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THE WEIGHTED MOORE–PENROSE INVERSE FOR SUM OF MATRICES

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Abstract. In this paper we exhibit that under the rank additivity condition $r(A+B) = r(A) + r(B)$, a neat relationship between the weighted Moore-Penrose inverse of $A+B$ and the weighted Moore-Penrose inverses of A and B.

1. Introduction

Throughout this paper $C^{m\times n}$ denotes the set of all $m\times n$ matrices over the complex field C. I_k denotes the identity matrix of order k, $O_{m\times n}$ is the $m\times n$ matrix of all zero entries (if no confusion occurs, we will drop the subscript). For a matrix $A\in C^{m\times n}$, the symbols A^*, $R(A)$, $N(A)$ and $r(A)$ denote the conjugate transpose, the range space, the null space and the rank of A, respectively.

Adopting the notations of [12], let $A\in C^{m\times n}$, a generalized inverse X of A is a matrix which satisfies some of the following four Penrose equations [9]:

$$(1)\ AXA = A,\ \ (2)\ XAX = X,\ \ (3)\ (AX)^* = AX,\ \ (4)\ (XA)^* = XA.$$

For a subset $\{i, j, \cdots, k\}$ of the set $\{1,2,3,4\}$, the set of $n\times m$ matrices satisfying the equations (i), (j), $\cdots$, (k) from among equations $(1)-(4)$ is denoted by $A\{i, j,\cdots, k\}$. A matrix in $A\{i, j,\cdots, k\}$ is called an $\{i, j,\cdots, k\}$-inverse of A and is denoted by $A^{(i,j,\cdots,k)}$. The unique $\{1,2,3,4\}$-inverse of A is denoted by $A^\dagger$, which is called the Moore-Penrose inverse of A. The weighted Moore-Penrose inverse $A^\dagger_{MN}$ with respect to a pair of Hermitian positive definite matrices $M\in C^{m\times m}$ and $N\in C^{n\times n}$ is defined to be the unique solution of the following four matrix equations [12]:

$$AXA = A,\ \ XAX = X,\ \ (MAX)^* = MAX,\ \ (NXA)^* = NXA. \tag{1}$$

When $M = I_m$, $N = I_n$, the matrix X satisfying (1.1) is called the Moore-Penrose inverse of A and is denoted by $X = A^\dagger_{I_mI_n} = A^\dagger$. Further, $A^\sharp = N^{-1}A^*M$ stands for the weighted conjugate transpose of A.

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Although the theory of generalized inverses has had a substantial development over the past several decades, there are lots of fundamental problems on generalized inverses of matrices that need further investigation. One such problem is concerned to the generalized inverse for a sum of matrices. Suppose A and B are a pair of matrices of the same size. In many situations, one wants to know the expressions of $(A+B)^{(i,j,\cdots,K)}$ and its properties. For example, give a decomposition $(A+B)^{\dagger}_{MN} = P_1 + P_2$ under various given conditions, where P_1 and P_2 have some close relationship with weighted Moore-Penrose inverses of A and B, respectively.

The generalized inverse for a sum of matrices was introduced by Penrose [9]. It has quite important applications in numerical linear algebra and applied fields, such as, linear control theory [1], statistics [8], projection algorithms [2] and perturbation analysis of matrix [6]. Moreover, as one of the fundamental research problems in matrix theory, the generalized inverse for a sum of matrices is a very useful tool in many algorithms for the computation of the generalized parallel sum of A and B. The generalized parallel sum originally arose in an attempt to generalize a network synthesis procedure of Duffin [4] and has been studied in the scalar case by Erickson [5]. Suppose that $A,B \in C^{m\times n}$, then we define the generalized parallel sum of A and B by $A(A+B)^{\dagger}B$ or $A(A+B)^{\dagger}_{MN}B$. One such problem concerns to the Moore-Penrose inverse or the weighted Moore-Penrose inverse of $A+B$.

Various generalized inverses for sums of two rectangular matrices were developed by Cline [3], Hartwig [7], Y.Tian [10], Z.Xiong [13], and so on, see [9, 12]. In this paper, we exhibit a neat relationship between the weighted Moore-Penrose inverse of $A+B$ and the weighted Moore-Penrose inverses of A and B under some rank additivity conditions.

As the main tools in our discussion, we first mention the following three lemmas, which will be used in this paper.

LEMMA 1.1. [12] *Let $A \in C^{m\times n}$, and let M and N be two Hermitian positive definite matrices of order m and n, respectively. Then*

(I) $(A^{\dagger}_{MN})^* = (A^*)^{\dagger}_{N^{-1}M^{-1}}$,

(II) $A^{\dagger}_{MN} = N^{-1}A^*(AN^{-1}A^*)^{\dagger}_{MM^{-1}} = (A^*MA)^{\dagger}_{N^{-1}N}A^*M$,

(III) $A^{\dagger}_{MN} = N^{-1/2}(M^{1/2}AN^{-1/2})^{\dagger}M^{1/2}$,

(IV) $R(AA^{\dagger}_{MN}) = R(A),\ \ N(AA^{\dagger}_{MN}) = M^{-1}N(A^*) = N(A^{\sharp})$,

(V) $R(A^{\dagger}_{MN}A) = N^{-1}R(A^*) = R(A^{\sharp}),\ \ N(A^{\dagger}_{MN}A) = N(A)$,

(VI) $A^{\dagger} = (A^*A)^{\dagger}A^* = A^*(AA^*)^{\dagger}$.

LEMMA 1.2. [12] *Let L and M be two complementary subspaces of C^n, and let $P_{L,M}$ be a projector on L along M. Then*

(I) $(I_m - P_{L,M})A = O$ *iff* $R(A) \subseteq L$,

(II) $A(I_n - P_{L,M}) = O$ *iff* $M \subseteq N(A)$,

(III) $I_n - P_{L,M} = P_{M,L}$.

LEMMA 1.3. [11] [Weighted Singular Value Decomposition] *Let* $A \in C^{m\times n}$ *with rank* r, $M \in C^{m\times m}$ *and* $N \in C^{n\times n}$ *be two Hermitian positive definite matrices. Then there exists an* M*-unitary matrix* U *and an* N^{-1}*-unitary matrix* V *(i.e.* $U^*MU = I_m$, $V^*N^{-1}V = I_n$*), such that*

$$A = U \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix} V^*, \tag{2}$$

where $\Sigma = diag(\mu_1, \mu_2, \cdots, \mu_r)$, $\mu_i = \sqrt{\lambda_i}$ *and* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > 0$ *are the nonzero eigenvalues of* $A^\sharp A = (N^{-1}A^*M)A$. *Then* $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r > 0$ *are the nonzero weighted singular values of* A *and (2) is called the Weighted Singular Value Decomposition of* A. *In this case, the weighted Moore-Penrose inverse of* A *with respect to the Hermitian positive definite matrices* $M \in C^{m\times m}$ *and* $N \in C^{n\times n}$ *is given by*

$$A^\dagger_{MN} = N^{-1}V \begin{pmatrix} \Sigma^{-1} & O \\ O & O \end{pmatrix} U^*M. \tag{3}$$

For the convenience of readers, we will adopt the following notation

$$W = A + (Y + Y_p)G(Z + Z_p)^*, \quad \tilde{A} = M^{1/2}AN^{-1/2}, \quad \tilde{Y} = M^{1/2}YS^{-1/2},$$

$$\tilde{Y_p} = M^{1/2}Y_pS^{-1/2}, \quad \tilde{Z} = N^{-1/2}ZS^{1/2}, \quad \tilde{Z_p} = N^{-1/2}Z_pS^{1/2},$$

where $A \in C^{m\times n}$, $Y, Y_p \in C^{m\times s}$, $G \in C^{s\times s}$, $Z, Z_p \in C^{n\times s}$, M, N and S are three Hermitian positive definite matrices of order m, n and s, respectively.

2. Main results

In this section, the weighted Moore-Penrose inverse of $A + B$ is discussed under the condition $r(A + B) = r(A) + r(B)$.

THEOREM 2.1. *Let* $A \in C^{m\times n}$, $Y, Y_p \in C^{m\times s}$, $G \in C^{s\times s}$, $Z, Z_p \in C^{n\times s}$, M, N *and* S *be three Hermitian positive definite matrices of order* m, n *and* s*, respectively. Suppose that* $R(\tilde{Y}) \subseteq R(\tilde{A})$, $R(\tilde{Y_p}) \perp R(\tilde{A})$, $R(\tilde{Z}) \subseteq R(\tilde{A}^*)$, $R(\tilde{Z_p}) \perp R(\tilde{A}^*)$, G *is invertible,* Y_p *is of full column rank and* Z_p *is of full column rank. Then*

$$\begin{aligned} W^\dagger_{MN} &= (A + (Y + Y_p)G(Z + Z_p)^*)^\dagger_{MN} \\ &= A^\dagger_{MN} - EZ^*A^\dagger_{MN} - A^\dagger_{MN}YC^* + E(G^{-1} + Z^*A^\dagger_{MN}Y)C^*, \end{aligned} \tag{4}$$

where $C = MY_p(Y_P{}^*MY_P)^{-1} = ((Y_P)^\dagger_{MS})^*$ *and* $E = N^{-1}Z_p(Z_p{}^*N^{-1}Z_p)^{-1} = (Z_p{}^*)^\dagger_{SN}$.

Proof. Let $T = A^{\dagger}_{MN} - EZ^*A^{\dagger}_{MN} - A^{\dagger}_{MN}YC^* + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^*$. Then it is only necessary to show that T and W satisfy the equations in (1). From the hypothesis $R(\tilde{Y}_p) \perp R(\tilde{A})$ and $R(\tilde{Z}_p) \perp R(\tilde{A}^*)$, we have

$$R(\tilde{Y}_p) \subseteq N(\tilde{A}^*), \quad R(\tilde{A}) \subseteq N(\tilde{Y}_p{}^*), \tag{5}$$

and

$$R(\tilde{Z}_p) \subseteq N(\tilde{A}), \quad R(\tilde{A}^*) \subseteq N(\tilde{Z}_p{}^*). \tag{6}$$

Combining the formulas (III), (VI) in Lemma 1.1 with (5), we have

$$\begin{aligned} N^{-1/2}A^*C &= N^{-1/2}A^*((Y_P)^{\dagger}_{MS})^* = N^{-1/2}A^*M^{1/2}((\widetilde{Y_P})^{\dagger})^*S^{-1/2} \\ &= \widetilde{A}^*((\widetilde{Y_P})^{\dagger})^*S^{-1/2} = \widetilde{A}^*([(\widetilde{Y_P})\widetilde{Y_P}]^{\dagger}(\widetilde{Y_P})^*)^*S^{-1/2} \\ &= \widetilde{A}^*\widetilde{Y_P}[(\widetilde{Y_P})^*\widetilde{Y_P}]^{\dagger}S^{-1/2} = O. \end{aligned} \tag{7}$$

According to (7), we get

$$N^{-1/2}A^*C = O \text{ and } C^*A = O. \tag{8}$$

On the other hand, by the formula (II) in Lemma 1.1 and (5), we have

$$\begin{aligned} S^{-1/2}Y_P^*(A^{\dagger}_{MN})^* &= S^{-1/2}Y_P^*MA(A^*MA)^{\dagger}_{N^{-1}N} \\ &= S^{-1/2}Y_P^*M^{1/2}M^{1/2}A(A^*MA)^{\dagger}_{N^{-1}N} \\ &= (\widetilde{Y_P})^*\widetilde{A}(A^*MA)^{\dagger}_{N^{-1}N} = O. \end{aligned} \tag{9}$$

From (9), we have

$$S^{-1/2}Y_P^*(A^{\dagger}_{MN})^* = O \text{ and } (A^{\dagger}_{MN})Y_P = O. \tag{10}$$

Similar to the proof of (8) and (10), we obtain the following formulas by the formulas (II), (III), (VI) in Lemma 1.1 and (6).

$$AE = O, \quad Z_p^*A^{\dagger}_{MN} = O. \tag{11}$$

According to the hypothesis of Theorem 2.1, we obtain

$$R(\tilde{Y}) \subseteq R(\tilde{A}) \Leftrightarrow R(Y) \subseteq R(A) \text{ and } R(\tilde{Z}) \subseteq R(\tilde{A}^*) \Leftrightarrow R(Z) \subseteq R(A^*). \tag{12}$$

Then, from definition of C and E in (4) and the formulas (8), (10), (11), (12), we have

$$Z^*E = O, \ C^*Y = O, \ C^*Y_p = I_s, \ Z_P{}^*E = I_s. \tag{13}$$

Combining (8), (10), (11), (12), (13) with the formulas (IV), (V) in Lemma 1.1 and the formulas (I), (II) in Lemma 1.2, we have

$$\begin{aligned} &WT \\ &= (A + (Y+Y_p)G(Z+Z_p)^*)(A^{\dagger}_{MN} - EZ^*A^{\dagger}_{MN} - A^{\dagger}_{MN}YC^* + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^*) \\ &= AA^{\dagger}_{MN} - YC^* - (Y+Y_p)GZ^*A^{\dagger}_{MN}YC^* + (Y+Y_p)G(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^* \\ &= AA^{\dagger}_{MN} - YC^* + (Y+Y_p)C^* \\ &= AA^{\dagger}_{MN} + Y_p(Y_p)^{\dagger}_{MS}. \end{aligned} \tag{14}$$

From (14), we obtain

$$MWT = MAA^{\dagger}_{MN} + MY_p(Y_p)^{\dagger}_{MS} = (MWT)^*. \tag{15}$$

Similar to the proof of (14) and (15), we have

$$\begin{aligned}
& TW \\
&= (A^{\dagger}_{MN} - EZ^*A^{\dagger}_{MN} - A^{\dagger}_{MN}YC^* + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^*)(A + (Y + Y_p)G(Z + Z_p)^*) \\
&= A^{\dagger}_{MN}A - EZ^* + A^{\dagger}_{MN}YG(Z + Z_p)^* - EZ^*A^{\dagger}_{MN}YG(Z + Z_p)^* \\
&- A^{\dagger}_{MN}YC^*(Y + Y_p)G(Z + Z_p)^* + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^*(Y + Y_p)G(Z + Z_p)^* \\
&= A^{\dagger}_{MN}A - EZ^* + E(Z + Z_p)^* \\
&= A^{\dagger}_{MN}A + (Z_p{}^*)^{\dagger}_{SN}Z_p{}^*,
\end{aligned} \tag{16}$$

and

$$NTW = NA^{\dagger}_{MN}A + N(Z_p{}^*)^{\dagger}_{SN}Z_p{}^* = (NTW)^*. \tag{17}$$

On the other hand, by (8), (10), (11), (12), (13) and the formulas (IV), (V) in Lemma 1.1 and the formulas (I), (II) in Lemma 1.2, we have

$$\begin{aligned}
& WTW \\
&= (AA^{\dagger}_{MN} + Y_p(Y_p)^{\dagger}_{MS})(A + (Y + Y_p)G(Z + Z_p)^*) \\
&= A + AA^{\dagger}_{MN}YG(Z + Z_p)^* + Y_p(Y_p)^{\dagger}_{MS}(Y + Y_p)G(Z + Z_p)^* \\
&= A + YG(Z + Z_p)^* + Y_pG(Z + Z_p)^* \\
&= W,
\end{aligned} \tag{18}$$

and

$$\begin{aligned}
& TWT \\
&= (A^{\dagger}_{MN} - EZ^*A^{\dagger}_{MN} - A^{\dagger}_{MN}YC^* + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^*)(AA^{\dagger}_{MN} + Y_p(Y_p)^{\dagger}_{MS}) \\
&= A^{\dagger}_{MN} - EZ^*A^{\dagger}_{MN} - A^{\dagger}_{MN}YC^*Y_p(Y_p)^{\dagger}_{MS} + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^*Y_p(Y_p)^{\dagger}_{MS} \\
&= A^{\dagger}_{MN} - EZ^*A^{\dagger}_{MN} - A^{\dagger}_{MN}YC^* + E(G^{-1} + Z^*A^{\dagger}_{MN}Y)C^* \\
&= T.
\end{aligned} \tag{19}$$

Combining (15), (17), (18) with (19), we obtain the results in Theorem 2.1. □

THEOREM 2.2. *Let $A \in C^{m\times n}$, $Y, Y_p \in C^{m\times s}$, $G \in C_s{}^{s\times s}$ and $Z, Z_p \in C^{n\times s}$. Assume $R(\tilde{Y}) \subseteq R(\tilde{A})$, $R(\tilde{Y_p}) \perp R(\tilde{A})$, $R(\tilde{Z}) \subseteq R(\tilde{A}^*)$ and $R(\tilde{Z_p}) \perp R(\tilde{A}^*)$. Then the following statements, (I) implies (II), conversely, (II) and (III) imply (1).*

(I) Y_p and Z_p are of full column rank,

(II) $r(A + (Y + Y_p)G(Z + Z_p)^*) = r(A) + r((Y + Y_p)G(Z + Z_p)^*)$,

(III) $r((Y+Y_p)G(Z+Z_p)^*)=s$.

Proof. (II) and (III)$\Rightarrow$(I): using the assumption $R(\tilde{Y})\subseteq R(\tilde{A})$, we have

$R(Y)\subseteq R(A)$ and $R(A+(Y+Y_p)G(Z+Z_p)^*)\subseteq R(A)+R(Y)+R(Y_p)=R(A)+R(Y_p)$.

Thus, if (II) and (III) in Theorem 2.2 hold, we have

$$r(A)+s=r(A+(Y+Y_p)G(Z+Z_p)^*)\leqslant r(A)+r(Y_p), \tag{20}$$

from (20) and the assumption, we conclude that

$$s\leqslant r(Y_p)\leqslant s,$$

that is Y_p (and similarly Z_p) is of full column rank.

(I)$\Rightarrow$(II): Suppose Y_p and Z_p are of full column rank, then from Theorem 2.1 and the formula (IV) in Lemma 1.1, we have

$$\begin{aligned} r(W) &= r(A+(Y+Y_p)G(Z+Z_p)^*) \\ &= r(WT) \\ &= r((A+(Y+Y_p)G(Z+Z_p)^*)T) \\ &= r(AA^{\dagger}_{MN}+Y_p(Y_p)^{\dagger}_{MS}) \\ &= r(M^{1/2}AN^{-1/2}N^{1/2}A^{\dagger}_{MN}M^{-1/2}+M^{1/2}Y_PS^{-1/2}S^{1/2}(Y_p)^{\dagger}_{MS}M^{-1/2}) \\ &= r(\tilde{A}(\tilde{A})^{\dagger}+\tilde{Y_p}(\tilde{Y_p})^{\dagger}). \end{aligned} \tag{21}$$

Since $R(\tilde{Y_p})\perp R(\tilde{A})$ and $r(Y_p)=s\geqslant r(G)\geqslant r((Y+Y_p)G(Z+Z_p)^*)$, we obtain

$$\begin{aligned} r(A+(Y+Y_p)G(Z+Z_p)^*) &= r(\tilde{A}(\tilde{A})^{\dagger}+\tilde{Y_p}(\tilde{Y_p})^{\dagger}) \\ &= r(\tilde{A})+r(\tilde{Y_p}) \\ &= r(A)+r(Y_p) \\ &\geqslant r(A)+r((Y+Y_p)G(Z+Z_p)^*). \end{aligned} \tag{22}$$

On the other hand, according to a trivially fact, we have

$$r(A+(Y+Y_p)G(Z+Z_p)^*)\leqslant r(A)+r((Y+Y_p)G(Z+Z_p)^*). \tag{23}$$

Combining (22) and (23), we conclude

$$r(A+(Y+Y_p)G(Z+Z_p)^*)=r(A)+r((Y+Y_p)G(Z+Z_p)^*), \tag{24}$$

that is the formula (II) in Theorem 2.2 holds. □

For positive integers s and m such that $s\leqslant m$, let $L_{m,s}$ denotes a matrix of size $m\times s$ with ones on the diagonal and zeros elsewhere. Similarly, let $L_{n,s}$ $(s\leqslant n)$ denotes a matrix of size $n\times s$ with ones on the diagonal and zeros elsewhere. To simplify notations and since n and m are fixed, we short $L_{m,s}$ to L_s and $L_{n,s}$ to $L_s{}'$.

Let $B \in C_s^{m\times n}$. Then from Lemma 1.3, we know that the matrices B has weighted singular value decompositions:

$$B = U_B D_B V_B^*,$$

where $U_B^* M U_B = I_m$, $V_B * N^{-1} V_B = I_n$ and

$$D_B = \begin{pmatrix} \Sigma_s & O \\ O & O \end{pmatrix}, \tag{25}$$

$\Sigma_s = diag(\mu_1, \mu_2, \cdots, \mu_s)$, and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_s > 0$, μ_i are the nonzero weighted singular values of B. In this case, the weighted Moore-Penrose inverse of B with respect to the Hermitian positive definite matrices $M \in C^{m\times m}$ and $N \in C^{n\times n}$ is given by

$$B_{MN}^{\dagger} = N^{-1} V_B \begin{pmatrix} \Sigma_s^{-1} & O \\ O & O \end{pmatrix} U_B^* M = N^{-1} V_B D_B^{\dagger} U_B^* M. \tag{26}$$

In the next of this section, we will consider the weighted Moore-Penrose inverse of $A+B$ under the condition $r(A+B) = r(A) + r(B)$, where $A \in C_r^{m\times n}$ and $B \in C_s^{m\times n}$ are given matrices.

THEOREM 2.3. *Let $A \in C_r^{m\times n}$, $B \in C_s^{m\times n}$, M, N and S be three Hermitian positive definite matrices of order m, n and s, respectively. If $r(A+B) = r(A) + r(B)$, then*

$$(A+B)_{MN}^{\dagger} = (I_n - F) A_{MN}^{\dagger} (I_m - J) + F B_{MN}^{\dagger} J, \tag{27}$$

where

$$F = (P_{R(B^\sharp),N(B)} P_{N(A),R(A^\sharp)})_{NN}^{\dagger},$$

$$J = (P_{N(A^\sharp),R(A)} P_{R(B),N(B^\sharp)})_{MM}^{\dagger}.$$

Proof. From the formula (25), we have

$$\begin{aligned} A+B &= M^{-1/2}(M^{1/2} A N^{-1/2} + M^{1/2} B N^{-1/2}) N^{1/2} \\ &= M^{-1/2}(M^{1/2} A N^{-1/2} + M^{1/2} U_B D_B V_B^* N^{-1/2}) N^{1/2} \\ &= M^{-1/2}(M^{1/2} A N^{-1/2} + M^{1/2} U_B L_s L_s^* D_B L_s' (L_s')^* V_B^* N^{-1/2}) N^{1/2}. \end{aligned} \tag{28}$$

Let

$$X = P_{R(\tilde{A})} M^{1/2} U_B L_s, \tag{29}$$

$$X_p = P_{R(\tilde{A})^{\perp}} M^{1/2} U_B L_s, \tag{30}$$

$$Q = P_{R(\tilde{A}^*)} N^{-1/2} V_B L_s{}', \tag{31}$$

$$Q_p = P_{R(\tilde{A}^*)^{\perp}} N^{-1/2} V_B L_s{}', \tag{32}$$

and

$$G_1 = L_s{}^* D_B L_s{}'. \tag{33}$$

Then, from (28–33) and the formula (III) in Lemma 1.2, we have

$$X + X_p = M^{1/2} U_B L_s \text{ and } Q + Q_p = N^{-1/2} V_B L_s{}',$$

and

$$A + B = A + (M^{-1/2}X + M^{-1/2}X_p) G_1 (N^{1/2}Q + N^{1/2}Q_p)^*, \tag{34}$$

where

$$G_1 = L_s{}^* D_B L_s{}' = \begin{pmatrix} \mu_1 & \cdots & O \\ \vdots & \ddots & \vdots \\ O & \cdots & \mu_s \end{pmatrix},$$

μ_i are the nonzero weighted singular values of B and G_1 is invertible.

Let $(\widetilde{M^{-1/2}X}) = M^{1/2}M^{-1/2}XS^{-1/2}$ and $(\widetilde{N^{1/2}Q}) = N^{-1/2}N^{1/2}QS^{1/2}$, then from (29) and (31), we have

$$\begin{aligned} R(\widetilde{M^{-1/2}X}) = R(XS^{-1/2}) \text{ and } R(\widetilde{M^{-1/2}X}) \subseteq R(\tilde{A}), \\ R(\widetilde{N^{1/2}Q}) = R(QS^{1/2}) \text{ and } R(\widetilde{N^{1/2}Q}) \subseteq R(\tilde{A}^*). \end{aligned} \tag{35}$$

Let $(\widetilde{M^{-1/2}X_p}) = M^{1/2}M^{-1/2}X_pS^{-1/2}$ and $(\widetilde{N^{1/2}Q_p}) = N^{-1/2}N^{1/2}Q_pS^{1/2}$, then from (30) and (32), we have

$$\begin{aligned} R(\widetilde{M^{-1/2}X_p}) = R(X_p) \text{ and } R(\widetilde{M^{-1/2}X_p}) \perp R(\tilde{A}), \\ R(\widetilde{N^{1/2}Q_p}) = R(Q_p) \text{ and } R(\widetilde{N^{1/2}Q_p}) \perp R(\tilde{A}^*). \end{aligned} \tag{36}$$

From the assumptions $r(B) = s$ and $r(A+B) = r(A) + r(B)$ in Theorem 2.3, and the results in Theorem 2.2, we know $M^{-1/2}X_p$ and $N^{1/2}Q_p$ are of full column rank.

Combining the formulas (29-36) with the results in Theorem 2.1, we have

$$\begin{aligned} (A+B)^{\dagger}_{MN} &= (A + (M^{-1/2}X + M^{-1/2}X_p) G_1 (N^{1/2}Q + N^{1/2}Q_p)^*)^{\dagger}_{MN} \\ &= A^{\dagger}_{MN} - K(N^{1/2}Q)^* A^{\dagger}_{MN} - A^{\dagger}_{MN}(M^{-1/2}X)H^* \\ &\quad + K(G_1{}^{-1} + (N^{1/2}Q)^* A^{\dagger}_{MN}(M^{-1/2}X))H^*, \end{aligned} \tag{37}$$

where

$$H = M(M^{-1/2}X_p)({X_p}^*M^{-1/2}MM^{-1/2}X_p)^{-1} = M^{1/2}X_p({X_p}^*X_p)^{-1},$$
$$K = N^{-1}(N^{1/2}Q_p)({Q_p}^*N^{1/2}N^{-1}N^{1/2}Q_p)^{-1} = N^{-1/2}Q_p({Q_p}^*Q_p)^{-1}.$$

According to (25), (26), (33), (34) and the formula (III) in Lemma 1.2, we have

$${G_1}^{-1} = (L_s')^*{D_B}^{\dagger}L_s \text{ and } {D_B}^{\dagger} = {V_B}^*B^{\dagger}_{MN}U_B = {V_B}^*N^{-1/2}(M^{1/2}BN^{-1/2})^{\dagger}M^{1/2}U_B.$$

Thus

$$\begin{aligned} K{G_1}^{-1}H^* &= K(L_s')^*{D_B}^{\dagger}L_sH^* \\ &= K(L_s')^*{V_B}^*N^{-1/2}(M^{1/2}BN^{-1/2})^{\dagger}M^{1/2}U_BL_sH^* \\ &= K(N^{1/2}Q+N^{1/2}Q_p)^*B^{\dagger}_{MN}(M^{-1/2}X+M^{-1/2}X_p)H^*. \end{aligned} \tag{38}$$

The last equation holds since

$$X+X_p = M^{1/2}U_BL_s \text{ and } Q+Q_p = N^{-1/2}V_BL_s'.$$

By (37) and (38), we have

$$\begin{aligned} (A+B)^{\dagger}_{MN} &= (I_n - KQ^*N^{1/2})A^{\dagger}_{MN}(I_m - M^{-1/2}XH^*) \\ &\quad + (KQ^*N^{1/2} + K{Q_p}^*N^{1/2})B^{\dagger}_{MN}(M^{-1/2}XH^* + M^{-1/2}X_pH^*). \end{aligned} \tag{39}$$

This is the basic form of $(A+B)^{\dagger}_{MN}$ that we seek. Next we proceed to compute $KQ^*N^{1/2}$, $M^{-1/2}XH^*$, $K{Q_p}^*N^{1/2}$ and $M^{-1/2}X_pH^*$.

Let $\tilde{B} = M^{1/2}BN^{-1/2}$. Then from (29–32) and (37), we have

$$\begin{aligned} KQ^*N^{1/2} &= N^{-1/2}Q_p({Q_p}^*Q_p)^{-1}Q^*N^{1/2} = N^{-1/2}({Q_p}^*)^{\dagger}Q^*N^{1/2} \\ &= N^{-1/2}((L_s')^*{V_B}^*N^{-1/2}P_{R(\tilde{A}^*)^{\perp}})^{\dagger}(L_s')^*{V_B}^*N^{-1/2}P_{R(\tilde{A}^*)}N^{1/2} \\ &= N^{-1/2}(N^{-1/2}V_BL_s'(L_s')^*{V_B}^*N^{-1/2}P_{R(\tilde{A}^*)^{\perp}})^{\dagger}P_{R(\tilde{A}^*)}N^{1/2} \\ &= N^{-1/2}(N^{-1/2}V_B{D_B}^{\dagger}D_B{V_B}^*N^{-1/2}P_{R(\tilde{A}^*)^{\perp}})^{\dagger}P_{R(\tilde{A}^*)}N^{1/2} \\ &= N^{-1/2}(N^{-1/2}V_B{D_B}^{\dagger}{U_B}^*M^{1/2}M^{1/2}U_BD_B{V_B}^*N^{-1/2}P_{R(\tilde{A}^*)^{\perp}})^{\dagger}P_{R(\tilde{A}^*)}N^{1/2} \\ &= N^{-1/2}((M^{1/2}U_BD_B{V_B}^*N^{-1/2})^{\dagger}M^{1/2}U_BD_B{V_B}^*N^{-1/2}P_{R(\tilde{A}^*)^{\perp}})^{\dagger}P_{R(\tilde{A}^*)}N^{1/2} \\ &= N^{-1/2}(P_{R(\tilde{B}^*)}P_{R(\tilde{A}^*)^{\perp}})^{\dagger}P_{R(\tilde{A}^*)}N^{1/2} \\ &= (P_{R(B^{\sharp}),N(B)}P_{N(A),R(A^{\sharp})})^{\dagger}_{NN}P_{R(A^{\sharp}),N(A)}, \end{aligned} \tag{40}$$

and

$$\begin{aligned}
M^{-1/2}XH^* &= M^{-1/2}P_{R(\tilde{A})}M^{1/2}U_BL_s({X_p}^*X_p)^{-1}{X_p}^*M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}M^{1/2}U_BL_s{X_p}^{\dagger}M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}M^{1/2}U_BL_s(P_{R(\tilde{A})^{\perp}}M^{1/2}U_BL_s)^{\dagger}M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}(P_{R(\tilde{A})^{\perp}}M^{1/2}U_BL_s{L_s}^*{U_B}^*M^{1/2})^{\dagger}M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}(P_{R(\tilde{A})^{\perp}}M^{1/2}U_BD_B{D_B}^{\dagger}{U_B}^*M^{1/2})^{\dagger}M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}(P_{R(\tilde{A})^{\perp}}M^{1/2}U_BD_B{V_B}^*N^{-1/2}N^{-1/2}V_B{D_B}^{\dagger}{U_B}^*M^{1/2})^{\dagger}M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}(P_{R(\tilde{A})^{\perp}}\tilde{B}\tilde{B}^{\dagger})^{\dagger}M^{1/2} \\
&= M^{-1/2}P_{R(\tilde{A})}(P_{R(\tilde{A})^{\perp}}P_{R(\tilde{B})})^{\dagger}M^{1/2} \\
&= P_{R(A),N(A^{\sharp})}(P_{N(A^{\sharp}),R(A)}P_{R(B),N(B^{\sharp})})^{\dagger}_{MM}.
\end{aligned} \tag{41}$$

Similarly, we get

$$\begin{aligned}
K{Q_p}^*N^{1/2} &= (P_{R(B^{\sharp}),N(B)}P_{N(A),R(A^{\sharp})})^{\dagger}_{NN}P_{N(A),R(A^{\sharp})}, \\
M^{-1/2}X_pH^* &= P_{N(A^{\sharp}),R(A)}(P_{N(A^{\sharp}),R(A)}P_{R(B),N(B^{\sharp})})^{\dagger}_{MM}.
\end{aligned} \tag{42}$$

From (40-42) and the formula (III) in Lemma 1.2, we have

$$\begin{aligned}
KQ^*N^{1/2} + K{Q_p}^*N^{1/2} &= (P_{R(B^{\sharp}),N(B)}P_{N(A),R(A^{\sharp})})^{\dagger}_{NN} = F, \\
M^{-1/2}XH^* + M^{-1/2}X_pH^* &= (P_{N(A^{\sharp}),R(A)}P_{R(B),N(B^{\sharp})})^{\dagger}_{MM} = J.
\end{aligned} \tag{43}$$

Substituting (40), (41), (42) and (43) into (39) yield

$$(A+B)^{\dagger}_{MN} = (I_n - FP_{R(A^{\sharp}),N(A)})A^{\dagger}_{MN}(I_m - P_{R(A),N(A^{\sharp})}J) + FB^{\dagger}_{MN}J. \tag{44}$$

From the formulas (IV), (V) in Lemma 1.1 and the formulas (I), (II) in Lemma 1.2, we have

$$P_{R(A^{\sharp}),N(A)}A^{\dagger}_{MN} = A^{\dagger}_{MN} \ \text{ and } \ A^{\dagger}_{MN}P_{R(A),N(A^{\sharp})} = A^{\dagger}_{MN}, \tag{45}$$

Combining (44) with (45), we have

$$(A+B)^{\dagger}_{MN} = (I_n - F)A^{\dagger}_{MN}(I_m - J) + FB^{\dagger}_{MN}J. \tag{46}$$

The proof of Theorem 2.3 is complete. □

COROLLARY 2.4. *Let $A \in {C_r}^{m\times n}$, $B \in {C_s}^{m\times n}$. If $r(A+B) = r(A) + r(B)$, then*

$$(A+B)^{\dagger} = (I_n - S')A^{\dagger}(I_m - T') + S'B^{\dagger}T', \tag{47}$$

where

$$S' = (P_{R(B^*)}P_{N(A)})^{\dagger} \quad \text{and} \quad T' = (P_{N(A^*)}P_{R(B)})^{\dagger}. \tag{48}$$

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