STAR PARTIAL ORDER–HEREDITARY SUBSPACES IN $\mathscr{B}(\mathscr{H})$

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(*Communicated by D. R. Farenick*)

Abstract. Let $\mathscr{B}(\mathscr{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. It is proved that a weak operator topology closed nonzero subspace $\mathscr{M}$ in $\mathscr{B}(\mathscr{H})$ is hereditary with respect to the star partial order, that is, for any $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{M}$, $A \in \mathscr{M}$ whenever $A^*A = A^*B$ and $AA^* = BA^*$, if and only if there is a unique pair of nonzero projections P and Q in $\mathscr{B}(\mathscr{H})$ such that $\mathscr{M} = P\mathscr{B}(\mathscr{H})Q$.

1. Introduction

In the last few decades, many researchers have studied properties of various partial orders on semigroups, such as minus partial order, star partial order, left and right star partial order, and so on (cf. [1, 2, 5, 6, 10]). Moreover, some of these partial orders have been extended to the matrix and operator algebras and many interesting results have been obtained.

Let M_n be the algebra of all $n \times n$ complex matrices. One of the orders on M_n is the star partial order $\overset{*}{\leqslant}$ defined by Drazin in [6]. Let $A, B \in M_n$. Then we say that $A \overset{*}{\leqslant} B$ if $A^*A = A^*B$ and $AA^* = BA^*$. We note that this definition can be extended to a C^*-algebra by the same way. In particular, it can be extended to the C^*-algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. For example, Dolinar and Marovt gave an equivalent definition (Definition 2 in [5]) of the star partial order and considered some properties of this partial order in [5].

As we known for a partial order on $\mathscr{B}(\mathscr{H})$, the heredity of the partial order is an important property to consider; for instance, the notion of hereditary subalgebras in a C^*-algebra ([4]). Similarly, we consider partial order-hereditary subspaces in $\mathscr{B}(\mathscr{H})$ for a partial order "$\leqslant$".

DEFINITION 1. Let "$\leqslant$" be a partial order on $\mathscr{B}(\mathscr{H})$ and $\mathscr{M}$ a subspace of $\mathscr{B}(\mathscr{H})$. For any $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{M}$, if $A \in \mathscr{M}$ whenever $A \leqslant B$, then we say that $\mathscr{M}$ is a hereditary subspace with respect to the partial order "$\leqslant$".

If $\mathcal{M}$ is a hereditary subspace with respect to the star partial order, then we say that $\mathcal{M}$ is a star partial order-hereditary subspace of $\mathcal{B}(\mathcal{H})$. It is clear $\mathcal{M}^* = \{X^* : X \in \mathcal{M}\}$ is also a star partial order-hereditary subspace whenever $\mathcal{M}$ is. We recall that on operator $P \in \mathcal{B}(\mathcal{H})$ is a projection if $P = P^* = P^2$. Let P and Q be two projections in $\mathcal{B}(\mathcal{H})$ and $\mathcal{M} = P\mathcal{B}(\mathcal{H})Q$. For any $A \in P\mathcal{B}(\mathcal{H})Q$ and $B \in \mathcal{B}(\mathcal{H})$, if $B \stackrel{*}{\leqslant} A$, by Corollary 2.4 in [1], we have $R(B) \subseteq R(A)$ and $R(B^*) \subseteq R(A^*)$, thus $B \in P\mathcal{B}(\mathcal{H})Q$. Then it is clear that $\mathcal{M}$ is a weak operator topology closed star partial order-hereditary subspace. Does the converse hold? We consider this problem in this paper. It shall be proven in Theorem 1 that if $\mathcal{M}$ is a norm closed star partial order-hereditary subspace in $\mathcal{B}(\mathcal{H})$, then there is a unique pair of projections P and Q in $\mathcal{B}(\mathcal{H})$ such that $\mathcal{M} \cap \mathcal{K}(\mathcal{H}) = P\mathcal{K}(\mathcal{H})Q$ and $\overline{\mathcal{M}}^w = P\mathcal{B}(\mathcal{H})Q$, where $\mathcal{K}(\mathcal{H})$ is the set of all compact operators in $\mathcal{B}(\mathcal{H})$ and $\overline{\mathcal{M}}^w$ is the weak operator topology closure of $\mathcal{M}$. We next recall some notions.

Let $\mathcal{H}$ and $\mathcal{K}$ be two complex Hilbert spaces and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We denote by $\mathcal{F}(\mathcal{H})$ and $\mathcal{P}(\mathcal{H})$ the set of all finite-rank operators and the set of all projections in $\mathcal{B}(\mathcal{H})$, respectively. For every pair of vectors $x, y \in \mathcal{H}$, $x \perp y$ means that $\langle x, y \rangle = 0$ and $x \otimes y$ stand for a rank-1 linear operator on $\mathcal{H}$ defined by $(x \otimes y)z = \langle z, y \rangle x$ for any $z \in \mathcal{H}$. For a subset S of $\mathcal{H}$ (resp. $\mathcal{B}(\mathcal{H})$), $[S]$ denotes the norm closed subspace of $\mathcal{H}$ (resp. $\mathcal{B}(\mathcal{H})$) spanned by S. For an operator $A \in \mathcal{B}(\mathcal{H})$, we write $R(A)$, $N(A)$ and $\sigma(A)$ for the range, the kernel and the spectrum of A, respectively. Throughout this paper, we will denote by I the identity operator on any Hilbert space.

2. Star partial order-hereditary subspaces

For a closed subspace $M \subseteq \mathcal{H}$, we denote by P_M the orthogonal projection on M. Let $T \in \mathcal{B}(\mathcal{H})$. We denote by $H_1 = \overline{R(T^*)}$, $H_2 = N(T)$, $K_1 = \overline{R(T)}$ and $K_2 = N(T^*)$ respectively. Then

$$\mathcal{H} = H_1 \oplus H_2 = K_1 \oplus K_2 \tag{1}$$

and

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} \tag{2}$$

with respect to the orthogonal decompositions (1), where $T_0 \in \mathcal{B}(H_1, K_1)$ is an injective operator with dense range.

LEMMA 1. *Let $\mathcal{M}$ be a norm closed star partial order-hereditary subspace in $\mathcal{B}(\mathcal{H})$. Then $P_{R(T)}\mathcal{B}(\mathcal{H})P_{N(T)^\perp} \subseteq \mathcal{M}$ for any operator $T \in \mathcal{M}$ with closed range.*

Proof. Let $T \in \mathcal{M}$ with closed range and has the matrix form (2). We have T_0 is invertible. Let $T_0 = UA$ be the polar decomposition of T_0. Then $A \in \mathcal{B}(H_1)$ is an invertible positive operator and $U \in \mathcal{B}(H_1, K_1)$ is a unitary operator. Now we let $A = \int_{\sigma(A)} \lambda dE_\lambda$ be the spectral decomposition of A. For any Borel subset $\Delta \subseteq \sigma(A)$, we have $H_1 = E(\Delta)H_1 \oplus (I - E(\Delta))H_1$. Then

$$\mathcal{H} = E(\Delta)H_1 \oplus (I - E(\Delta))H_1 \oplus H_2 = UE(\Delta)H_1 \oplus U(I - E(\Delta))H_1 \oplus K_2. \tag{3}$$

Put $U_1 = U|_{E(\Delta)H_1}$, $A_1 = E(\Delta)A$, $U_2 = U|_{(I-E(\Delta))H_1}$ and $A_2 = (I - E(\Delta))A$ on H_1 respectively. Then

$$T = \begin{pmatrix} U_1A_1 & 0 & 0 \\ 0 & U_2A_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

according to the orthogonal decompositions (3). Let

$$T_\Delta = \begin{pmatrix} U_1A_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

according to the decompositions (3) again. It follows that $T_\Delta \overset{*}{\leqslant} T$ from Lemma 3 in [5]. Hence $T_\Delta \in \mathscr{M}$ for all Borel subset $\Delta \subset \sigma(A)$. Thus for any simple function $f = \sum_{k=1}^{n} \alpha_k \chi_{\Delta_k}$ on $\sigma(A)$, where $\chi_\Delta(\cdot)$ is the characteristic function of Δ, we have $T_f = \sum_{k=1}^{n} \alpha_k T_{\Delta_k} \in \mathscr{M}$. In fact, we have

$$T_f = \sum_{k=1}^{n} \alpha_k T_{\Delta_k} = \begin{pmatrix} Uf(A)A & 0 \\ 0 & 0 \end{pmatrix}$$

according to the decompositions (1).

Let $f(\lambda) = \lambda^{-1}$ on $\sigma(A)$. Then f is continuous on $\sigma(A)$ and hence there exists a sequence of simple function $\{f_n\}$ such that $|f_n - f| \to 0$ $(n \to \infty)$ uniformly and thus $\|f_n(A) - f(A)\| = \|f_n(A) - A^{-1}\| \to 0$ $(n \to \infty)$. Since $\mathscr{M}$ is norm closed and

$$\begin{pmatrix} Uf_n(A)A & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{M},$$

it follows that

$$\lim_{n \to \infty} \begin{pmatrix} Uf_n(A)A & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix} \in \mathscr{M}.$$

It is known that $(UE)^*(UE) = (UE)^*U$ and $(UE)(UE)^* = U(UE)^*$ for every projection $E \in \mathscr{B}(H_1)$. Then $UE \overset{*}{\leqslant} U$ and thus

$$\begin{pmatrix} UE & 0 \\ 0 & 0 \end{pmatrix} \overset{*}{\leqslant} \begin{pmatrix} U & 0 \\ 0 & 0 \end{pmatrix},$$

which implies that $\begin{pmatrix} UE & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{M}$. Then for every $X \in \mathcal{B}(H_1)$, we have $\begin{pmatrix} UX & 0 \\ 0 & 0 \end{pmatrix} \in$ $\mathcal{M}$ since X is a linear combination of finitely many projections from Theorem 3 in [9]. Take any $B \in P_{K_1}\mathcal{B}(\mathcal{H})P_{H_1}$. Then

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} U(U^*B_0) & 0 \\ 0 & 0 \end{pmatrix}$$

for some $B_0 \in \mathcal{B}(H_1, K_1)$. Then $B \in \mathcal{M}$ and hence $P_{K_1}\mathcal{B}(\mathcal{H})P_{H_1} \subseteq \mathcal{M}$. $\quad\square$

We note that if there is an invertible $T \in \mathcal{M}$, then $\mathcal{M} = \mathcal{B}(\mathcal{H})$.

LEMMA 2. *Let $\mathcal{M}$ be a nonzero norm closed star partial order-hereditary subspace in $\mathcal{B}(\mathcal{H})$. Then there exist rank-1 operators in $\mathcal{M}$.*

Proof. Take any nonzero $T \in \mathcal{M}$ and let $T = UA$ be the polar decomposition of T, where U is a partial isometry with initial space H_1 and final space K_1, and $A = \int_{\sigma(A)} \lambda dE_\lambda$ the spectral decomposition of A as in the proof of Lemma 1. For any $t \in (0, \|T\|)$, put $T_t = UE[t, \|T\|]A$. It is easy to show that $T_t \neq 0$ has closed range. Furthermore, according to the decompositions $\mathcal{H} = H_1 \oplus H_2 = E[0, \|T\|]H_1 \oplus (I - E[0, \|T\|])H_1 \oplus H_2 = UE[0, \|T\|]H_1 \oplus U(I - E[0, \|T\|])H_1 \oplus K_2 = K_1 \oplus K_2$, we have

$$T_t = \begin{pmatrix} UE[t, \|T\|]A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } T = \begin{pmatrix} UE[t, \|T\|]A & 0 & 0 \\ 0 & U(I - E[t, \|T\|])A & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that $T_t \overset{*}{\leqslant} T$ by Lemma 3 in [5] again. Then $T_t \in \mathcal{M}$ for all $t \in (0, \|T\|)$. Now the desired result follows from Lemma 1. $\quad\square$

THEOREM 1. *Let $\mathcal{M}$ be a nonzero norm closed star partial order-hereditary subspace in $\mathcal{B}(\mathcal{H})$. Then there exists a unique pair of projections $P, Q \in \mathcal{B}(\mathcal{H})$ such that $\mathcal{M} \cap \mathcal{K}(\mathcal{H}) = P\mathcal{K}(\mathcal{H})Q$ and $\overline{\mathcal{M}}^w = P\mathcal{B}(\mathcal{H})Q$.*

Proof. The uniqueness is evident. We recall that $\mathcal{P}(\mathcal{H})$ is the set of all projections in $\mathcal{B}(\mathcal{H})$. For any two projections $P, Q \in \mathcal{P}(\mathcal{H})$, $P \leqslant Q$ whenever $PQ = QP = P$. Let $\mathcal{T} = \{(P, Q) : P, Q \in \mathcal{P}(\mathcal{H}) \text{ and } P\mathcal{K}(\mathcal{H})Q \subseteq \mathcal{M}\}$. We define an partial order on $\mathcal{T}$ such that $\mathcal{T}$ is a poset. We say that $(P_1, Q_1) \leqslant (P_2, Q_2)$ if $P_1 \leqslant P_2$ and $Q_1 \leqslant Q_2$ for any pair of $(P_1, Q_1), (P_2, Q_2) \in \mathcal{T}$. Note that $(0, 0) \in \mathcal{P}(\mathcal{H})$. We claim that $\mathcal{T}$ has a maximal element with respect to this partial order. For an arbitrary totally order subset $\{(P_i, Q_i)\} \subseteq \mathcal{T}$, $\{P_i\}$ and $\{Q_i\}$ are also totally order subsets of $\mathcal{P}(\mathcal{H})$. Then both $\{P_i\}$ and $\{Q_i\}$ have supremums in $\mathcal{P}(\mathcal{H})$, which are denoted by P_0 and Q_0, respectively. Then $(P_i, Q_i) \leqslant (P_0, Q_0)$ for all i. For any pair of vectors $x, y \in \mathcal{H}$, since P_i and Q_i converge to P_0 and Q_0 in the strong operator topology respectively, we have $\|P_i x - P_0 x\| \to 0$ and $\|Q_i y - Q_0 y\| \to 0$. Hence $\|P_i x \otimes Q_i y - P_0 x \otimes Q_0 y\| \to 0$. Since $P_i x \otimes Q_i y = P_i(x \otimes y)Q_i \in \mathcal{M}$ for all i and $\mathcal{M}$ is norm closed, we have $P_0 x \otimes Q_0 y \in \mathcal{M}$.

Therefore $P_0 \mathscr{K}(\mathscr{H})Q_0 \subseteq \mathscr{M}$, that is $(P_0, Q_0) \in \mathscr{T}$. It follows that $\mathscr{T}$ contains a maximal element (P, Q) from $Zorn's$ Lemma, and then $P\mathscr{K}(\mathscr{H})Q \subseteq \mathscr{M}$. It is clear that both P and Q are nonzero from Lemma 2.

Next we prove that $\mathscr{M} \cap \mathscr{K}(\mathscr{H}) \subseteq P\mathscr{K}(\mathscr{H})Q$. We firstly note that $\mathscr{M} \cap \mathscr{K}(\mathscr{H})$ is the closed linear span of all rank-1 operators in $\mathscr{M}$.

Let $T \in \mathscr{M} \cap \mathscr{K}(\mathscr{H})$ be a nonzero operator with the matrix form (2). It is known that both H_1 and K_1 are separable. As in the proof of Lemma 1, if $T_0 = UA$ is the polar decomposition of T_0, then $A \in \mathscr{B}(H_1)$ is injective positive compact operator and $U \in \mathscr{B}(H_1, K_1)$ is a unitary operator. Thus there are an orthonormal basis $\{e_i : 1 \leqslant i \leqslant n\}$ (n is finite or ∞) of H_1 and a sequence of decreasing positive numbers $\{\lambda_i : 1 \leqslant i \leqslant n\}$ such that $A = \sum_{i=1}^{n} \lambda_i(e_i \otimes e_i)$, where the series converges in norm if n is infinite. It is trivial that $\lambda_i e_i \otimes e_i \overset{*}{\leqslant} A$ for all i. Then

$$\lambda_i U e_i \otimes e_i = \begin{pmatrix} \lambda_i U e_i \otimes e_i & 0 \\ 0 & 0 \end{pmatrix} \overset{*}{\leqslant} \begin{pmatrix} UA & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} = T.$$

Hence $\lambda_i U e_i \otimes e_i \in \mathscr{M}$ for all i. It follows that

$$T = \begin{pmatrix} T_0 & 0 \\ 0 & 0 \end{pmatrix} = \sum_{i=1}^{n} \begin{pmatrix} \lambda_i U e_i \otimes e_i & 0 \\ 0 & 0 \end{pmatrix}.$$

If $\mathscr{M} \cap \mathscr{K}(\mathscr{H}) \nsubseteq P\mathscr{K}(\mathscr{H})Q$, then there exist two nonzero vectors ξ and η such that $T = \xi \otimes \eta \in \mathscr{M}$ and $T \notin P\mathscr{K}(\mathscr{H})Q$. Note that

$$T = \xi \otimes \eta = P\xi \otimes Q\eta + P\xi \otimes (1-Q)\eta + (1-P)\xi \otimes Q\eta + (1-P)\xi \otimes (1-Q)\eta.$$

Put $\xi_1 = (1-P)\xi$ and $\eta_1 = (1-Q)\eta$. Then there is at least one nonzero vector in $\{\xi_1, \eta_1\}$ by the assumption. The proof will be divided into three cases.

Case 1. $P\xi = 0$ and $Q\eta = 0$. We may assume that ξ and η are unit vectors. For any unit vectors $x_1 \in P\mathscr{H}$ and $y_1 \in Q\mathscr{H}$, we have $x_1 \otimes y_1 \in P\mathscr{K}(\mathscr{H})Q \subseteq \mathscr{M}$. Put $A = \frac{1}{2}(x_1 \otimes y_1 + x_1 \otimes \eta + \xi \otimes y_1 + \xi \otimes \eta)$ and $B = x_1 \otimes y_1 + \xi \otimes \eta$. Note that $B \in \mathscr{M}$. Since $\langle \xi, x_1 \rangle = \langle \eta, y_1 \rangle = 0$, we have $A^*A = A^*B$ and $AA^* = BA^*$. That is, $A \overset{*}{\leqslant} B$. Thus $A \in \mathscr{M}$. It follows that both $x_1 \otimes \eta + \xi \otimes y_1 = 2A - B$ and $x_1 \otimes y_1 + x_1 \otimes \eta + \xi \otimes y_1 - \xi \otimes \eta = 2A - B + x_1 \otimes y_1 - \xi \otimes \eta$ are in $\mathscr{M}$. We put $A_1 = x_1 \otimes y_1 + x_1 \otimes \eta$ and $B_1 = x_1 \otimes y_1 + x_1 \otimes \eta + \xi \otimes y_1 - \xi \otimes \eta$ again. Then we similarly have $A_1 \overset{*}{\leqslant} B_1$, which implies that $A_1 \in \mathscr{M}$. Thus $x_1 \otimes \eta = A_1 - x_1 \otimes y_1 \in \mathscr{M}$. Denote by P_x the projection onto the space $[x]$ for any vector $x \in \mathscr{H}$. Then $P\mathscr{K}(\mathscr{H})P_\eta \subseteq \mathscr{M}$. Hence $P\mathscr{K}(\mathscr{H})(Q + P_\eta) \subseteq \mathscr{M}$ and $(P, Q) \leqslant (P, Q + P_\eta) \in \mathscr{T}$. This is a contradiction since (P, Q) is a maximal element of $\mathscr{T}$.

Case 2. $P\xi \neq 0$ and $Q\eta \neq 0$. If there is only one nonzero element in $\{\xi_1, \eta_1\}$, say, $\eta_1 \neq 0$, then $\xi_1 = 0$, $P\xi = \xi$ and $T_1 = \xi \otimes \eta_1 = T - \xi \otimes Q\eta \in \mathscr{M}$. If $\dim P\mathscr{H} = 1$, then $P\mathscr{K}(\mathscr{H})P_{\eta_1} \subseteq \mathscr{M}$ and hence $P\mathscr{K}(\mathscr{H})(Q + P_{\eta_1}) \subseteq \mathscr{M}$. This is a contradiction too. Assume that $\dim P\mathscr{H} \geqslant 2$. Take any unit vectors $x_1 \in (P - P_\xi)\mathscr{H}$ and $y_1 \in Q\mathscr{H}$.

Then we have $\xi \otimes y_1, x_1 \otimes y_1 \in \mathcal{M}$. As above, we may assume that ξ and η_1 are also unit vectors. Let $A = \frac{1}{2}(\xi \otimes y_1 + \xi \otimes \eta_1 + x_1 \otimes y_1 + x_1 \otimes \eta_1)$ and $B = \xi \otimes \eta_1 + x_1 \otimes y_1$. We can show that $A^*A = A^*B$ and $AA^* = BA^*$, which means that $A \stackrel{*}{\leqslant} B$. Note that $B \in \mathcal{M}$ and so $A \in \mathcal{M}$. Thus $x_1 \otimes \eta_1 = 2A - \xi \otimes y_1 - \xi \otimes \eta_1 - x_1 \otimes y_1 \in \mathcal{M}$. For any $x \in P\mathcal{H}$, we have $x = x_1 + \alpha\xi$ for some $x_1 \in (P - P_\xi)\mathcal{H}$ and $\alpha \in \mathbb{C}$. It follows that $x \otimes \eta_1 = x_1 \otimes \eta_1 + \alpha T_1 \in \mathcal{M}$. Hence $P\mathcal{K}(\mathcal{H})(Q + P_{\eta_1}) \subseteq \mathcal{M}$, a contradiction. We similarly have a contradiction if $\xi_1 \neq 0$ and $\eta_1 = 0$. Thus both ξ_1 and η_1 are nonzero. Since $P\xi \otimes Q\eta \in \mathcal{M}$, $P\xi \otimes \eta_1 + \xi_1 \otimes Q\eta + \xi_1 \otimes \eta_1 = \xi \otimes \eta - P\xi \otimes Q\eta \in \mathcal{M}$. Let $A = P\xi \otimes \eta_1 + \xi_1 \otimes \eta_1$ and $B = -\frac{\|\xi_1\|^2}{\|P\xi\|^2}P\xi \otimes Q\eta + P\xi \otimes \eta_1 + \xi_1 \otimes Q\eta + \xi_1 \otimes \eta_1 \in \mathcal{M}$.

We can also show that $A^*A = A^*B$ and $AA^* = BA^*$. Thus $A \stackrel{*}{\leqslant} B$ and $A \in \mathcal{M}$. It follows that $\xi_1 \otimes Q\eta = (P\xi \otimes \eta_1 + \xi_1 \otimes Q\eta + \xi_1 \otimes \eta_1) - A \in \mathcal{M}$ and thus $\xi \otimes Q\eta = P\xi \otimes Q\eta + \xi_1 \otimes Q\eta \in \mathcal{M}$. That is, there is a rank-1 operator $\xi \otimes Q\eta \in \mathcal{M}$ such that $(I - P)\xi \neq 0$ and $(I - Q)Q\eta = 0$. We now have a contradiction again.

Case 3. Either $P\xi$ or $Q\eta$ is 0. Without loss of generality we assume that $Q\eta = 0$. Then $T = P\xi \otimes \eta_1 + \xi_1 \otimes \eta_1 \in \mathcal{M}$. If $\xi_1 = 0$, then $T = P\xi \otimes \eta_1 = \xi \otimes \eta_1 = T_1 \in \mathcal{M}$ as in Case 2. We can get a contradiction too.

Now we assume that $\xi_1 \neq 0$. Take any nonzero vector $y \in Q\mathcal{H}$ and put $T_y = P\xi \otimes y + T \in \mathcal{M}$. Note that $T_y \notin P\mathcal{K}(\mathcal{H})Q$. Since $(P\xi \otimes y)^*(P\xi \otimes y) = \|P\xi\|^2 y \otimes y$ and $\|P\xi\|^2 y \otimes y + \|P\xi\|^2 y \otimes \eta_1 = (P\xi \otimes y)^*(T_y)$, we have $(P\xi \otimes y)^*(P\xi \otimes y) \neq (P\xi \otimes y)^*(T_y)$. It follows that $P\xi \otimes y \stackrel{*}{\leqslant} T_y$ does not hold. We know that there are two rank-1 operators $T_1 = x_1 \otimes y_1$ and $T_2 = x_2 \otimes y_2$ for some $x_i, y_i \in \mathcal{H}$ such that $T_i \stackrel{*}{\leqslant} T_y$ for $i = 1, 2$ and $T_y = T_1 + T_2$ from Lemma 3 in [5]. Then $T_i \in \mathcal{M}(i = 1, 2)$. If $PT_1 Q = 0$, then $PT_2 Q = PT_y Q = P\xi \otimes y \neq T_2$ since $T_2 \stackrel{*}{\leqslant} T_y$. This implies that $T_2 \notin P\mathcal{K}(\mathcal{H})Q$ as well as $Px_2 \neq 0$ and $Qy_2 \neq 0$. Hence by Case 2 this is a contradiction. We can similarly get a contradiction if $PT_2 Q = 0$. Thus $PT_i Q \neq 0$ for $i = 1, 2$. Since $T_y \notin P\mathcal{K}(\mathcal{H})Q$, there is at least one T_i, for example T_2, such that $T_2 \notin P\mathcal{K}(\mathcal{H})Q$. Note that $T_2 \in \mathcal{M}$ and $Px_2 \neq 0$ and $Qy_2 \neq 0$. Then T_2 satisfies the condition of Case 2. This is a contradiction again. We similarly have a contradiction if $P\xi = 0$. Consequently, we have $\mathcal{M} \cap \mathcal{K}(\mathcal{H}) = P\mathcal{K}(\mathcal{H})Q$.

Lastly, we prove that $\overline{\mathcal{M}}^w = P\mathcal{B}(\mathcal{H})Q$. Since $\mathcal{M} \cap \mathcal{K}(\mathcal{H}) = P\mathcal{K}(\mathcal{H})Q$, it is obvious that $P\mathcal{B}(\mathcal{H})Q \subseteq \overline{\mathcal{M}}^w$. It is sufficient to show that $\overline{\mathcal{M}}^w \subseteq P\mathcal{B}(\mathcal{H})Q$. Let $T \in \mathcal{M}$ with closed range. By Lemma 1, we have

$$P_{R(T)}\mathcal{B}(\mathcal{H})P_{N(T)^\perp} \subseteq \mathcal{M}.$$

Hence,

$$P_{R(T)}\mathcal{B}(\mathcal{H})P_{N(T)^\perp} = \overline{P_{R(T)}\mathcal{K}(\mathcal{H})P_{N(T)^\perp}}^w$$
$$\subseteq \overline{\mathcal{M} \cap \mathcal{K}(\mathcal{H})}^w = \overline{P\mathcal{K}(\mathcal{H})Q}^w = P\mathcal{B}(\mathcal{H})Q,$$

which implies that $R(T) \subseteq R(P)$. For any operator $T \in \mathcal{M}$, let $T_t(0 < t < \|T\|)$ be operators defined as in the proof of Lemma 2. We note that $T_t \in \mathcal{M}$ with closed range such that $\lim_{t \to 0}\|T_t - T\| = 0$. We then have $R(T) \subseteq R(P)$ since $R(T_t) \subseteq R(P)$. It follows

that $[Ax : A \in \mathcal{M}, x \in \mathcal{H}] \subseteq R(P)$. Since $Tx \in [Ax : A \in \mathcal{M}]$ for all $T \in \overline{\mathcal{M}}^w$ and any $x \in \mathcal{H}$, we have $R(T) \subseteq R(P)$. Similarly, $R(T^*) \subseteq R(Q)$ for all $T \in \overline{\mathcal{M}}^w$. Therefore $T \in P\mathcal{B}(\mathcal{H})Q$ and $\overline{\mathcal{M}}^w \subseteq P\mathcal{B}(\mathcal{H})Q$. $\square$

COROLLARY 1. *Let $\mathcal{M}$ be a nonzero star partial order-hereditary subspace in M_n. Then there exists a unique pair of projections $P, Q \in M_n$ such that $\mathcal{M} = PM_nQ$.*

EXAMPLE 1. Let $\mathcal{N}$ be an infinite dimensional Hilbert space and $\mathcal{H} = \mathcal{N} \oplus \mathcal{N}$. Let

$$\mathcal{M} = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : \quad X_{11}, X_{12} \in \mathcal{B}(\mathcal{N}) \text{ and } X_{21}, X_{22} \in \mathcal{K}(\mathcal{N}) \right\}.$$

Then $\mathcal{M}$ is a norm closed star partial order-hereditary subspace such that $\mathcal{K}(\mathcal{H}) \subsetneqq \mathcal{M} \subsetneqq \mathcal{B}(\mathcal{H})$.

In fact, this is easily proved by Douglas's Range Inclusion Theorem (cf. Theorem 17.1 in [4]). It is also known that both $\mathcal{M}^*$ and $\mathcal{M} \cap \mathcal{M}^*$ are also star partial order-hereditary subspaces containing $\mathcal{K}(\mathcal{H})$. However $\mathcal{M} \bigvee \mathcal{M}^*$ is not star partial order-hereditary. In fact, we have

$$\mathcal{M} \bigvee \mathcal{M}^* = \left\{ \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix} : \quad X_{11}, X_{12}, X_{21} \in \mathcal{B}(\mathcal{N}) \text{ and } X_{22} \in \mathcal{K}(\mathcal{N}) \right\}.$$

Let $A = \frac{1}{2} \begin{pmatrix} I & I \\ I & I \end{pmatrix}$ and $B = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where $I \in \mathcal{B}(\mathcal{N})$ is the identity operator. It it trivial that $A \overset{*}{\leqslant} B$ and $B \in \mathcal{M} \bigvee \mathcal{M}^*$. However $A \notin \mathcal{M} \bigvee \mathcal{M}^*$.

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(Received August 21, 2012)

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