GREEN'S FUNCTION, RESOLVENT, PARSEVAL EQUALITY OF DIFFERENTIAL OPERATOR WITH BLOCK–TRIANGULAR MATRIX COEFFICIENTS

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We dedicate this work to our dear teacher,
Academician Vladimir Aleksandrovich Marchenko
in honour of his glorious anniversary,
with deep respect and the best wishes

(*Communicated by F. Gesztesy*)

Abstract. Green's function for the Sturm-Liouville operator with a block-triangular matrix potential growing at infinity is constructed. For this function, a series expansion is obtained and the Parseval equality is proved.

V. A. Marchenko introduced a notion of generalized spectral function R for a Sturm-Liouville operator with arbitrary complex valued potential on the semiaxis [9, 10], which was transferred under his scientific supervision to the case of non-selfadjoint systems [8, 11]. The distribution (the matrix in the case of systems) R acts on the topological space of test functions. The spectral distribution R determines formulas of expansion in eigenfunctions and also allows one to solve the inverse problem of spectral analysis in the non-selfadjoint case. In the case of selfadjoint problems R is generated by a non-negative measure (either scalar one or matricial in the case systems). We are interested in clarifying a specific form of spectrum and a spectral matricial distribution R for some classes of non-selfadjoint systems. While solving in [3] the inverse scattering problem on semiaxis, in the case of triangular matrix potentials, the Parseval equality is produced, and thus a form of spectral matrix distribution of V. A. Marchenko type is found (for the selfadjoint case see [1]). In the work [6] we obtain conditions which guarantee discreteness of spectrum for a wide class of Sturm-Liouville operators on the semiaxis with a triangular matrix potential which diagonal blocks are Hermitian matrices. For those potentials, a form of V. A. Marchenko type spectral matricial distribution is found. Note that this distribution, under presence of multiple poles for the resolvent, does not reduce to a matrix measure, even non-selfadjoint one. These results are applied to transferring the Sturm type oscillation theory from selfadjoint systems [13] onto systems with a triangular matrix potential [5].

I. Consider the equation with a block-triangular matrix potential:

$$l[\bar{y}] = -\bar{y}'' + V(x)\bar{y} = \lambda\bar{y}, \qquad 0 \leqslant x < \infty, \tag{1}$$

where

$$V(x) = v(x) \cdot I_m + U(x), \qquad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & \dots & U_{1r}(x) \\ 0 & U_{22}(x) & \dots & U_{2r}(x) \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ 0 & 0 & \dots & U_{rr}(x) \end{pmatrix}, \tag{2}$$

$v(x)$ is a real scalar function, $0 < v(x) \to \infty$ monotonically as $x \to \infty$, $v(x)$ has monotone absolutely continuous derivative, and $U(x)$ is a sufficiently small perturbation, for example, $U(x) \cdot v^{-1}(x) \to 0$ as $x \to \infty$, or $|U| \cdot v^{-1} \in L^\infty(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x)$, $k = \overline{1,r}$ are Hermitian matrices of order $m_k \geqslant 1$ (in particular, for $m_k = 1$ they are real scalar functions). Let $\sum_{k=1}^{r} m_k = m$, and let I_m be a unit matrix of order m.

Denote by H_m a finite-dimensional Hilbert space of order m. A vector $\bar{h} \in H_m$ will be written as $\bar{h} = \mathrm{col}\left(\bar{h}_1, \bar{h}_2, \dots, \bar{h}_r\right)$, where $\bar{h}_k$, $k = \overline{1,r}$ is a vector from H_{m_k}. Thus, $\bar{y} = \mathrm{col}(\bar{y}_1, \bar{y}_2, \dots, \bar{y}_r)$, where $\bar{y}_k \in H_{m_k}$.

In the case of

$$v(x) \geqslant Cx^{2\alpha}, \qquad C > 0, \quad \alpha > 1, \tag{3}$$

we suppose that the coefficients of the equation (1) satisfy the relations:

$$\int_0^\infty |U(t)| \cdot v^{-\frac{1}{2}}\, dt < \infty, \tag{4}$$

$$\int_0^\infty v'^2(t) \cdot v^{-\frac{5}{2}}(t)\, dt < \infty, \qquad \int_0^\infty v''(t) \cdot v^{-\frac{3}{2}}(t)\, dt < \infty. \tag{5}$$

(Note that spectral properties of a one-dimensional Schrödinger operator with polynomial potential were studied in [4].)

Consider the functions

$$\gamma_0(x, \lambda) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(-\int_0^x \sqrt{v(u)}\, du\right), \tag{6}$$

$$\gamma_\infty(x, \lambda) = \frac{1}{\sqrt[4]{4v(x)}} \cdot \exp\left(\int_0^x \sqrt{v(u)}\, du\right). \tag{7}$$

It can be easily seen that

$$\gamma_0(x, \lambda) \to 0, \qquad \gamma_\infty(x, \lambda) \to \infty \qquad \text{as} \qquad x \to \infty.$$

These solutions form the fundamental system of solutions of the scalar differential equation

$$-z'' + (v(x) + q(x))z = 0, \tag{8}$$

where $q(x)$ is defined by

$$q(x) = \frac{5}{16}\left(\frac{v'(x)}{v(x)}\right)^2 - \frac{1}{4}\frac{v''(x)}{v(x)}. \tag{9}$$

(cf. [14, 12]).

In this case,

$$W\{\gamma_0, \gamma_\infty\} = \gamma_0(x,\lambda)\cdot\gamma_\infty'(x,\lambda) - \gamma_0'(x,\lambda)\cdot\gamma_\infty(x,\lambda) = 1 \qquad \text{for all} \qquad x \in [0,\infty). \tag{10}$$

In the case of $v(x) = x^{2\alpha}$, $0 < \alpha \leqslant 1$, suppose that the coefficients of the equation (1) satisfy the relation

$$\int_0^\infty |U(t)|\cdot t^{-\alpha}\,dt < \infty. \tag{11}$$

Now define functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ as follows:

$$\gamma_0(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha}-\lambda)}}\cdot\exp\left(-\int_a^x\sqrt{u^{2\alpha}-\lambda}\,du\right),$$

$$\gamma_\infty(x,\lambda) = \frac{1}{\sqrt[4]{4(x^{2\alpha}-\lambda)}}\cdot\exp\left(\int_a^x\sqrt{u^{2\alpha}-\lambda}\,du\right).$$

These functions form the fundamental system of solutions of the scalar differential equation

$$-z'' + (x^{2\alpha} - \lambda + q(x,\lambda))z = 0 \tag{12}$$

as well, where $q(x,\lambda)$ is defined similarly as the function $q(x)$ in the formula (9):

$$q(x,\lambda) = \frac{5}{16}\left(\frac{v'(x)}{v(x)-\lambda}\right)^2 - \frac{1}{4}\frac{v''(x)}{v(x)-\lambda}.$$

In this case, the relation (10) holds true.

If $\dfrac{\alpha+1}{2\alpha} = n \in \mathbb{N}$, i.e., $\alpha = \dfrac{1}{2n-1}$, the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ will have the following asymptotics as $x \to \infty$ (see [6]; in the monograph [14], by Langer's technique, the asymptotics was established in another form, with the use of Henkel's functions):

$$\gamma_0(x,\lambda) = c\cdot\exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1}\frac{1\cdot3\cdot\ldots\cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\times x^{\frac{1\cdot3\cdot\ldots\cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n - \frac{\alpha}{2}}\cdot(1+o(1)), \tag{13}$$

$$\gamma_\infty(x,\lambda) = c\cdot\exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2}\cdot\frac{x^{1-\alpha}}{1-\alpha} - \sum_{k=2}^{n-1}\frac{1\cdot3\cdot\ldots\cdot(2k-3)}{k!}\cdot\left(\frac{\lambda}{2}\right)^k\cdot\frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\times x^{-\left(\frac{1\cdot3\cdot\ldots\cdot(2n-3)}{n!}\cdot\left(\frac{\lambda}{2}\right)^n + \frac{\alpha}{2}\right)}\cdot(1+o(1)). \tag{14}$$

In particular, for $\alpha = 1$ $(n = 1)$ these expressions have the form[1]

$$\gamma_0(x,\lambda) = c \cdot x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)(1+o(1)), \tag{15}$$

$$\gamma_\infty(x,\lambda) = c \cdot x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right)(1+o(1)). \tag{16}$$

If $\dfrac{\alpha+1}{2\alpha} \notin \mathbb{N}$, then, setting $n = \left[\dfrac{\alpha+1}{2\alpha}\right] + 1$, where $[\beta]$ stands for an integral part of a number β, we obtain the following asymptotics for the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ at infinity:

$$\gamma_0(x,\lambda) = c \cdot x^{-\frac{\alpha}{2}} \exp\left(-\frac{x^{1+\alpha}}{1+\alpha} + \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} + \sum_{k=2}^{n-1} \frac{1\cdot 3\cdot\ldots\cdot(2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\times \exp\left(-\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \frac{x^{-\alpha}}{\alpha}\right) \cdot \left(1+o\left(x^{-\alpha}\right)\right), \tag{17}$$

$$\gamma_\infty(x,\lambda) = c \cdot x^{-\frac{\alpha}{2}} \exp\left(\frac{x^{1+\alpha}}{1+\alpha} - \frac{\lambda}{2} \cdot \frac{x^{1-\alpha}}{1-\alpha} - \sum_{k=2}^{n-1} \frac{1\cdot 3\cdot\ldots\cdot(2k-3)}{k!} \cdot \left(\frac{\lambda}{2}\right)^k \cdot \frac{x^{1-(2k-1)\alpha}}{1-(2k-1)\alpha}\right)$$

$$\times \exp\left(\frac{1\cdot 3\cdot\ldots\cdot(2n-3)}{n!} \cdot \left(\frac{\lambda}{2}\right)^n \cdot \frac{x^{-\alpha}}{\alpha}\right) \cdot \left(1+o\left(x^{-\alpha}\right)\right). \tag{18}$$

In particular, for $\alpha = \dfrac{1}{2}$ $(n = 2)$ these formulas have the form

$$\gamma_0(x,\lambda) = cx^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3}x^{\frac{3}{2}} + \lambda x^{\frac{1}{2}} - \left(\frac{\lambda}{2}\right)^2 x^{-\frac{1}{2}}\right) \cdot \left(1+o\left(x^{-\frac{1}{2}}\right)\right), \tag{19}$$

$$\gamma_\infty(x,\lambda) = cx^{-\frac{1}{4}} \cdot \exp\left(\frac{2}{3}x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}} + \left(\frac{\lambda}{2}\right)^2 x^{-\frac{1}{2}}\right) \cdot \left(1+o\left(x^{-\frac{1}{2}}\right)\right). \tag{20}$$

In [6], there were proved both Theorem 1 and Corollaries 1, 2 cited below.

THEOREM 1. *Suppose that, for the equation* (1), *there hold either the conditions* (3), (4), (5) *for* $\alpha > 1$, *or condition* (11) *for* $0 < \alpha \leqslant 1$. *Then the equation* (1) *has a unique* $m \times m$ *matrix solution* $\Phi(x,\lambda)$ *decreasing at infinity and satisfying the relation*

$$\lim_{x\to\infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I_m \tag{21}$$

such that

$$\lim_{x\to\infty} \frac{\Phi'(x,\lambda)}{\gamma_0'(x,\lambda)} = I_m. \tag{22}$$

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[1]For $\alpha = 1$ and $\alpha = \dfrac{1}{2}$, i.e., for $v(x) = x^2$ and $v(x) = x$, the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$ is known.

Also, this equation has $m \times m$ matrix solution $\Psi(x,\lambda)$ growing at infinity and satisfying the relation

$$\lim_{x \to \infty} \frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_m \tag{23}$$

such that

$$\lim_{x \to \infty} \frac{\Psi'(x,\lambda)}{\gamma'_\infty(x,\lambda)} = I_m. \tag{24}$$

COROLLARY 1. *If $\alpha = 1$, i.e., the coefficient $v(x) = x^2$, then, under the condition (11), the equation (1) has a unique $m \times m$ matrix solution $\Phi(x,\lambda)$ decreasing at infinity and satisfying the relation*

$$\lim_{x \to \infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I_m, \tag{25}$$

where $\gamma_0(x,\lambda) = x^{\frac{\lambda-1}{2}} \cdot \exp\left(-\frac{x^2}{2}\right)$, such that

$$\lim_{x \to \infty} \frac{\Phi'(x,\lambda)}{\gamma'_0(x,\lambda)} = I_m.$$

Also, this equation has $m \times m$ matrix solution $\Psi(x,\lambda)$ growing at infinity and satisfying the relation

$$\lim_{x \to \infty} \frac{\Psi(x,\lambda)}{\gamma_\infty(x,\lambda)} = I_m$$

such that

$$\lim_{x \to \infty} \frac{\Psi'(x,\lambda)}{\gamma'_\infty(x,\lambda)} = I_m,$$

where $\gamma_\infty(x,\lambda) = x^{-\frac{\lambda+1}{2}} \cdot \exp\left(\frac{x^2}{2}\right)$.

COROLLARY 2. *If $\alpha = \frac{1}{2}$, i.e., the coefficient $v(x) = x$, then, under the condition (11), the equation (1) has a unique $m \times m$ matrix solution $\Phi(x,\lambda)$ decreasing at infinity and satisfying the relation*

$$\lim_{x \to \infty} \frac{\Phi(x,\lambda)}{\gamma_0(x,\lambda)} = I_m,$$

where $\gamma_0(x,\lambda) = x^{-\frac{1}{4}} \cdot \exp\left(-\frac{2}{3}x^{\frac{3}{2}} + \lambda x^{\frac{1}{2}}\right)$, such that

$$\lim_{x \to \infty} \frac{\Phi'(x,\lambda)}{\gamma'_0(x,\lambda)} = I_m.$$

Also, this equation has $m \times m$ matrix solution $\Psi(x, \lambda)$ growing at infinity and satisfying the relation

$$\lim_{x \to \infty} \frac{\Psi(x, \lambda)}{\gamma_\infty(x, \lambda)} = I_m$$

such that

$$\lim_{x \to \infty} \frac{\Psi'(x, \lambda)}{\gamma'_\infty(x, \lambda)} = I_m,$$

where $\gamma_\infty(x, \lambda) = x^{-\frac{1}{4}} \cdot \exp\left(\frac{2}{3} x^{\frac{3}{2}} - \lambda x^{\frac{1}{2}}\right).$

II. Let the following boundary condition be given at $x = 0$:

$$\cos A \cdot \bar{y}'(0) - \sin A \cdot \bar{y}(0) = 0, \tag{26}$$

where A is a block-triangular matrix of a similar structure as the coefficients of the differential equation,

$$A = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1r} \\ 0 & A_{22} & \dots & A_{2r} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A_{rr} \end{pmatrix}, \tag{27}$$

and A_{kk}, $k = \overline{1, r}$ are Hermitian matrices of order $m_k \geqslant 1$, $\sum_{k=1}^{r} m_k = m$.

Together with the problem (1), (26), we consider the separated system

$$l_k[\bar{y}_k] = -\bar{y}_k'' + \left(v(x) I_{m_k} + U_{kk}(x)\right) \bar{y}_k = \lambda \bar{y}_k, \qquad k = \overline{1, r},$$

with the boundary conditions

$$\cos A_{kk} \cdot \bar{y}_k'(0) - \sin A_{kk} \cdot \bar{y}_k(0) = 0, \qquad k = \overline{1, r}. \tag{28}$$

Denote by L_0 the minimal differential operator generated by the differential expression $l[\bar{y}]$ and the boundary condition (26), and denote by L_k, $k = \overline{1, r}$ the minimal symmetric operators on $L_2\left(H_{m_k}, (0, \infty)\right)$ generated by the differential expressions $l_k[\bar{y}_k]$ and the boundary conditions (28). Taking into account the conditions on coefficients, as well as a sufficient smallness of the perturbations $U_{kk}(x)$ $(U_{kk}(x) \cdot v^{-1}(x) \to 0$ as $x \to \infty$, or $|U_{kk}| \cdot v^{-1} \in L^\infty(\mathbb{R}_+))$, we conclude that, for every symmetric operator L_k, $k = \overline{1, r}$, there is the case of a limit point at infinity. Hence their self-adjoint extensions[2] $\widetilde{L}_k$ are the closures of operators L_k respectively. The operators $\widetilde{L}_k$ are semi-bounded, and their spectra are discrete.

Denote by L the extension of the operator L_0 generated by the requirement on functions from the domain of the operator L to belong to $L_2\left(H_m, (0, \infty)\right)$.

The following theorem is proved in [6].

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[2]A self-adjointness of the general Sturm-Liouville differential equations (with four terms) having matrix coefficients were studied in [7].

THEOREM 2. *Suppose that, for the equation* (1)*, there hold either the conditions* (3)*,* (4)*,* (5) *for* $\alpha > 1$*, or the condition* (11) *for* $0 < \alpha \leqslant 1$*. Then the discrete spectrum of the operator L is real and coincides with the union of spectra of the self-adjoint operators* $\tilde{L}_k$*,* $k = \overline{1,r}$*, i.e.,*

$$\sigma_d(L) = \bigcup_{k=1}^{r} \sigma\left(\tilde{L}_k\right). \tag{29}$$

Let us number the eigenvalues of the operator L in ascending order and with regard to their multiplicities:

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n \leqslant \ldots$$

Along with the equation (1), we will consider the left equation

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda\tilde{y}, \qquad \tilde{y} = (\tilde{y}_1 \ \tilde{y}_2 \ \ldots \ \tilde{y}_r). \tag{30}$$

Matrix solutions of the equation (30) decaying and growing at infinity will be denoted by $\widetilde{\Phi}(x,\lambda)$ and $\widetilde{\Psi}(x,\lambda)$ respectively.

Denote by $Y(x,\lambda)$ and $\widetilde{Y}(x,\lambda)$ the solutions of the equations (1) and (30) respectively, satisfying the initial conditions

$$Y(0,\lambda) = \cos A, \quad Y'(0,\lambda) = \sin A, \quad \widetilde{Y}(0,\lambda) = \cos A, \quad \widetilde{Y}'(0,\lambda) = \sin A, \quad \lambda \in \mathbb{C}. \tag{31}$$

Put

$$G(x,t,\lambda) = \begin{cases} Y(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1}\widetilde{\Phi}(t,\lambda), & 0 \leqslant x \leqslant t \\ -\Phi(x,\lambda)\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(t,\lambda), & x \geqslant t \end{cases}. \tag{32}$$

THEOREM 3. *The matrix function* $G(x,t,\lambda)$ *is the Green function of the differential operator L, i.e.:*

1. *The function* $G(x,t,\lambda)$ *is continuous for all* $x,t \in [0,\infty)$*.*

2. *For any fixed* t*, the function* $G(x,t,\lambda)$ *has a derivative with respect to* x *such that it is continuous on both intervals* $[0,t)$ *and* (t,∞)*, and at* $x = t$ *it has the jump*
$$G'_x(x+0,x,\lambda) - G'_x(x-0,x,\lambda) = -I_m. \tag{33}$$

3. *For a fixed* t*, the function* $G(x,t,\lambda)$ *with respect to* x *is a matrix solution of the equation* (1) *on both intervals* $[0,t)$ *and* (t,∞)*, and satisfies the boundary condition* (26)*. For a fixed* x*, the function* $G(x,t,\lambda)$ *with respect to* t *is a matrix solution of the equation* (30) *on both intervals* $[0,x)$ *and* (x,∞)*, and satisfies the boundary condition* $\tilde{y}'(0) \cdot \cos A - \tilde{y}(0) \cdot \sin A = 0$*.*

Proof. The function $G(x,t,\lambda)$ is continuous with respect to x on the both intervals $[0,t)$ and (t,∞), and the similar is fulfilled with respect to t. To prove the continuity of the function $G(x,t,\lambda)$ for all $x,t \geqslant 0$, it is sufficient to verify that the identity

$$Y(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1}\widetilde{\Phi}(x,\lambda) + \Phi(x,\lambda)\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(x,\lambda) \equiv 0 \qquad (34)$$

holds true for all $x \geqslant 0$. By definition of the Wronskian, this identity can be rewritten in the form

$$Y(x,\lambda)\left(\widetilde{\Phi}(x,\lambda)Y'(x,\lambda) - \widetilde{\Phi}'(x,\lambda)Y(x,\lambda)\right)^{-1}\widetilde{\Phi}(x,\lambda) -$$

$$-\Phi(x,\lambda)\left(\widetilde{Y}'(x,\lambda)\Phi(x,\lambda) - \widetilde{Y}(x,\lambda)\Phi'(x,\lambda)\right)^{-1}\widetilde{Y}(x,\lambda) \equiv 0,$$

or

$$\left(Y'(x,\lambda)Y^{-1}(x,\lambda) - \widetilde{\Phi}^{-1}(x,\lambda)\widetilde{\Phi}'(x,\lambda)\right)^{-1} \equiv \left(\widetilde{Y}^{-1}(x,\lambda)\widetilde{Y}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda)\right)^{-1},$$

i.e.,

$$Y'(x,\lambda)Y^{-1}(x,\lambda) - \widetilde{\Phi}^{-1}(x,\lambda)\widetilde{\Phi}'(x,\lambda) \equiv \widetilde{Y}^{-1}(x,\lambda)\widetilde{Y}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda),$$

which is equivalent to

$$Y'(x,\lambda)Y^{-1}(x,\lambda) - \widetilde{Y}^{-1}(x,\lambda)\widetilde{Y}'(x,\lambda) \equiv \widetilde{\Phi}^{-1}(x,\lambda)\widetilde{\Phi}'(x,\lambda) - \Phi'(x,\lambda)\Phi^{-1}(x,\lambda),$$

or to

$$\widetilde{Y}^{-1}(x,\lambda)\left(\widetilde{Y}(x,\lambda)Y'(x,\lambda) - \widetilde{Y}'(x,\lambda)Y(x,\lambda)\right)Y^{-1}(x,\lambda) \equiv$$

$$\equiv -\widetilde{\Phi}^{-1}(x,\lambda)\left(\widetilde{\Phi}(x,\lambda)\Phi'(x,\lambda) - \widetilde{\Phi}'(x,\lambda)\Phi^{-1}(x,\lambda)\right)\Phi^{-1}(x,\lambda).$$

But the latter follows from the relation $W\{\widetilde{Y},Y\} = W\{\widetilde{\Phi},\Phi\} = 0$.

To verify that the jump of the first derivative at $t = x$ equals $(-I_m)$, i.e., that the equality (33) holds, it suffices to prove the identity

$$Y'(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1}\widetilde{\Phi}(x,\lambda) + \Phi'(x,\lambda)\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(x,\lambda) \equiv I_m. \qquad (35)$$

Consider the function

$$C(x,t,\lambda) = Y(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1}\widetilde{\Phi}(t,\lambda) + \Phi(x,\lambda)\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(t,\lambda),$$

which presents an analogue of the Cauchy function. This function is a solution of the equation (1) with respect to x, and it is a solution of the equation (30) with respect to t. By (34), we have $C(x,x,\lambda) \equiv 0$. But in this case $C''_{xx}\big|_{t=x} = (V(x) - \lambda I_m)C\big|_{t=x} \equiv 0$, and, therefore, $C'_x(x,t,\lambda)\big|_{t=x} \equiv \Omega_1(\lambda)$, i.e.,

$$Y'(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1}\widetilde{\Phi}(x,\lambda) + \Phi'(x,\lambda)\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(x,\lambda) \equiv \Omega_1(\lambda). \qquad (36)$$

By showing that $\Omega_1(\lambda) = I_m$, we get the formula (33).

Since the matrix solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ form the fundamental system of solutions of the equation (1), we conclude that the matrix solution $Y(x,\lambda)$ of the equation (1) satisfying the initial conditions (31), can be rewritten as

$$Y(x,\lambda) = \Phi(x,\lambda)A(\lambda) + \Psi(x,\lambda)B(\lambda),$$

where $A(\lambda) = -W\{\widetilde{\Psi},Y\}$, $B(\lambda) = W\{\widetilde{\Phi},Y\}$, i.e.,

$$Y(x,\lambda) = \Psi(x,\lambda)W\{\widetilde{\Phi},Y\} - \Phi(x,\lambda)W\{\widetilde{\Psi},Y\}. \tag{37}$$

The matrix solution $\widetilde{Y}(x,\lambda)$ of the equation (30) admits the representation in the form

$$\widetilde{Y}(x,\lambda) = \widetilde{W}\{\widetilde{\Phi},Y\}\widetilde{\Psi}(x,\lambda) - \widetilde{W}\{\widetilde{\Psi},Y\}\widetilde{\Phi}(x,\lambda), \tag{38}$$

where

$$\widetilde{W}\{\widetilde{\Phi},Y\} = \sin A \cdot \Phi(0,\lambda) - \cos A \cdot \Phi'(0,\lambda) = -\Omega(0,\lambda) = -W\{\widetilde{Y},\Phi\}. \tag{39}$$

In the same way we get $\widetilde{W}\{\widetilde{\Psi},Y\} = -W\{\widetilde{Y},\Psi\}$. Thus,

$$\widetilde{Y}(x,\lambda) = W\{\widetilde{Y},\Psi\}\widetilde{\Phi}(x,\lambda) - W\{\widetilde{Y},\Phi\}\widetilde{\Psi}(x,\lambda). \tag{40}$$

Substituting (37) and (40) in the formula (36) and taking into account that the equality (36) holds identically with respect to x, we get

$$\Omega_1(\lambda) = \lim_{x\to\infty}\left[\Psi'(x,\lambda)\widetilde{\Phi}(x,\lambda) - \Phi'(x,\lambda)\widetilde{\Psi}(x,\lambda)\right].$$

By Theorem 1 and by Corollaries 1, 2 on the asymptotic behaviour of the functions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ at infinity, we have

$$\Omega_1(\lambda) = \lim_{x\to\infty}\left[\gamma_0(x,\lambda)\gamma'_\infty(x,\lambda) - \gamma'_0(x,\lambda)\gamma_\infty(x,\lambda)\right]\cdot I_m = W\{\gamma_0,\gamma_\infty\}\cdot I_m = I_m.$$

This completes the proof of the formula (33) as well as the proof of Theorem 3. $\quad\square$

In view of the definition (32), the function $G(x,t,\lambda)$ is meromorphic with respect to the parameter λ which poles coincide with the eigenvalues of the operator L.

Consider the operator R_λ defined on $L_2(H_m,(0,\infty))$ by the relation

$$\left(R_\lambda\bar{f}\right)(x) = \int_0^\infty G(x,t,\lambda)\bar{f}(t)\,dt = \tag{41}$$

$$= -\int_0^x \Phi(x,\lambda)\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(t,\lambda)\bar{f}(t)\,dt + \int_x^\infty Y(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1}\widetilde{\Phi}(t,\lambda)\bar{f}(t)\,dt.$$

THEOREM 4. *The operator R_λ is the resolvent of the operator L.*

Proof. One can directly verify that, for any function $\bar{f}(x) \in L_2(H_m, (0, \infty))$, the vector function $\bar{y}(x, \lambda) = (R_\lambda \bar{f})(x)$ is a solution of the equation $l[\bar{y}] - \lambda \bar{y} = \bar{f}$ whenever $\lambda \notin \sigma(L)$. We will prove that $\bar{y}(x, \lambda) \in L_2(H_m, (0, \infty))$.

By using formulas (37) and (40), we can rewrite the relation (41) in the form

$$\left(R_\lambda \bar{f}\right)(x) = -\int_0^a \Phi(x, \lambda) \left(W\{\widetilde{Y}, \Phi\}\right)^{-1} \widetilde{Y}(t, \lambda) \bar{f}(t) \, dt$$
$$-\bar{\chi}_1(x, \lambda) + \bar{\chi}_2(x, \lambda) - \bar{\chi}_3(x, \lambda) + \bar{\chi}_4(x, \lambda),$$

where $a > 0$ and

$$\bar{\chi}_1(x, \lambda) = \Phi(x, \lambda) \left(W\{\widetilde{Y}, \Phi\}\right)^{-1} W\{\widetilde{Y}, \Psi\} \int_a^x \widetilde{\Phi}(t, \lambda) \bar{f}(t) \, dt,$$

$$\bar{\chi}_2(x, \lambda) = \Phi(x, \lambda) \int_a^x \widetilde{\Psi}(t, \lambda) \bar{f}(t) \, dt,$$

$$\bar{\chi}_3(x, \lambda) = \Phi(x, \lambda) W\{\widetilde{\Psi}, Y\} \left(W\{\widetilde{\Phi}, Y\}\right)^{-1} \int_x^\infty \widetilde{\Phi}(t, \lambda) \bar{f}(t) \, dt,$$

$$\bar{\chi}_4(x, \lambda) = \Psi(x, \lambda) \int_x^\infty \widetilde{\Phi}(t, \lambda) \bar{f}(t) \, dt.$$

Let us show that each of these vector functions $\bar{\chi}_1(x, \lambda)$, $\bar{\chi}_2(x, \lambda)$, $\bar{\chi}_3(x, \lambda)$, $\bar{\chi}_4(x, \lambda)$ belongs to $L_2(H_m, (a, \infty))$. By Theorem 1 and by formulas for $\gamma_0(x, \lambda)$, the matrix solution $\Phi(x, \lambda)$ decays rather quickly, therefore, $|\Phi(x, \lambda)| \in L_2(0, \infty)$. It follows that $\bar{\chi}_1(x, \lambda) \in L_2(H_m, (0, \infty))$ and $\bar{\chi}_3(x, \lambda) \in L_2(H_m, (0, \infty))$. First we prove the assertion for the function $\bar{\chi}_2(x, \lambda)$ if $\alpha > 1$ and the coefficients of the equation (1) satisfy the conditions (3), (4), (5). In this case, we have

$$|\bar{\chi}_2(x, \lambda)| \leqslant |\Phi(x, \lambda)| \int_a^x \left|\widetilde{\Psi}(t, \lambda)\right| |\bar{f}(t)| \, dt.$$

In view of the asymptotic formulas for the matrix solutions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ (see Theorem 1), we obtain that

$$|\bar{\chi}_2(x, \lambda)| \leqslant c_1(\lambda) \gamma_0(x, \lambda) \int_a^x \gamma_\infty(t, \lambda) \left|\bar{f}(t)\right| \, dt. \tag{42}$$

Let us rewrite this relation in the form

$$|\bar{\chi}_2(x, \lambda)| \leqslant c_1(\lambda) \gamma_0(x, \lambda) \gamma_\infty(x, \lambda) \int_a^x \frac{\gamma_\infty(t, \lambda)}{\gamma_\infty(x, \lambda)} \left|\bar{f}(t)\right| \, dt.$$

By using formulas (6) and (7) and by applying the Cauchy-Bunyakovskii inequality, we obtain

$$|\bar{\chi}_2(x, \lambda)| \leqslant \frac{1}{2} c_1(\lambda) \frac{1}{\sqrt{v(x)}} \left(\int_a^x \sqrt{\frac{v(x)}{v(t)}} \exp\left(-2 \int_t^x \sqrt{v(u)} \, du\right) dt\right)^{\frac{1}{2}} \left(\int_0^\infty \left|\bar{f}(t)\right|^2 dt\right)^{\frac{1}{2}}.$$

Since $t \leqslant x$, we get $\exp\left(-2\int_t^x \sqrt{v(u)}\,du\right) \leqslant 1$, and then the latter estimate for $\chi_2(x,\lambda)$ can be rewritten in the form

$$|\bar{\chi}_2(x,\lambda)| \leqslant c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^x \frac{1}{\sqrt{v(t)}}\,dt\right)^{\frac{1}{2}} \leqslant c_2(\lambda)\frac{1}{\sqrt[4]{v(x)}}\left(\int_a^\infty \frac{1}{\sqrt{v(t)}}\,dt\right)^{\frac{1}{2}}.$$

By formula (3), we get

$$|\bar{\chi}_2(x,\lambda)| \leqslant \frac{c_3(\lambda)}{\sqrt[4]{v(x)}},$$

and hence, if $\alpha > 1$ and coefficients of the equation (1) satisfy the conditions (3), (4), (5), we have $\bar{\chi}_2(x,\lambda) \in L_2(H_m,(a,\infty))$. In the case of $v(x) = x^{2\alpha}$, $0 < \alpha \leqslant 1$, the assertion can be proved similarly.

For the function $\bar{\chi}_4(x,\lambda)$, we prove our statement in the case of $v(x) = x^{2\alpha}$, $0 < \alpha \leqslant 1$ and coefficients of the equation (1) satisfy the condition (11). As in (42), in this case we have

$$|\bar{\chi}_4(x,\lambda)| \leqslant c_1(\lambda)\gamma_\infty(x,\lambda)\int_x^\infty \gamma_0(t,\lambda)\left|\bar{f}(t)\right|\,dt,$$

which can be rewritten in the form

$$|\bar{\chi}_4(x,\lambda)| \leqslant c_1(\lambda)\gamma_0(x,\lambda)\gamma_\infty(x,\lambda)\int_x^\infty \frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|\bar{f}(t)\right|\,dt.$$

Let us use the asymptotics of the functions $\gamma_0(x,\lambda)$ and $\gamma_\infty(x,\lambda)$, for example, in the case $\dfrac{\alpha+1}{2\alpha} = n \in \mathbb{N}$, i.e., $\alpha = \dfrac{1}{2n-1}$ (see (13), (14)). Putting $a(\alpha,\lambda) = \dfrac{1\cdot 3\cdot\ldots\cdot(2n-3)}{n!}\cdot\left(\dfrac{\lambda}{2}\right)^n$, we obtain

$$|\bar{\chi}_4(x,\lambda)| \leqslant c_2(\lambda)x^{-\alpha}\int_x^\infty \frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\left|\bar{f}(t)\right|\,dt$$

$$\leqslant c_2(\lambda)x^{-\alpha}\left(\int_a^x \left(\frac{\gamma_0(t,\lambda)}{\gamma_0(x,\lambda)}\right)^2 dt\right)^{\frac{1}{2}}\left(\int_0^\infty \left|\bar{f}(t)\right|^2 dt\right)^{\frac{1}{2}},$$

$$|\bar{\chi}_4(x,\lambda)| \leqslant c_3(\lambda)x^{-\alpha}\left(\int_x^\infty \left(\frac{t}{x}\right)^{2a(\alpha,\lambda)-\alpha}\exp\frac{-2x^{\alpha+1}\left(\left(\frac{t}{x}\right)^{\alpha+1}-1\right)}{1+\alpha}\,dt\right)^{\frac{1}{2}}.$$

Making the change of variables $t = xu$, we get

$$|\bar{\chi}_4(x,\lambda)| \leqslant c_3(\lambda)x^{-\alpha+\frac{1}{2}}\left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha}\exp\frac{-2x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha}\,du\right)^{\frac{1}{2}}.$$

Since the inequality $\exp\dfrac{-x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha} \leqslant x^{-2}$ holds for all $\alpha \in (0,1]$ and $u \in [1,\infty)$ whenever x is sufficiently large x, we have

$$|\bar{\chi}_4(x,\lambda)| \leqslant c_3(\lambda)x^{-\alpha-\frac{1}{2}} \left(\int_1^\infty u^{2a(\alpha,\lambda)-\alpha} \exp\frac{-x^{\alpha+1}(u^{\alpha+1}-1)}{1+\alpha} du \right)^{\frac{1}{2}}.$$

This implies that $|\bar{\chi}_4(x,\lambda)| \leqslant c_4(\alpha,\lambda)x^{-\alpha-\frac{1}{2}}$, and hence $\bar{\chi}_4(x,\lambda) \in L_2(H_m,(a,\infty))$. In the case if $0 < \alpha \leqslant 1$ and $\dfrac{\alpha+1}{2\alpha} \notin \mathbb{N}$, and also $\alpha > 1$, the proof is similar.

Thus, $R_\lambda \bar{f} \in L_2(H_m,(0,\infty))$ for any function $\bar{f} \in L_2(H_m,(0,\infty))$. $\square$

Since the resolvent R_λ is a meromorphic function with respect to λ which poles coincide with the eigenvalues of the operator L, the formulation of Theorem 2 can be refined.

THEOREM 5. *Suppose that, for the equation* (1)*, there either hold the conditions* (3)*,* (4)*,* (5) *for* $\alpha > 1$ *or the condition* (11) *for* $0 < \alpha \leqslant 1$*. Then the spectrum of the operator L is real, discrete and coincides with the union of spectra of self-adjoint operators* $\widetilde{L}_k$*,* $k = \overline{1,r}$*, i.e.,*

$$\sigma(L) = \bigcup_{k=1}^r \sigma\left(\widetilde{L}_k\right).$$

III. As above, we denote by $Y(x,\lambda)$ the matrix solution of the equation (1) satisfying the initial conditions $Y(0,\lambda) = \cos A$, $Y'(0,\lambda) = \sin A$, and by $Z(x,\lambda)$ the matrix solution of the equation (1) satisfying the initial conditions $Z(0,\lambda) = -\sin A$, $Z'(0,\lambda) = \cos A$. Then the solutions $\Phi(x,\lambda)$, $\widetilde{\Phi}(x,\lambda)$ admit the representation in the form:

$$\Phi(x,\lambda) = Z(x,\lambda)W\{\widetilde{Y},\Phi\} - Y(x,\lambda)W\{\widetilde{Z},\Phi\}, \tag{43}$$

$$\widetilde{\Phi}(x,\lambda) = W\{\widetilde{\Phi},Z\}\widetilde{Y}(x,\lambda) - W\{\widetilde{\Phi},Y\}\widetilde{Z}(z,\lambda). \tag{44}$$

The Green function (32) can be rewritten as

$$G(x,t,\lambda) = Y(x,\lambda)\left(W\{\widetilde{\Phi},Y\}\right)^{-1} W\{\widetilde{\Phi},Z\}\widetilde{Y}(t,\lambda) + \ldots =$$

$$= Y(x,\lambda)W\{\widetilde{Z},\Phi\}\left(W\{\widetilde{Y},\Phi\}\right)^{-1}\widetilde{Y}(t,\lambda) + \ldots.$$

Here the dots means the entire function with respect to λ. Consider the disk on the complex plane bounded by the circle C_{R_n} of radius R_n with the center at the origin such that, for a sufficiently large n, there hold conditions $|\lambda_n| < R_n$ and $\lambda_{n+1} > R_n$. By

integrating the function $\dfrac{G(x,t,\lambda)}{\lambda - z}$ over the contour indicated, we get:

$$\frac{1}{2\pi i} \int_{C_{R_n}} \frac{G(x,t,\lambda)}{\lambda - z} d\lambda = G(x,t,\lambda)$$

$$+ \sum_{j=1}^{n} \text{Res}_{\lambda_j} \left\{ \frac{1}{\lambda - z} Y(x,\lambda) W\{\widetilde{Z},\Phi\} \left(W\{\widetilde{Y},\Phi\} \right)^{-1} \widetilde{Y}(t,\lambda) \right\}.$$

By letting n tend to infinity, we conclude that

$$G(x,t,z) = - \sum_{j=1}^{\infty} \text{Res}_{\lambda_j} \left\{ \frac{1}{\lambda - z} Y(x,\lambda) W\{\widetilde{Z},\Phi\} \left(W\{\widetilde{Y},\Phi\} \right)^{-1} \widetilde{Y}(t,\lambda) \right\}. \qquad (45)$$

If the matrices $A(\lambda)$ and $C(\lambda)$ are entire functions and the matrix $B(\lambda)$ has a pole of order r_j at λ_j, then the residue of the matrix $A(\lambda)B(\lambda)C(\lambda)$ at λ_j can be calculated as follows:

$$\text{Res}_{\lambda_j}\{A(\lambda)B(\lambda)C(\lambda)\} = \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} A(\lambda) \Bigg|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \text{Res}_{\lambda_j} \left\{ B(\lambda)(\lambda - \lambda_j)^{k+l} \right\}$$

$$\times \frac{1}{l!} \frac{d^l}{d\lambda^l} C(\lambda) \Bigg|_{\lambda=\lambda_j}.$$

If $A(\lambda) = I$, we get

$$\text{Res}_{\lambda_j}\{B(\lambda)C(\lambda)\} = \sum_{l=0}^{r_j-1} \text{Res}_{\lambda_j} \left\{ B(\lambda)(\lambda - \lambda_j)^l \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} C(\lambda) \Bigg|_{\lambda=\lambda_j}. \qquad (46)$$

The formula (45) takes the form

$$G(x,t,z) = - \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{1}{\lambda - z} Y(x,\lambda) W\{\widetilde{Z},\Phi\} \right) \Bigg|_{\lambda=\lambda_j} \qquad (47)$$

$$\times \sum_{l=0}^{r_j-(k+1)} \text{Res}_{\lambda_j} \left\{ \left(W\{\widetilde{Y},\Phi\} \right)^{-1} (\lambda - \lambda_j)^{k+l} \right\} \cdot \frac{1}{l!} \frac{d^l}{d\lambda^l} \widetilde{Y}(t,\lambda) \Bigg|_{\lambda=\lambda_j}.$$

As in [1], [2], [3], define the normalizing polynomials by

$$N_j(t) = e^{-\lambda_j t} \text{Res}_{\lambda_j} \left\{ e^{\lambda t} \left(W\{\widetilde{Y},\Phi\} \right)^{-1} W\{\widetilde{Y},\Psi\} \right\},$$

or

$$N_j(t) = \sum_{k=0}^{r_j-1} \left(\sum_{l=0}^{r_j-(k+1)} \text{Res}_{\lambda_j} \left\{ \left(W\{\widetilde{Y},\Phi\} \right)^{-1} (\lambda - \lambda_j)^{l+k} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W\{\widetilde{Y},\Psi\} \Bigg|_{\lambda=\lambda_j} \right) \frac{t^k}{k!}.$$

Note that

$$\frac{d^k}{dt^k}\left(N_j(t)\right)\Big|_{t=0} = \sum_{l=0}^{r_j-(k+1)} \text{Res}_{\lambda_j}\left\{\left(W\{\widetilde{Y},\Phi\}\right)^{-1}(\lambda-\lambda_j)^{l+k}\right\}\frac{1}{l!}\frac{d^l}{d\lambda^l}W\{\widetilde{Y},\Psi\}\Big|_{\lambda=\lambda_j},$$
(48)

or

$$\frac{d^k}{dt^k}\left(N_j(t)\right)\Big|_{t=0} = \text{Res}_{\lambda_j}\left\{\left(W\{\widetilde{Y},\Phi\}\right)^{-1}(\lambda-\lambda_j)^k W\{\widetilde{Y},\Psi\}\right\}.$$

LEMMA 1. *Assume that $\Omega(\lambda)$ is an entire matrix function and $\det\Omega(\lambda_0)=0$ at some point λ_0, and that the matrix $\Omega^{-1}(\lambda)$ has a pole of order r at λ_0. Then*

$$\sum_{l=0}^{k}\text{Res}_{\lambda_0}\left\{\Omega^{-1}(\lambda)(\lambda-\lambda_0)^{r-(k+1)+l}\right\}\frac{1}{l!}\frac{d^l}{d\lambda^l}\Omega(\lambda)\Big|_{\lambda=\lambda_0}=0, \qquad k=0,1,\dots,r-1,$$

or

$$\sum_{l=0}^{r-(k+1)}\text{Res}_{\lambda_0}\left\{\Omega^{-1}(\lambda)(\lambda-\lambda_0)^{k+l}\right\}\frac{1}{l!}\frac{d^l}{d\lambda^l}\Omega(\lambda)\Big|_{\lambda=\lambda_0}=0, \qquad k=0,1,\dots,r-1.$$
(49)

As in [1] and [2] for $r=1$, the proof follows from the expansion of the functions $\Omega(\lambda)$ and $\Omega^{-1}(\lambda)$ in a neighborhood at λ_0:

$$\Omega(\lambda) = \sum_{l=0}^{\infty}\frac{1}{l!}\frac{d^l}{d\lambda^l}\Omega(\lambda)\Big|_{\lambda=\lambda_0}\cdot(\lambda-\lambda_0)^l,$$

$$\Omega^{-1}(\lambda) = \frac{A_{-r}}{(\lambda-\lambda_0)^r}+\frac{A_{-(r-1)}}{(\lambda-\lambda_0)^{r-1}}+\dots+\frac{A_{-1}}{\lambda-\lambda_0}+A_0+\dots,$$

where $A_{-l}=\text{Res}_{\lambda_0}\left\{\Omega^{-1}(\lambda)(\lambda-\lambda_0)^{l-1}\right\}$.

REMARK 1. *If $\Omega(\lambda)=W\{\widetilde{Y},\Phi\}$ and λ_j is the pole of multiplicity r_j, then the formula (49) takes the form*

$$\sum_{l=0}^{r_j-(k+1)}\text{Res}_{\lambda_j}\left\{\left(W\{\widetilde{Y},\Phi\}\right)^{-1}(\lambda-\lambda_0)^{k+l}\right\}\frac{1}{l!}\frac{d^l}{d\lambda^l}W\{\widetilde{Y},\Phi\}\Big|_{\lambda=\lambda_j}=0,$$

$$k=0,1,\dots,r-1. \qquad (50)$$

LEMMA 2. *The following relations hold for all* $k = 0, 1, \ldots, r_j - 1$[3]:

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} W\{\tilde{\Phi}, Z\} \bigg|_{\lambda=\lambda_j} = \mathrm{Res}_{\lambda_j} \left\{ (W\{\tilde{Y}, \Phi\})^{-1} (\lambda - \lambda_j)^k \right\}, \tag{51}$$

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} W\{\tilde{\Phi}, Y\} \bigg|_{\lambda=\lambda_j} = 0, \tag{52}$$

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^l}{d\lambda^l} W\{\tilde{Y}, \Phi\} \bigg|_{\lambda=\lambda_j} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} = 0. \tag{53}$$

Proof. Due to the formula (48), we get:

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} W\{\tilde{\Phi}, Z\} \bigg|_{\lambda=\lambda_j} =$$

$$= \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \left(\sum_{s=0}^{r_j-(k+l+1)} \mathrm{Res}_{\lambda_j} \left\{ (W\{\tilde{Y}, \Phi\})^{-1} (\lambda - \lambda_j)^{k+l+s} \right\} \right.$$

$$\left. \times \frac{1}{s!} \frac{d^s}{d\lambda^s} W\{\tilde{Y}, \Psi\} \bigg|_{\lambda=\lambda_j} \right) \frac{d^l}{d\lambda^l} W\{\tilde{\Phi}, Z\} \bigg|_{\lambda=\lambda_j}.$$

In what follows, we will consider values of the function $N_j(t)$ as well as its derivatives at $t = 0$, and values of the Wronskians and their derivatives at $\lambda = \lambda_j$. Therefore, in order to simplify the notation, we will not mention the calculation of functions' values at a point.

Making the change $l + s = m$, one can reduce the latter sum to the form

$$\sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \left(\sum_{m=l}^{r_j-(k+1)} \mathrm{Res}_{\lambda_j} \left\{ (W\{\tilde{Y}, \Phi\})^{-1} (\lambda - \lambda_j)^{k+m} \right\} \right.$$

$$\left. \times \frac{1}{(m-l)!} \frac{d^{m-l}}{d\lambda^{m-l}} W\{\tilde{Y}, \Psi\} \right) \frac{d^l}{d\lambda^l} W\{\tilde{\Phi}, Z\}.$$

Then, making the change of the summation limits, we obtain

$$\sum_{m=0}^{r_j-(k+1)} \frac{1}{m!} \mathrm{Res}_{\lambda_j} \left\{ (W\{\tilde{Y}, \Phi\})^{-1} (\lambda - \lambda_j)^{k+m} \right\} \left(\sum_{l=0}^{m} C_m^l \frac{d^{m-l}}{d\lambda^{m-l}} W\{\tilde{Y}, \Psi\} \frac{d^l}{d\lambda^l} W\{\tilde{\Phi}, Z\} \right)$$

$$= \sum_{m=0}^{r_j-(k+1)} \frac{1}{m!} \mathrm{Res}_{\lambda_j} \left\{ (W\{\tilde{Y}, \Phi\})^{-1} (\lambda - \lambda_j)^{k+m} \right\} \frac{d^m}{d\lambda^m} \left(W\{\tilde{Y}, \Psi\} W\{\tilde{\Phi}, Z\} \right). \tag{54}$$

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[3]The formulas (52), (53) are similar to the formula (25) from [3] while they are proved here in a different way.

By using the asymptotics of the matrix solutions $\Phi(x,\lambda)$ and $\Psi(x,\lambda)$ at infinity, for any λ we have

$$W\{\tilde{Y},\Psi\}W\{\tilde{\Phi},Z\} = \lim_{x\to\infty}(\gamma'_\infty\tilde{Y} - \gamma_\infty\tilde{Y}')(\gamma_0 Z' - \gamma'_0 Z) = I + \lim_{x\to\infty}(\gamma_0\tilde{Y}' - \gamma'_0\tilde{Y})(\gamma'_\infty Z - \gamma_\infty Z')$$

$$= I + \lim_{x\to\infty}\left(\tilde{Y}'\Phi - \tilde{Y}\Phi'\right)\left(\tilde{\Psi}'Z - \tilde{\Psi}Z'\right) = I + W\{\tilde{Y},\Phi\}W\{\tilde{\Psi},Z\}. \tag{55}$$

Similarly,

$$W\{\tilde{Y},\Psi\}W\{\tilde{\Phi},Y\} = W\{\tilde{Y},\Phi\}W\{\tilde{\Psi},Y\}. \tag{56}$$

In view of (55), the right-hand side of the formula (54) takes the form

$$\mathrm{Res}_{\lambda_j}\left\{\left(W\{\tilde{Y},\Phi\}\right)^{-1}(\lambda - \lambda_j)^k\right\} +$$

$$+ \sum_{m=0}^{r_j-(k+1)}\frac{1}{m!}\mathrm{Res}_{\lambda_j}\left\{\left(W\{\tilde{Y},\Phi\}\right)^{-1}(\lambda-\lambda_j)^{k+m}\right\}\frac{d^m}{d\lambda^m}\left(W\{\tilde{Y},\Phi\}W\{\tilde{\Psi},Z\}\right).$$

From here, using the formula (46) we obtain that the latter sum equals 0. This completes the proof of the formula (51). The relations (52) and (53) can be proved similarly, with the use of the formula (56) instead of (55) by the completion of the proof. $\square$

THEOREM 6. *Let L_0 be a minimal differential operator generated by the differential expression* (1) *which coefficients satisfy either the conditions* (3), (4), (5) *for $\alpha > 1$, or the condition* (11) *for $0 < \alpha \leqslant 1$, as well as the boundary condition* (26). *Let also L be the extension of the operator L_0 generated by the requirement for functions from the domain of the operator L to belong to $L_2(H_m,(0,\infty))$. Then the Green function of the operator L has the form*

$$G(x,t,z) = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\frac{1}{\lambda-z}\Phi(x,\lambda)\right)\Bigg|_{\lambda=\lambda_j} \tag{57}$$

$$\times \sum_{l=0}^{r_j-(k+1)}\frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}}N_j(t)\Bigg|_{t=0}\frac{d^l}{d\lambda^l}\left(\tilde{\Phi}(t,\lambda)\right)\Bigg|_{\lambda=\lambda_j}.$$

Proof. Let us transform the formula (47) for the Green function $G(x,t,z)$. Consider the interior sum and substitute the values of residues in it according to (51):

$$\sum_{l=0}^{r_j-(k+1)}\mathrm{Res}_{\lambda_j}\left\{\left(W\{\tilde{Y},\Phi\}\right)^{-1}(\lambda-\lambda_j)^{k+l}\right\}\frac{1}{l!}\frac{d^l}{d\lambda^l}\tilde{Y}(t,\lambda)\Bigg|_{\lambda=\lambda_j} =$$

$$= \sum_{l=0}^{r_j-(k+1)}\left(\sum_{s=0}^{r_j-(k+l+1)}\frac{1}{s!}\frac{d^{k+l+s}}{dt^{k+l+s}}N_j(t)\frac{d^s}{d\lambda^s}W\{\tilde{\Phi},Z\}\right)\frac{1}{l!}\frac{d^l}{d\lambda^l}\tilde{Y}(t,\lambda).$$

Making the change $l + s = u$, we conclude that the latter sum will take the form

$$\sum_{l=0}^{r_j-(k+1)} \left(\sum_{u=l}^{r_j-(k+1)} \frac{1}{(u-l)!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^{u-l}}{d\lambda^{u-l}} W\{\tilde{\Phi}, Z\} \right) \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda).$$

If we interchange the summation limits, we obtain

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \left(\sum_{l=0}^{u} C_u^l \frac{d^{u-l}}{d\lambda^{u-l}} W\{\tilde{\Phi}, Z\} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda) \right) =$$

$$= \sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^u}{d\lambda^u} \left(W\{\tilde{\Phi}, Z\} \tilde{Y}(t, \lambda) \right). \tag{58}$$

Using the formula (44) and proving that

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^u}{d\lambda^u} \left(W\{\tilde{\Phi}, Y\} \tilde{Z}(t, \lambda) \right) = 0, \tag{59}$$

we conclude that the right-hand side of the formula (58) takes the form

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^u}{d\lambda^u} \left(\tilde{\Phi}(t, \lambda) \right).$$

By rewriting the left-hand side of (59) as

$$\sum_{u=0}^{r_j-(k+1)} \frac{1}{u!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \sum_{s=0}^{u} C_u^s \frac{d^{u-s}}{d\lambda^{u-s}} \left(W\{\tilde{\Phi}, Y\} \right) \frac{d^s}{d\lambda^s} \tilde{Z}(t, \lambda)$$

and by interchanging the summation limits, we get

$$\sum_{s=0}^{r_j-(k+1)} \frac{1}{s!} \left(\sum_{u=s}^{r_j-(k+1)} \frac{1}{(u-s)!} \frac{d^{k+u}}{dt^{k+u}} N_j(t) \frac{d^{u-s}}{d\lambda^{u-s}} \left(W\{\tilde{\Phi}, Y\} \right) \right) \frac{d^s}{d\lambda^s} \tilde{Z}(t, \lambda).$$

Then, making the change $u - s = m$, we obtain

$$\sum_{s=0}^{r_j-(k+1)} \frac{1}{s!} \left(\sum_{m=0}^{r_j-(k+s+1)} \frac{1}{m!} \frac{d^{k+s+m}}{dt^{k+s+m}} N_j(t) \frac{d^m}{d\lambda^m} \left(W\{\tilde{\Phi}, Y\} \right) \right) \frac{d^s}{d\lambda^s} \tilde{Z}(t, \lambda).$$

In view of (52), the interior sum equals 0, which implies (59). It follows that

$$\sum_{l=0}^{r_j-(k+1)} \mathrm{Res}_{\lambda_j} \left\{ (W\{\tilde{Y}, \Phi\})^{-1} (\lambda - \lambda_j)^{k+l} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} \tilde{Y}(t, \lambda) \bigg|_{\lambda=\lambda_j} =$$

$$= \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{\Phi}(t, \lambda) \right) \bigg|_{\lambda=\lambda_j},$$

and the right-hand side of (47) takes the form

$$-\sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\frac{1}{\lambda-z}Y(x,\lambda)W\{\tilde{Z},\Phi\}\right)\Bigg|_{\lambda=\lambda_j}$$

$$\sum_{l=0}^{r_j-(k+1)}\frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}}N_j(t)\Bigg|_{t=0}\frac{d^l}{d\lambda^l}\left(\tilde{\Phi}(t,\lambda)\right)\Bigg|_{\lambda=\lambda_j}.$$

Finally, by interchanging the summation limits with respect to k and l and by using the formulas (43) and (53), we will pass to the formula (57) for the Green function. $\quad\square$

If all the eigenvalues λ_j of the operator L are simple, i.e., the poles λ_j of the matrix $\left(W\{\tilde{Y},\Phi\}\right)^{-1}$ are simple, then the matrix $N_j(t)$ is constant and can be calculated by the formula

$$N_j=\text{Res}_{\lambda_j}\left\{\left(W\{\tilde{Y},\Phi\}\right)^{-1}\right\}W\{\tilde{Y},\Psi\}\big|_{\lambda=\lambda_j}.$$

It follows that

$$\text{Res}_{\lambda_j}\left\{\left(W\{\tilde{Y},\Phi\}\right)^{-1}\right\}=N_jW\{\tilde{\Phi},Z\}\big|_{\lambda=\lambda_j}.$$

Hence in the case if all the eigenvalues of the operator L are simple, the formula (57) can be substantially refined:

$$G(x,t,z)=-\sum_{j=1}^{\infty}\frac{1}{\lambda_j-z}Y(x,\lambda_j)W\{\tilde{Z},\Phi\}\big|_{\lambda=\lambda_j}N_jW\{\tilde{\Phi},Z\}\big|_{\lambda=\lambda_j}\tilde{Y}(t,\lambda_j)$$

$$=\sum_{j=1}^{\infty}\frac{1}{\lambda_j-z}\Phi(x,\lambda_j)N_j\tilde{\Phi}(t,\lambda_j).$$

This formula is equivalent to (57) for the Green function that was constructed in [2] in the case of a potential decaying at infinity and such that its first moment is bounded.

IV. Let $S(x)$ and $T(x)$ be arbitrary matrix functions from $L_2(H_m,(0,\infty))$. Denote

$$E(S,\lambda)=\int_0^{\infty}S(t)\Phi(t,\lambda)dt, \tag{60}$$

$$\tilde{E}(S,\lambda)=\int_0^{\infty}\tilde{\Phi}(t,\lambda)S(t)dt. \tag{61}$$

THEOREM 7. *Suppose that, for the problem* (1), (26), *coefficients either satisfy the conditions* (3), (4), (5) *for* $\alpha>1$, *or the condition* (11) *for* $0<\alpha\leqslant 1$. *Then, for arbitrary matrix functions* $S(x)$, $T(x)\in L_2(H_m,(0,\infty))$, *there holds the expansion in*

the solutions $\Phi(x, \lambda)$ and $\tilde{\Phi}(x, \lambda)$ of the equations (1) *and* (30) *respectively:*

$$S(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(E\left(S, \lambda \right) \right) \bigg|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{\Phi}(x, \lambda) \right) \bigg|_{\lambda=\lambda_j},$$
(62)

$$S(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\Phi(x, \lambda) \right) \bigg|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{E}\left(S, \lambda \right) \right) \bigg|_{\lambda=\lambda_j},$$
(63)

and holds the Parseval equality

$$\int_0^{\infty} S(x) T(x)\, dx = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} E\left(S, \lambda \right) |_{\lambda=\lambda_j}$$
(64)

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \tilde{E}\left(T, \lambda \right) \bigg|_{\lambda=\lambda_j}.$$

Proof. Since $(\tilde{l} - zI) \left[\tilde{\Phi}(x, \lambda) \right] = (\lambda - z) \tilde{\Phi}(x, \lambda)$, we conclude that $\tilde{\Phi}(x, \lambda) =$
$= \dfrac{1}{\lambda - z} (\tilde{l} - zI) \left[\tilde{\Phi}(x, \lambda) \right]$ for $\lambda \neq z$. It follows that

$$\tilde{E}\left(R_z[T], \lambda \right) = \int_0^{\infty} \tilde{\Phi}(x, \lambda) R_z[T](x)\, dx = \frac{1}{\lambda - z} \int_0^{\infty} (\tilde{l} - zI) \left[\tilde{\Phi}(x, \lambda) \right] R_z[T](x)\, dx.$$

For a finite function $T(x) \in L_2\left(H_m, (0, \infty) \right)$, by integrating by parts twice, we get

$$\tilde{E}\left(R_z[T], \lambda \right) = \frac{1}{\lambda - z} \int_0^{\infty} \tilde{\Phi}(x, \lambda) \left(l - zI \right) R_z[T](x)\, dx$$

$$= \frac{1}{\lambda - z} \int_0^{\infty} \tilde{\Phi}(x, \lambda) T(x)\, dx = \frac{1}{\lambda - z} \tilde{E}\left(T, \lambda \right).$$
(65)

Taking into account the formulas (41), (57) and (61), for an arbitrary matrix $T(x) \in$
$\in L_2\left(H_m, (0, \infty) \right)$ we have

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} \left(\frac{1}{\lambda - z} \Phi(x, \lambda) \right) \bigg|_{\lambda=\lambda_j}$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \tilde{E}\left(T, \lambda \right) \bigg|_{\lambda=\lambda_j}.$$

Introducing the temporary notation $a_k(\lambda_j)$ for the summation with respect to l, we can rewrite the latter formula as

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \sum_{s=0}^{k} \frac{1}{s!} \frac{1}{(\lambda_j - z)^{k-s+1}} \frac{d^s}{d\lambda^s}(\Phi(x,\lambda))\bigg|_{\lambda=\lambda_j} a_k(\lambda_j).$$

Interchange the summation limits with respect to k and s:

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{s=0}^{r_j-1} \frac{1}{s!} \frac{d^s}{d\lambda^s}(\Phi(x,\lambda))\bigg|_{\lambda=\lambda_j} \sum_{k=s}^{r_j-1} \frac{1}{(\lambda_j - z)^{k-s+1}} a_k(\lambda_j).$$

In what follows, we will consider values of the function $\Phi(x,\lambda)$ as well as its derivatives in λ at $\lambda = \lambda_j$, and values of the function $N_j(t)$ and its derivatives at $t = 0$. Hence, as above, we simplify the notation of functions by no mentioning the calculation of functions' values at a point. Putting $k - s = u$, we obtain:

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{s=0}^{r_j-1} \frac{1}{s!} \frac{d^s}{d\lambda^s}(\Phi(x,\lambda)) \sum_{u=0}^{r_j-(s+1)} \frac{1}{(\lambda_j - z)^{u+1}}$$
$$\times \sum_{l=0}^{r_j-(s+u+1)} \frac{1}{l!} \frac{d^{s+u+l}}{dt^{s+u+l}} N_j(t) \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda).$$

Interchange the summation limits with respect to u and l:

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k}(\Phi(x,\lambda))$$
$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \left(\sum_{u=0}^{r_j-(k+l+1)} \frac{1}{(\lambda_j - z)^{u+1}} \frac{d^{k+u+l}}{dt^{k+u+l}} N_j(t) \right) \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda).$$

By making the change $u + l = p$, we get

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k}(\Phi(x,\lambda))$$
$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \left(\sum_{p=l}^{r_j-(k+1)} \frac{1}{(\lambda_j - z)^{p-l+1}} \frac{d^{k+p}}{dt^{k+p}} N_j(t) \right) \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda).$$

Interchange the summation limits with respect to l and p:

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k}(\Phi(x,\lambda))$$
$$\times \sum_{p=0}^{r_j-(k+1)} \frac{d^{k+p}}{dt^{k+p}} N_j(t) \left(\sum_{l=0}^{p} \frac{1}{l!} \frac{1}{(\lambda_j - z)^{p-l+1}} \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda) \right).$$

This yields that

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \sum_{p=0}^{r_j-(k+1)} \frac{1}{p!} \frac{d^{k+p}}{dt^{k+p}} N_j(t) \frac{d^p}{d\lambda^p} \left(\frac{1}{\lambda-z} \tilde{E}(T,\lambda) \right).$$

In view of the formula (65), we have

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \bigg|_{\lambda=\lambda_j}$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{E}(R_z[T],\lambda) \right) \bigg|_{\lambda=\lambda_j}.$$

It follows that

$$\int_0^{\infty} S(x)(R_z[T])(x)\,dx =$$

$$= \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (E(S,\lambda)) \bigg|_{\lambda=\lambda_j} \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{E}(R_z[T],\lambda) \right) \bigg|_{\lambda=\lambda_j}.$$

Thus, for any finite function $T(x) \in L_2(H_m,(0,\infty))$, we get

$$\int_0^{\infty} S(x)(R_z[T])(x)\,dx = \int_0^{\infty} \left(\sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (E(S,\lambda)) \bigg|_{\lambda=\lambda_j} \right.$$

$$\times \left. \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \bigg|_{t=0} \frac{d^l}{d\lambda^l} (\tilde{\Phi}(x,\lambda)) \bigg|_{\lambda=\lambda_j} \right) (R_z[T])(x)\,dx.$$

Since the range of the resolvent is dense in $L_2(H_m,(0,\infty))$, we pass to the formula (62). In the similar way we can prove the formula (63). Finally, by multiplying the both sides of the relation (62) by $T(x)$ and by integrating, we obtain the Parseval equality (64). □

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