the following inclusions are strict:

$$\begin{aligned} &\text{hyponormal operator} \subset p-\text{hyponormal operator} \subset \text{class}\,A\,\text{operator} \\ &\subset \text{quasi-class}\,A\,\text{operator} \subset k-\text{quasi-class}\,A\,\text{operator}. \end{aligned}$$

An operator $T \in B(H)$ is said to have the single-valued extension property (or SVEP) if for every open subset G of $\mathbb{C}$ and any analytic function $f : G \to H$ such that $(T-z)f(z) \equiv 0$ on G, we have $f(z) \equiv 0$ on G. For $T \in B(H)$ and $x \in H$, the set $\rho_T(x)$

is defined to consist of elements $z_0 \in \mathbb{C}$ such that there exists an analytic function $f(z)$ defined in a neighborhood of z_0, with values in H, which verifies $(T-z)f(z) = x$, and it is called the local resolvent set of T at x. We denote the complement of $\rho_T(x)$ by $\sigma_T(x)$, called the local spectrum of T at x, and define the local spectral subspace of T, $H_T(F) = \{x \in H : \sigma_T(x) \subset F\}$ for each subset F of $\mathbb{C}$. An operator $T \in B(H)$ is said to have Bishop's property (β) if for every open subset G of $\mathbb{C}$ and every sequence $f_n : G \to H$ of H-valued analytic functions such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of G, $f_n(z)$ converges uniformly to 0 in norm on compact subsets of G. An operator $T \in B(H)$ is said to have Dunford's property (C) if $H_T(F)$ is closed for each closed subset F of $\mathbb{C}$. It is well known that

$$\text{Bishop's property } (\beta) \Rightarrow \text{ Dunford's property}(C) \Rightarrow \text{ SVEP.}$$

For more details about Bishop's property (β). The interested reader is referred to [23, 24] for more details.

A closed subspace of H is said to be hyperinvariant for T if it is invariant under every operator in the commutant $\{T\}'$ of T. The question whether every operator on H has an hyperinvariant subspace is one of the most difficult problems in operator theory. Our principal objective in the present paper is to derive the existence of nontrivial hyperinvariant subspace of k-quasi-class A operators. It is known that Every operator which commutes with a (nonzero) compact operator has a (proper closed) hyperinvariant subspace [29]. In [17] it is shown that every non scalar n-normal operators has nontrivial hyperinvariant subspace (cf. also [26, p.76] and [20]). The corresponding problem for subnormal operators remains unsolved (cf.[20]). (Recall that an n-normal operator may be defined as an $n \times n$ operator matrix with entries are mutually commuting normal operators, and a subnormal operator is the restriction of a normal operator to an invariant subspace.) But, in [12] the authors study the hyperinvariant subspace problem for subnormal operators. They showed that every normalized subnormal operator such that either $\{S^{*n}S^n\}^{\frac{1}{n}}$ does not converge in the SOT to the identity operator or $\{S^nS^{*n}\}^{\frac{1}{n}}$ does not converge in the SOT to zero has a nontrivial hyperinvariant subspace. The purpose of this paper is to make a beginning on the hyperinvariant subspace problem for another class of operators closely related to the normal operators namely, the class of k-quasi-class A operators.

2. Main Results

In the sequel we need the following lemmas.

LEMMA 2.1. [14] *Let $T \in B(H)$ be k-quasi-class A operator, the range of T^k be not dense and*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad H = [\operatorname{ran} T^k] \oplus N(T^{*k}).$$

Then T_1 is a class A operator, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

LEMMA 2.2. [14] *Let $\mathscr{M}$ be a closed T-invariant subspace of H. Then the restriction $T|_{\mathscr{M}}$ of a k-quasi-class A operator T to $\mathscr{M}$ is a k-quasi-class A operator.*

THEOREM 2.1. *Every k-quasi-class A operator has Bishop's property (β).*

Proof. If the range of T^k is dense, then T is a class A operator. Hence, T has Bishop's property (β) by [19]. So, we assume that the range of T^k is not dense. Let $(T-z)f_n(z) \to 0$ uniformly on every compact subset of D for analytic functions $f_n(z)$ on D. Then we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3 - z)f_{n2}(z) \end{pmatrix} \to 0.$$

Since T_3 is nilpotent, T_3 has Bishop's property (β). Hence $f_{n2}(z) \to 0$ uniformly on every compact subset of D. Then $(T_1 - z)f_{n1}(z) \to 0$. Since T_1 is a class A operator, T_1 has Bishop's property (β) by [19]. Hence $f_{n1}(z) \to 0$ uniformly on every compact subset of D. Thus T has Bishop's property (β). $\square$

For $k > 1$, a nilpotent operator is k-quasi-class A. This shows that operators in this class need not be normaloid. But a quasi-class A operator is normaloid as we will show in the following theorem. For this we need the following lemma.

LEMMA 2.3. *If T is k-quasi-class A, then $r(T) \geqslant \frac{||T^n||}{||T^{n-1}||}$ for every positive integer $n \geqslant k+1$.*

Proof. Since

$$\frac{||T^{k+n}||}{||T^{k+n-1}||} \geqslant \frac{||T^{k+n-1}||}{||T^{k+n-2}||} \geqslant \ldots \geqslant \frac{||T^{k+1}||}{||T^k||},$$

$$\frac{||T^{k+n}||}{||T^{k+n-1}||} \geqslant \left(\frac{||T^{k+1}||}{||T^k||}\right)^n.$$

Thus

$$||T^n|| \geqslant \left(\frac{||T^{k+1}||}{||T^k||}\right)^{n-k}$$

or

$$||T^n||^{\frac{1}{n}} \geqslant \left(\frac{||T^{k+1}||}{||T^k||}\right)^{1-\frac{k}{n}}.$$

Letting $n \to \infty$, we get

$$r(T) \geqslant \frac{||T^{k+1}||}{||T^k||}. \tag{2.1}$$

Similarly

$$r(T) \geqslant \frac{||T^{k+2}||}{||T^{k+1}||}.$$

In general,

$$r(T) \geqslant \frac{||T^n||}{||T^{n-1}||}$$

for every positive integer $n \geqslant k+1$. $\square$

THEOREM 2.2. *Every quasi-class A operator is normaloid, that is, $||T|| = r(T)$ (the spectral radius of T).*

Proof. It suffices to take $k = 1$ in (2.1). $\square$

Recall that an operator $X \in B(H,K)$ is called a quasiaffinity if it has trivial kernel and dense range. An operator $S \in B(H)$ is said to be a quasiaffine transform of $T \in B(K)$ if there is a quasiaffinity $X \in B(H,K)$ such that $XS = TX$. Furthermore, S and T are quasisimilar if there are quasiaffinities X and Y such that $XS = TX$ and $SY = YT$.

Now we will show that a k-quasi-class A operator with certain conditions has a nontrivial hyperinvariant subspace.

THEOREM 2.3. *Let $T \in B(H)$ be a k-quasi-class A operator such that $T \neq zI$ for all $z \in \mathbb{C}$. If S is a decomposable quasiaffine transform of T, then T has a nontrivial hyperinvariant subspace.*

Proof. If S is a decomposable quasiaffine transform of T, then there exists a quasiaffinity X such that $XS = TX$, where S is decomposable. Assume that T has no nontrivial hyperinvariant subspace. Then $\sigma_p(T) = \emptyset$ and $H_T(F) = \{0\}$ for each closed set F proper in $\sigma(T)$ [21, Lemma 3.6.1]. Let $\{U,V\}$ be an open cover of $\mathbb{C}$ such that $\sigma(T)\setminus\overline{U} \neq \emptyset$ and $\sigma(T)\setminus\overline{V} \neq \emptyset$. Now if $x \in H_S(\overline{U})$, then $\sigma_S(x) \subset \overline{U}$. Hence there exists an analytic H-valued function f defined on $\mathbb{C}\setminus\overline{U}$ such that $(S-z)f(z) \equiv x$ for all $z \in \mathbb{C}\setminus\overline{U}$. So $(T-z)Xf(z) = X(S-z)f(z) = Xx$. Therefore, $\mathbb{C}\setminus\overline{U} \subset \rho_T(Xx)$. This implies that $Xx \in H_T(\overline{U})$, that is, $XH_S(\overline{V}) \subset H_T(\overline{V})$. Since S is decomposable,

$$XH = XH_S(\overline{U}) + XH_S(\overline{V}) \subseteq H_T(\overline{U}) + H_T(\overline{V}) = \{0\}.$$

This is a contradiction. Hence T has a nontrivial hyperinvariant subspace. $\square$

Note that Theorem 2.3 should be compared with the Theorem 4.5 on p. 56 of the Colojoara-Foias Book [7].

COROLLARY 2.1. *Let $T \in B(H)$ be a class A operator such that $T \neq zI$ for all $z \in \mathbb{C}$. If S is a decomposable quasiaffine transform of T, then T has a nontrivial hyperinvariant subspace.*

THEOREM 2.4. *Let $T \in B(H)$ be a k-quasi-class A operator such that $T \neq zI$ for all $z \in \mathbb{C}$. If $\lim_{n\to\infty} ||T^n x||^{\frac{1}{n}} < ||T||$ for some nonzero $x \in H$, then T has a nontrivial hyperinvariant subspace.*

Proof. Assume that $\lim_{n\to\infty}||T^n x||^{\frac{1}{n}} < ||T||$ for some nonzero $x \in H$. Since T is a k-quasi-class A operator,

$$||T^n x||^2 \leqslant ||T^{n+1}x||||T^{n-1}x||$$

for every positive integer n and every $x \in H$ from [14]. Hence, [5, Proposition 4.6]and [5, Remark] imply that T has a nontrivial hyperinvariant subspace. $\square$

COROLLARY 2.2. *Let $T \in B(H)$ be a class A operator such that $T \neq zI$ for all $z \in \mathbb{C}$. If $\lim_{n\to\infty}||T^n x||^{\frac{1}{n}} < ||T||$ for some nonzero $x \in H$, then T has a nontrivial hyperinvariant subspace.*

THEOREM 2.5. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(H \oplus H).$$

If T has Bishop's property β and there exists a non zero $x \in H \oplus H$ such that $\sigma_T(x) \subsetneqq \sigma(T)$. Then T has a nontrivial hyperinvariant subspace.

Proof. Assume that

$$\mathscr{M} = \{y \in H \oplus H : \sigma_T(y) \subseteq \sigma_T(x)\},$$

that is, $\mathscr{M} = H_T(\sigma_T(x))$. Since T has Bishop's property β, hence T has Dunford's property (C). It follows from [7] that $\mathscr{M}$ is a T-hyperinvariant subspace. Since $x \in \mathscr{M}$, we have $\mathscr{M} \neq \{0\}$. Now, set $\mathscr{M} = H \oplus H$. Since T has the single extension property, we get $\sigma(T) = \bigcup\{\sigma_T(y) : y \in H \oplus H\} \subseteq \sigma_T(x) \subsetneqq \sigma(T)$ from [22]. This is a contradiction. Hence $\mathscr{M}$ is a nontrivial T-hyperinvariant subspace. $\square$

Since a k-quasi-class A operator has Bishop's property β by Theorem 2.1, by applying Lemma 2.1 and Theorem 2.5 we get the following corollary.

COROLLARY 2.3. *Let*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \in B(H \oplus H).$$

be k-quasi-class A. If there exists a non zero $x \in H \oplus H$ such that $\sigma_T(x) \subsetneqq \sigma(T)$, then T has a nontrivial hyperinvariant subspace.

Let $T \in B(H)$ and $x \in H$. Then $\{T^n x\}_{n=0}^{\infty}$ is called the orbit of x under T, and is denoted by $O(x,T)$. If $O(x,T)$ is dense in H, then x is called a hypercyclic vector for T

Now we are ready to prove a necessary and sufficient condition for the hypercyclicity of the adjoint of a quasi-class A operator. Recall that if T is an invertible quasi-class A operator, then T and T^{-1} are class A operators [14, 15].

THEOREM 2.6. *Let $T \in B(H)$ be a quasi-class A operator. Then T^* is hypercyclic if and only if $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in H$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

Proof. Assume that T^* is hypercyclic. By [13, Proposition 2.3], it suffices to show that $\sigma(T)$ meets both $\mathbb{D}$ and $\mathbb{C} \setminus \overline{D}$. Let $S = T|_M$ for some closed T-invariant subspace M and let x be a hypercyclic vector for T^*. Since $(S^*)^n Px = P(T^*)^n x$ for each nonnegative integer n where P is the orthogonal projection of H onto M, we have

$$\overline{\{(S^*)^n(Px)\}_{n=0}^{\infty}} = P(\overline{\{(T^*)^n x\}_{n=0}^{\infty}}) = P(H) = M.$$

Thus Px is hypercyclic for S^*. Since S is quasi-class A and normaloid by Theorem 2.2 and Lemma 2.2, $r(S) = ||S|| = ||S^*|| > 1$ [25]. Hence $\sigma(T) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$. Now we have to prove that $\sigma(S) \cap \mathbb{D} \neq \emptyset$. For this, assume that $\sigma(S) \subset \mathbb{C} \setminus \overline{\mathbb{D}}$. Since S^{-1} is a class A operator [14] and $\sigma(S^{-1}) \subset \overline{\mathbb{D}}$, we have, $||S^{-1}|| = r(S^{-1}) \leqslant 1$. Since S^* is hypercyclic and invertible, $(S^*)^{-1}$ is hypercyclic [25]. Hence $||S^{-1}|| = ||(S^*)^{-1}|| > 1$ [25]. This is a contradiction, and so $\sigma(S) \cap \mathbb{D} \neq \emptyset$. For the converse, assume that $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in H$. Then $H_T(\mathbb{C} \setminus \mathbb{D}) = \{0\}$ and $H_T(\overline{\mathbb{D}}) = \{0\}$. Since T has property (β) Theorem 2.1, T^* has the property (δ). Thus both $H_{T^*}(\mathbb{C} \setminus \overline{\mathbb{D}})$ and $H_{T^*}(\overline{\mathbb{D}})$ are dense in H by [22, Proposition 2.5.14]. Hence T^* is hypercyclic by [13, Theorem 3.2]. $\square$

COROLLARY 2.4. *Let $T \in B(H)$ be a class A operator. Then T^* is hypercyclic if and only if $\sigma_T(x) \cap \mathbb{D} \neq \emptyset$ and $\sigma_T(x) \cap (\mathbb{C} \setminus \overline{\mathbb{D}}) \neq \emptyset$ for all nonzero $x \in H$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.*

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