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ON DERIVATIONS AND JORDAN DERIVATIONS THROUGH ZERO PRODUCTS

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Abstract. Let $\mathscr{A}$ be a unital complex (Banach) algebra and $\mathscr{M}$ be a unital (Banach) $\mathscr{A}$-bimodule. The main results describe (continuous) derivations or Jordan derivations $D:\mathscr{A}\to\mathscr{M}$ through the action on zero products, under certain conditions on $\mathscr{A}$ and $\mathscr{M}$. The proof is based on the consideration of a (continuous) bilinear map satisfying a related condition.

1. Introduction

Throughout this paper all algebras and vector spaces will be over the complex field $\mathbb{C}$ and all algebras are associative with unity, unless indicated otherwise. All modules are unital. Let $\mathscr{A}$ be an algebra and $\mathscr{M}$ be an $\mathscr{A}$-bimodule. Recall that a linear map $D:\mathscr{A}\to\mathscr{M}$ is said to be a *Jordan derivation* (or *generalized Jordan derivation*) if $D(a\circ b)=D(a)\bullet b+a\bullet D(b)$ (or $D(a\circ b)=D(a)\bullet b+a\bullet D(b)-aD(1)b-bD(1)a$) for all $a,b\in\mathscr{A}$.

Here and subsequently, $'\circ'$ denotes the Jordan product $a\circ b=ab+ba$ on $\mathscr{A}$ and $'\bullet'$ denotes the Jordan product on $\mathscr{M}$:

$$a\bullet m=m\bullet a=am+ma,\quad a\in\mathscr{A},m\in\mathscr{M}.$$

D is called a *derivation* (or *generalized derivation*) if $D(ab)=D(a)b+aD(b)$ (or $D(ab)=D(a)b+aD(b)-aD(1)b$) for all $a,b\in\mathscr{A}$. Clearly, each (generalized) derivation is a (generalized) Jordan derivation. The converse is, in general, not true.

The question of characterizing derivations or Jordan derivations on algebras through the action on zero products has attracted the attention of many authors over the last few years. We refer the reader to [2, 8] for a full account of the topic and a list of references.

In this paper, we consider the subsequent conditions on a linear map D from an algebra $\mathscr{A}$ into an $\mathscr{A}$-bimodule $\mathscr{M}$:

(d1) $ab=0\Rightarrow aD(b)+D(a)b=0.$

(d2) $ab=ba=0\Rightarrow aD(b)+D(a)b=0.$

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(d3) $a \circ b = 0 \Rightarrow a \bullet D(b) + D(a) \bullet b = 0$.

(d4) $ab = ba = 0 \Rightarrow a \bullet D(b) + D(a) \bullet b = 0$.

Our purpose is to investigate whether these conditions characterizes derivations or Jordan derivations.

The above questions and the question of characterizing linear maps that preserve zero products, Jordan product, etc. on algebras can be sometimes effectively solved by considering bilinear maps that preserve certain zero product properties (for instance, see [1, 2, 3, 5, 7]). Motivated by these reasons Brešar et al. [4] introduced the concept of zero product (Jordan product) determined algebras, which can be used to study the linear maps preserving zero product (Jordan product) and derivable (Jordan derivable) maps at zero point.

In this context one is usually involved with the following condition on a bilinear map $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$, where $\mathscr{X}$ is an arbitrary linear space:

$$a, b \in \mathscr{A}, \quad ab = ba = 0 \Rightarrow \phi(a,b) = 0. \qquad (G)$$

A way to unify and generalize both of the concepts of zero product determined and zero Jordan product determined consists in considering bilinear maps satisfying (G).

The paper is organized as follows. In section 2 we introduce the notation and terminology, and then a class of (Banach) $\mathscr{A}$-bimodules satisfying a condition $\mathbb{M}$ ($\mathbb{M}'$). Also we give several classes of bimodules which satisfy this condition. Section 3 is concerned with bilinear maps. We will consider the condition (G) for bilinear maps in this section. Also we present some results concerning the notions of zero (Jordan) product determined algebras. In section 4 we study the linear maps satisfying (d1)–(d4) for modules with property $\mathbb{M}$ ($\mathbb{M}'$), by using the results of section 3.

2. Preliminaries

In this section we introduce the notation and terminology, and then a special class of (Banach) bimodules.

Let $\mathscr{A}$ be an algebra, then $\mathfrak{I}(\mathscr{A})$ denotes the set of all linear combinations of idempotents in $\mathscr{A}$. Let $\mathscr{M}$ be an $\mathscr{A}$-bimodule. We say that $\mathscr{M}$ satisfies $\mathbb{M}$, if there is an ideal $\mathscr{J}$ in $\mathscr{A}$ such that $\mathscr{J} \subseteq \mathfrak{I}(\mathscr{A})$ and

$$\{m \in \mathscr{M} \mid xmx = 0 \; for all \; x \in \mathscr{J}\} = \{0\}. \qquad (2.1)$$

If $\mathscr{A}$ is a Banach algebra, $\mathscr{M}$ is a Banach $\mathscr{A}$-bimodule and there is an ideal $\mathscr{J}$ in $\mathscr{A}$ such that $\mathscr{J} \subseteq \overline{\mathfrak{I}(\mathscr{A})}$ and (2.1) holds, then we say that $\mathscr{M}$ satisfies $\mathbb{M}'$.

Note that if (Banach) $\mathscr{A}$-bimodule $\mathscr{M}$ satisfies $\mathbb{M}$ ($\mathbb{M}'$), then we have

$$\{m \in \mathscr{M} \mid xm = mx = 0 \; for all \; x \in \mathscr{J}\} = \{0\}.$$

Now we introduce the class of (Banach) bimodules with the property $\mathbb{M}$ ($\mathbb{M}'$).

PROPOSITION 2.1. *Let $\mathscr{A}$ be an (Banach) algebra with $\mathscr{A} = \mathfrak{I}(\mathscr{A})$ ($\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$). Then every (Banach) $\mathscr{A}$-bimodule $\mathscr{M}$ satisfies $\mathbb{M}$ ($\mathbb{M}'$)*

Proof. Let $m \in \mathscr{M}$ and $ama = 0$ for all $a \in \mathscr{A}$. Since $\mathscr{A}$ is unital, it follows that $m = 0$. Now if we consider $\mathscr{A}$ as an ideal, then by hypothesis any (Banach) $\mathscr{A}$-bimodule $\mathscr{M}$ satisfies $\mathbb{M}$ ($\mathbb{M}'$). $\square$

If $\mathscr{A}$ is a W^*-algebra, then the linear span of projections is norm dense in $\mathscr{A}$, so $\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$.

Let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ denotes the algebra of all bounded linear operators on $\mathscr{H}$. Then from [9, Lemma 3.2] and [11, Theorem 1], we have $B(\mathscr{H}) = \mathfrak{I}(B(\mathscr{H}))$. Recall that a W^*-algebra is called *properly infinite* if it contains no nonzero finite central projection. Since every element in a properly infinite W^*-algebra $\mathscr{A}$ is a sum of at most five idempotents [11, Theorem 4], it follows that $\mathscr{A} = \mathfrak{I}(\mathscr{A})$.

Let $\mathscr{A}$ be an algebra. Recall that a non-zero ideal $\mathscr{I}$ of $\mathscr{A}$ is called *essential* if it has non-zero intersection with every non-zero ideal of $\mathscr{A}$. The *socle* of $\mathscr{A}$, $Soc(A)$, is the sum of all minimal left ideals of $\mathscr{A}$, or minimal right ideals of $\mathscr{A}$, if they exists; otherwise it is zero. From Remark 2 of [6] we have the next proposition.

PROPOSITION 2.2. *Let $\mathscr{A}$ be a semisimple Banach algebra with non-zero socle. If $Soc(\mathscr{A})$ is essential, then $\mathscr{A}$ as an $\mathscr{A}$-bimodule satisfies* $\mathbb{M}$.

Let $\mathscr{X}$ be a Banach space. We denote by $\mathscr{B}(\mathscr{X})$ the algebra of all bounded linear operators on $\mathscr{X}$, and $\mathscr{F}(\mathscr{X})$ denotes the algebra of all finite rank operators in $\mathscr{B}(\mathscr{X})$. Recall that a subalgebra $\mathscr{A}$ of the algebra $\mathscr{B}(\mathscr{X})$ is called *standard* if $\mathscr{A}$ contains the identity and the ideal $\mathscr{F}(\mathscr{X})$. If $\mathscr{A}$ is a standard operator algebra on a Banach space $\mathscr{X}$, then $\mathscr{A}$ is primitive and $Soc(\mathscr{A}) = \mathscr{F}(\mathscr{X})$ is essential. Thus, Proposition 2.2 applied for standard operator algebras.

A *nest* $\mathscr{N}$ on a Banach space $\mathscr{X}$ is a chain of closed (under norm topology) subspaces of $\mathscr{X}$ which is closed under the formation of arbitrary intersection and closed linear span (denoted by $\vee$), and which includes $\{0\}$ and $\mathscr{X}$. The *nest algebra* associated to the nest $\mathscr{N}$, denoted by $Alg\mathscr{N}$, is the weak closed operator algebra of the form

$$Alg\mathscr{N} = \{T \in \mathscr{B}(\mathscr{X}) \,|\, T(N) \subseteq N \, for \, all \, N \in \mathscr{N}\}.$$

When $\mathscr{N} \neq \{\{0\}, X\}$, we say that $\mathscr{N}$ is non-trivial. It is clear that if $\mathscr{N}$ is trivial, then $Alg\mathscr{N} = \mathscr{B}(\mathscr{X})$. Denote $Alg_{\mathscr{F}}\mathscr{N} := Alg\mathscr{N} \cap \mathscr{F}(\mathscr{X})$, the set of all finite rank operators in $Alg\mathscr{N}$ and for $N \in \mathscr{N}$, let $N_- = \vee\{M \in \mathscr{N} \,|\, M \subset N\}$.

PROPOSITION 2.3. *Let $\mathscr{N}$ be a nest on a Banach space $\mathscr{X}$. If $N \in \mathscr{N}$ is complemented in $\mathscr{X}$ whenever $N_- = N$, then $\mathscr{B}(\mathscr{X})$ as a $Alg\mathscr{N}$-bimodule satisfies* $\mathbb{M}$.

Proof. $Alg_{\mathscr{F}}\mathscr{N}$ is an ideal of $Alg\mathscr{N}$ and from [9], it is contained in the $\mathfrak{I}(Alg\mathscr{N})$. Suppose that $T \in \mathscr{B}(\mathscr{X})$ and $FTF = 0$ for each $F \in Alg_{\mathscr{F}}\mathscr{N}$. So we have $(F_1 + F_2)T(F_1 + F_2) = 0$ and hence $F_1TF_2 + F_2TF_1 = 0$, for any $F_1, F_2 \in Alg_{\mathscr{F}}\mathscr{N}$. By [12] we have $\overline{Alg_{\mathscr{F}}\mathscr{N}}^{SOT} = Alg\mathscr{N}$. Therefore there is a net $(F_\gamma)_{\gamma\in\Gamma}$ in $Alg_{\mathscr{F}}\mathscr{N}$ converges to the identity operator I with respect to the strong operator topology. So $FTF_\gamma + F_\gamma TF = 0$ for each $\gamma \in \Gamma$ and $F \in Alg_{\mathscr{F}}\mathscr{N}$. Thus $FT + TF = 0$ for all

$F \in Alg_{\mathscr{F}}\mathscr{N}$ and hence $F_\gamma T + TF_\gamma = 0$ for all $\gamma \in \Gamma$. So $T = 0$ and $\mathscr{B}(\mathscr{X})$ satisfies $\mathbb{M}$. □

It is obvious that the nests on Hilbert spaces, finite nests and the nests having order-type $\omega+1$ or $1+\omega^*$, where ω is the order-type of the natural numbers, satisfy the condition in Proposition 2.3 automatically.

3. Bilinear maps vanishing on zero products

In this section we concern with bilinear maps on algebras. From this point up to the last section $\mathscr{A}$ is an algebra.

The algebra $\mathscr{A}$ is called *zero product determined* if for every linear space $\mathscr{X}$ and every bilinear map $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$, the following holds. If $\phi(a,b) = 0$ whenever $ab = 0$, then there exists a linear map $T : \mathscr{A} \to \mathscr{X}$ such that $\phi(a,b) = T(ab)$ for all $a,b \in \mathscr{A}$. If the ordinary product is replaced by the Jordan product, then it is said that $\mathscr{A}$ is *zero Jordan product determined*.

We will show that any unital Banach algebra spanned by idempotents is zero product determined and zero Jordan product determined.

THEOREM 3.1. *Let $\mathscr{X}$ be a linear space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a bilinear map satisfying*

$$a,b \in \mathscr{A}, \quad ab = 0 \Rightarrow \phi(a,b) = 0.$$

Then

$$\phi(a,x) = \phi(ax,1) \quad \textit{and} \quad \phi(x,a) = \phi(1,xa)$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. Indeed, if $\mathscr{A} = \mathfrak{J}(\mathscr{A})$, then $\mathscr{A}$ is zero product determined.

Proof. Let $a \in \mathscr{A}$. For arbitrary idempotent $p \in \mathscr{A}$, let $q = 1-p$. We have

$$\phi(a,p) = \phi(ap,p) + \phi(aq,p) = \phi(ap,p),$$

since $(aq)p = 0$. On the other hand we have

$$\phi(ap,1) = \phi(ap,p) + \phi(ap,q) = \phi(ap,p).$$

By comparing the two expressions for $\phi(ap,p)$, we arrive at $\phi(a,p) = \phi(ap,1)$. Since every $x \in \mathfrak{J}(\mathscr{A})$ is a linear combination of idempotent elements in $\mathscr{A}$, we get

$$\phi(a,x) = \phi(ax,1)$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. Similarly, we get $\phi(x,a) = \phi(1,xa)$ for all $a \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$.

Now suppose that $\mathscr{A} = \mathfrak{J}(\mathscr{A})$. Let $\mathscr{X}$ be a linear space, and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a bilinear map such that for all $a,b \in \mathscr{A}$, $ab = 0$ implies $\phi(a,b) = 0$. From above identity we have

$$\phi(a,b) = \phi(ab,1)$$

for all $a,b \in \mathscr{A}$, since $\mathscr{A} = \mathfrak{I}(\mathscr{A})$. If we define the linear map $T : \mathscr{A} \to \mathscr{X}$ by $T(a) = \phi(a,1)$, then T satisfies all the requirements in the definition of zero product determined algebras. Thus $\mathscr{A}$ is a zero product determined algebra. □

PROPOSITION 3.2. *Let $\mathscr{A}$ be a Banach algebra, let $\mathscr{X}$ be a Banach space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a continuous bilinear map satisfying*

$$a,b \in \mathscr{A}, \quad ab = 0 \Rightarrow \phi(a,b) = 0.$$

Then

$$\phi(a,x) = \phi(ax,1) \quad \text{and} \quad \phi(x,a) = \phi(1,xa)$$

for all $a \in \mathscr{A}$ and $x \in \overline{\mathfrak{I}(\mathscr{A})}$. If $\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$, then there exists a continuous linear map $T : \mathscr{A} \to \mathscr{X}$ such that $\phi(a,b) = T(ab)$ for all $a,b \in \mathscr{A}$.

Proof. A similar proof as that of Theorem 3.1 and the fact that ϕ is continuous, shows that $\phi(a,x) = \phi(ax,1)$ for all $a \in \mathscr{A}$ and $x \in \overline{\mathfrak{I}(\mathscr{A})}$. If $\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$, we find

$$\phi(a,b) = \phi(ab,1)$$

for all $a,b \in \mathscr{A}$. Now we define the linear mapping $T : \mathscr{A} \to \mathscr{X}$ by $T(a) = \phi(a,1)$. So we have $\phi(a,b) = T(ab)$ for all $a,b \in \mathscr{A}$, and since ϕ is continuous, T is continuous. □

THEOREM 3.3. *Let $\mathscr{X}$ be a linear space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a bilinear map satisfying*

$$a,b \in \mathscr{A}, \quad a \circ b = 0 \Rightarrow \phi(a,b) = 0.$$

Then

$$\phi(a,x) = \frac{1}{2}\phi(ax,1) + \frac{1}{2}\phi(xa,1)$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{I}(\mathscr{A})$. Indeed, if $\mathscr{A} = \mathfrak{I}(\mathscr{A})$, then $\mathscr{A}$ is zero Jordan product determined.

Proof. Let $a,p \in \mathscr{A}$ with $p^2 = p$ and let $q = 1-p$. We have $(p-q) \circ paq = 0$ and $(p-q) \circ qap = 0$. So $\phi(paq, p-q) = 0$ and $\phi(qap, p-q) = 0$. Hence $\phi(paq,p) = \phi(paq,q)$ and $\phi(qap,p) = \phi(qap,q)$. Therefore

$$\phi(paq,p) = \frac{1}{2}\phi(paq,1); \text{ and}$$
$$\phi(qap,p) = \frac{1}{2}\phi(qap,1).$$

By these identities and the fact that $pap \circ q = 0$ and $qaq \circ p = 0$, we have

$$\begin{aligned}
&\frac{1}{2}\phi(ap,1) + \frac{1}{2}\phi(pa,1) \\
&= \frac{1}{2}\phi(pap,p) + \frac{1}{2}\phi(qap,p) + \frac{1}{2}\phi(qap,q) + \frac{1}{2}\phi(pap,p) + \frac{1}{2}\phi(paq,p) + \frac{1}{2}\phi(paq,q) \\
&= \phi(pap,p) + \frac{1}{2}\phi(qap,1) + \frac{1}{2}\phi(paq,1) = \phi(pap+qap+paq+qaq,p) = \phi(a,p).
\end{aligned}$$

Since every $x \in \mathfrak{I}(\mathscr{A})$ is a linear combination of idempotent elements in $\mathscr{A}$, we get

$$\phi(a,x) = \frac{1}{2}\phi(ax,1) + \frac{1}{2}\phi(xa,1)$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{I}(\mathscr{A})$.

Now let $\mathscr{A} = \mathfrak{I}(\mathscr{A})$, $\mathscr{X}$ be a linear space, and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a bilinear map such that for all $a,b \in \mathscr{A}$, $a \circ b = 0$ implies $\phi(a,b) = 0$. If we define the linear map $T : \mathscr{A} \to \mathscr{X}$ by $T(a) = \frac{1}{2}\phi(a,1)$, then T satisfies all the requirements in the definition of zero Jordan product determined algebras. Thus $\mathscr{A}$ is a zero Jordan product determined algebra. □

PROPOSITION 3.4. *Let $\mathscr{A}$ be a Banach algebra, let $\mathscr{X}$ be a Banach space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a continuous bilinear map satisfying*

$$a,b \in \mathscr{A}, \quad a \circ b = 0 \Rightarrow \phi(a,b) = 0.$$

Then

$$\phi(a,x) = \frac{1}{2}\phi(ax,1) + \frac{1}{2}\phi(xa,1)$$

for all $a \in \mathscr{A}$ and $x \in \overline{\mathfrak{I}(\mathscr{A})}$. If $\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$, then there exists a continuous linear map $T : \mathscr{A} \to \mathscr{X}$ such that $\phi(a,b) = T(a \circ b)$ for all $a,b \in \mathscr{A}$.

Proof. By using similar arguments as that in the proof of Theorem 3.3 and the fact that ϕ is continuous, it follows that $\phi(a,x) = \frac{1}{2}\phi(ax,1) + \frac{1}{2}\phi(xa,1)$ for all $a \in \mathscr{A}$ and $x \in \overline{\mathfrak{I}(\mathscr{A})}$. If $\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$, we get

$$\phi(a,b) = \frac{1}{2}\phi(ab,1) + \frac{1}{2}\phi(ba,1)$$

for all $a,b \in \mathscr{A}$. Define $T : \mathscr{A} \to \mathscr{X}$ by $T(a) = \frac{1}{2}\phi(a,1)$. Then T is continuous and $\phi(a,b) = T(a \circ b)$ for all $a,b \in \mathscr{A}$. □

We continue by studying the condition (G).

THEOREM 3.5. *Let $\mathscr{X}$ be a linear space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a bilinear map satisfying (G). Then*

$$\phi(a,x) + \phi(x,a) = \phi(ax,1) + \phi(1,xa) \quad \textit{and} \quad \phi(x,1) = \phi(1,x)$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{I}(\mathscr{A})$. Indeed, if $\mathscr{A} = \mathfrak{I}(\mathscr{A})$, then

$$\phi(a,b) + \phi(b,a) = \phi(ab,1) + \phi(1,ba) \quad \textit{and} \quad \phi(a,1) = \phi(1,a)$$

for all $a,b \in \mathscr{A}$.

Proof. Let $a, p \in \mathscr{A}$ with $p^2 = p$ and let $q = 1 - p$. Since $pq = qp = 0$, we see that

$$\phi(p,q) = \phi(p,1) - \phi(p,p) = 0 \quad \textit{and} \quad \phi(q,p) = \phi(1,p) - \phi(p,p) = 0.$$

So $\phi(p,1) = \phi(1,p)$. By linearity, it shows

$$\phi(x,1) = \phi(1,x)$$

for all $x \in \mathfrak{I}(\mathscr{A})$. Now we have $(p+paq)(q-paq) = (q-paq)(p+paq) = 0$ and $(p+qap)(q-qap) = (q-qap)(p+qap) = 0$. So $\phi(p+paq, q-paq) = 0$ and $\phi(p+qap, q-qap) = 0$. Hence

$$\phi(paq,p) = \phi(q,paq) \quad \textit{and} \quad \phi(p,qap) = \phi(qap,q).$$

By these identities and the fact that $(pap)q = q(pap) = 0$ and $(qaq)p = p(qaq) = 0$, we have

$$\begin{aligned}\phi(a,p) + \phi(p,a) &= \phi(pap,p) + \phi(paq,p) + \phi(qap,p) \\ &\quad + \phi(p,pap) + \phi(p,paq) + \phi(p,qap) \\ &= \phi(pap,p) + \phi(q,paq) + \phi(qap,p) \\ &\quad + \phi(p,pap) + \phi(p,paq) + \phi(qap,q) \\ &= \phi(ap,1) + \phi(1,pa)\end{aligned}$$

Since every $x \in \mathfrak{I}(\mathscr{A})$ is a linear combination of idempotent elements in $\mathscr{A}$, we get

$$\phi(a,x) + \phi(x,a) = \phi(ax,1) + \phi(1,xa)$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{I}(\mathscr{A})$. $\square$

COROLLARY 3.6. *Let $\mathscr{A}$ be a Banach algebra, let $\mathscr{X}$ be a Banach space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a continuous bilinear map satisfying (G). Then*

$$\phi(a,x) + \phi(x,a) = \phi(ax,1) + \phi(1,xa) \quad \textit{and} \quad \phi(x,1) = \phi(1,x)$$

for all $a \in \mathscr{A}$ and $x \in \overline{\mathfrak{I}(\mathscr{A})}$. Indeed, if $\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$, then

$$\phi(a,b) + \phi(b,a) = \phi(ab,1) + \phi(1,ba) \quad \textit{and} \quad \phi(a,1) = \phi(1,a)$$

for all $a, b \in \mathscr{A}$.

Recall that a bilinear map $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$, where $\mathscr{X}$ is a linear space, is called *symmetric* if $\phi(a,b) = \phi(b,a)$ holds for all $a, b \in \mathscr{A}$.

PROPOSITION 3.7. *Let $\mathscr{A} = \mathfrak{I}(\mathscr{A})$ ($\mathscr{A} = \overline{\mathfrak{I}(\mathscr{A})}$), let $\mathscr{X}$ be a linear (Banach) space and let $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{X}$ be a (continuous) bilinear map. The following conditions are equivalent:*

(i) ϕ *is a symmetric bilinear map satisfying the condition*

$$a,b \in \mathscr{A}, \quad ab = ba = 0 \Rightarrow \phi(a,b) = 0;$$

(ii) ϕ *satisfies*

$$a,b \in \mathscr{A}, \quad a \circ b = 0 \Rightarrow \phi(a,b) = 0;$$

(iii) there exists a (continuous) linear map $T : \mathscr{A} \to \mathscr{X}$ *such that* $\phi(a,b) = T(a \circ b)$ *for all* $a,b \in \mathscr{A}$.

Proof. $(iii) \Rightarrow (i)$ and $(iii) \Rightarrow (ii)$ are clear. $(ii) \Rightarrow (iii)$ obtains from Theorem 3.3 (Proposition 3.4). We show that $(i) \Rightarrow (ii)$ holds.

By Theorem 3.5 (Corollary 3.6), we have

$$\phi(a,b) + \phi(b,a) = \phi(ab,1) + \phi(1,ba)$$

for all $a,b \in \mathscr{A}$. So $\phi(a,b) = \frac{1}{2}\phi(ab+ba,1)$, since ϕ is symmetric. If we define the linear mapping $T : \mathscr{A} \to \mathscr{X}$ by $T(a) = \frac{1}{2}\phi(a,1)$, then $\phi(a,b) = T(a \circ b)$ for all $a,b \in \mathscr{A}$. (It is obvious if ϕ is continuous, then T is continuous). □

4. Characterizing derivations and Jordan derivations through zero products

In this section for $\mathscr{M}$ bimodule over $\mathscr{A}$, and $D : \mathscr{A} \to \mathscr{M}$ a linear map, we will consider the following conditions:

(d1) $ab = 0 \Rightarrow aD(b) + D(a)b = 0.$

(d2) $ab = ba = 0 \Rightarrow aD(b) + D(a)b = 0.$

(d3) $a \circ b = 0 \Rightarrow a \bullet D(b) + D(a) \bullet b = 0.$

(d4) $ab = ba = 0 \Rightarrow a \bullet D(b) + D(a) \bullet b = 0.$

THEOREM 4.1. *Let* $\mathscr{A}$ *be an (Banach) algebra,* $\mathscr{M}$ *be an (Banach)* $\mathscr{A}$*-bimodule and* $\mathscr{J}$ *be an ideal of* $\mathscr{A}$ *such that* $\mathscr{J} \subseteq \mathfrak{I}(\mathscr{A})$ *(* $\mathscr{J} \subseteq \overline{\mathfrak{I}(\mathscr{A})}$ *) and*

$$\{m \in \mathscr{M} \mid xm = mx = 0 \ for\, all\ x \in \mathscr{J}\} = \{0\}.$$

Assume that $D : \mathscr{A} \to \mathscr{M}$ *is a (continuous) linear map satisfying* (d1). *Then* D *is a generalized derivation and* $aD(1) = D(1)a$ *for all* $a \in \mathscr{A}$.

Proof. Define a bilinear map $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{M}$ by $\phi(a,b) = aD(b) + D(a)b$. Then $\phi(a,b) = 0$ for all $a,b \in \mathscr{A}$ with $ab = 0$. By applying Theorem 3.1, we obtain $\phi(a,x) = \phi(ax,1)$ for all $a \in \mathscr{A}$ and $x \in \mathfrak{I}(\mathscr{A})$. So

$$aD(x) + D(a)x = axD(1) + D(ax), \tag{4.1}$$

for all $a \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. Letting $a = 1$ in (4.1), we arrive at $D(1)x = xD(1)$, for all $x \in \mathscr{J}$. So we have $aD(1)x = axD(1) = D(1)ax$ and $xaD(1) = D(1)xa = xD(1)a$, for all $a \in \mathscr{A}$ and $x \in \mathscr{J}$. Hence $(aD(1) - D(1)a)\mathscr{J} = \mathscr{J}(aD(1) - D(1)a) = \{0\}$, for each $a \in \mathscr{A}$. From hypothesis it follows that

$$D(1)a = aD(1), \tag{4.2}$$

for all $a \in \mathscr{A}$.

Let $a, b \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. By applying (4.1) and (4.2), we obtain

$$D(abx) = abD(x) + D(ab)x - aD(1)bx,$$

and on the other hand

$$\begin{aligned} D(abx) &= aD(bx) + D(a)bx - abxD(1) \\ &= abD(x) + aD(b)x + D(a)bx - 2aD(1)bx. \end{aligned}$$

By comparing the two expressions for $D(abx)$, we arrive at

$$(D(ab) - aD(b) - D(a)b + aD(1)b)x = 0 \tag{4.3}$$

for all $a, b \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. By Theorem 3.1, we have $\phi(x,a) = \phi(1,xa)$ for all $a \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. Now by this identity and using similar arguments as above it follows that

$$x(D(ab) - aD(b) - D(a)b + aD(1)b) = 0 \tag{4.4}$$

for all $a, b \in \mathscr{A}$ and $x \in \mathfrak{J}(\mathscr{A})$. Hence from (4.3) and (4.4), we find that $(D(ab) - aD(b) - D(a)b + aD(1)b)\mathscr{J} = \mathscr{J}(D(ab) - aD(b) - D(a)b + aD(1)b) = \{0\}$, for each $a, b \in \mathscr{A}$. From hypothesis it follows that

$$D(ab) = aD(b) + D(a)b - aD(1)b,$$

for all $a, b \in \mathscr{A}$.

By Proposition 3.2 and using similar arguments as that in the above proof , we get the result in case of Banach algebras. □

In order to prove next theorem we will adopt the following notational convention

$$[a,m,b] = amb + bma \quad \textit{and} \quad [a,b,m] = [m,b,a] = abm + mba$$

for all $a, b \in \mathscr{A}$ and $m \in \mathscr{M}$, where $\mathscr{A}$ is an algebra and $\mathscr{M}$ is an $\mathscr{A}$-bimodule. Also we need the following lemma, the proof of which is routine and will be omitted.

LEMMA 4.2. *Let $\mathscr{A}$ be an algebra and $\mathscr{M}$ be an $\mathscr{A}$-bimodule. For all $a, b, c \in \mathscr{A}$ and $m \in \mathscr{M}$ we have*

(i)

$$2[a,m,b] = a \bullet (b \bullet m) + b \bullet (a \bullet m) - (a \circ b) \bullet m$$

and

$$2[a,b,m] = a \bullet (b \bullet m) + (a \circ b) \bullet m - b \bullet (a \bullet m);$$

(ii)

$$[m, a\circ b, c] = [b\bullet m, a, c] + [m, a, b\circ c] - [m, a, c]\bullet b$$

and

$$\begin{aligned}[a, b\bullet m, c] &= [a\bullet m, b, c] + [a, b, c\bullet m] - [a, b, c]\bullet m \\ &= [a\circ b, m, c] + [a, m, b\circ c] - [a, m, c]\bullet b.\end{aligned}$$

THEOREM 4.3. *Let $\mathscr{A}$ be an (Banach) algebra, $\mathscr{M}$ be an (Banach) $\mathscr{A}$-bimodule satisfying $\mathbb{M}$ ($\mathbb{M}'$). Suppose that $D : \mathscr{A} \to \mathscr{M}$ is a (continuous) linear map. Then the following conditions are equivalent:*

(i) *D is a generalized Jordan derivation and $aD(1) = D(1)a$ for all $a \in \mathscr{A}$;*

(ii) *D satisfies* (d3)*;*

(iii) *D satisfies* (d4).

Proof. Clearly (i) implies (ii) and (ii) implies (iii). We show that (iii) implies (i).

Let $\mathscr{J}$ be an ideal of $\mathscr{A}$ such that $\mathscr{J} \subseteq \mathfrak{J}(\mathscr{A})$ (if $\mathscr{A}$ is a Banach algebra we assume that $\mathscr{J} \subseteq \overline{\mathfrak{J}(\mathscr{A})}$) and

$$\{m \in \mathscr{M} \,|\, xmx = 0 \; for\, all\; x \in \mathscr{J}\} = \{0\}.$$

Let p be a idempotent of $\mathscr{A}$. As $p(1-p) = (1-p)p = 0$ it follows that

$$2D(p) + pD(1) + D(1)p = 2pD(p) + 2D(p)p.$$

By multiplying this identity on the left and right by p, respectively, we arrive at

$$\begin{aligned}pD(1)p + D(1)p &= 2pD(p)p, \\ pD(1) + pD(1)p &= 2pD(p)p,\end{aligned}$$

which implies $pD(1) = D(1)p$. By linearity, it shows $xD(1) = D(1)x$ for all $x \in \mathscr{J}$. Hence $aD(1)x = D(1)ax$ and $xD(1)a = xaD(1)$ for each $a \in \mathscr{A}$ and $x \in \mathscr{J}$. Therefore $x(aD(1) - D(1)a)x = 0$ for all $x \in \mathscr{J}$ and by hypothesis we have

$$aD(1) = D(1)a$$

for all $a \in \mathscr{A}$.

Define $\Delta : \mathscr{A} \to \mathscr{M}$ by $\Delta(a) = D(a) - aD(1)$. Then Δ is a linear map which satisfies (d4) and $\Delta(1) = 0$. We will show that Δ is a Jordan derivation. So D is a generalized Jordan derivation.

Now define a bilinear map $\phi : \mathscr{A} \times \mathscr{A} \to \mathscr{M}$ by $\phi(a,b) = a\bullet\Delta(b) + \Delta(a)\bullet b$. So $\phi(a,b) = 0$ for all $a, b \in \mathscr{A}$ with $ab = ba = 0$, and by Theorem 3.5, we get $\phi(a,x) + \phi(x,a) = \phi(ax,1) + \phi(1,xa)$ for all $a \in \mathscr{A}$ and $x \in \mathscr{J}$. Hence

$$\Delta(a\circ x) = a\bullet\Delta(x) + \Delta(a)\bullet x \tag{4.5}$$

for all $a \in \mathscr{A}$ and $x \in \mathscr{J}$.

Claim 1. For all $a \in \mathscr{A}$ and $x,y \in \mathscr{J}$, we have

$$\Delta([x,a,y]) = [\Delta(x),a,y] + [x,\Delta(a),y] + [x,a,\Delta(y)]$$

Reason. Let $x,y \in \mathscr{J}$ and $a \in \mathscr{A}$. From Lemma 4.2 and (4.5), we obtain

$$\begin{aligned}
2\Delta([x,a,y]) &= \Delta(x\circ(a\circ y)) + \Delta(y\circ(a\circ x)) - \Delta((x\circ y)\circ a) \\
&= x\bullet\Delta(a\circ y) + \Delta(x)\bullet(a\circ y) + y\bullet\Delta(a\circ x) \\
&\quad + \Delta(y)\bullet(a\circ x) - (x\circ y)\bullet\Delta(a) - \Delta(x\circ y)\bullet a \\
&= x\bullet(y\bullet\Delta(a)) + x\bullet(\Delta(y)\bullet a) + \Delta(x)\bullet(y\circ a) \\
&\quad + y\bullet(\Delta(x)\bullet a) + y\bullet(\Delta(a)\bullet x) + \Delta(y)\bullet(x\circ a) \\
&\quad - (x\circ y)\bullet\Delta(a) - (x\bullet\Delta(y))\bullet a - (\Delta(x)\bullet y)\bullet a \\
&= 2[\Delta(x),a,y] + 2[x,\Delta(a),y] + 2[x,a,\Delta(y)].
\end{aligned}$$

Claim 2. For all $a \in \mathscr{A}$ and $x,y \in \mathscr{J}$, we have

$$\Delta([x,a^2,y]) = [\Delta(x),a^2,y] + [x,a\bullet\Delta(a),y] + [x,a^2,\Delta(y)]$$

Reason. Let $x,y \in \mathscr{J}$ and $a \in \mathscr{A}$. From Lemma 4.2, Claim 1 and (4.5), it follows that

$$\begin{aligned}
2\Delta([x,a^2,y]) &= \Delta([x,a\circ a,y]) \\
&= \Delta([x\circ a,a,y]) + \Delta([x,a,y\circ a]) - \Delta([x,a,y]\circ a) \\
&= [\Delta(x\circ a),a,y] + [x\circ a,\Delta(a),y] + [x\circ a,a,\Delta(y)] \\
&\quad + [\Delta(x),a,y\circ a] + [x,\Delta(a),y\circ a] + [x,a,\Delta(y\circ a)] \\
&\quad - a\bullet\Delta([x,a,y]) - \Delta(a)\bullet[x,a,y].
\end{aligned}$$

So

$$\begin{aligned}
2\Delta([x,a^2,y]) &= [a\bullet\Delta(x),a,y] + [\Delta(x),a,y\circ a] - [\Delta(x),a,y]\bullet a \\
&\quad + [\Delta(a)\bullet x,a,y] + [x,a,y\bullet\Delta(a)] - [x,a,y]\bullet\Delta(a) \\
&\quad + [x,a,a\bullet\Delta(y)] + [a\circ x,a,\Delta(y)] - [x,a,\Delta(y)]\bullet a \\
&\quad + [x\circ a,\Delta(a),y] + [x,\Delta(a),y\circ a] - [x,\Delta(a),y]\bullet a \\
&= 2[\Delta(x),a^2,y] + 2[x,a\bullet\Delta(a),y] + 2[x,a^2,\Delta(y)].
\end{aligned}$$

Now by applying Claim 1, we have

$$\Delta([x,a^2,x]) = [\Delta(x),a^2,x] + [x,\Delta(a^2),x] + [x,a^2,\Delta(x)]$$

for all $a \in \mathscr{A}$ and $x \in \mathscr{J}$. On the other hand from Claim 2, we see that

$$\Delta([x,a^2,x]) = [\Delta(x),a^2,x] + [x,a\bullet\Delta(a),x] + [x,a^2,\Delta(x)]$$

for all $a \in \mathscr{A}$ and $x \in \mathscr{J}$. By comparing the two expressions for $\Delta([x,a^2,x])$, we arrive at

$$x(\Delta(a^2) - a \bullet \Delta(a))x = 0$$

for all $a \in \mathscr{A}$ and $x \in \mathscr{J}$. Therefore by hypothesis we have $\Delta(a^2) = a \bullet \Delta(a)$ for each $a \in \mathscr{A}$ and so Δ is a Jordan derivation.

Similarly, by Corollary 3.6 we have the result in case of Banach algebras and continuous linear maps. □

THEOREM 4.4. *Let $\mathscr{A}$ be an (Banach) algebra, $\mathscr{M}$ be an (Banach) $\mathscr{A}$-bimodule satisfying $\mathbb{M}$ ($\mathbb{M}'$). Suppose that $D : \mathscr{A} \to \mathscr{M}$ is a (continuous) linear map satisfying* (d2). *Then D is a generalized Jordan derivation and $aD(1) = D(1)a$ for all $a \in \mathscr{A}$.*

Proof. Let $a,b \in \mathscr{A}$ with $ab = ba = 0$. So

$$aD(b) + D(a)b = 0 \quad \textit{and} \quad bD(a) + D(b)a = 0.$$

Hence $aD(b) + D(a)b + bD(a) + D(b)a = 0$ and D satisfies (d4). Therefore by Theorem 4.3, D is a generalized Jordan derivation and $aD(1) = D(1)a$ for all $a \in \mathscr{A}$. □

REMARK 4.5. In Theorem 4.4 it is not necessarily true that any linear mapping $D : \mathscr{A} \to \mathscr{M}$ satisfying (d2) is a generalized derivation. Indeed, if D is a *anti-derivation*, i.e. $D(ab) = D(b)a + bD(a)$ for all $a,b \in \mathscr{A}$, then D satisfies (d2). There are simple examples on some algebras and their (special) bimodules with anti-derivations such that they are not derivations. An example is given on the algebra T_2 of 2×2 upper triangular matrices over $\mathbb{C}$ [10]. Let us recall it. We make $\mathbb{C}$ an T_2-bimodule by defining $a\gamma = a_{22}\gamma$ and $\gamma a = \gamma a_{11}$ for all $\gamma \in \mathbb{C}$, $a \in T_2$. A map $D : T_2 \to \mathbb{C}$ defined by $D(a) = a_{12}$ is an anti-derivation which is not a derivation. Note that if $\mathscr{A} = T_2$ and $\mathscr{M} = \mathbb{C}$, then $\mathscr{A}$, $\mathscr{M}$ and D satisfy all the requirements in Theorem 4.4.

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