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APPROXIMATE DOUBLE COMMUTANTS IN VON NEUMANN ALGEBRAS AND C*–ALGEBRAS

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Dedicated to Eric Nordgren,
a great mathematician
and a great friend

(*Communicated by D. R. Farenick*)

Abstract. Richard Kadison showed that not every commutative von Neumann subalgebra of a factor von Neumann algebra is equal to its relative double commutant. We prove that every commutative C*-subalgebra of a centrally prime C*-algebra $\mathscr{B}$ equals its relative approximate double commutant. If $\mathscr{B}$ is a von Neumann algebra, there is a related distance formula.

One of the fundamental results in the theory of von Neumann algebras is von Neumann's classical *double commutant theorem*, which says that if $\mathscr{S}=\mathscr{S}^*\subseteq B(H)$, then $\mathscr{S}''=W^*(\mathscr{S})$. In 1978 [3] the author proved an asymptotic version of von Neumann's theorem, the *approximate double commutant theorem*. For the asymptotic version, we define the *approximate double commutant* of $\mathscr{S}\subseteq B(H)$, denoted by $\operatorname{Appr}(S)''$, to be the set of all operators T such that

$$\|A_\lambda T-TA_\lambda\|\to 0$$

for every bounded net $\{A_\lambda\}$ in $B(H)$ for which

$$\|A_\lambda S-SA_\lambda\|\to 0$$

for every $S\in\mathscr{S}$. More generally, if $\mathscr{B}$ is a unital C*-algebra and $\mathscr{S}\subseteq\mathscr{B}$, we define the *relative approximate double commutant of* S *in* $\mathscr{B}$, denoted by $\operatorname{Appr}(S,\mathscr{B})''$, in the same way but insisting that the T's and the A_λ's be in $\mathscr{B}$. The approximate double commutant theorem in $B(H)$ [3] says that if $\mathscr{S}=\mathscr{S}^*$, then $\operatorname{Appr}(\mathscr{S})''=C^*(\mathscr{S})$. Moreover, if we restrict the $\{A_\lambda\}$'s to be nets of unitaries or nets of projections that asymptotically commute with every element of $\mathscr{S}$, the resulting approximate double commutant is still $C^*(\mathscr{S})$.

A von Neumann algebra $\mathscr{B}$ is *hyperreflexive* if there is a constant $K\geqslant 1$ such that, for every $T\in B(H)$

$$dist(T,\mathscr{B})\leqslant K\sup\left\{\|TP-PT\|:P\in\mathscr{B}',P\text{ a projection}\right\}.$$

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The smallest such K is called the *constant of hyperreflexivity* for $\mathscr{B}$. The inequality

$$\sup\left\{\|TP-PT\| : P\in\mathscr{M}', P \text{ a projection}\right\} \leqslant dist\,(T,\mathscr{M})$$

is always true. The question of whether every von Neumann algebra is hyperreflexive is still open and is equivalent to a number of other important problems in von Neumann algebras (see [6]). It was proved by the author [4] that every unital C*-subalgebra $\mathscr{A}$ of $B(H)$ is approximately hyperreflexive; more precisely, if $T\in B(H)$, then there is a net $\{P_\lambda\}$ of projections such that

$$\|AP_\lambda - P_\lambda A\| \to 0$$

for every $A\in\mathscr{A}$, and

$$dist\,(T,\mathscr{A}) \leqslant 29\lim_{\lambda}\|TP_\lambda - P_\lambda T\|.$$

If we replace the role of $B(H)$ with a factor von Neumann algebra, then the double commutant theorem fails, even when the subalgebra is commutative. Suppose $\mathscr{S}$ is a subset of a ring $\mathscr{R}$. We define the *relative commutant* of $\mathscr{S}$ in $\mathscr{R}$, the *relative double commutant* of $\mathscr{S}$ in $\mathscr{R}$, and the *relative triple commutant* of $\mathscr{S}$ in $\mathscr{R}$, respectively, by

$$(\mathscr{S},\mathscr{R})' = \{T\in\mathscr{R} : \forall S\in\mathscr{S}, TS=ST\},$$

$$(\mathscr{S},\mathscr{R})'' = \left\{T\in\mathscr{R} : \forall A\in(\mathscr{S},\mathscr{R})', TA=AT\right\},$$

and

$$(\mathscr{S},\mathscr{R})''' = \left\{T\in\mathscr{R} : \forall A\in(\mathscr{S},\mathscr{R})'', TA=AT\right\}.$$

It is clear from general Galois nonsense that

$$(\mathscr{S},\mathscr{R})''' = (\mathscr{S},\mathscr{R})'.$$

Following R. Kadison [8] we will say a subring $\mathscr{M}$ of a unital ring $\mathscr{B}$ is *normal* if

$$\mathscr{M} = (\mathscr{M},\mathscr{B})'' = \left(\mathscr{M}'\cap\mathscr{B}\right)'\cap\mathscr{B}.$$

R. Kadison [8] proved that if $\mathscr{M}$ is type I von Neumann subalgebra of a von Neumann algebra $\mathscr{B}$, then $\mathscr{M}$ is normal in $\mathscr{B}$ if and only if its center $\mathscr{Z}(\mathscr{M}) = \mathscr{M}\cap\mathscr{M}'$ is normal if and only if $\mathscr{Z}(\mathscr{M})$ is an intersection of masas (maximal abelian selfadjoint subalgebras) of $\mathscr{B}$. See the paper of B. J. Vowden [14] for more examples. We see that the part of Kadison's result concerning abelian C*-subalgebras is true in the C*-algebraic setting. We prove a general version for rings, which applies to commutative nonselfadjoint subalgebras of a C*-algebra or von Neumann algebra.

LEMMA 1. *Suppose $\mathscr{M}$ is a unital abelian subring of a unital ring $\mathscr{B}$. The following are equivalent:*

1. $\mathscr{M} = (\mathscr{M},\mathscr{B})''$.

2. *$\mathscr{M}$ is an intersection of maximal abelian subrings of $\mathscr{B}$.*

3. *$\mathscr{M}$ is an intersection of subrings of the form $(\mathscr{S},\mathscr{B})'$ for subsets $\mathscr{S}$ of $\mathscr{B}$.*

Proof. First note that every maximal abelian subring $\mathscr{E}$ has the property that $\mathscr{E} = (\mathscr{E},\mathscr{B})'$, which implies $\mathscr{E} = (\mathscr{E},\mathscr{B})''$ and the implication $(2) \Longrightarrow (3)$. It is also clear that if $\{\mathscr{S}_i : i \in I\}$ is a collection of nonempty subsets of $\mathscr{B}$, then

$$\bigcup_{i\in I}(\mathscr{S}_i,\mathscr{B})' \subseteq (\cap_{i\in I}\mathscr{S}_i,\mathscr{B})',$$

and

$$(\cap_{i\in I}\mathscr{S}_i,\mathscr{B})'' \subseteq \bigcap_{i\in I}(\mathscr{S}_i,\mathscr{B})''.$$

This, and the fact that $(\mathscr{S},\mathscr{B})''' = (\mathscr{S},\mathscr{B})'$ always holds, yields $(3) \Longrightarrow (1)$.

To prove $(1) \Longrightarrow (2)$, suppose (1) holds, and let $\mathscr{W}$ be a maximal abelian subring of $\mathscr{B}$ such that $\mathscr{M} \subseteq \mathscr{W}$. For each $W \in \mathscr{W}\setminus\mathscr{M}$, by (1), there is a $T_W \in (\mathscr{M},\mathscr{B})'$ such that $T_W W \neq W T_W$. Since the ring generated by $\mathscr{M} \cup \{T_W\}$ is abelian, it is contained in a maximal abelian subring $\mathscr{S}_W$, and $W \notin \mathscr{S}_W$. Hence

$$\mathscr{M} = \mathscr{W} \cap \bigcap_{W\in\mathscr{W}\setminus\mathscr{M}} \mathscr{S}_W,$$

which proves (2) holds. $\square$

If in the statement and proof of the preceding lemma we replace "ring" with "C*-algebra", and the ring generated by $\mathscr{M}\cup\{T_W\}$ with $C^*(\mathscr{M}\cup\{T_W\})$, we obtain the following result for C*-algebras.

COROLLARY 1. *Suppose $\mathscr{M}$ is a unital commutative C*-subalgebra of a unital C*-algebra $\mathscr{B}$. The following are equivalent:*

1. *$\mathscr{M}$ is normal in $\mathscr{B}$.*

2. *$\mathscr{M}$ is an intersection of maximal abelian subalgebras of $\mathscr{B}$.*

3. *$\mathscr{M}$ is an intersection of masas in $\mathscr{B}$.*

4. *$\mathscr{M}$ is an intersection of algebras of the form $(\mathscr{S},\mathscr{B})'$ for subsets $\mathscr{S}$ of $\mathscr{B}$.*

We now know that every masa in a C*-algebra is normal. If $\mathscr{M}$ is a masa in a von Neumann algebra $\mathscr{B}$, then the double commutant theorem holds even with a distance formula. The proof is a simple adaptation of the proof of Lemma 3.1 in [13].

LEMMA 2. *Suppose $\mathscr{M}$ is a masa in a von Neumann algebra $\mathscr{B}$ and $T \in \mathscr{B}$. Then*

$$\begin{aligned} dist\,(T,\mathscr{M}) &\leqslant \sup\left\{\|UT - TU\| : U = U^* \in \mathscr{B}, U^2 = 1\right\} \\ &= 2\sup\left\{\|TP - PT\| : P = P^* = P^2 \in \mathscr{B}\right\} \end{aligned}$$

Proof. Let R denote the right-hand side of the inequality, and let D be the closed ball in $\mathscr{B}$ centered at T with radius R. Suppose $\mathscr{F}$ is a finite orthogonal set of projections in $\mathscr{M}$ whose sum is 1. Let $G(\mathscr{F})$ be the set of all sums of the form

$$\sum_{P\in\mathscr{F}} \lambda_P P$$

with each λ_P in $\{-1,1\}$. Then $G(\mathscr{F})$ is a finite group of unitaries and each $U\in G(\mathscr{F})$ has the form $2Q-1$ with Q a finite sum of elements in $\mathscr{F}$. Moreover, if $U=2Q-1$,

$$2\|TQ-QT\| = \|TU-UT\| = \|T-UTU^*\|.$$

It follows that $UTU^*\in D$ for every $U\in G(\mathscr{F})$. Define

$$S_{\mathscr{F}} = \frac{1}{card G(\mathscr{F})} \sum_{U\in G(F)} UTU^*.$$

Since $G(\mathscr{F})$ is a group, it easily follows that, for every $U_0\in G(\mathscr{F})$,

$$U_0 S_{\mathscr{F}} U_0^* = S_{\mathscr{F}}.$$

This implies that $S_{\mathscr{F}} = \sum_{P\in\mathscr{F}} PTP \in (\mathscr{F},\mathscr{B})' = (G(\mathscr{F}),\mathscr{B})'$. Choose a subnet $\{S_{\mathscr{F}_\lambda}\}$ that converges in the weak operator topology to $S\in D$. Then $S\in(\mathscr{M},\mathscr{B})'\cap D$. Since $(\mathscr{M},\mathscr{B})' = \mathscr{M}$, we conclude

$$dist(T,\mathscr{M}) \leqslant \|T-S\| \leqslant R. \quad \square$$

We now address the approximate double commutant relative to a C*-algebra. If $\mathscr{S}$ is a subset of a C*-algebra $\mathscr{B}$, we know that $\mathrm{Appr}(\mathscr{S},\mathscr{B})''$ must contain the center $\mathscr{Z}(\mathscr{B}) = \mathscr{B}\cap\mathscr{B}'$. Hence if $\mathscr{A}$ is a unital C*-subalgebra of a C*-algebra $\mathscr{B}$, then

$$C^*(\mathscr{A}\cup\mathscr{Z}(\mathscr{B})) \subseteq \mathrm{Appr}(\mathscr{A},\mathscr{B})''.$$

When $\mathscr{A}$ is commutative, we will prove that equality holds in certain cases, including when $\mathscr{B}$ is a von Neumann algebra.

The following result is based on S. Macado's generalization [11] of the Bishop-Stone-Weierstrass theorem. If K is a compact Hausdorff space and $\mathscr{G}$ is a unital closed subalgebra of $C(K)$, a subset E of K is called $\mathscr{G}$*-antisymmetric* if, for every $g\in\mathscr{G}$, the restriction $g|_E$ is real-valued implies $g|_E$ is constant. Machados's theorem [11] says that if $h\in C(K)$, then there is a closed $\mathscr{G}$-antisymmetric set $E\subseteq K$ such that

$$dist(h,\mathscr{G}) = dist(h|_E, \mathscr{G}|_E),$$

where $\mathscr{G}|_E = \{g|_E : g\in\mathscr{G}\}$. A beautiful, short, elementary proof of Machado's theorem was given by T. J. Ransford in [12].

LEMMA 3. *Suppose $\mathscr{W}$ is a unital C*-subalgebra of a commutative C*-algebra $\mathscr{D}$, and $S=S^*\in\mathscr{D}$ and $S\notin\mathscr{W}$. Then there are multiplicative linear functionals α,β on $\mathscr{D}$ and nets $\{A_\lambda\},\{B_\lambda\},\{X_\lambda\}$ and $\{Y_\lambda\}$ in $\mathscr{D}$ such that*

1. $0 \leqslant X_\lambda \leqslant A_\lambda \leqslant 1, 0 \leqslant Y_\lambda \leqslant B_\lambda \leqslant 1$,
2. $X_\lambda Y_\lambda = 0, A_\lambda X_\lambda = X_\lambda,\ Y_\lambda B_\lambda = Y_\lambda$,
3. $\|DA_\lambda - \alpha(D)A_\lambda\| \to 0$ *and* $\|DB_\lambda - \beta(D)B_\lambda\| \to 0$ *for every* $D \in \mathscr{D}$,
4. $\alpha(A) = \beta(A)$ *for every* $A \in \mathscr{W}$,
5. $\alpha(X_\lambda) = \beta(Y_\lambda) = 1$ *for every* λ,
6. $\beta(S) - \alpha(S) = 2\mathit{dist}(S, \mathscr{W})$.

Proof. Let K be the maximal ideal space of $\mathscr{D}$ and let $\Gamma : \mathscr{D} \to C(K)$ be the Gelfand map, which must be a $*$-isomorphism since $\mathscr{D}$ is a commutative C*-algebra. Let $g = \Gamma(S) = \Gamma(S^*) = \bar{g}$. It follows from Machado's theorem [11] that there is a $\Gamma(\mathscr{W})$-antisymmetric set $E \subseteq K$ such that

$$\mathit{dist}(S, \mathscr{W}) = \mathit{dist}(g, \Gamma(\mathscr{W})) = \mathit{dist}(g|_E, \Gamma(\mathscr{W})|_E).$$

Since $\Gamma(\mathscr{W})$ is self-adjoint and E is $\Gamma(\mathscr{W})$-antisymmetric, every function in $\Gamma(\mathscr{W})$ is constant. Hence $\mathit{dist}(g|_E, \Gamma(\mathscr{W})|_E)$ is the distance from $g|_E$ to the constant functions. It is clear that the closest constant function to $g|_E$ is

$$\frac{g(\beta) + g(\alpha)}{2},$$

where $\alpha, \beta \in E,\ g(\beta) = \max_{x \in E} g(x)$ and $g(\alpha) = \min_{x \in E} g(x)$. Let Λ be the directed set of all pairs $\lambda = (U_\lambda, V_\lambda)$ of disjoint open sets with $\alpha \in U_\lambda$ and $\beta \in V_\lambda$, ordered by $\lambda_1 \leqslant \lambda_2$ if and only if $U_{\lambda_2} \subseteq U_{\lambda_1}$ and $V_{\lambda_2} \subseteq V_{\lambda_1}$. For each $\lambda \in \Lambda$ choose continuous functions $r_\lambda, s_\lambda, t_\lambda, u_\lambda : K \to [0,1]$ such that

a. $r_\lambda(\alpha) = t_\lambda(\beta) = 1$,

b. $0 \leqslant r_\lambda = r_\lambda s_\lambda \leqslant s_\lambda \leqslant 1$,

c. $0 \leqslant t_\lambda = t_\lambda u_\lambda \leqslant u_\lambda \leqslant 1$,

d. $\operatorname{supp} s_\lambda \subseteq U_\lambda$ and $\operatorname{supp} u_\lambda \subseteq V_\lambda$.

If we choose $A_\lambda, B_\lambda, X_\lambda, Y_\lambda \in \mathscr{A}$ such that $\Gamma(X_\lambda) = r_\lambda,\ \Gamma(A_\lambda) = s_\lambda,\ \Gamma(Y_\lambda) = t_\lambda$, and $\Gamma(B_\lambda) = u_\lambda$, then statements (1)-(6) are clear. $\square$

A C*-algebra $\mathscr{B}$ is *primitive* if it has a faithful irreducible representation. A C*-algebra $\mathscr{B}$ is *prime* if, for every $x, y \in \mathscr{B}$, we have

$$x\mathscr{B}y = \{0\} \Longrightarrow x = 0 \text{ or } y = 0.$$

Every primitive C*-algebra is prime, and it was proved by Dixmier [2] that every separable prime C*-algebra is primitive. N. Weaver [15] gave an example of a nonseparable prime C*-algebra that is not primitive.

We define $\mathscr{B}$ to be *centrally prime* if, whenever $x,y \in \mathscr{B}$, $0 \leqslant x,y \leqslant 1$ and $x\mathscr{B}y = \{0\}$, there is an $e \in \mathscr{Z}(\mathscr{B})$ such that $x \leqslant e \leqslant 1$ and $y \leqslant 1-e \leqslant 1$. The centrally prime algebras include the prime ones, von Neumann algebras, and $\prod_{i\in I}\mathscr{B}_i / \sum_{i\in I}\mathscr{B}_i$ or a C*-ultraproduct $\prod_{i\in I}^{\alpha}\mathscr{B}_i$ when $\{\mathscr{B}_i : i \in I\}$ is a collection of unital primitive C*-algebras (see the proof of Theorem 4).

We characterize $\mathrm{Appr}(\mathscr{A},\mathscr{B})''$ for every commutative C*-subalgebra $\mathscr{A}$ of a centrally prime C*-algebra $\mathscr{B}$, and we show that there is a distance formula for every commutative unital C*-subalgebra if and only if every masa in $\mathscr{B}$ has a distance formula. In particular, when $\mathscr{B}$ is a von Neumann algebra, we obtain a distance formula.

REMARK 1. Here is a useful comment on distance formulas. If $\mathscr{B}$ is a unital C*-algebra and $\mathscr{S} = \mathscr{S}^* \subseteq \mathscr{B}$, then $(\mathscr{S},\mathscr{B})'$ is a unital C*-algebra, so, by the Russo-Dye theorem, the closed unit ball of $(\mathscr{S},\mathscr{B})'$ is the norm-closed convex hull of the set of unitary elements in $(\mathscr{S},\mathscr{B})'$. Hence, for any $T \in \mathscr{B}$,

$$\sup\left\{\|TW - WT\| : W \in (\mathscr{S},\mathscr{B})', \|W\| \leqslant 1\right\}$$

$$= \sup\left\{\|TU - UT\| : U \in (\mathscr{S},\mathscr{B})', U \text{ is unitary}\right\}.$$

A similar result holds in the approximate case. Suppose $(\Lambda,\leqslant)$ is a directed set. Then $\prod_{\lambda\in\Lambda}\mathscr{B}$ is a unital C*-algebra and the set

$$\mathscr{E} = \left\{ \{W_\lambda\} \in \prod_{\lambda\in\Lambda}\mathscr{B} : \forall S \in \mathscr{S}, \lim_{\lambda}\|W_\lambda S - SW_\lambda\| = 0 \right\}$$

is a unital C*-algebra and the closed unit ball $\mathscr{E}_1$ of $\mathscr{E}$ is the closed convex hull of its unitary group. Hence

$$\sup\left\{ \limsup_{\lambda}\|TW_\lambda - W_\lambda T\| : W = \{W_\lambda\} \in \mathscr{E}, \|W\| \leqslant 1 \right\}$$

$$= \sup\left\{ \limsup_{\lambda}\|TU_\lambda - U_\lambda T\| : U = \{U_\lambda\} \in \mathscr{E}, U \text{ is unitary} \right\}.$$

THEOREM 1. *Suppose $\mathscr{B}$ is a centrally prime unital C*-algebra and $\mathscr{Z}(\mathscr{B}) \subseteq \mathscr{W} \subseteq \mathscr{D}$ are unital commutative C*-subalgebras of $\mathscr{B}$. Suppose $S = S^* \in \mathscr{D}$. Then there is a net $\{W_\lambda\}$ in $\mathscr{B}$ such that*

1. *W_λ is unitary for every λ,*
2. *$\lim_\lambda \|AW_\lambda - W_\lambda A\| = 0$ for every $A \in \mathscr{W}$,*
3. *$\lim_\lambda \|SW_\lambda - W_\lambda S\| = 2\mathit{dist}(S,\mathscr{W})$.*

Moreover, if $\mathscr{B}$ is a von Neumann algebra, then there is a net $\{P_\lambda\}$ of projections in $\mathscr{B}$ such that

4. $\lim_\lambda \|AP_\lambda - P_\lambda A\| = 0$ *for every* $A \in \mathscr{W}$,
5. $\lim_\lambda \ \|SP_\lambda - P_\lambda S\| = dist\,(S, \mathscr{W})$.

Proof. Let $\mathscr{W} = C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B}))$, $\mathscr{D} = C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B}) \cup \{S\})$. Now choose α, β and nets $\{A_\lambda\}, \{B_\lambda\}, \{X_\lambda\}$ and $\{Y_\lambda\}$ in $\mathscr{D}$ as in Lemma 3. We first show that $X_\lambda \mathscr{B} Y_\lambda \neq \{0\}$; otherwise, since $\mathscr{B}$ is centrally prime, there is an $e \in \mathscr{Z}(\mathscr{B})$ such that $X_\lambda \leqslant e \leqslant 1$ and $Y_\lambda \leqslant 1 - e \leqslant 1$. Hence $\alpha(e) = 1$ and $\beta(1-e) = 1$, or $\beta(e) = 0$. However, $e \in \mathscr{W}$ and, by part (4) of Lemma 3, we get $\alpha(e) = \beta(e)$. This contradiction shows that $X_\lambda \mathscr{B} Y_\lambda \neq \{0\}$. Hence there is a $C_\lambda \in \mathscr{B}$ such that $\|X_\lambda C_\lambda Y_\lambda\| = 1$. Define $W_\lambda = X_\lambda C_\lambda Y_\lambda = A_\lambda W_\lambda = W_\lambda B_\lambda$. Lemma 3 implies that, for every $D \in \mathscr{D}$,

$$\|DW_\lambda - \alpha(D) W_\lambda\| = \|DA_\lambda W_\lambda - \alpha(D) A_\lambda W_\lambda\| \leqslant \|[D - \alpha(D)] A_\lambda\| \, \|W_\lambda\| \to 0,$$

and

$$\|W_\lambda D - \beta(D) W_\lambda\| = \|W_\lambda B_\lambda D - \beta(D) W_\lambda B_\lambda\| \leqslant \|W_\lambda\| \, \|B_\lambda [D - \alpha(D)]\| \to 0.$$

Since $\alpha(A) = \beta(A)$ for every $A \in \mathscr{W}$, it follows that $\|AW_\lambda - W_\lambda A\| \to 0$. It also follows that

$$\lim_\lambda \|W_\lambda S - SW_\lambda\| = \lim_\lambda |\beta(S) - \alpha(S)| \, \|W_\lambda\| = |\beta(S) - \alpha(S)| = 2dist\,(S, \mathscr{W}).$$

We now appeal to Remark 1 to replace the net $\{W_\lambda\}$ with a net of unitaries.

Now suppose $\mathscr{B}$ is a von Neumann algebra. Once we get $X_\lambda \mathscr{B} Y_\lambda \neq 0$ we know that there is a partial isometry V_λ in $\mathscr{B}$ whose final space is contained in the closure of $ranX_\lambda$ and whose initial space is contained in $(\ker Y_\lambda)^\perp$. Then (3) holds with $\{W_\lambda\}$ replaced with $\{V_\lambda\}$. Also, $V_\lambda^2 = 0$ (since $X_\lambda Y_\lambda = 0$), so $P_\lambda = \frac{1}{2}\left(V_\lambda + V_\lambda^* + V_\lambda V_\lambda^* + V_\lambda^* V_\lambda\right)$ is a projection. Using the above arguments gives us

$$\left\|DV_\lambda^* V_\lambda - \beta(D) V_\lambda^* V_\lambda\right\| \to 0, \left\|V_\lambda^* V_\lambda D - \beta(D) V_\lambda^* V_\lambda\right\| \to 0$$

and

$$\left\|DV_\lambda V_\lambda^* - \alpha(D) V_\lambda V_\lambda^*\right\| \to 0, \left\|V_\lambda V_\lambda^* D - \alpha(D) V_\lambda V_\lambda^*\right\| \to 0,$$

which implies

$$\left\|DV_\lambda^* V_\lambda - V_\lambda^* V_\lambda D + DV_\lambda V_\lambda^* - V_\lambda V_\lambda^* D\right\| \to 0$$

for every $D \in \mathscr{B}$. Thus

$$\begin{aligned}\lim_\lambda \|SP_\lambda - P_\lambda S\| &= \frac{1}{2} \lim \left\|(\alpha(S) V_\lambda - V_\lambda \beta(S)) + \left(\beta(S) V_\lambda^* - V_\lambda^* \alpha(S)\right)\right\| \\ &= \lim_\lambda \frac{1}{2} |\beta(S) - \alpha(S)| \left\|V_\lambda^* - V_\lambda\right\| = \frac{1}{2} |\beta(S) - \alpha(S)| = dist\,(S, \mathscr{W}),\end{aligned}$$

since $\left\|V_\lambda^* - V_\lambda\right\| = 1$ for every λ. $\square$

THEOREM 2. *Suppose $\mathscr{A}$ is a unital commutative C*-subalgebra of a centrally prime unital C*-algebra $\mathscr{B}$. Then*

$$\mathrm{Appr}(\mathscr{A},\mathscr{B})'' = C^*(\mathscr{A}\cup\mathscr{Z}(\mathscr{B})).$$

Hence $\mathscr{A}$ is normal if and only if $\mathscr{Z}(\mathscr{B})\subseteq\mathscr{A}$.

Proof. It is clear that $\mathscr{W} = C^*(\mathscr{A}\cup\mathscr{Z}(\mathscr{B}))\subseteq \mathrm{Appr}(\mathscr{A},\mathscr{B})''$. Choose a masa $\mathscr{D}$ of $\mathscr{B}$ with $\mathscr{A}\subseteq\mathscr{D}$. Then

$$\mathscr{W}\subseteq \mathrm{Appr}(\mathscr{A},\mathscr{B})''\subseteq \mathrm{Appr}(\mathscr{D},\mathscr{B})'' = \mathscr{D}.$$

If we choose $S = S^*\in \mathrm{Appr}(\mathscr{A},\mathscr{B})''$ and apply Theorem 1 we see that $S\in\mathscr{W}$. Since $\mathrm{Appr}(\mathscr{A},\mathscr{B})''$ is a C*-algebra, we have proved that $\mathrm{Appr}(\mathscr{A},\mathscr{B})''\subseteq\mathscr{W}$. $\square$

COROLLARY 2. *If $\mathscr{B}$ is a centrally prime C*-algebra with trivial center, e.g., a factor von Neumann algebra or the Calkin algebra, then $\mathscr{A} = \mathrm{Appr}(\mathscr{A},\mathscr{B})''$ for every commutative unital C*-subalgebra $\mathscr{A}$ of $\mathscr{B}$.*

In the von Neumann algebra setting, we get a distance formula. We have not tried to get the best constant.

THEOREM 3. *Suppose $\mathscr{A}$ is a unital commutative C*-subalgebra of a von Neumann algebra $\mathscr{B}$ and $T\in\mathscr{B}$. Then there is a net $\{P_\lambda\}$ of projections in $\mathscr{B}$ such that,*

1. *for every $A\in\mathscr{A}$,*
$$\|AP_\lambda - P_\lambda A\|\to 0,$$
and

2. $$dist(T, C^*(\mathscr{A}\cup\mathscr{Z}(\mathscr{B})))\leqslant 10\lim_\lambda\|TP_\lambda - P_\lambda T\|.$$

Proof. Let $\mathscr{W} = C^*(\mathscr{A}\cup\mathscr{Z}(\mathscr{B}))$. We define the seminorm Δ on $\mathscr{B}$ by setting $\Delta(V)$ to be the supremum of $\lim_\lambda\|VP_\lambda - P_\lambda V\|$ taken over all nets $\{P_\lambda\}$ of projections in $\mathscr{B}$ for which $\|AP_\lambda - P_\lambda A\|\to 0$ for every $A\in\mathscr{A}$ and $\lim_\lambda\|VP_\lambda - P_\lambda V\|$ exists. Let $\mathscr{D}$ be a masa in $\mathscr{B}$ such that $\mathscr{W}\subseteq\mathscr{D}$.

We first assume $T = T^*$. It follows from Lemma 2 that there is an $S\in\mathscr{D}$ such that

$$\|S-T\|\leqslant 2\sup\left\{\|TP-PT\| : P = P^* = P^2\in\mathscr{D}\right\}\leqslant 2\Delta(T).$$

If we apply Theorem 1, we obtain a net $\{P_\lambda\}$ of projections in $\mathscr{B}$ such that

$$\lim_\lambda\|WP_\lambda - P_\lambda W\| = 0$$

for every $W \in \mathscr{W}$, and such that

$$\lim_{\lambda} \|P_\lambda S - SP_\lambda\| = dist\,(S, \mathscr{W}).$$

It follows that

$$\begin{aligned} dist\,(T,\mathscr{W}) &\leqslant dist\,(S,\mathscr{W}) + \|S-T\| \leqslant \Delta(S) + 2\Delta(T) \\ &\leqslant \Delta(S-T) + \Delta(T) + 2\Delta(T) \leqslant \|S-T\| + 3\Delta(T) \leqslant 5\Delta(T). \end{aligned}$$

whenever $T = T^*$.

For the general case,

$$\begin{aligned} dist\,(T,\mathscr{A}) &\leqslant dist\,(\mathrm{Re}\,T,\mathscr{A}) + dist\,(\mathrm{Im}\,T,\mathscr{A}) \\ &\leqslant 5\Delta(\mathrm{Re}\,T) + 5\Delta(\mathrm{Im}\,T) \leqslant 5\left[\frac{1}{2}\Delta(T+T^*) + \frac{1}{2}\Delta(T-T^*)\right] \\ &\leqslant 5\,[\Delta(T) + \Delta(T^*)] = 10\Delta(T), \end{aligned}$$

since $\Delta(T) = \Delta(T^*)$. $\square$

In some cases our results yield information on relative double commutants.

THEOREM 4. *Suppose $\{\mathscr{B}_n\}$ is a sequence of primitive C*-algebras and $\mathscr{B} = \prod_{n\geqslant 1} \mathscr{B}_n / \sum_{n\geqslant 1}^{\oplus} \mathscr{B}_n$. If $\mathscr{A}$ is a separable commutative unital C*-subalgebra of $\mathscr{B}$, then*

$$(\mathscr{A},\mathscr{B})'' = C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B})),$$

i.e., $C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B}))$ is normal.

Proof. We first show that $\mathscr{B}$ is centrally prime. Since each $\mathscr{B}_n$ is primitive, we can assume, for each $n \in \mathbb{N}$, that there is a Hilbert space H_n such that $\mathscr{B}_n$ is an irreducible unital C*-subalgebra of $B(H_n)$. Suppose $A, B \in \mathscr{B}$, $0 \leqslant A, B \leqslant 1$ and $A\mathscr{B}B = 0$. We can lift A, B, respectively to a sequences $\{A_n\}, \{B_n\}$ in $\prod_{n\geqslant 1} \mathscr{B}_n$. Hence, for every bounded sequence $\{T_n\} \in \prod_{n\geqslant 1} \mathscr{B}_n$, we have

$$\lim_{n\to\infty} \|A_n T_n B_n\| = 0.$$

Choose unit vectors $e_n, f_n \in H_n$ so that $\|A_n e_n\| \geqslant \|A_n\|/2$ and $\|B_n f_n\| \geqslant \|B_n\|/2$. It follwos from the irreducibility of $\mathscr{B}_n$ and Kadison's transitivity theorem [9] that there is a $T_n \in \mathscr{B}_n$ such that $\|T_n\| = 1$ and $T_n B_n f_n = \|B_n f_n\|\, e_n$. It follows that

$$0 = \lim_{n\to\infty} \|A_n T_n B_n\| \geqslant \lim_{n\to\infty} \|A_n T_n B_n f_n\| \geqslant \lim \frac{1}{4}\|A_n\|\,\|B_n\|.$$

Hence

$$\lim_{n\to\infty} \min\left(\|A_n\|, \|B_n\|\right)^2 \leqslant \lim_{n\to\infty} \|A_n\|\,\|B_n\| = 0.$$

For each $n \in \mathbb{N}$ we define

$$P_n = \begin{cases} 1 \text{ if } \|B_n\| \leqslant \|A_n\| \\ 0 \text{ if } \|A_n\| < \|B_n\| \end{cases}.$$

Then $\{P_n\}$ is in the center of $\prod_{n \geqslant 1} \mathscr{B}_n$ and

$$\lim_{n \to \infty} \|P_n B_n\| = \lim_{n \to \infty} \|(1 - P_n) A_n\| = 0.$$

If P is the image of $\{P_n\}$ in the quotient $\mathscr{B}$, then P is a central projection and $PA = P$ and $(1 - P)B = B$. Hence $\mathscr{B}$ is centrally prime. So it follows that

$$\operatorname{Appr}(\mathscr{A}, \mathscr{B})'' = C^*(\mathscr{A} \cup \mathscr{Z}(\mathscr{B})).$$

The proof will be completed with proof of the following claim: If $\mathscr{S}$ is a norm-separable subset of $\mathscr{B}$, then

$$\operatorname{Appr}(\mathscr{S}, \mathscr{B})'' = (\mathscr{S}, \mathscr{B})''.$$

It is clear from considering constant sequences that the inclusion $\operatorname{Appr}(\mathscr{S}, \mathscr{B})'' \subseteq (\mathscr{S}, \mathscr{B})''$ holds for every unital C*-algebra $\mathscr{B}$. To prove the reverse inclusion, suppose $T \notin appr(\mathscr{S}, \mathscr{B})''$. Then there is and $\varepsilon > 0$ and a net $\{A_\lambda\}$ in $\mathscr{B}$ such that $\|A_\lambda S - SA_\lambda\| \to 0$ for every $S \in \mathscr{S}$, and such that $\|A_\lambda T - TA_\lambda\| \geqslant \varepsilon$ for every λ. Let $\mathscr{S}_0 = \{S_1, S_2, \ldots\}$ be a dense subset of $\mathscr{S}$. We can lift each S_n to $\{S_n(j)\}_{j \geqslant 1} \in \prod_{k \geqslant 1} \mathscr{B}_k$ and lift T to $\{T(j)\}_{j \geqslant 1}$. It follows that, for every $n \in \mathbb{N}$, there is an $A_n \in \mathscr{B}$ with $\|A_n\| = 1$ such that

a. $\|A_n S_k - S_k A_n\| < 1/n$ for $1 \leqslant k \leqslant n$,

b. $\|A_n T - TA_n\| > \varepsilon/2$.

Note that if $B \in \mathscr{B}$ lifts to $\{B(j)\}_{j \geqslant 1} \in \prod_{k \geqslant 1} \mathscr{B}_k$, then $\|B\| = \limsup_{j \to \infty} \|B(j)\|$. If we lift each A_n to $\{A_n(j)\}$, it follows that we can find an arbitrarily large $j_n \in \mathbb{N}$ such that $\|A_n(j_n) S_k(j_n) - S_k(j_n) A_n(j_n)\| < 1/n$ for $1 \leqslant k \leqslant n$ and $\|A_n(j_n) T(j_n) - T(j_n) A_n(j_n)\| > \varepsilon/2$. Since j_n can be chosen to be arbitrarily large, we can choose $\{j_n\}$ so that $j_1 < j_2 < \cdots$. We now define $A \in \mathscr{B}$ by defining

$$A(j) = \begin{cases} A_n(j_n) & \text{if } j = j_n \text{ for some } n \geqslant 1 \\ 0 & \text{otherwise} \end{cases}.$$

We see that $AS_k = S_k A$ for all $k \geqslant 1$ and $\|AT - TA\| \geqslant \varepsilon/2$. Hence $T \notin (\mathscr{S}, \mathscr{B})''$. $\square$

We conclude with some questions.

1. If $\mathscr{M}$ is a normal von Neumann subalgebra of a factor von Neumann algebra $\mathscr{B}$, is there a constant $K \geqslant 1$ such that, for every $T \in \mathscr{B}$,

$$dist(T, \mathscr{M}) \leqslant K \sup\left\{\|TP - PT\| : P = P^2 = P^* \in \mathscr{M}' \cap \mathscr{B}\right\}?$$

When $\mathscr{B} = B(H)$, this question is equivalent to Kadison's similarity problem. What about factors not of type I?

2. Is there an analog of Theorem 3 for arbitrary C*-subalgebras of a factor von Neumann algebra?

3. It seems likely that a version of parts (4) and (5) of Theorem 1 might hold under assumptions weaker than $\mathscr{B}$ being a von Neumann algebra. Is it true when $\mathscr{B}$ has real-rank zero? What if we include nuclear and simple? The key is getting the partial isometries V_λ in the proof of Theorem 1. When does a unital C*-algebra $\mathscr{B}$ have the property that whenever $X,Y,A,B \geqslant 0$ are in $\mathscr{B}$ and $AX = X$, $BY = Y$, $AB = 0$ and $X\mathscr{B}Y \neq \{0\}$, there is a nonzero partial isometry $V \in \mathscr{B}$ such that $AV = VB = V$?

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