RANK–ONE PERTURBATIONS OF NORMAL
OPERATORS AND HYPONORMALITY

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(*Communicated by H. Radjavi*)

Abstract. Let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N acting on a separable, infinite dimensional, complex Hilbert space $\mathscr{H}$. It is proved that the hyponormality of T is equivalent to the normality of T. Some characterizations of hyponormality[normality] of T are obtained.

1. Introduction and notation

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators on $\mathscr{H}$. For nonzero vectors u and v in $\mathscr{H}$ we shall write $u \otimes v$ for the rank-one operator in $\mathscr{L}(\mathscr{H})$ defined by $(u \otimes v)(x) = \langle x, v \rangle u$, $x \in \mathscr{H}$. For $X, Y \in \mathscr{L}(\mathscr{H})$, we denote by $[X, Y] = XY - YX$. An operator $T \in \mathscr{L}(\mathscr{H})$ is *normal* if $[T^*, T] = 0$, and $T \in \mathscr{L}(\mathscr{H})$ is *hyponormal* if $[T^*, T]$ is positive, i.e., $[T^*, T] \geqslant 0$. An operator T in $\mathscr{L}(\mathscr{H})$ is called a *rank-one perturbation of a normal operator* if there exist nonzero vectors u, v in $\mathscr{H}$ and a bounded normal operator $N \in \mathscr{L}(\mathscr{H})$ such that T equals the operator $N + u \otimes v$. The rank-one perturbations of a bounded (unbounded) operators can be applied to several related areas in mathematical physics ([1], [8], [11]). Since an invariant subspace problem about rank-one perturbations of diagonal operators was introduced in [10], several operator theorists have been studied the structure of rank-one perturbations of diagonal operators to detect their invariant subspaces ([3], [4], [5]). The rank-one perturbations of diagonal operators have been developed well by several authors ([6], [3], [4], [5]). Especially, E. Ionascu ([6]) obtained some general properties of rank-one perturbations of diagonal operators. In [12], J. Stampfli characterized the isometry of rank-one perturbations of normal operators in $\mathscr{L}(\mathscr{H})$, and proved that certain such operators of small norm split off a unitary piece from the shift. Also, in [9] one detected the structure of rank-one perturbations of unilateral shifts operators. Moreover, in [2] one considered rank-one perturbations of weighted shifts to examine distinctions between various sorts of weak hyponormalities; see [7] for weak hyponormalities.

In this note we prove that the hyponormality and normality of a rank-one perturbation of normal operator $T = N + u \otimes v$ in $\mathscr{L}(\mathscr{H})$ are equivalent. In addition, we obtain some characterizations of hyponormality[normality] of such an operator T with vectors u and v in $\mathscr{H}$.

Throughout this note, we write $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ for the sets of positive integers, real numbers, and complex numbers, respectively. For $A \in \mathscr{L}(\mathscr{H})$, $\mathrm{ran}A$ denotes the range of A as usual. Since $(Au) \otimes v = A(u \otimes v)$, we denote it by $Au \otimes v$. For a subspace $\mathscr{M}$ of $\mathscr{H}$, $\vee \mathscr{M}$ is the subspace of $\mathscr{H}$ spanned by $\mathscr{M}$, and $P_{\mathscr{M}}$ denotes an orthogonal projection to $\mathscr{M}$. Here, subspace means always "closed subspace". For a rank-one perturbation $T = N + u \otimes v$ of a normal operator N in $\mathscr{L}(\mathscr{H})$, without loss of generality we assume that $\|u\| = \|v\| = 1$ in this note.

2. Equivalence

Let u and v be vectors in $\mathscr{H}$ and let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N in $\mathscr{L}(\mathscr{H})$. Then it can be obtained easily that

$$[T^*, T] = N^*u \otimes v + v \otimes N^*u + v \otimes v - Nv \otimes u - u \otimes Nv - u \otimes u. \tag{2.1}$$

For brevity, we denote the subspace by

$$\mathscr{R}_{u,v} := \vee\{u, v, N^*u, Nv\}. \tag{2.2}$$

Then, by (2.1) we have that $\mathrm{ran}([T^*, T]) \subset \mathscr{R}_{u,v}$ obviously.

We first discuss a matrix structure of the commutator $[T^*, T]$ of T under the independence condition of u and v as following.

LEMMA 2.1. *Let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N in $\mathscr{L}(\mathscr{H})$ and let u and v be linearly independent. Then there exists an orthonormal system $\{e_i\}_{i=1}^4$ in $\mathscr{H}$ with $e_1 = u$ such that*
  (i) $\vee\{u, v\} = \vee\{e_1, e_2\}$,
  (ii) $\mathscr{R}_{u,v} \subset \vee\{e_i\}_{i=1}^4 \ (:= \mathscr{M}_{u,v})$, where $\mathscr{R}_{u,v}$ is as in (2.2),
  (iii) $[T^*, T] \cong A \oplus 0_{\mathscr{H} \ominus \mathscr{M}_{u,v}}$ relative to $\mathscr{M}_{u,v} \oplus (\mathscr{H} \ominus \mathscr{M}_{u,v})$, where

$$A \cong \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \overline{a_{12}} & a_{22} & a_{23} & a_{24} \\ \overline{a_{13}} & \overline{a_{23}} & 0 & 0 \\ \overline{a_{14}} & \overline{a_{24}} & 0 & 0 \end{pmatrix} \tag{2.3}$$

with

$$a_{11} = 2Re(\langle N^*u, u \rangle \langle u, v \rangle - \langle Nv, u \rangle) + |\langle u, v \rangle|^2 - 1; \tag{2.3a}$$

$$a_{12} = \langle e_2, v \rangle \langle N^*u, u \rangle + \langle e_2, N^*u + v \rangle \langle v, u \rangle - \langle e_2, Nv \rangle; \tag{2.3b}$$

$$a_{13} = \langle e_3, N^*u \rangle \langle v, u \rangle - \langle e_3, Nv \rangle; \tag{2.3c}$$

$$a_{14} = \langle e_4, N^*u \rangle \langle v, u \rangle - \langle e_4, Nv \rangle; \tag{2.3d}$$

$$a_{22} = 2Re(\langle N^*u, e_2 \rangle \langle e_2, v \rangle) + |\langle v, e_2 \rangle|^2; \tag{2.3e}$$

$$a_{23} = \langle e_3, N^*u \rangle \langle v, e_2 \rangle; \tag{2.3f}$$

$$a_{24} = \langle e_4, N^*u \rangle \langle v, e_2 \rangle. \tag{2.3g}$$

Proof. Since u and v are linearly independent, by Gram-Schmidt orthogonal process ([13, Th. 3.5]), we get an orthonormal system $\{e_1, e_2\}$ with $e_1 = u$ such that

$\vee\{e_1,e_2\} = \vee\{u,v\}$. Let $\{e_i\}_{i=1}^4$ be an orthonormal system in $\mathscr{H}$ such that $\mathscr{R}_{u,v} \subset \vee\{e_i\}_{i=1}^4$. We denote by $\mathscr{M}_{u,v} := \vee\{e_i\}_{i=1}^4$. Recall from (2.1) that

$$[T^*,T]h = \langle h,v\rangle N^*u + \langle h, N^*u+v\rangle v - \langle h,u\rangle Nv - \langle h, Nv+u\rangle u, \quad \forall h \in \mathscr{H}. \quad (2.4)$$

Thus $[T^*,T]\mathscr{M}_{u,v} \subset \mathscr{R}_{u,v} \subset \mathscr{M}_{u,v}$, and so $\mathscr{M}_{u,v}$ is a reducing subspace for $[T^*,T]$. We now consider an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $\mathscr{H}$ containing $\mathscr{M}_{u,v}$. Since $\mathscr{R}_{u,v} \subset \mathscr{M}_{u,v}$, by using (2.4), we get $[T^*,T]e_j = 0$, $j \geqslant 5$. Hence we have a decomposition $[T^*,T] \cong A \oplus 0_{\mathscr{H}\ominus\mathscr{M}_{u,v}}$ relative to $\mathscr{M}_{u,v} \oplus (\mathscr{H} \ominus \mathscr{M}_{u,v})$, where A is unitarily equivalent to a 4×4 complex matrix $(a_{ij})_{1\leqslant i,j\leqslant 4}$. Without loss of generality, we can consider as $A = (a_{ij})_{1\leqslant i,j\leqslant 4}$ now. Substituting e_j for h in (2.4), $1 \leqslant j \leqslant 4$, we obtain that, for $1 \leqslant i,j \leqslant 4$,

$$a_{ij} = \langle [T^*,T]e_j, e_i\rangle \quad\quad\quad\quad\quad\quad\quad\quad\quad (2.5)$$
$$= \langle e_j,v\rangle\langle N^*u,e_i\rangle + \langle e_j, N^*u+v\rangle\langle v,e_i\rangle - \langle e_j,u\rangle\langle Nv,e_i\rangle - \langle e_j, Nv+u\rangle\langle u,e_i\rangle.$$

Simplifying a_{ij} in (2.5) for $i,j = 1,2,3,4$, the equalities a_{ij} in (2.5) can be expressed as (2.3a-g) easily. It is obvious that $a_{ij} = \overline{a_{ji}}$. Thus the proof is complete. $\quad\square$

The following is a parallel result of Lemma 2.1 when u and v are linearly dependent.

LEMMA 2.2. *Let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N in $\mathscr{L}(\mathscr{H})$. Let u and v be linearly dependent and suppose $v = e^{i\theta}u$ for some $\theta \in [0,2\pi)$. Then there exists an orthonormal system $\{f_i\}_{i=1}^3$ in $\mathscr{H}$ with $u = f_1$ such that*
(i) $\mathscr{R}_{u,v} \subset \vee\{f_i\}_{i=1}^3$ $(:= \mathscr{N}_u)$,
(ii) $[T^*,T] \cong B \oplus 0_{\mathscr{H}\ominus\mathscr{N}_u}$ relative to $\mathscr{N}_u \oplus (\mathscr{H} \ominus \mathscr{N}_u)$, where

$$B \cong \begin{pmatrix} 0 & b_{12} & b_{13} \\ \overline{b_{12}} & 0 & 0 \\ \overline{b_{13}} & 0 & 0 \end{pmatrix} \quad\quad\quad\quad (2.6)$$

with

$$b_{12} = e^{i\theta}\langle f_2, N^*u\rangle - e^{-i\theta}\langle f_2, Nu\rangle, \quad\quad\quad\quad (2.6a)$$
$$b_{13} = e^{i\theta}\langle f_3, N^*u\rangle - e^{-i\theta}\langle f_3, Nu\rangle. \quad\quad\quad\quad (2.6b)$$

Proof. By the same method as the proof of Lemma 2.1, we may prove that there exists an orthonormal system $\{f_i\}_{i=1}^3$ in $\mathscr{H}$ with $f_1 = u$ such that $\mathscr{R}_{u,v} \subset \vee\{f_i\}_{i=1}^3$. For brevity, we set $\mathscr{N}_u := \vee\{f_i\}_{i=1}^3$. Considering an orthonormal basis $\{f_i\}_{i=1}^\infty$ of $\mathscr{H}$ containing $\mathscr{N}_u$, we see that $[T^*,\overline{T}] \cong B \oplus 0_{\mathscr{H}\ominus\mathscr{N}_u}$ relative to a decomposition $\mathscr{N}_u \oplus (\mathscr{H} \ominus \mathscr{N}_u)$, where B is unitarily equivalent to a 3×3 complex matrix $(b_{ij})_{1\leqslant i,j\leqslant 3}$. Without loss of generality, we say $B = (b_{ij})_{1\leqslant i,j\leqslant 3}$ as in the proof of Lemma 2.1. Then it follows from (2.4) that, for $1 \leqslant i,j \leqslant 3$,

$$b_{ij} = e^{-i\theta}\langle f_j,u\rangle\langle N^*u,f_i\rangle + e^{i\theta}\langle f_j, N^*u\rangle\langle u,f_i\rangle - e^{i\theta}\langle f_j,u\rangle\langle Nu,f_i\rangle - e^{-i\theta}\langle f_j, Nu\rangle\langle u,f_i\rangle.$$
$$(2.7)$$

A direct computation shows that the matrix B and its coefficient b_{ij} can be represented as in (2.6) and (2.6a,b), respectively. $\square$

Now we are ready to prove the main theorem of this section.

THEOREM 2.3. *Let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N in $\mathscr{L}(\mathscr{H})$. Then T is hyponormal if and only if T is normal.*

Proof. It is sufficient to see that if T is hyponormal, then T is normal. So we suppose that T is hyponormal. To claim the normality of T, we consider two cases of linear independence and dependence of u and v.

First we assume that u and v are linearly independent. Then it follows from Lemma 2.1 that there exists an orthonormal system $\{e_i\}_{i=1}^4$ in $\mathscr{H}$ with $e_1 = u$ such that $[T^*, T] \cong A \oplus 0_{\mathscr{H} \ominus \mathscr{M}_{u,v}}$, where A and $\mathscr{M}_{u,v}$ are as in Lemma 2.1. It is obvious that $A \geqslant 0$. We denote $\mathscr{K}_{ij}$ for a subspace spanned by vectors e_i and e_j, and denote $P_{ij} := P_{\mathscr{K}_{ij}}[T^*, T]|_{\mathscr{K}_{ij}}$ for a compression of $[T^*, T]$ to $\mathscr{K}_{ij}$, where $1 \leqslant i, j \leqslant 4$. Then the matrix form of P_{13} is represented by

$$P_{13} \cong \begin{pmatrix} a_{11} & a_{13} \\ \overline{a_{13}} & 0 \end{pmatrix}. \tag{2.8}$$

Obviously P_{13} is positive definite if and only if $a_{11} \geqslant 0$ and $a_{13} = 0$. Since matrices corresponding to P_{14}, P_{23}, and P_{24} are positive definite, we have that $a_{22} \geqslant 0$ and $a_{14} = a_{23} = a_{24} = 0$. Considering P_{12}, we obtain the condition

$$a_{11}a_{22} - |a_{12}|^2 \geqslant 0 \tag{2.9}$$

by the positivity of P_{12}. In particular, if we take e_2 in $\mathscr{H}$ such that

$$e_2 = \frac{v - \langle v, u \rangle u}{\|v - \langle v, u \rangle u\|} \tag{2.10}$$

via the Gram-Schmidt orthogonal process as in the proof of Lemma 2.1, we may use the same conclusion of Lemma 2.1. Hence, by using (2.10), the formula of (2.3e) can be represented by

$$a_{22} = 2\mathrm{Re}(\langle N^*u, v \rangle - \langle u, v \rangle \langle N^*u, u \rangle) + \delta, \tag{2.11}$$

where $\delta := 1 - |\langle u, v \rangle|^2$. Since $a_{11} \geqslant 0$ and $a_{22} \geqslant 0$, by (2.3a) and (2.11), we have

$$2\mathrm{Re}(\langle N^*u, u \rangle \langle u, v \rangle - \langle Nv, u \rangle) \geqslant \delta \geqslant 2\mathrm{Re}(\langle u, v \rangle \langle N^*u, u \rangle - \langle N^*u, v \rangle).$$

By the property of complex numbers that $\mathrm{Re}(z + w) = \mathrm{Re}(z + \overline{w})$ for all $z, w \in \mathbb{C}$, we have that

$$\delta = 2\mathrm{Re}(\langle N^*u, u \rangle \langle u, v \rangle - \langle Nv, u \rangle) = 2\mathrm{Re}(\langle u, v \rangle \langle N^*u, u \rangle - \langle N^*u, v \rangle).$$

By (2.3a) and (2.11), $a_{11} = a_{22} = 0$. It follows from (2.9) that $a_{12} = 0$. Thus we obtain finally that $A = 0$, i.e., $[T^*, T] = 0_{\mathscr{H}}$.

Next we consider the case that u and v are linearly dependent. Using Lemma 2.2, there exists an orthonormal system $\{f_i\}_{i=1}^3$ in $\mathscr{H}$ with $u = f_1$ such that $[T^*, T] \cong B \oplus 0_{\mathscr{H} \ominus \mathscr{N}_u}$, where B and $\mathscr{N}_u$ are as in Lemma 2.2. Since $(b_{ij})_{1 \leqslant i,j \leqslant 3}$ in (2.6) is positive definite, obviously $b_{12} = b_{13} = 0$, i.e., $[T^*, T] = 0_{\mathscr{H}}$. Hence the hyponormality of T is equivalent to the normality and the proof is complete. $\square$

3. Characterizations

In this section we characterize the hyponormality[normality] of rank-one perturbations of normal operators in $\mathscr{L}(\mathscr{H})$ by using notations in Section 2. We first consider the case of linear dependence of u and v as following.

THEOREM 3.1. *Let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N in $\mathscr{L}(\mathscr{H})$. Let u and v be linearly dependent with $v = e^{i\theta}u$ for some $\theta \in [0, 2\pi)$. Then the following conditions are equivalent:*
  (i) *T is hyponormal [normal],*
  (ii) *one of the following conditions a) and b) holds:*
   a) *$\vee\{u\} \supset \{N^*u, Nu\}$,*
   b) *$Nu \in \vee\{u, N^*u\}$ and*

$$\langle N^*u, Nu \rangle - \langle N^*u, u \rangle^2 = e^{2i\theta} \|N^*u - \langle N^*u, u \rangle u\|^2, \tag{3.1}$$

  (iii) *there exists $\alpha \in \mathbb{C}$ such that $e^{-i\theta}N^*u - e^{i\theta}Nu = \alpha u$.*

Proof. (i) $\Rightarrow$ (ii) It follows from Lemma 2.2 that there exists an orthonormal system $\{f_i\}_{i=1}^3$ in $\mathscr{H}$ with $u = f_1$ such that $\mathscr{R}_{u,v} \subset \mathscr{N}_u = \vee\{f_i\}_{i=1}^3$ and $[T^*, T] \cong B \oplus 0_{\mathscr{H} \ominus \mathscr{N}_u}$, where B as in (2.6) and b_{ij} are as in (2.6a,b). Since T is normal via Theorem 2.3, obviously $b_{12} = b_{13} = 0$. We consider two cases of the linear dependence and independence of u and N^*u.

If u and N^*u are linearly dependent, i.e., $N^*u = \langle N^*u, u \rangle u$, then $b_{12} = -e^{-i\theta}\langle f_2, Nu \rangle = 0$ and $b_{13} = -e^{-i\theta}\langle f_3, Nu \rangle = 0$. Thus $Nu \in (\{f_2, f_3\})^\perp$. Since $Nu \in \mathscr{N}_u$, we have $Nu \in \vee\{u\}$. On the other hand, if u and N^*u are linearly independent, then $\|N^*u - \langle N^*u, u \rangle u\| \neq 0$. Consider $f_1 = u$ as usual. By the Gram-Schmidt orthogonal process, we may take $f_2 \in \mathscr{H}$ such that

$$f_2 = \frac{N^*u - \langle N^*u, u \rangle u}{\|N^*u - \langle N^*u, u \rangle u\|}. \tag{3.2}$$

Then there exists an orthonormal basis $\{f_i\}_{i=1}^\infty$ in $\mathscr{H}$ such that $\mathscr{R}_{u,v} \subset \vee\{f_i\}_{i=1}^3$ and $[T^*, T] \cong B \oplus 0_{\mathscr{H} \ominus \mathscr{N}_u}$, where B is as in (2.6) and its coefficients b_{ij} satisfy (2.6a,b). Since $b_{13} = 0$, by (2.6b), we have that

$$b_{13} = e^{i\theta}\langle f_3, N^*u \rangle - e^{-i\theta}\langle f_3, Nu \rangle = -e^{-i\theta}\langle f_3, Nu \rangle = 0. \tag{3.3}$$

Hence Nu is a linear combination of f_1 and f_2, which proves the first part of (ii)-b). According to (2.6a), we may get

$$b_{12} = e^{i\theta}\langle f_2, N^*u \rangle - e^{-i\theta}\langle f_2, Nu \rangle = 0. \tag{3.4}$$

Substituting equation (3.4) for f_2 in (3.2), we get the condition (3.1).

(ii) $\Rightarrow$ (iii) If (ii)-a) holds, obviously $e^{-i\theta}N^*u - e^{i\theta}Nu = \alpha u$ for some $\alpha \in \mathbb{C}$. So we only consider the case (ii)-b). We know that there exists an orthonormal system $\{f_i\}_{i=1}^3$ in $\mathscr{H}$ with $u = f_1$ such that f_2 satisfies (3.2), $\mathscr{R}_{u,v} \subset \mathscr{N}_u = \vee\{f_i\}_{i=1}^3$, and $[T^*, T] \cong B \oplus 0_{\mathscr{H} \ominus \mathscr{N}_u}$ with (2.6) and (2.6a,b). By (3.1) and the definition of f_2, we see that

$$e^{i\theta}\langle f_2, N^*u \rangle = e^{-i\theta}\langle f_2, Nu \rangle. \tag{3.5}$$

By using the first part of condition b), we have

$$e^{-i\theta}N^*u - e^{i\theta}Nu = e^{-i\theta}\sum_{j=1}^{2}\langle N^*u, f_j\rangle f_j - e^{i\theta}\sum_{j=1}^{2}\langle Nu, f_j\rangle f_j$$

$$\overset{(3.5)}{=} (e^{-i\theta}\langle N^*u, u\rangle - e^{i\theta}\langle Nu, u\rangle)u.$$

The number $\alpha = e^{-i\theta}\langle N^*u, u\rangle - e^{i\theta}\langle Nu, u\rangle$ is the value required in (iii).

(iii)$\Rightarrow$(i) Suppose that $e^{-i\theta}N^*u - e^{i\theta}Nu = \alpha u$ for some $a \in \mathbb{C}$. Recall from Lemma 2.2 that there exists an orthonormal system $\{f_1, f_2, f_3\}$ in $\mathscr{H}$ with $u = f_1$ such that, $\mathscr{R}_{u,v} \subset \mathscr{N}_u = \vee\{f_i\}_{i=1}^{3}$ and $[T^*, T] \cong B \oplus 0_{\mathscr{H}\ominus\mathscr{N}_u}$ with (2.6) and (2.6a,b). If u and N^*u are linear dependent, since $\mathscr{R}_{u,v} = \vee\{u\}$, we have $b_{12} = 0 = b_{13}$. If u and N^*u are linear independent, we can take an orthonormal system $\{f_1, f_2, f_3\}$ in $\mathscr{H}$ such that $u = f_1$, f_2 satisfies (3.2), $\mathscr{R}_{u,v} \subset \vee\{f_i\}_{i=1}^{3}$, and $[T^*, T] \cong B \oplus 0_{\mathscr{H}\ominus\mathscr{N}_u}$ with (2.6) and (2.6a,b), as in the proof of "(i)$\Rightarrow$(ii)". Note that

$$\alpha u = e^{-i\theta}N^*u - e^{i\theta}Nu$$
$$= e^{-i\theta}\langle N^*u, u\rangle u - e^{i\theta}\langle Nu, u\rangle u + e^{-i\theta}\langle N^*u, f_2\rangle f_2 - e^{i\theta}\langle Nu, f_2\rangle f_2$$
$$= \alpha u + e^{-i\theta}\langle N^*u, f_2\rangle f_2 - e^{i\theta}\langle Nu, f_2\rangle f_2.$$

Then $e^{i\theta}\langle f_2, N^*u\rangle = e^{-i\theta}\langle f_2, Nu\rangle$, which implies that $b_{12} = 0$. It follows from the hypothesis of (iii) that $b_{13} = 0$. Hence $[T^*, T] = 0_{\mathscr{H}}$, i.e., T is normal. $\quad\square$

We now discuss the case of linear independence of u and v for the hyponormality [normality] of rank-one perturbations of normal operators.

THEOREM 3.2. *Let $T = N + u \otimes v$ be a rank-one perturbation of a normal operator N in $\mathscr{L}(\mathscr{H})$. Suppose that u and v are linearly independent. Then T is hyponormal [normal] if and only if the following two conditions hold:*
(i) $\vee\{u, v\} \supset \{N^*u, Nv\}$ and
(ii) $\alpha = \delta$ and $\beta = 0$, where $\delta := 1 - |\langle u, v\rangle|^2$ and

$$\alpha = 2Re(\langle N^*u, u\rangle\langle u, v\rangle - \langle Nv, u\rangle), \tag{3.6a}$$

$$\beta = 2\langle v, u\rangle \cdot Re(\langle v, N^*u\rangle) - \langle v, u\rangle^2\langle u, N^*u\rangle - \langle v, Nv\rangle + \delta(\langle N^*u, u\rangle + \langle v, u\rangle). \tag{3.6b}$$

Proof. ($\Rightarrow$) Suppose that T is normal. Consider $e_1 = u$ as usual. By the Gram-Schmidt orthogonal process, take

$$e_2 = \frac{v - \langle v, u\rangle u}{\|v - \langle v, u\rangle u\|}. \tag{3.7}$$

Then there exists an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ such that $\mathscr{R}_{u,v} \subset \mathscr{M}_{u,v} := \vee\{e_i\}_{i=1}^{4}$, and $[T^*, T] \cong A \oplus 0_{\mathscr{H}\ominus\mathscr{M}_{u,v}}$, where A is as in (2.3) and the entries a_{ij} of A satisfy (2.3a-g). Since T is normal, obviously $a_{ij} = 0$ for all i, j. Since $a_{13} = a_{14} = a_{23} = a_{24} = 0$, (i) holds. On the other hand, since $a_{11} = a_{22} = a_{12} = 0$, we see easily that $\alpha = \delta$ and $\beta = 0$.

($\Leftarrow$) Suppose that conditions (i) and (ii) hold. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal basis for $\mathcal{H}$ such that $e_1 = u$, e_2 is as in (3.7), $\mathcal{R}_{u,v} \subset \mathcal{M}_{u,v} := \vee\{e_i\}_{i=1}^4$, and $[T^*,T] \cong A \oplus 0_{\mathcal{H} \ominus \mathcal{M}_{u,v}}$ with (2.3) and (2.3a-g), as usual. By using (i), (2.3c,d), and (2.3f,g), it is obvious that $a_{13} = a_{14} = a_{23} = a_{24} = 0$. Observe that two conditions $\alpha = \delta$ and $\beta = 0$ imply $a_{11} = a_{22} = a_{12} = 0$. Hence $[T^*,T] = 0_{\mathcal{H}}$. Thus the proof is complete. $\square$

We now give an example for a normal[hyponormal] operator $T = N + u \otimes v$ by using Theorems 3.1 and 3.2 as following.

EXAMPLE 3.3. Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space and let $\{e_j\}_{j=0}^\infty$ be an orthonormal basis of $\mathcal{H}$. Let $u = \frac{1}{\sqrt{x^2+1}} \sum_{i=0}^\infty \alpha_i e_i$ with $\alpha_0 = x$ and $\alpha_j = (1/\sqrt{2})^j$, $j \geqslant 1$, and let $v = \frac{1}{\sqrt{y^2+1}} \sum_{i=0}^\infty \beta_i e_i$ with $\beta_0 = y$ and $\beta_j = \alpha_j$, $j \geqslant 1$, where x and y are real variables. Consider $N = \frac{1}{\sqrt{x^2+1}\sqrt{y^2+1}} \text{Diag}\{z,1,1,...\}$, where z is a real variable, and define $T = N + u \otimes v$. We consider two cases of linear dependence and independence of u and v as following:

1° Dependent case. Note that if $x = y$, then u and v are linearly dependent. Setting
$$\mathcal{D}_1 = \{(x,1) : x \in \mathbb{R}\} \cup \{(0,z) : z \in \mathbb{R}\},$$
it follows from a direct computation that T satisfies the condition (ii)-a) [resp., (ii)-b)] in Theorem 3.1 if and only if $(x,z) \in \mathcal{D}_1$ [resp., $(x,z) \in \mathbb{R}^2 \backslash \mathcal{D}_1$]. Hence T is normal[hyponormal] if and only if $(x,y,z) \in \{(t,t,s) : t,s \in \mathbb{R}\}$ in this case.

2° Independent case. Note that if $x \neq y$, then u and v are linearly independent. In this case we can see that $T := N + u \otimes v$ satisfies Theorem 3.2(i) easily. By a direct computation, we obtain that

$$\alpha = \delta \Longleftrightarrow x^3 - yx^2 + (3 - 2z)x - y = 0, \qquad (3.8a)$$

$$\beta = 0 \Longleftrightarrow yx^3 + zx^2 + yx - z + 2 = 0, \qquad (3.8b)$$

and that the common roots of two equations in (3.8a,b) are exactly the line $\mathcal{D}_2 := \{(t, -t, t^2 + 2) : t \in \mathbb{R} \backslash \{0\}\}$ in $\mathbb{R}^3$. Hence T is normal[hyponormal] if and only if $(x,y,z) \in \mathcal{D}_2$ in this case.

We close this note with the following remark.

REMARK 3.4. Changing the order of the vectors u, v, N^*u, and Nv, the conditions in Theorems 3.1 and 3.2 for hyponormality of rank-one perturbation $T = N + u \otimes v$ will be changed slightly; for example, considering u and N^*u as pivots instead of u and v, we can obtain analogue conditions in Theorems 3.1 and 3.2 for normality[hyponormality] of T. Of course, we may also consider as pivots "u and Nv", (or "v and N^*u", "v and Nv", "N^*u and Nv") instead of "u and v". We leave them for interesting readers.

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