Operators and Matrices
Volume 8, Number 3 (2014), 635–650 doi:10.7153/oam-08-34

COMMUTING OF BLOCK DUAL TOEPLITZ OPERATORS

BO ZHANG AND YUFENG LU

(*Communicated by V. V. Peller*)

Abstract. In this paper, we characterize the commuting (semi-commuting) and the essentially commuting (semi-commuting) of block dual Toeplitz operators.

1. Introduction

Let dA denote Lebesgue area measure on the unit disk $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equals 1. The scalar valued Bergman space $L_a^2(\mathbb{D})$ is the Hilbert space consisting of all analytic functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D},dA)$. The scalar valued Bergman space $L_a^2(\mathbb{D})$ is a Hilbert space with the inner product given by $\langle f,g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z)$, for f, $g\in L_a^2(\mathbb{D})$. Let P_0 denote the orthogonal projection of $L^2(\mathbb{D},dA)$ onto $L_a^2(\mathbb{D})$. For a function $f\in L^\infty(\mathbb{D})$, the Toeplitz operator $T_f:\ L_a^2(\mathbb{D})\to L_a^2(\mathbb{D})$ and the Hankel operator $H_f:\ L_a^2(\mathbb{D})\to (L_a^2(\mathbb{D}))^\perp$ are respectively defined by

$$T_f(g)=P_0(fg),\quad H_f(g)=(I-P_0)(fg),\quad g\in L_a^2(\mathbb{D}).$$

T_f and H_f are clearly bounded linear operators for every function $f\in L^\infty(\mathbb{D})$. We can also define the dual Toeplitz operator $S_f:\ (L_a^2(\mathbb{D}))^\perp\to(L_a^2(\mathbb{D}))^\perp$, $S_fg=(I-P_0)(fg)$, $g\in(L_a^2(\mathbb{D}))^\perp$. Clearly, S_f is a bounded linear operator on $(L_a^2(\mathbb{D}))^\perp$ for every function $f\in L^\infty(\mathbb{D})$.

For a measurable function $f:\mathbb{D}\to\mathbb{C}^n$ with $\int_{\mathbb{D}}\|f(z)\|^2_{\mathbb{C}^n}dA(z)<\infty$, we say that $f\in L^2(\mathbb{D},\mathbb{C}^n)$. The space $L^2(\mathbb{D},\mathbb{C}^n)$ is a Hilbert space with the inner product given by $\langle f,g\rangle=\int_{\mathbb{D}}\langle f(z),g(z)\rangle_{\mathbb{C}^n}dA(z)$. The vector valued Bergman space $L_a^2(\mathbb{D},\mathbb{C}^n)$ is the Hilbert space consisting of all analytic $\mathbb{C}^n-$ valued functions on $\mathbb{D}$ that are also in $L^2(\mathbb{D},\mathbb{C}^n)$. Let $M_{n\times n}$ be the set of $n\times n$ complex matrices, $L^\infty_{n\times n}(\mathbb{D})$ denote the space of $M_{n\times n}$-valued essentially bounded Lebesgue measurable functions on $\mathbb{D}$ and $H^\infty_{n\times n}(\mathbb{D})$ denote the space of $M_{n\times n}$-valued bounded analytic functions on $\mathbb{D}$.

Mathematics subject classification (2010): 47B35, 47B47.
Keywords and phrases: Vector valued Bergman space, dual Toeplitz operator, commutator, semi-commutator, compact.
This research is supported by NSFC, Item Numbers: 11271059, 11326097 and 11301046.

Paper OaM-08-34

For $F=(f_{ij})_{n\times n}\in L^{\infty}_{n\times n}(\mathbb{D})$, the block Toeplitz operator T_F and the block Hankel operator H_F on $L^2(\mathbb{D},\mathbb{C}^n)$, the block dual Toeplitz operator S_F on $(L^2(\mathbb{D},\mathbb{C}^n))^{\perp}$ with symbol F are defined respectively, as follows:

$$T_F=\begin{pmatrix} T_{f_{11}} & T_{f_{12}} & \cdots & T_{f_{1n}} \\ T_{f_{21}} & T_{f_{22}} & \cdots & T_{f_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ T_{f_{n1}} & T_{f_{n2}} & \cdots & T_{f_{nn}} \end{pmatrix},\ H_F=\begin{pmatrix} H_{f_{11}} & H_{f_{12}} & \cdots & H_{f_{1n}} \\ H_{f_{21}} & H_{f_{22}} & \cdots & H_{f_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ H_{f_{n1}} & H_{f_{n2}} & \cdots & H_{f_{nn}} \end{pmatrix} \text{ and } S_F=\begin{pmatrix} S_{f_{11}} & S_{f_{12}} & \cdots & S_{f_{1n}} \\ S_{f_{21}} & S_{f_{22}} & \cdots & S_{f_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ S_{f_{n1}} & S_{f_{n2}} & \cdots & S_{f_{nn}} \end{pmatrix},$$

where each $T_{f_{ij}}(1\leqslant i,j\leqslant n)$ is a Toeplitz operator on $L^2_a(\mathbb{D})$, each $H_{f_{ij}}(1\leqslant i,j\leqslant n)$ is a Hankel operator on $L^2_a(\mathbb{D})$ and each $S_{f_{ij}}(1\leqslant i,j\leqslant n)$ is a dual Toeplitz operator on $(L^2_a(\mathbb{D}))^{\perp}$. Let P be the orthogonal projection from $L^2(\mathbb{D},\mathbb{C}^n)$ onto $L^2_a(\mathbb{D},\mathbb{C}^n)$. It is easy to see that $T_Fu=P(Fu)$ and $S_Fv=(I-P)(Fv)$, where $u=(u_1,u_2,\cdots,u_n)^T$, $v=(v_1,v_2,\cdots,v_n)^T$ and $u_i\in L^2_a(\mathbb{D})$, $v_i\in(L^2_a(\mathbb{D}))^{\perp}$. Clearly, T_F and S_F are bounded linear operators for every function $F=(f_{ij})_{n\times n}\in L^{\infty}_{n\times n}(\mathbb{D})$.

The problem on commuting (dual) Toeplitz operators has attracted special attention over the years, particularly in view of the implications that commutativity has for the study of the associated structural and spectral theories. In the setting of the Hardy space over the unit disk, the celebrated theorem of Brown and Halmos [4] gives concrete necessary and sufficient conditions on the symbols to guarantee commutativity and Gorkin and Zheng [5] completely characterized the essentially commuting Toeplitz operators.

In the case of the Bergman space of the unit disk, the first complete result was obtained by S. Axler and Željko. Čučković [2], who characterized commuting Toeplitz operators with harmonic symbols. K. Stroethoff later extended that result to essentially commuting Toeplitz operators in [9], and S. Axler, Željko. Čučković and N. V. Rao [3] subsequently proved that if two Toeplitz operators commute and the symbol of one of them is nonconstant analytic, then the other one must be analytic.

Recently, many mathematicians have paid more attention to dual Toeplitz operators.

On the unit disk, commutativity of dual Toeplitz operators were studied by K. Stroethoff and D. Zheng in [11]. On the vector valued Bergman space, Kerr [7] gave a necessary and sufficient condition for the boundedness of the Toeplitz product $T_FT_{G^*}$ on $L^2_a(\mathbb{D},\mathbb{C}^n)$, and Lu, Zhang and Shi [8] gave some necessary and sufficient conditions for the product of two block dual Toeplitz operators still to be a block dual Toeplitz operator. Because two matrix-valued functions may not be commuting, the commuting problems of block (dual) Toeplitz operators are much more complicated and interesting. On the vector valued Hardy space $H^2(\mathbb{C}^n)$, Gu and Zheng [6] gave necessary and sufficient conditions for two block Toeplitz operators commuting or essentially commuting. In this paper, we investigate when two block dual Toeplitz operators on the vector Bergman space commute or essentially commute and give some necessary and sufficient conditions.

2. Preliminaries

On the vector valued Bergman space, it is easy to check that

$$T_{FG} = T_F T_G + H^*_{F^*} H_G$$

and

$$S_{FG} = S_F S_G + H_F H^*_{G^*}. \tag{2.1}$$

If we write $f = f_+ + f_-$ for each $f \in L^2(\mathbb{D})$ where $f_+ \in L^2_a(\mathbb{D})$ and $f_- \in (L^2_a(\mathbb{D}))^\perp$, then $F = (f_{ij})_{n\times n}$ can also be written as $F = F_+ + F_-$ with $F_+ = ((f_{ij})_+)_{n\times n}$ and $F_- = ((f_{ij})_-)_{n\times n}$. For $A = (a_{ij}) \in M_{n\times n}$, we define

$$\|A\|_\infty = \sup_{1\leqslant i,j\leqslant n} |a_{ij}|.$$

Let $(M_{n\times n})_1$ denote the closed unit ball of $M_{n\times n}$ in the above norm. Let $\mathscr{P}_n$ be the set of $n\times n$ permutation matrices and $E_j = (a_{ik})_{n\times n}$, where $a_{jj} = 1$ and $a_{ik} = 0 (i \neq j \text{ or } k \neq j)$. Note that for a $n\times n$ matrix B, BE_j and B have the same $j-$th column and all other columns of BE_j equal to zero.

The Bergman space $L^2_a(\mathbb{D})$ has reproducing kernels K_w given by

$$K_w(z) = \frac{1}{(1-\overline{w}z)^2}, \quad z,\ w \in \mathbb{D}.$$

For every $h \in L^2_a(\mathbb{D})$, we have $\langle h, K_w\rangle = h(w)$, $w \in \mathbb{D}$. In particular, $\|K_w\|_2 = \langle K_w, K_w\rangle^{\frac{1}{2}}$ $= (1-|w|^2)^{-1}$. The functions

$$k_w(z) = \frac{1-|w|^2}{(1-\overline{w}z)^2}$$

are the normalized reproducing kernels for $L^2_a(\mathbb{D})$.

For $w \in \mathbb{D}$, the fractional linear transformation φ_w, defined by

$$\varphi_w(z) = \frac{w-z}{1-\overline{w}z}, \ \ z \in \mathbb{D},$$

is an automorphism of the unit disk. In fact, $\varphi_w^{-1} = \varphi_w$. The real Jacobian for the change of variable $\zeta = \varphi_w(z)$ is equal to $|\varphi_w'(z)|^2 = (1-|w|^2)^2/|1-\overline{w}z|^4$. Thus we have the change-of-variable formula

$$\int_{\mathbb{D}} h(\varphi_w(z))dA(z) = \int_{\mathbb{D}} h(z)\frac{(1-|w|^2)^2}{|1-\overline{w}z|^4}dA(z) = \int_{\mathbb{D}} h(z)|k_w(z)|^2 dA(z), \tag{2.2}$$

where h is a positive measurable or integrable function on $\mathbb{D}$.

For f, $g \in L^2(\mathbb{D})$, define the rank 1 operator $f\otimes g\colon L^2(\mathbb{D}) \to L^2(\mathbb{D})$ by

$$(f\otimes g)h = \langle h, g\rangle f$$

for $h \in L^2(\mathbb{D})$.

Also for $F = (f_{ij})_{n\times n}, G = (g_{ij})_{n\times n} \in M_{n\times n}(L^2(\mathbb{D}))$, define the operator $F \otimes G$: $L^2(\mathbb{D},\mathbb{C}^n) \to L^2(\mathbb{D},\mathbb{C}^n)$ by

$$(F \otimes G)h = \begin{pmatrix} \sum_{k=1}^{n} f_{1k} \otimes g_{k1} & \sum_{k=1}^{n} f_{1k} \otimes g_{k2} & \cdots & \sum_{k=1}^{n} f_{1k} \otimes g_{kn} \\ \sum_{k=1}^{n} f_{2k} \otimes g_{k1} & \sum_{k=1}^{n} f_{2k} \otimes g_{k2} & \cdots & \sum_{k=1}^{n} f_{2k} \otimes g_{kn} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^{n} f_{nk} \otimes g_{k1} & \sum_{k=1}^{n} f_{nk} \otimes g_{k2} & \cdots & \sum_{k=1}^{n} f_{nk} \otimes g_{kn} \end{pmatrix} h$$

for $h \in L^2(\mathbb{D},\mathbb{C}^n)$.

Given a linear operator T on $(L_a^2(\mathbb{D}))^\perp$ and $w \in \mathbb{D}$, we define the operator $S_w(T)$ by

$$S_w(T) = T - S_{\varphi_w} T S_{\overline{\varphi}_w}.$$

Note that

$$S_w^2(T) = S_w(S_w(T)) = T - 2S_{\varphi_w} T S_{\overline{\varphi}_w} + S_{\varphi_w}^2 T S_{\overline{\varphi}_w}^2.$$

From the proof of Proposition 4.8 in [10], we have

$$(H_f k_w) \otimes (H_{\overline{g}} k_w) = H_f (k_w \otimes k_w) H_{\overline{g}}^* = S_w^2 (H_f H_{\overline{g}}^*).$$

Let T be a linear operator on $(L_a^2(\mathbb{D}))^\perp \otimes \mathbb{C}^n$, we can also define the operator $S_W(T) = T - S_{\psi_w} T S_{\overline{\psi}_w}$, where

$$S_{\psi_w} = \begin{pmatrix} S_{\varphi_w} & 0 & \cdots & 0 \\ 0 & S_{\varphi_w} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{\varphi_w} \end{pmatrix}_{n\times n}.$$

It follows that

$$S_W^2(T) = T - 2S_{\psi_w} T S_{\overline{\psi}_w} + S_{\psi_w}^2 T S_{\overline{\psi}_w}^2.$$

3. Commuting block dual Toeplitz operators

Stroethoff and Zheng [11] showed that the semi-commutator $S_f S_g - S_{fg}$ is zero exactly when either f or $\overline{g}$ is analytic for the scalar functions f and g, and they also characterized when the commutator $S_f S_g - S_g S_f$ is zero. In this section, we will characterize when the semi-commutator $(S_F, S_G] = S_F S_G - S_{FG}$ or the commutator $[S_F, S_G] = S_F S_G - S_G S_F$ is zero for block dual Toeplitz operators with matrix symbols F and G.

THEOREM 3.1. *Let* $F=(f_{ij})_{n\times n}$, $G=(g_{ij})_{n\times n}\in L^\infty_{n\times n}(\mathbb{D})$. *Then:*
(i) $(S_F,S_G]=0$ *if and only if* $F_-\otimes\overline{G}_-=0$*;*
(ii) $(S_F,S_G]=0$ *if and only if there exist matrix* $A_j\in(M_{n\times n})_1$ *and* $R_j\in\mathscr{P}_n$ *for each* $j=1,\cdots,n$*, such that*

$$(R_j-A_j)F^T\in H^\infty_{n\times n}(\mathbb{D}) \quad \textit{and} \quad A_j^*\overline{G}E_j\in H^\infty_{n\times n}(\mathbb{D}), \quad j=1,\cdots,n.$$

Proof. (i) If $(S_F,S_G]=0$, that is $S_FS_G=S_{FG}$, applying (2.1), then we have $H_FH^*_{G^*}=0$. It is easy to check that $S^2_W(H_FH^*_{G^*})=\left(\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w)\right)_{n\times n}$. Therefore, $\sum_{k=1}^{n}(H_{f_{ik}}1)\otimes(H_{\overline{g}_{kj}}1)=0$, that is $F_-\otimes\overline{G}_-=0$.

Conversely, suppose $F_-\otimes\overline{G}_-=0$. Claim 1: S_FS_G is a dual Toeplitz operator.

We use induction to prove Claim 1. If $H_{f_{11}}1\otimes H_{\overline{g}_{11}}1=0$, then f_{11} or $\overline{g}_{11}$ is analytic on $\mathbb{D}$. Therefore, the result is true for 1.

Now assume the result is true for $n-1$. That is, if $\sum_{k=1}^{n-1}(H_{f_{ik}}1)\otimes(H_{\overline{g}_{kj}}1)=0$, then $\sum_{k=1}^{n-1}S_{f_{ik}}S_{g_{kj}}$ is a dual Toeplitz operator.

In the following, we prove that the result is true for n.

If $\sum_{k=1}^{n}(H_{f_{ik}}1)\otimes(H_{\overline{g}_{kj}}1)=0$, then we get

$$\langle u,H_{\overline{g}_{1j}}1\rangle H_{f_{i1}}1+\langle u,H_{\overline{g}_{2j}}1\rangle H_{f_{i2}}1+\cdots+\langle u,H_{\overline{g}_{nj}}1\rangle H_{f_{in}}1=0 \tag{3.1}$$

for all $u\in(L^2_a)^\perp$. Let $Q_j=V\{H_{\overline{g}_{kj}}1;1\leqslant k\leqslant n-1\}$. For every $u\in Q_j^\perp$, we have $\langle u,H_{\overline{g}_{nj}}1\rangle H_{f_{in}}1=0$. It follows that $H_{f_{in}}1=0$ (Case 1) or $\langle u,H_{\overline{g}_{nj}}1\rangle=0$ (Case 2).

Case 1. If $H_{f_{in}}1=0$, then f_{in} is analytic on $\mathbb{D}$ and $\sum_{k=1}^{n-1}(H_{f_{ik}}1)\otimes(H_{\overline{g}_{kj}}1)=0$. By the induction hypothesis, we obtain that $\sum_{k=1}^{n-1}S_{f_{ik}}S_{g_{kj}}$ is a dual Toeplitz operator. Since $\sum_{k=1}^{n}S_{f_{ik}}S_{g_{kj}}=\sum_{k=1}^{n-1}S_{f_{ik}}S_{g_{kj}}+S_{f_{in}}S_{g_{nj}}$ and f_{in} is analytic, we can conclude that $\sum_{k=1}^{n}S_{f_{ik}}S_{g_{kj}}$ is also a dual Toeplitz operator.

Case 2. If $\langle u,H_{\overline{g}_{nj}}1\rangle=0$ for each $u\in Q_j^\perp$, then $H_{\overline{g}_{nj}}1\in Q_j$. So we get that there exist λ_1, λ_2, $\cdots$, $\lambda_{n-1}\in\mathbb{C}$ such that

$$H_{\overline{g}_{nj}}1=\sum_{i=1}^{n-1}\lambda_iH_{\overline{g}_{ij}}1.$$

So $\overline{g}_{nj}-\lambda_1\overline{g}_{1j}-\lambda_2\overline{g}_{2j}-\cdots-\lambda_{n-1}\overline{g}_{(n-1)j}$ is an analytic function, denoted by h. Replacing g_{nj} by $\overline{\lambda}_1g_{1j}+\overline{\lambda}_2g_{2j}+\cdots+\overline{\lambda}_{n-1}g_{(n-1)j}+\overline{h}$ in (3.1), we get

$$\sum_{k=1}^{n-1}\langle u,H_{\overline{g}_{kj}}1\rangle H_{f_{ik}}1+\langle u,\sum_{k=1}^{n-1}\lambda_kH_{\overline{g}_{kj}}1\rangle H_{f_{in}}1=0.$$

Thus

$$\langle u, H_{\overline{g}_{1j}}1\rangle H_{f_{i1}+\overline{\lambda}_1 f_{in}}1+\langle u, H_{\overline{g}_{2j}}1\rangle H_{f_{i2}+\overline{\lambda}_2 f_{in}}1+\cdots+\langle u, H_{\overline{g}_{(n-1)j}}1\rangle H_{f_{i(n-1)}+\overline{\lambda}_{n-1} f_{in}}1=0.$$

It is equal to

$$\sum_{k=1}^{n-1}(H_{f_{ik}+\overline{\lambda}_k f_{in}}1)\otimes(H_{\overline{g}_{kj}}1)=0.$$

By the induction hypothesis, we get $\sum\limits_{k=1}^{n-1} S_{(f_{ik}+\overline{\lambda}_k f_{in})}S_{g_{kj}}$ is a dual Toeplitz operator. Note that

$$\begin{aligned}\sum_{k=1}^{n} S_{f_{ik}}S_{g_{kj}} &= \sum_{k=1}^{n-1} S_{f_{ik}}S_{g_{kj}} + S_{f_{in}}S_{g_{nj}}\\ &= \sum_{k=1}^{n-1} S_{f_{ik}}S_{g_{kj}} + S_{f_{in}}S_{\overline{\lambda}_1 g_{1j}+\overline{\lambda}_2 g_{2j}+\cdots+\overline{\lambda}_{n-1}g_{(n-1)j}+\overline{h}}\\ &= \sum_{k=1}^{n-1} S_{f_{ik}}S_{g_{kj}} + \sum_{k=1}^{n-1}\overline{\lambda}_k S_{f_{in}}S_{g_{kj}} + S_{f_{in}\overline{h}},\end{aligned}$$

where the last equality follows from that h is an analytic function. So $\sum\limits_{k=1}^{n} S_{f_{ik}}S_{g_{kj}}$ is also a dual Toeplitz operator. By the arbitrary of i and j, we conclude that $S_F S_G$ is a dual Toeplitz operator. Claim 1 is proved.

If $S_F S_G$ is a dual Toeplitz operator, then there exists an $H=(h_{ij})_{n\times n}\in L^{\infty}_{n\times n}(\mathbb{D})$, such that $S_F S_G = S_H$. Applying (2.1), we have $S_{FG-H}=H_F H^*_{G^*}$. It is easy to check that $S^2_W(H_F H^*_{G^*})=\big(\sum\limits_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w)\big)_{n\times n}$. Thus

$$S_{(1-|\varphi_w|^2)^2(\sum\limits_{k=1}^{n} f_{ik}g_{kj}-h_{ij})}=\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w).$$

In particular, $S_{(1-|z|^2)^2(\sum\limits_{k=1}^{n} f_{ik}g_{kj}-h_{ij})}=\sum\limits_{k=1}^{n}(H_{f_{ik}}1)\otimes(H_{\overline{g}_{kj}}1)$.

Since $\mathrm{rank}(S_{(1-|z|^2)^2(\sum\limits_{k=1}^{n} f_{ik}g_{kj}-h_{ij})})\leqslant n$, there exist not all zero complex numbers a_1, a_2, $\cdots$, a_{n+1}, such that

$$S_{(1-|z|^2)^2(\sum\limits_{k=1}^{n} f_{ik}g_{kj}-h_{ij})}(a_1\overline{z}+a_2\overline{z}^2+\cdots+a_{n+1}\overline{z}^{n+1})=0.$$

Combining the above equality with facts that $\varphi=(1-|z|^2)^2(\sum\limits_{k=1}^{n} f_{ik}g_{kj}-h_{ij})(a_1\overline{z}+a_2\overline{z}^2+\cdots+a_{n+1}\overline{z}^{n+1})$ is analytic and $\lim\limits_{|z|\to 1-}\varphi(z)=0$, we get $\varphi(z)=0$ for all $z\in\mathbb{D}$.

Thus $\sum_{k=1}^{n} f_{ik}g_{kj} = h_{ij}$ with the exception of at most $n+1$ points. Therefore, $S_F S_G = S_{FG}$.

(ii) From the proof of (i), we have if $S_F S_G = S_{FG}$, then $(\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w))_{n\times n} = 0$. For each $j = 1,\cdots,n$, by Proposition 4 in [6], there exist matrix $A_j \in (M_{n\times n})_1$ and permutation matrix R_j such that

$$(R_j - A_j)[H_{f_{i1}}1,\cdots,H_{f_{in}}1]^T = 0,\ i = 1,\cdots,n$$

and

$$A_j^*[H_{\overline{g}_{1j}}1,\cdots,H_{\overline{g}_{nj}}1]^T = 0.$$

Since $(R_j - A_j)(I - P_0) = (I - P_0)(R_j - A_j)$, we have

$$(R_j - A_j)[f_{i1},\cdots,f_{in}]^T \in H^\infty_{n\times 1}(\mathbb{D}),\ i = 1,\cdots,n$$

and

$$A_j^*[\overline{g}_{1j},\cdots,\overline{g}_{nj}]^T \in H^\infty_{n\times 1}(\mathbb{D}).$$

So we get

$$(R_j - A_j)F^T \in H^\infty_{n\times n}(\mathbb{D}) \quad \text{and} \quad A_j^*\overline{G}E_j \in H^\infty_{n\times n}(\mathbb{D}).$$

Next, we prove the sufficiency. For $A_j \in (M_{n\times n})_1$ and permutation matrix R_j. Let

$$x_i = (x_{i1},\cdots,x_{in})^T = (R_j - A_j)f_{i\sigma} \quad \text{and} \quad y = (y_1,\cdots,y_n)^T = A_j^* g_\sigma,$$

where σ is permutation and $f_{i\sigma} = (f_{i\sigma(1)},\cdots,f_{i\sigma(n)})$, then we have

$$\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w) = H^*_{x_{i1}}H_{\overline{g}_{ij}} + \cdots + H^*_{x_{in}}H_{\overline{g}_{nj}} + H^*_{f_{i1}}H_{y_1} + \cdots + H^*_{f_{in}}H_{y_n},\ i = 1,\cdots,n,$$

where $j = 1,\cdots,n$. Since $x_{ik},\ y_k \in H^\infty(\mathbb{D})$, $1 \leqslant k \leqslant n$, $\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w) = 0$ for $1 \leqslant i,\ j \leqslant n$. It is easy to check that $S^2_W(H_F H^*_{G^*}) = \big(\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w)\big)_{n\times n}$. Therefore, we have $H_F H^*_{G^*} = 0$. That is $S_F S_G = S_{FG}$.

So we complete the proof. □

In the following, we study the commutator $[S_F, S_G] = S_F S_G - S_G S_F$ by reducing it to the semi-commutator case. Let

$$B = \begin{pmatrix} G & F \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} F & 0 \\ -G & 0 \end{pmatrix}.$$

A simple calculation gives that

$$S_B S_C = \begin{pmatrix} S_G & S_F \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S_F & 0 \\ -S_G & 0 \end{pmatrix} = \begin{pmatrix} S_G S_F - S_F S_G & 0 \\ 0 & 0 \end{pmatrix}.$$

Combining this with (2.1), we see that $S_F S_G = S_G S_F$ if and only if $S_{BC} = H_B H^*_{C^*}$. By Theorem 3.1, we have $BC = 0$, that is $GF - FG = 0$ and $H_B H^*_{C^*} = 0$. This leads to the following result.

THEOREM 3.2. *Let $F=(f_{ij})_{n\times n}$, $G=(g_{ij})_{n\times n}\in L^\infty_{n\times n}(\mathbb{D})$. Then:*
(i) $[S_F,S_G]=0$ if and only if $FG=GF$ and $F_-\otimes\overline{G}_- - G_-\otimes\overline{F}_-=0$.
(ii) $[S_F,S_G]=0$ if and only if $FG=GF$ and there exist matrix $A_j\in(M_{2n\times 2n})_1$ and $R_j\in\mathscr{P}_{2n}$ for each $j=1,\cdots,n$, such that

$$(R_j-A_j)\begin{pmatrix}G^T\\F^T\end{pmatrix}\in H^\infty_{2n\times n}(\mathbb{D}) \quad and \quad A_j^*\begin{pmatrix}\overline{F}\\-\overline{G}\end{pmatrix}E_j\in H^\infty_{2n\times n}(\mathbb{D}), \quad j=1,\cdots,n.$$

An operator A is said to be normal if $AA^*-A^*A=0$. Taking $G=F^*$ in Theorem 3.2 and noting that $S_F^*=S_{F^*}$, we have the following characterization of normal block Toeplitz operators.

COROLLARY 3.1. *Let $F\in L^\infty_{n\times n}(\mathbb{D})$. Then:*
(i) T_F is normal if and only if $FF^*=F^*F$ and $F_-\otimes F_-^T-F_-^*\otimes\overline{F}_-=0$.
(ii) T_F is normal if and only if $FF^*=F^*F$ and there exist matrix $A_j\in(M_{2n\times 2n})_1$ and $R_j\in\mathscr{P}_{2n}$ for each $j=1,\cdots,n$, such that

$$(R_j-A_j)\begin{pmatrix}\overline{F}\\F^T\end{pmatrix}\in H^\infty_{2n\times n}(\mathbb{D}) \quad and \quad A_j^*\begin{pmatrix}\overline{F}\\-F^T\end{pmatrix}E_j\in H^\infty_{2n\times n}(\mathbb{D}), \quad j=1,\cdots,n.$$

4. Essentially commuting block dual Toeplitz operators

Stroethoff and Zheng [11] proved that the commutator $[S_f,S_g]$ is compact if and only if $\|(H_gk_w)\otimes(H_{\overline{f}}k_w)-(H_fk_w)\otimes(H_{\overline{g}}k_w)\|\to 0$ as $|w|\to 1-$, where f and g be bounded measurable functions on $\mathbb{D}$. For block dual Toeplitz operators, we will give a necessary and a sufficient condition for the semi-commutator $(S_F,S_G]=S_FS_G-S_{FG}$ and the commutator $[S_F,S_G]=S_FS_G-S_GS_F$ to be compact, analogous to the result of Stroethoff and Zheng.

For $F=(f_{ij})_{n\times n}\in L^\infty_{n\times n}(\mathbb{D})$, we define

$$H_Fk_w=\begin{pmatrix}H_{f_{11}}k_w & H_{f_{12}}k_w & \cdots & H_{f_{1n}}k_w\\ H_{f_{21}}k_w & H_{f_{22}}k_w & \cdots & H_{f_{2n}}k_w\\ \vdots & \vdots & \ddots & \vdots\\ H_{f_{n1}}k_w & H_{f_{n2}}k_w & \cdots & H_{f_{nn}}k_w\end{pmatrix},$$

then we get our main result.

THEOREM 4.1. *Let $F=(f_{ij})_{n\times n}$, $G=(g_{ij})_{n\times n}\in L^\infty_{n\times n}(\mathbb{D})$.*
(i) If $(S_F,S_G]$ is compact, then $\|(H_Fk_w)\otimes(H_{\overline{G}}k_w)\|\to 0$, as $|w|\to 1-$.
(ii) If $\|(H_{F(z)}k_w(z))\overline{(H_{\overline{G(\zeta)}}k_w(\zeta))^T}\|\to 0$, as $|w|\to 1-$, then $(S_F,S_G]$ is compact.

THEOREM 4.2. *Let F, $G\in L^\infty_{n\times n}(\mathbb{D})$.*
(i) If $[S_F,S_G]$ is compact, then $FG=GF$ and $\|(H_Fk_w)\otimes(H_{\overline{G}}k_w)-(H_Gk_w)\otimes(H_{\overline{F}}k_w)\|\to 0$ as $|w|\to 1-$.
(ii) If $\|(H_{F(z)}k_w(z))\overline{(H_{\overline{G(\zeta)}}k_w(\zeta))^T}-(H_{G(\zeta)}k_w(\zeta))\overline{(H_{\overline{F(z)}}k_w(z))^T}\|\to 0$, as $|w|\to 1-$, then $[S_F,S_G]$ is compact and $FG=GF$.

To prove the main theorems we will make use of the following lemmas.

LEMMA 4.1. *If $f_k,\ g_k \in L^2(\mathbb{D}),\ k=1,2,\cdots,n$, then*

$$\|\sum_{k=1}^{n} f_k \otimes g_k\| \leqslant \|\sum_{k=1}^{n} \overline{g}_k(\zeta) f_k(z)\|.$$

Proof. Let $u \in L^2(\mathbb{D})$. Then it is easy to check that

$$\begin{aligned}\|\sum_{k=1}^{n}(f_k \otimes g_k)u\|_2^2 &= \int_{\mathbb{D}} |\sum_{k=1}^{n}\langle u, g_k\rangle f_k(z)|^2 dA(z)\\ &= \int_{\mathbb{D}} \Big|\sum_{k=1}^{n}\int_{\mathbb{D}} u(\zeta)\overline{g}_k(\zeta) f_k(z) dA(\zeta)\Big|^2 dA(z)\\ &= \int_{\mathbb{D}} \Big|\int_{\mathbb{D}} u(\zeta)\big(\sum_{k=1}^{n}\overline{g}_k(\zeta) f_k(z)\big) dA(\zeta)\Big|^2 dA(z).\end{aligned}$$

Applying Hölder's inequality we have

$$\Big|\int_{\mathbb{D}} u(\zeta)\big(\sum_{k=1}^{n}\overline{g}_k(\zeta) f_k(z)\big) dA(\zeta)\Big|^2 \leqslant \int_{\mathbb{D}} |u(\zeta)|^2 dA(\zeta) \int_{\mathbb{D}} \Big|\sum_{k=1}^{n}\overline{g}_k(\zeta) f_k(z)\Big|^2 dA(\zeta).$$

Thus

$$\begin{aligned}\|\sum_{k=1}^{n}(f_k \otimes g_k)u\|_2 &\leqslant \|u\|_2 \Big(\int_{\mathbb{D}}\int_{\mathbb{D}} |\sum_{k=1}^{n}\overline{g}_k(\zeta) f_k(z)|^2 dA(\zeta) dA(z)\Big)^{1/2}\\ &= \|u\|_2 \cdot \|\sum_{k=1}^{n}\overline{g}_k(\zeta) f_k(z)\|.\end{aligned}$$

So we get the desired result. $\square$

In the following we write P_1 for the integral operator on $L^2(\mathbb{D}, dA)$ with kernel $1/|1-\overline{w}z|^2$. It is well-known that P_1 is L^p-bounded for $1 < p < \infty$ (see [1] or [12]).

LEMMA 4.2. *Let $\varepsilon > 0$ and $\delta = (2+\varepsilon)/(1+\varepsilon)$. Then*

$$\begin{aligned}&\Big|\sum_{k=1}^{n}(H_{\overline{g}_k}^* u)'(w)\overline{(H_{f_k}^* v)'(w)}\Big|\\ &\leqslant n^{\frac{1}{\delta}} C_\varepsilon \sup_{1\leqslant k\leqslant n}\|g_k\|_\infty^{\frac{1}{\delta}} \sup_{1\leqslant k\leqslant n}\|f_k\|_\infty^{\frac{1}{\delta}} \frac{4}{(1-|w|^2)^2} P_1[|u|^\delta](w)^{1/\delta} P_1[|v|^\delta](w)^{1/\delta}\\ &\quad\times \|\sum_{k=1}^{n}(H_{f_k(z)}k_w(z))\overline{(H_{\overline{g_k(\zeta)}}k_w(\zeta))}\|^{\frac{1}{2+\varepsilon}}\end{aligned}$$

where $w \in \mathbb{D}$, $C_\varepsilon \in \mathbb{C}$, $f_k,\ g_k \in L^\infty(\mathbb{D}, dA)$ and $u,\ v \in (L_a^2(\mathbb{D}))^\perp$ with $k=1,2,\cdots,n$.

Proof. Note that for every function $u \in L_a^2(\mathbb{D})$, the derivative $u'(w) = \langle u, K_w' \rangle$. So

$$
\begin{aligned}
|\sum_{k=1}^{n} (H_{\overline{g}_k}^* u)'(w)\overline{(H_{f_k}^* v)'(w)}| &= |\sum_{k=1}^{n} \langle H_{\overline{g}_k}^* u, K_w' \rangle \langle K_w', H_{f_k}^* v \rangle| \\
&= |\sum_{k=1}^{n} \int_{\mathbb{D}} \frac{u(z)\overline{z\overline{g_k(z)}}}{(1-\overline{z}w)^3} dA(z) \overline{\int_{\mathbb{D}} \frac{v(\zeta)\overline{\zeta f_k(\zeta)}}{(1-\overline{\zeta}w)^3} dA(\zeta)}|.
\end{aligned}
$$

Letting $G_{k,w}$ denote $P_0(g_k \circ \varphi_w) \circ \varphi_w$ and $\widetilde{G}_{k,w}$ denote $P_0(f_k \circ \varphi_w) \circ \varphi_w$, $1 \leqslant k \leqslant n$, the functions $z \to zG_{k,w}(z)/(1-\overline{w}z)^3$ and $\zeta \to \zeta \widetilde{G}_{k,w}(\zeta)/(1-\overline{w}\zeta)^3$ are in $L_a^2(\mathbb{D})$. Since u, $v \in (L_a^2(\mathbb{D}))^{\perp}$ we have

$$
\int_{\mathbb{D}} \frac{u(z)\overline{zG_{k,w}(z)}}{(1-\overline{z}w)^3} dA(z) = 0 \quad \text{and} \quad \int_{\mathbb{D}} \frac{v(\zeta)\overline{\zeta \widetilde{G}_{k,w}(\zeta)}}{(1-\overline{\zeta}w)^3} dA(\zeta) = 0.
$$

Thus

$$
\begin{aligned}
&|\sum_{k=1}^{n} (H_{\overline{g}_k}^* u)'(w)\overline{(H_{f_k}^* v)'(w)}| \\
=&|\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{u(z)\overline{z}}{(1-\overline{z}w)^3} \frac{\overline{v(\zeta)}\zeta}{(1-\zeta\overline{w})^3} [\sum_{k=1}^{n} \overline{(\overline{g_k(z)}-G_{k,w}(z))(f_k(\zeta)-\widetilde{G}_{k,w}(\zeta))}] dA(\zeta)dA(z)| \\
\leqslant& \Big(\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|u(z)|^{\delta}}{|1-\overline{z}w|^{4-\delta}} \frac{|v(\zeta)|^{\delta}}{|1-\overline{\zeta}w|^{4-\delta}} dA(\zeta)dA(z)\Big)^{\frac{1}{\delta}} \\
&\times \Big(\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{|\sum_{k=1}^{n} \overline{(\overline{g_k(z)}-G_{k,w}(z))(f_k(\zeta)-\widetilde{G}_{k,w}(\zeta))}|^{2+\varepsilon}}{|1-\overline{z}w|^4|1-\overline{\zeta}w|^4} dA(\zeta)dA(z)\Big)^{\frac{1}{2+\varepsilon}} \\
=&\frac{1}{(1-|w|^2)^2} \Big(\int_{\mathbb{D}} \frac{|u(z)|^{\delta}}{|1-\overline{z}w|^2} \frac{(1-|w|^2)^{\varepsilon/(1+\varepsilon)}}{|1-\overline{z}w|^{\varepsilon/(1+\varepsilon)}} dA(z)\Big)^{\frac{1}{\delta}} \Big(\int_{\mathbb{D}} \frac{|v(\zeta)|^{\delta}}{|1-\overline{\zeta}w|^2} \frac{(1-|w|^2)^{\varepsilon/(1+\varepsilon)}}{|1-\overline{\zeta}w|^{\varepsilon/(1+\varepsilon)}} dA(\zeta)\Big)^{\frac{1}{\delta}} \\
&\times \Big(\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|w|^2)^4|\sum_{k=1}^{n} \overline{(\overline{g_k(z)}-G_{k,w}(z))(f_k(\zeta)-\widetilde{G}_{k,w}(\zeta))}|^{2+\varepsilon}}{|1-\overline{z}w|^4|1-\overline{\zeta}w|^4} dA(\zeta)dA(z)\Big)^{\frac{1}{2+\varepsilon}}.
\end{aligned}
$$

Since $(1-|w|^2)/|1-\overline{z}w| < 2$, $(1-|w|^2)/|1-\overline{\zeta}w| < 2$ and $2^{\varepsilon/(1+\varepsilon)} < 2$, we get

$$
\begin{aligned}
&|\sum_{k=1}^{n} (H_{\overline{g}_k}^* u)'(w)\overline{(H_{f_k}^* v)'(w)}| \\
\leqslant& \frac{4}{(1-|w|^2)^2} P_1[|u|^{\delta}](w)^{1/\delta} P_1[|v|^{\delta}](w)^{1/\delta} \\
&\times \Big(\int_{\mathbb{D}}\int_{\mathbb{D}} \frac{(1-|w|^2)^4|\sum_{k=1}^{n} \overline{(\overline{g_k(z)}-G_{k,w}(z))(f_k(\zeta)-\widetilde{G}_{k,w}(\zeta))}|^{2+\varepsilon}}{|1-\overline{z}w|^4|1-\overline{\zeta}w|^4} dA(\zeta)dA(z)\Big)^{\frac{1}{2+\varepsilon}}.
\end{aligned}
\tag{4.1}
$$

By the change-of-variable formula (2.2), we have

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\frac{(1-|w|^2)^4|\sum_{k=1}^{n}\overline{(\overline{g_k(z)}-G_{k,w}(z))}(f_k(\zeta)-\widetilde{G}_{k,w}(\zeta))|^{2+\varepsilon}}{|1-\overline{z}w|^4|1-\overline{\zeta}w|^4}dA(\zeta)dA(z)$$
$$=\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(\overline{g}_k\circ\varphi_w(z)-P_0(\overline{g}_k\circ\varphi_w(z)))}(f_k\circ\varphi_w(\zeta)-P_0(f_k\circ\varphi_w(\zeta)))\Big|^{2+\varepsilon}dA(\zeta)dA(z)$$
$$=\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^{2+\varepsilon}dA(\zeta)dA(z). \tag{4.2}$$

Using (2.2), it is easy to check that

$$\Big(\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^2dA(\zeta)dA(z)\Big)^{1/2}$$
$$=\Big(\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{H_{\overline{g}_k}k_w(\zeta)}H_{f_k}k_w(z)\Big|^2dA(\zeta)dA(z)\Big)^{1/2} \tag{4.3}$$
$$=\Big\|\sum_{k=1}^{n}(H_{f_k(z)}k_w(z))\overline{(H_{\overline{g_k(\zeta)}}k_w(\zeta))}\Big\|.$$

Applying Hölder's inequality and (4.3), we get

$$\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^{2+\varepsilon}dA(\zeta)dA(z)$$
$$\leqslant\Big(\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^2dA(\zeta)dA(z)\Big)^{1/2}$$
$$\times\Big(\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^{2(1+\varepsilon)}dA(\zeta)dA(z)\Big)^{1/2} \tag{4.4}$$
$$\leqslant\Big(\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^{2(1+\varepsilon)}dA(\zeta)dA(z)\Big)^{1/2}$$
$$\times\Big\|\sum_{k=1}^{n}(H_{f_k(z)}k_w(z))\overline{(H_{\overline{g_k(\zeta)}}k_w(\zeta))}\Big\|.$$

Since P_0 is $L^{2+2\varepsilon}$-bounded, there exist constants $C_{k,\varepsilon}>0$, $1\leqslant k\leqslant n$, such that

$$\Big(\int_{\mathbb{D}}\int_{\mathbb{D}}\Big|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big|^{2(1+\varepsilon)}dA(\zeta)dA(z)\Big)^{\frac{1}{2+2\varepsilon}}$$
$$=\Big\|\sum_{k=1}^{n}\overline{(I-P_0)(\overline{g}_k\circ\varphi_w(z))}(I-P_0)(f_k\circ\varphi_w(\zeta))\Big\|_{L^{2(1+\varepsilon)}(\mathbb{D}^2,dA(\zeta)dA(z))}$$
$$\leqslant\sum_{k=1}^{n}\big\|(I-P_0)(\overline{g}_k\circ\varphi_w(z))\big\|_{L^{2(1+\varepsilon)}(\mathbb{D},dA(z))}\big\|(I-P_0)(f_k\circ\varphi_w(\zeta))\big\|_{L^{2(1+\varepsilon)}(\mathbb{D},dA(\zeta))}$$

$$\leqslant \sum_{k=1}^{n} C_{k,\varepsilon}\|g_k\|_\infty\|f_k\|_\infty \tag{4.5}$$
$$\leqslant nC'_\varepsilon \sup_{1\leqslant k\leqslant n}\|g_k\|_\infty \sup_{1\leqslant k\leqslant n}\|f_k\|_\infty,$$

where $C'_\varepsilon = \max\{C_{k,\varepsilon}, 1\leqslant k\leqslant n\}$.

From (4.1), (4.3), (4.4) and (4.5), we conclude that there exists a constant C_ε, such that

$$|\sum_{k=1}^{n}(H^*_{\overline{g}_k}u)'(w)\overline{(H^*_{f_k}v)'(w)}|$$
$$\leqslant n^{\frac{1}{\delta}}C_\varepsilon \sup_{1\leqslant k\leqslant n}\|g_k\|_\infty^{\frac{1}{\delta}} \sup_{1\leqslant k\leqslant n}\|f_k\|_\infty^{\frac{1}{\delta}} \frac{4}{(1-|w|^2)^2} P_1[|u|^\delta](w)^{1/\delta}P_1[|v|^\delta](w)^{1/\delta}$$
$$\times \|\sum_{k=1}^{n}(H_{f_k(z)}k_w(z))\overline{(H_{\overline{g_k(\zeta)}}k_w(\zeta))}\|^{1/(2+\varepsilon)}. \quad \square$$

Proof of Theorem 4.1. (i) If $(S_F, S_G]$ is compact, then $H_F H^*_{G^*}$ is compact by (2.1). It is obvious that $H_F H^*_{G^*}$ is compact if and only if each entry of $(\sum_{k=1}^{n} H_{f_{ik}}H^*_{\overline{g}_{kj}})_{ij}$ is compact. Then Lemma 6.2 in [10] implies that $\|\sum_{k=1}^{n} H_{f_{ik}}H^*_{\overline{g}_{kj}} - S_{\varphi_w}(\sum_{k=1}^{n} H_{f_{ik}}H^*_{\overline{g}_{kj}})S_{\overline{\varphi_w}}\| \to 0$ as $|w|\to 1-$. From Proposition 4.1 in [10], we know that $k_w\otimes k_w = I - 2T_{\varphi_w}T_{\overline{\varphi}_w} + T^2_{\varphi_w}T^2_{\overline{\varphi}_w}$. Using identities (4.6) and (4.7) in [10], we have

$$\|\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w)\|$$
$$=\|\sum_{k=1}^{n}H_{f_{ik}}(k_w\otimes k_w)H^*_{\overline{g}_{kj}}\|$$
$$=\|\sum_{k=1}^{n}(H_{f_{ik}}H^*_{\overline{g}_{kj}} - 2H_{f_{ik}}T_{\varphi_w}T_{\overline{\varphi}_w}H^*_{\overline{g}_{kj}} + H_{f_{ik}}T^2_{\varphi_w}T^2_{\overline{\varphi}_w}H^*_{\overline{g}_{kj}})\|$$
$$=\|\sum_{k=1}^{n}(H_{f_{ik}}H^*_{\overline{g}_{kj}} - 2S_{\varphi_w}H_{f_{ik}}H^*_{\overline{g}_{kj}}S_{\overline{\varphi}_w} + S^2_{\varphi_w}H_{f_{ik}}H^*_{\overline{g}_{kj}}S^2_{\overline{\varphi}_w})\|$$
$$\leqslant\|\sum_{k=1}^{n}(H_{f_{ik}}H^*_{\overline{g}_{kj}}-S_{\varphi_w}H_{f_{ik}}H^*_{\overline{g}_{kj}}S_{\overline{\varphi}_w})\|+\|\sum_{k=1}^{n}S_{\varphi_w}(H_{f_{ik}}H^*_{\overline{g}_{kj}}-S_{\varphi_w}H_{f_{ik}}H^*_{\overline{g}_{kj}}S_{\overline{\varphi}_w})S_{\overline{\varphi}_w}\|$$
$$\leqslant 2\|\sum_{k=1}^{n}(H_{f_{ik}}H^*_{\overline{g}_{kj}}-S_{\varphi_w}H_{f_{ik}}H^*_{\overline{g}_{kj}}S_{\overline{\varphi}_w})\|$$
$$\leqslant 2\|\sum_{k=1}^{n}H_{f_{ik}}H^*_{\overline{g}_{kj}}-S_{\varphi_w}(\sum_{k=1}^{n}H_{f_{ik}}H^*_{\overline{g}_{kj}})S_{\overline{\varphi}_w}\|.$$

Thus $\lim_{|w|\to 1-}\|\sum_{k=1}^{n}(H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w)\| = 0$ for any $1\leqslant i,j\leqslant n$. Therefore $\|(H_F k_w)\otimes(H_{\overline{G}}k_w)\|\to 0$ as $|w|\to 1-$. We get the conclusion.

(ii) If $\|(H_{F(z)}k_w(z))\overline{(H_{\overline{G(\zeta)}}k_w(\zeta))^T}\| \to 0$ as $|w| \to 1-$, then for each $1 \leqslant i, j \leqslant n$, we have $\|\sum_{k=1}^{n}(H_{f_{ik}(z)}k_w(z))\overline{(H_{\overline{g}_{kj}(\zeta)}k_w(\zeta))}\| \to 0$ as $|w| \to 1-$. In the following, we only need to prove $\sum_{k=1}^{n} H_{f_{ik}}H^*_{\overline{g}_{kj}}$ is a compact operator, for each $1 \leqslant i, j \leqslant n$.

For u, $v \in C_c(\mathbb{D}) \cap (L^2_a(\mathbb{D}))^\perp$, as Theorem 6.3 in [10] we have

$$\langle \sum_{k=1}^{n} H_{f_{ik}}H^*_{\overline{g}_{kj}}u, v\rangle = \sum_{k=1}^{n}\langle H^*_{\overline{g}_{kj}}u, H^*_{f_{ik}}v\rangle = I + II + III,$$

where

$$I = 3(\sum_{k=1}^{n}\int_{\mathbb{D}}(1-|w|^2)^2(H^*_{\overline{g}_{kj}}u)(w)\overline{(H^*_{f_{ik}}v)(w)}dA(w)),$$

$$II = \frac{1}{2}(\sum_{k=1}^{n}\int_{\mathbb{D}}(1-|w|^2)^2(H^*_{\overline{g}_{kj}}u)'(w)\overline{(H^*_{f_{ik}}v)'(w)}dA(w)),$$

$$III = \frac{1}{3}(\sum_{k=1}^{n}\int_{\mathbb{D}}(1-|w|^2)^3(H^*_{\overline{g}_{kj}}u)'(w)\overline{(H^*_{f_{ik}}v)'(w)}dA(w)).$$

For $0 < s < 1$ we write $I = I_s + I'_s$, $II = II_s + II'_s$ and $III = III_s + III'_s$, where

$$I_s = 3(\sum_{k=1}^{n}\int_{s<|w|<1}(1-|w|^2)^2(H^*_{\overline{g}_{kj}}u)(w)\overline{(H^*_{f_{ik}}v)(w)}dA(w)),$$

$$II_s = \frac{1}{2}(\sum_{k=1}^{n}\int_{s<|w|<1}(1-|w|^2)^2(H^*_{\overline{g}_{kj}}u)'(w)\overline{(H^*_{f_{ik}}v)'(w)}dA(w)),$$

$$III_s = \frac{1}{3}(\sum_{k=1}^{n}\int_{s<|w|<1}(1-|w|^2)^3(H^*_{\overline{g}_{kj}}u)'(w)\overline{(H^*_{f_{ik}}v)'(w)}dA(w)).$$

It is easy to see that there exist compact operators K^I_s, K^{II}_s and K^{III}_s on $(L^2_a(\mathbb{D}))^\perp$ such that $\langle K^I_s u, v\rangle = I'_s$, $\langle K^{II}_s u, v\rangle = II'_s$ and $\langle K^{III}_s u, v\rangle = III'_s$. Observing that the operator $K_s = K^I_s + K^{II}_s + K^{III}_s$ is compact, and $\langle(\sum_{k=1}^{n} H_{f_{ik}}H^*_{\overline{g}_{kj}} - K_s)u, v\rangle = I_s + II_s + III_s$, we will estimate each of the terms I_s, II_s and III_s. Note that

$$\begin{aligned}
I_s &= 3(\sum_{k=1}^{n}\int_{s<|w|<1}(1-|w|^2)^2(H^*_{\overline{g}_{kj}}u)(w)\overline{(H^*_{f_{ik}}v)(w)}dA(w)) \\
&= 3\int_{s<|w|<1}(1-|w|^2)^2(\sum_{k=1}^{n}(H^*_{\overline{g}_{kj}}u)(w)\overline{(H^*_{f_{ik}}v)(w)})dA(w) \\
&= 3\int_{s<|w|<1}(1-|w|^2)^2(\sum_{k=1}^{n}\langle H^*_{\overline{g}_{kj}}u, K_w\rangle\overline{\langle H^*_{f_{ik}}v, K_w\rangle})dA(w) \\
&= 3\int_{s<|w|<1}(1-|w|^2)^2(\sum_{k=1}^{n}\langle u, H_{\overline{g}_{kj}}K_w\rangle\langle H_{f_{ik}}K_w, v\rangle)dA(w) \\
&= 3\int_{s<|w|<1}\langle\sum_{k=1}^{n}((H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w))u, v\rangle dA(w).
\end{aligned}$$

It follows that

$$|I_s| \leqslant 3 \sup_{s<|w|<1} \|\sum_{k=1}^{n}((H_{f_{ik}}k_w)\otimes(H_{\overline{g}_{kj}}k_w))\| \cdot \|u\|_2\|v\|_2.$$

Using Lemma 4.2 we have

$$\begin{aligned}|II_s| \leqslant & \frac{1}{2}\int_{s<|w|<1}(1-|w|^2)^2|\sum_{k=1}^{n}(H^*_{\overline{g}_{kj}}u)'(w)\overline{(H^*_{f_{ik}}v)'(w)}|dA(w)\\ \leqslant & \frac{n^{\frac{1}{\delta}}}{2}C_\varepsilon \sup_{1\leqslant k\leqslant n}\|g_{kj}\|_\infty^{\frac{1}{\delta}} \sup_{1\leqslant k\leqslant n}\|f_{ik}\|_\infty^{\frac{1}{\delta}}\int_{s<|w|<1}P_1[|u|^\delta](w)^{1/\delta}P_1[|v|^\delta](w)^{1/\delta}dA(w)\\ & \times \sup_{s<|w|<1}\|\sum_{k=1}^{n}(H_{f_{ik}(z)}k_w(z))\overline{(H_{\overline{g}_{kj}(\zeta)}k_w(\zeta))}\|^{1/(2+\varepsilon)}.\end{aligned}$$

Since $p=2/\delta>1$ and P_1 is L^p-bounded, there exists a constant C such that

$$\int_{s<|w|<1}P_1[|u|^\delta](w)^{2/\delta}dA(w) \leqslant C\int_{s<|w|<1}[|u|^\delta(w)]^{2/\delta}dA(w) = C\|u\|_2^2.$$

By the Cauchy-Schwarz inequality,

$$\int_{s<|w|<1}P_1[|u|^\delta](w)^{1/\delta}P_1[|v|^\delta](w)^{1/\delta}dA(w) \leqslant C\|u\|_2\|v\|_2.$$

Thus

$$\begin{aligned}|II_s| \leqslant & \frac{Cn^{\frac{1}{\delta}}}{2}C_\varepsilon \sup_{1\leqslant k\leqslant n}\|g_{kj}\|_\infty^{\frac{1}{\delta}} \sup_{1\leqslant k\leqslant n}\|f_{ik}\|_\infty^{\frac{1}{\delta}}\\ & \times \sup_{s<|w|<1}\|\sum_{k=1}^{n}(H_{f_{ik}(z)}k_w(z))\overline{(H_{\overline{g}_{kj}(\zeta)}k_w(\zeta))}\|^{1/(2+\varepsilon)}\|u\|_2\|v\|_2.\end{aligned}$$

Term III_s is estimated similar to II_s. From the estimates of the three terms I_s, II_s and III_s, we obtain

$$|\langle(\sum_{k=1}^{n}H_{f_{ik}}H^*_{\overline{g}_{kj}}-K_s)u,v\rangle| \leqslant C' \sup_{s<|w|<1}\|\sum_{k=1}^{n}(H_{f_{ik}(z)}k_w(z))\overline{(H_{\overline{g}_{kj}(\zeta)}k_w(\zeta))}\|^{1/(2+\varepsilon)}\|u\|_2\|v\|_2$$

for some constant $C'>0$, combining with Lemma 4.1, we conclude that

$$\|\sum_{k=1}^{n}H_{f_{ik}}H^*_{\overline{g}_{kj}}-K_s\| \leqslant C' \sup_{s<|w|<1}\|\sum_{k=1}^{n}(H_{f_{ik}(z)}k_w(z))\overline{(H_{\overline{g}_{kj}(\zeta)}k_w(\zeta))}\|^{1/(2+\varepsilon)}.$$

So if $\|\sum_{k=1}^{n}(H_{f_{ik}(z)}k_w(z))\overline{(H_{\overline{g}_{kj}(\zeta)}k_w(\zeta))}\|\to 0$ as $|w|\to 1-$, then it follows from the above inequality that $K_s \to \sum_{k=1}^{n}H_{f_{ik}}H^*_{\overline{g}_{kj}}$ in operator norm. Since each of the operators

K_s is compact, we conclude that the operator $\sum_{k=1}^{n} H_{f_{ik}} H^*_{\overline{g}_{kj}}$ is compact. This completes the proof. □

By Theorem 4.1, it is easy to prove Theorem 4.2.

Proof of Theorem 4.2. Let

$$B = \begin{pmatrix} G & F \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} F & 0 \\ -G & 0 \end{pmatrix}.$$

Then $S_F S_G - S_G S_F$ is compact if and only if $S_B S_C$ is compact.

If $S_B S_C$ is compact, then (2.1) implies that the operator $S_{BC} - H_B H^*_{C^*} = S_B S_C$ is compact. Let $u_{w,s}$ be defined as in Corollary 6.2 in [11] and $B = (b_{ij})_{2n\times 2n}$, $C = (c_{ij})_{2n\times 2n}$, it follows that $(S_{(\sum_{k=1}^{2n} b_{ik}c_{kj})} - \sum_{k=1}^{2n} H_{b_{ik}} H^*_{\overline{c}_{kj}})_{2n\times 2n}$ is compact and

$$\|(S_{(\sum_{k=1}^{2n} b_{ik}c_{kj})} - \sum_{k=1}^{2n} H_{b_{ik}} H^*_{\overline{c}_{kj}}) u_{w,s}\|_2 \to 0, \quad s \to 0+.$$

By Lemma 7.1 in [11] we also have

$$\|\sum_{k=1}^{2n} H_{b_{ik}} H^*_{\overline{c}_{kj}} u_{w,s}\|_2 \to 0, \quad s \to 0+.$$

Since $(S_{(\sum_{k=1}^{2n} b_{ik}c_{kj})} u_{w,s}) \perp (\sum_{k=1}^{2n} H_{b_{ik}} H^*_{\overline{c}_{kj}} u_{w,s})$, we get

$$\|S_{(\sum_{k=1}^{2n} b_{ik}c_{kj})} u_{w,s}\|_2 = \|(S_{(\sum_{k=1}^{2n} b_{ik}c_{kj})} - \sum_{k=1}^{2n} H_{b_{ik}} H^*_{\overline{c}_{kj}}) u_{w,s}\|_2 + \|\sum_{k=1}^{2n} H_{b_{ik}} H^*_{\overline{c}_{kj}} u_{w,s}\|_2 \to 0,\ s \to 0+.$$

Thus Lemma 7.2 in [11] implies that

$$|\sum_{k=1}^{2n} b_{ik}(w) c_{kj}(w)|^2 = \lim_{s\to 0+} \|S_{(\sum_{k=1}^{2n} b_{ik}c_{kj})} u_{w,s}\|_2^2 = 0,$$

for a.e. w on $\mathbb{D}$, that is $B(w)C(w) = 0$ for almost all $w \in \mathbb{D}$. So we get that if $S_B S_C$ is a compact operator, then $H_B H^*_{C^*}$ is compact and $BC = 0$.

Using Theorem 4.1, and combining with the fact that $S_F S_G - S_G S_F$ is compact if and only if $H_B H^*_{C^*}$ is compact and $BC = 0$, we can get Theorem 4.2. This completes the proof. □

An operator A is said to be essentially normal if $AA^* - A^*A$ is compact. By taking $G = F^*$, we immediately get the following characterization of essentially normal block Toeplitz operators.

COROLLARY 4.1. *Let $F \in L^\infty_{n\times n}(\mathbb{D})$.*
(i) If S_F is a essentially normal block Toeplitz operators, then $FF^* = F^*F$ and $\|(H_F k_w) \otimes (H_{F^T} k_w) - (H_{F^*} k_w) \otimes (H_{\overline{F}} k_w)\| \to 0$ as $|w| \to 1-$.
(ii) If $\|(H_{F(z)} k_w(z))\overline{(H_{\overline{F^*(\zeta)}} k_w(\zeta))^T} - (H_{F^*(\zeta)} k_w(\zeta))\overline{(H_{\overline{F(z)}} k_w(z))^T}\| \to 0$ as $|w| \to 1-$, then S_F is a essentially normal block Toeplitz operators and $FF^* = F^*F$.

Acknowledgements. The suggestions given by the editor and the referee for the improvement of the original paper are gratefully acknowledged.

REFERENCES

[1] S. AXLER, *Bergman spaces and their operators. Surveys of some recent results in operator theory*, Vol. I, 1–50, Pitman Res. Notes Math. Ser., 171, Longman Sci. Tech., Harlow, 1988.
[2] S. AXLER AND Ž. ČUČKOVIĆ, *Commuting Toeplitz operators with harmonic symbols*, Integral Equations Operator Theory **14** (1991), no. 1, 1–12.
[3] S. AXLER, Ž. ČUČKOVIĆ AND N. V. RAO, *Commutants of analytic Toeplitz operators on the Bergman space*, Proc. Amer. Math. Soc. **128** (2000), no. 7, 1951–1953.
[4] A. BROWN AND P. R. HALMOS, *Algebraic Properties of Toeplitz Operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.
[5] P. GORKIN AND D. ZHENG, *Essentially commuting Toeplitz operators*, Pacific J. Math. **190** (1999), no. 1, 87–109.
[6] C. GU AND D. ZHENG, *Products of block Toeplitz operators*, Pacific J. Math. **185** (1998), no. 1, 115–148.
[7] R. KERR, *Products of Toeplitz operators on a vector valued Bergman space*, Integral Equations Operator Theory **66** (2010), no. 3, 367–395.
[8] B. ZHANG, Y. SHI AND Y. LU, *Products of block dual Toeplitz operators*, Acta Mathematica Sinica, Chinese Series, Nov., 2011, Vol 54, no. 6, pp. 901–912.
[9] K. STROETHOFF, *Essentially commuting Toeplitz operators with harmonic symbols*, Canad. J. Math. **45** (1993), no. 5, 1080–1093.
[10] K. STROETHOFF AND D. ZHENG, *Products of Hankel and Toeplitz operators on the Bergman space*, J. Funct. Anal. **169** (1999), no. 1, 289–313.
[11] K. STROETHOFF AND D. ZHENG, *Algebraic and spectral properties of dual Toeplitz operators*, Trans. Amer. Math. Soc. **354** (2002), no. 6, 2495–2520.
[12] K. ZHU, *Operator theory in function spaces*, Second edition. Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.

(Received January 9, 2012)

Bo Zhang
College of Science
Dalian Ocean University
Dalian 116023, China
e-mail: zhangdlut@aliyun.com, zhangdlut@163.com

Yufeng Lu
School of Mathematical Sciences
Dalian University of Technology
Dalian 116024, China
e-mail: lyfdlut@dlut.edu.cn