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STRONG COMMUTATIVITY PRESERVING GENERALIZED DERIVATIONS ON TRIANGULAR RINGS

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Abstract. Let $\mathscr{U} = Tri(A,M,B)$ be a triangular ring such that either A or B has no nonzero central ideals. It is shown that every pair of strong commutativity preserving generalized derivations g_1, g_2 of $\mathscr{U}$ (i.e., $[g_1(x), g_2(y)] = [x,y]$ for all $x,y \in \mathscr{U}$) is of the form $g_1(x) = \lambda^{-1}x + [x,u]$ and $g_2(x) = \lambda^2 g_1(x)$, where $\lambda \in Z(\mathscr{U})$, the center of $\mathscr{U}$, and $u \in \mathscr{U}$ with $u[\mathscr{U},\mathscr{U}] = 0 = [\mathscr{U},\mathscr{U}]u$. As consequences, every pair of strong commutativity preserving generalized derivations on upper triangular matrix rings and nest algebras is determined.

1. Introduction

Let R be a ring with center $Z(R)$. For $x,y \in R$, we set $[x,y] = xy - yx$. By $[R,R]$ we denote the additive subgroup of R generated by all $[x,y]$, where $x,y \in R$. An additive map $g : R \to R$ is called a *generalized derivation* of R if there exists a derivation d of R such that $g(xy) = g(x)y + xd(y)$ for all $x,y \in R$. Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \mapsto ax + xb$ for some $a,b \in R$). The notion of generalized derivations was introduced by Brešar in [5] and these maps have been studied extensively in rings and operator algebras (see [1, 4, 13, 14, 15, 16, 17]).

Let S be a subset of R. A map $f : S \to R$ is said to be *strong commutativity preserving* on S if $[f(x), f(y)] = [x,y]$ for all $x,y \in S$. In [2] Bell and Daif investigated strong commutativity preserving derivations on semiprime rings. In [6] Brešar and Miers proved that if f is a strong commutativity preserving map on a semiprime ring R, then there exist an invertible element $\lambda \in C$ with $\lambda^2 = 1$ and additive map $\xi : R \to C$ such that $f(x) = \lambda x + \xi(x)$ for all $x \in R$, where C is the extended centroid of R. They also proved that if $f, g : R \to R$ is a pair of additive maps of a semiprime ring R such that f is onto and $[f(x), g(x)] = [x,y]$ for all $x \in R$, then there exist an invertible element $\lambda \in C$ and additive maps $\xi, \eta : R \to C$ such that $f(x) = \lambda x + \xi(x)$ and $g(x) = \lambda^{-1}x + \eta(x)$ for all $x \in R$ [6, Theorem 2]. Strong commutativity preserving maps on rings have been discussed in several directions (see [10, 18, 19, 20, 21]).

In 2001, Cheung [7] initiated the study of mapping problems on triangular algebras; he described commuting maps of these algebras. This result has been extended in

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[3, 11, 12]. Recently, Qi and Hou [22] investigated surjective additive strong commutativity preserving maps of triangular rings.

In the present paper, we shall investigate strong commutativity preserving generalized derivations on triangular rings. As consequences strong commutativity preserving generalized derivations on upper triangular matrix rings and nest algebras are determined.

2. The main results

Let A and B be unital rings with unit elements 1_A and 1_B, respectively. Let M be a unital (A,B)-bimodule, which is faithful as a left A-module and also as a right B-module. The ring

$$\mathscr{U} = \mathrm{Tri}(A,M,B) := \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} \mid a \in A, m \in M, b \in B \right\}$$

under the usual matrix operations is said to be a *triangular ring* (see [12, 22, 23]). Let us define two natural projections $\pi_A : \mathscr{A} \to A$ and $\pi_B : \mathscr{A} \to B$ by

$$\pi_A : \begin{pmatrix} a & m \\ & b \end{pmatrix} \mapsto a \quad \text{and} \quad \pi_B : \begin{pmatrix} a & m \\ & b \end{pmatrix} \mapsto b.$$

Any element of the form

$$\begin{pmatrix} a & 0 \\ & b \end{pmatrix} \in \mathscr{U}$$

will be denoted by $a \oplus b$. By [23, Proposition 1.1] we know that the center $Z(\mathscr{U})$ of $\mathscr{U}$ coincides with

$$\{a \oplus b \mid am = mb \quad \text{for all } m \in M\}.$$

Moreover, $\pi_A(Z(\mathscr{A})) \subseteq Z(A)$ and $\pi_B(Z(\mathscr{A})) \subseteq Z(B)$, and there exists a unique ring isomorphism $\tau : \pi_A(Z(\mathscr{A})) \to \pi_B(Z(\mathscr{A}))$ such that $am = m\tau(a)$ for all $m \in M$. The most important examples of triangular rings are upper triangular matrix rings and nest algebras.

We begin with a description of generalized derivations of triangular rings.

PROPOSITION 2.1. *Let $\mathscr{U}$ be a triangular ring. Let g be a generalized derivation of $\mathscr{U}$. Then*

$$g\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a_0a + p_A(a) & as + tb + a_0m + f(m) \\ & b_0b + p_B(b) \end{pmatrix}$$

for all $a \in A$, $b \in B$, $m \in M$, where $a_0 \in A$, $b_0 \in B$, $s,t \in M$, and

(i) p_A is a derivation of A, $f(am) = p_A(a)m + af(m)$;

(ii) p_B is a derivation of B, $f(mb) = mp_B(b) + f(m)b$.

Proof. Since g be a generalized derivation of $\mathscr{U}$ we have that

$$g(xy) = g(x)y + xd(y)$$

for all $x,y \in \mathscr{U}$, where d is a derivation of $\mathscr{U}$. Let $x = 1$ we get $g(y) = g(1)y + d(y)$ for all $y \in \mathscr{U}$. In view of [8, Theorem 2.2.1] we have that

$$d\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} p_A(a) & as - sb + f(m) \\ & p_B(b) \end{pmatrix}$$

for all $a \in A$, $b \in B$, $m \in M$, where $s \in M$ and

(i) p_A is a derivation of A, $f(am) = p_A(a)m + af(m)$;

(ii) p_B is a derivation of B, $f(mb) = mp_B(b) + f(m)b$.

Set

$$g(1) = \begin{pmatrix} a_0 & m_0 \\ & b_0 \end{pmatrix}.$$

Then

$$\begin{aligned} g\begin{pmatrix} a & m \\ & b \end{pmatrix} &= \begin{pmatrix} a_0 & m_0 \\ & b_0 \end{pmatrix}\begin{pmatrix} a & m \\ & b \end{pmatrix} + \begin{pmatrix} p_A(a) & as - sb + f(m) \\ & p_B(b) \end{pmatrix} \\ &= \begin{pmatrix} a_0a + p_A(a) & as + tb + a_0m + f(m) \\ & b_0b + p_B(b) \end{pmatrix} \end{aligned}$$

for all $a \in A$, $b \in B$, $m \in M$, where $t = m_0 - s$. $\square$

We are in a position to present the main result of this paper.

THEOREM 2.1. *Let $\mathscr{U}$ be a triangular ring such that either A or B has no nonzero central ideals. If g_1, g_2 are a pair of generalized derivations such that*

$$[g_1(x), g_2(y)] = [x,y]$$

for all $x,y \in \mathscr{U}$, then $g_1(x) = \lambda^{-1}x + [x,u]$ and $g_2(x) = \lambda^2 g_1(x)$ for all $x \in \mathscr{U}$, where $\lambda \in Z(\mathscr{U})$ and $u \in \mathscr{U}$ with $u[\mathscr{U},\mathscr{U}] = 0 = [\mathscr{U},\mathscr{U}]u$.

Proof. We assume without loss of generality that A has no nonzero central ideals. In view of Proposition 2.1 we assume that

$$g_1\begin{pmatrix} a & m \\ & b \end{pmatrix} = \begin{pmatrix} a_0a + p_A(a) & as + tb + a_0m + f(m) \\ & b_0b + p_B(b) \end{pmatrix}$$

and

$$g_2\begin{pmatrix} a' & m' \\ & b' \end{pmatrix} = \begin{pmatrix} a_0'a' + p_A'(a') & a's' + t'b' + a_0'm' + f'(m') \\ & b_0'b' + p_B'(b') \end{pmatrix}$$

for all $a,a' \in A$, $b,b' \in B$, $m,m' \in M$, where $a_0,a_0' \in A$, $b_0,b_0' \in B$, $s,s',t,t' \in M$ and

(i) p_A, p'_A are derivations of A, $f(am) = p_A(a)m + af(m)$, and $f'(a'm') = p'_A(a')m' + a'f'(m')$;

(ii) p_B, p'_B are derivations of B, $f(mb) = mp_B(b) + f(m)b$, and $f'(m'b') = m'p'_B(b') + f'(m')b'$.

By our assumption we have that

$$\left[g_1\begin{pmatrix} a & m \\ & b \end{pmatrix}, g_2\begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right] = \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} a' & m' \\ & b' \end{pmatrix}\right] \tag{1}$$

for all $a, a' \in A$, $b, b' \in B$, and $m, m' \in M$. We prove the result in the following five steps:

Step 1. we prove that

$$a_0(a'_0m' + f'(m')) = m', \tag{2}$$
$$a'_0(a_0m + f(m)) = m \tag{3}$$

for all $m, m' \in M$. Setting $a = 1_A$, $b = m = 0$, and $a' = b' = 0$ in (1) we get that

$$\left[\begin{pmatrix} a_0 & s \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & a'_0m' + f'(m') \\ & 0 \end{pmatrix}\right] = \left[\begin{pmatrix} 1_A & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} 0 & m' \\ & 0 \end{pmatrix}\right]$$

for all $m' \in M$. This implies that

$$a_0(a'_0m' + f'(m')) = m'$$

for all $m' \in M$. Similarly, setting $a = b = 0$, $a' = 1_A$, $b' = m' = 0$ in (1) we get that

$$a'_0(a_0m + f(m)) = m$$

for all $m \in M$.

Step 2. We prove that $a_0 \oplus b_0, a'_0 \oplus b'_0 \in Z(\mathscr{U})$. Setting $a = 1_A$, $b = 1_B$, $m = 0$, $a' = b' = 0$ in (1) we get that

$$\left[\begin{pmatrix} a_0 & s+t \\ & b_0 \end{pmatrix}, \begin{pmatrix} 0 & a'_0m' + f'(m') \\ & 0 \end{pmatrix}\right] = 0$$

for all $m' \in M$. This implies that

$$a_0(a'_0m' + f'(m')) - (a'_0m' + f'(m'))b_0 = 0$$

for all $m \in M$. Multiplying the last relation by a_0 from the left hand side we get

$$a_0(a_0(a'_0m' + f'(m'))) = (a_0(a'_0m' + f'(m')))b_0$$

for all $m' \in M$. Substituting (2) into the last relation yields $a_0 m' = m' b_0$ for all $m' \in M$. Hence, $a_0 \oplus b_0 \in Z(\mathscr{U})$. By the symmetry of g_1 and g_2 we obtain that $a_0' \oplus b_0' \in Z(\mathscr{U})$.

Step 3. We prove that

$$a_0 p_A'(a) = 0, \quad b_0 p_B'(b) = 0, \quad a_0' p_A(a) = 0, \quad b_0' p_B(b) = 0$$

for all $a \in A$ and $b \in B$. Replacing m' by $m'b$ in (2) yields

$$a_0(a_0' m' b + f'(m')b + m' p_B'(b)) = m'b$$

for all $b \in B$, $m' \in M$. Multiplying (2) by $b \in B$ from the right hand side we obtain

$$a_0(a_0' m' + f'(m'))b = m'b$$

for all $b \in B$, $m' \in M$. Comparing the last two relations yields $a_0 m' p_B'(b) = 0$. Since $a_0 \oplus b_0 \in Z(\mathscr{U})$ we get that $m' b_0 p_B'(b) = 0$ for all $b \in B$ and $m' \in M$. The faithfulness of right B-module M yields that $b_0 p_B'(b) = 0$ for all $b \in B$. Similarly, replacing m' by am' in (2) yields

$$a_0(a_0' am' + af'(m') + p_A'(a)m') = am'$$

for all $a \in A$, $m' \in M$. Multiplying (2) by $a \in A$ from the left hand side we get

$$a_0(a_0' am' + af'(m')) = am'$$

for all $a \in A$, $m' \in M$ as $a_0, a_0' \in Z(A)$. Comparing the last two relations yields $a_0 p_A'(a)m' = 0$ for all $a \in A$ and $m' \in M$. The faithfulness of left A-module M yields that $a_0 p_A'(a) = 0$ for all $a \in A$. In view of the symmetry of g_1 and g_2 we obtain that $a_0' p_A(a) = 0$ and $b_0' p_B(b) = 0$ for all $a \in A$ and $b \in B$.

Step 4. We prove that $a_0' = a_0^{-1}$ and $b_0' = b_0^{-1}$ and

$$f = f' = 0, \quad p_A = p_A' = 0, \quad p_B = p_B' = 0.$$

Setting $m = m' = 0$ and $b = b' = 0$ in (1) we get that

$$\left[\begin{pmatrix} a_0 a + p_A(a) & as \\ & 0 \end{pmatrix}, \begin{pmatrix} a_0' a' + p_A(a') & a's' \\ & 0 \end{pmatrix} \right] = \left[\begin{pmatrix} a & 0 \\ & 0 \end{pmatrix}, \begin{pmatrix} a' & 0 \\ & 0 \end{pmatrix} \right] \tag{4}$$

for all $a, a' \in A$. It follows from (4) that

$$[a_0 a + p_A(a), a_0' a' + p_A'(a')] = [a, a'] \tag{5}$$

for all $a, a' \in A$. Multiplying (5) with $a_0 \in Z(A)$ we get that

$$a_0[a_0 a + p_A(a), a_0' a' + p_A'(a')] = a_0[a, a']$$

for all $a,a' \in A$. Since $a_0,a_0' \in Z(A)$, $a_0p_A'(a') = a_0'p_A(a) = 0$ for all $a,a' \in A$, we get from the last relation that

$$\begin{aligned}
[a_0a,a'] &= a_0[a_0a + p_A(a), a_0'a' + p_A'(a')] \\
&= [a_0a + p_A(a), a_0a_0'a' + a_0p_A'(a')] \\
&= [a_0a + p_A(a), a_0a_0'a'] \\
&= [a_0'a_0a + a_0'p_A(a), a_0a'] \\
&= [a_0'a_0a, a_0a'] \\
&= [a_0'a_0^2a, a']
\end{aligned}$$

and so $[a_0a - a_0'a_0^2a, a'] = 0$ for all $a,a' \in A$. This implies that

$$a_0(1_A - a_0'a_0)a \in Z(A)$$

for all $a \in A$. That is, $a_0(1_A - a_0'a_0)A$ is a central ideal of A. By our assumption we infer that $a_0(1_A - a_0'a_0) = 0$. Multiplying (2) with $(1_A - a_0'a_0)$ we get that

$$(1_A - a_0'a_0)m' = (1_A - a_0'a_0)a_0(a_0'm' + f'(m')) = 0$$

for all $m' \in M$. That is, $(1_A - a_0'a_0)M = 0$. The faithfulness of left A-module M yields $1_A - a_0'a_0 = 0$ and so $a_0'a_0 = 1_A$. Hence $a_0' = a_0^{-1}$ is an invertible element of $\pi_A(Z(\mathscr{U}))$. Since $a_0 \oplus b_0, a_0' \oplus b_0' \in Z(\mathscr{U})$ we easily check that $b_0' = b_0^{-1}$ is an invertible element of $\pi_B(Z(\mathscr{U}))$.

Thus, the relations (2) and (3) can be rewritten as

$$m' + a_0f'(m') = m' \quad \text{and} \quad m + a_0'f(m) = m$$

for all $m,m' \in M$. Hence $a_0f'(m') = 0$ and $a_0'f(m) = 0$ and so $f(m) = f'(m') = 0$ for all $m' \in M$. Since $a_0' = a_0^{-1}$ and $b_0' = b_0^{-1}$ we get from Step 3 that

$$p_A = p_A' = 0 \quad \text{and} \quad p_B = p_B' = 0.$$

Step 5. We prove that $s = -t$, $s' = t'$, $s' = (a_0')^2s$ and

$$[A,A]s = 0 = s[B,B].$$

Setting $m = m' = 0$ in (1) we get that

$$\begin{aligned}
&\left[\begin{pmatrix} a_0a + p_A(a) & as + tb \\ & b_0b + p_B(b) \end{pmatrix}, \begin{pmatrix} a_0'a' + p_A(a') & a's' + t'b' \\ & b_0'b' + p_B'(b') \end{pmatrix}\right] \\
&= \left[\begin{pmatrix} a & 0 \\ & b \end{pmatrix}, \begin{pmatrix} a' & 0 \\ & b \end{pmatrix}\right]
\end{aligned} \tag{6}$$

for all $a,a' \in A$ and $b,b' \in B$. It follows from (6) that

$$a_0a(a's' + t'b') + (as + tb)b_0'b' - a_0'a'(as + tb) - (a's' + t'b')b_0b = 0 \tag{7}$$

for all $a,a' \in A$ and $b,b' \in B$. Setting $b = b' = 0$ in (7) we get that

$$a_0aa's' - a_0'a'as = 0 \tag{8}$$

for all $a,a' \in A$. Setting $a = a' = 1_A$ in (8) we get $a_0s' = a_0's$. Thus, the relation (8) becomes $a_0(aa' - a'a)s' = 0$ and then $(aa' - a'a)s' = 0$ as a_0 is an invertible element of A. That is, $[A,A]s' = 0$. Recall that $a_0s' = a_0's$. It is easy to check that $[A,A]s = 0$. Setting $a = a' = 0$ in (7) we get that

$$tbb_0'b' - t'b'b_0b = 0 \tag{9}$$

for all $b,b' \in B$. Setting $b = b' = 1_B$ in (9) we get $tb_0' = t'b_0$. Thus, the relation (9) becomes $t[b,b']b_0' = 0$ and so $t[b,b'] = 0$ as b_0' is an invertible element of B. Hence $t[B,B] = 0$. Setting $a = 0$, $b' = 0$, $a' = 1_A$, and $b = 1_B$ in (7) we get $a_0't = -s'b_0$. Setting $a' = 0$, $b = 0$, $a = 1_A$, and $b' = 1_B$ in (7) we get $a_0t' = -sb_0'$. Recall that $a_0s' = a_0's$. It is easy to check that $s = -t$, $s' = -t'$, and $s' = (a_0')^2s$.

Set $\lambda = a_0' \oplus b_0'$. Then $\lambda^{-1} = a_0 \oplus b_0$. Using the relations in steps 2, 4, and 5 we obtain that

$$\begin{aligned} g_1\begin{pmatrix} a & m \\ & b \end{pmatrix} &= \begin{pmatrix} a_0a & as + tb + a_0m \\ & b_0b \end{pmatrix} \\ &= \lambda^{-1}\begin{pmatrix} a & m \\ & b \end{pmatrix} + \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} 0 & s \\ & 0 \end{pmatrix}\right] \end{aligned}$$

and

$$\begin{aligned} g_2\begin{pmatrix} a & m \\ & b \end{pmatrix} &= \lambda\begin{pmatrix} a & m \\ & b \end{pmatrix} + \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \begin{pmatrix} 0 & (a_0')^2s \\ & 0 \end{pmatrix}\right] \\ &= \lambda\begin{pmatrix} a & m \\ & b \end{pmatrix} + \left[\begin{pmatrix} a & m \\ & b \end{pmatrix}, \lambda^2\begin{pmatrix} 0 & s \\ & 0 \end{pmatrix}\right] \\ &= \lambda^2 g_1\begin{pmatrix} a & m \\ & b \end{pmatrix} \end{aligned}$$

for all $a \in A$, $b \in B$, $m \in M$. Set $u = \begin{pmatrix} 0 & s \\ & 0 \end{pmatrix}$. In view of Step 5 it is easy to check that $u[\mathscr{U},\mathscr{U}] = 0 = [\mathscr{U},\mathscr{U}]u$. This proves the result. □

REMARK 2.1. Let $\mathscr{U}$ be a triangular ring. Suppose that $u \in \mathscr{U}$ such that $u[\mathscr{U},\mathscr{U}] = 0 = [\mathscr{U},\mathscr{U}]u$. Then

$$u = \begin{pmatrix} 0 & m_0 \\ & 0 \end{pmatrix}$$

for some $m_0 \in M$ with $[A,A]m_0 = 0 = m_0[B,B]$.

Applying Theorem 2.1 and Remark 2.1 we have the following result:

COROLLARY 2.1. *Let $\mathscr{U}$ be a triangular ring such that either $1_A \in [A,A]$ or $1_B \in [B,B]$. If g_1, g_2 are a pair of generalized derivations such that*

$$[g_1(x), g_2(y)] = [x,y]$$

for all $x, y \in \mathscr{U}$, then there exists $\lambda \in Z(\mathscr{U})$ such that $g_1(x) = \lambda^{-1}x$ and $g_2(x) = \lambda x$ for all $x \in \mathscr{U}$.

Proof. We assume without loss of generality that $1_A \in [A,A]$. We claim that A has no nonzero central ideals. Indeed, if I is a central ideal of A, then $I = I1_A \subseteq I[A,A] = [IA,A] = 0$. By Theorem 2.1 we get that $g_1(x) = \lambda^{-1}x + [x,u]$ and $g_2(x) = \lambda^2 g_1(x)$ for all $x \in \mathscr{U}$, where $\lambda \in Z(\mathscr{U})$ and $u \in \mathscr{U}$ with $u[\mathscr{U},\mathscr{U}] = 0 = [\mathscr{U},\mathscr{U}]u$. It suffices to show $u = 0$. By Remark 2.1 we get that

$$u = \begin{pmatrix} 0 & m_0 \\ & 0 \end{pmatrix}$$

for some $m_0 \in M$ with $[A,A]m_0 = 0 = m_0[B,B]$. Since $1_A \in [A,A]$ we get $m_0 = 0$ and so $u = 0$. $\square$

3. Applications

Let $n \geqslant 2$ be an integer. Let $\mathscr{T}_n(S)$ be an upper upper triangular matrix ring over a unital ring S. Then $\mathscr{T}_n(S)$ can be viewed as the triangular ring

$$\begin{pmatrix} S & S^{n-1} \\ & \mathscr{T}_{n-1}(S) \end{pmatrix}.$$

Applying Theorem 2.1 we have the following result:

COROLLARY 3.1. *Let $\mathscr{T}_n(S)$ be an upper triangular matrix ring with $n \geqslant 3$. If g_1, g_2 are a pair of generalized derivations of $\mathscr{T}_n(S)$ such that*

$$[g_1(x), g_2(y)] = [x,y]$$

for all $x, y \in \mathscr{T}_n(S)$, then there exist $\lambda \in Z(\mathscr{T}_n(S))$ and $A \in \mathscr{T}_n(S)$ with the property

$$A[\mathscr{T}_n(S), \mathscr{T}_n(S)] = 0 = [\mathscr{T}_n(S), \mathscr{T}_n(S)]A$$

such that $g_1(x) = \lambda^{-1}x + [x,A]$ and $g_2(x) = \lambda^2 g_1(x)$ for all $x \in \mathscr{T}_n(S)$.

Proof. It is easy to check that $\mathscr{T}_{n-1}(S)$ has no nonzero central ideals. Consequently, Theorem 2.1 yields the conclusion. $\square$

As a consequence of Corollary 2.1 we have the following result:

COROLLARY 3.2. *Let S be a unital noncommutative ring with $1 \in [S,S]$. Let $\mathscr{T}_n(S)$ be an upper triangular matrix ring with $n \geqslant 2$. If g_1, g_2 are a pair of generalized derivations of $\mathscr{T}_n(S)$ such that*

$$[g_1(x), g_2(y)] = [x, y]$$

for all $x, y \in \mathscr{T}_n(S)$, then there exists $\lambda \in Z(\mathscr{T}_n(S))$ such that $g_1(x) = \lambda^{-1}x$ and $g_2(x) = \lambda x$ for all $x \in \mathscr{T}_n(S)$.

Applying Theorem 2.1 we have the following result:

COROLLARY 3.3. *Let S be a unital noncommutative prime ring. Let $\mathscr{T}_n(S)$ be an upper triangular matrix ring with $n \geqslant 2$. If g_1, g_2 are a pair of generalized derivations of $\mathscr{T}_n(S)$ such that*

$$[g_1(x), g_2(y)] = [x, y]$$

for all $x, y \in \mathscr{T}_n(S)$, then there exists $\lambda \in Z(\mathscr{T}_n(S))$ such that $g_1(x) = \lambda^{-1}x$ and $g_2(x) = \lambda x$ for all $x \in \mathscr{T}_n(S)$.

Proof. Since S is a noncommutative prime ring we see that S has no nonzero central ideals and so the condition of Theorem 2.1 is met. It follows from Theorem 2.1 that there exists an invertible element $\lambda \in Z(\mathscr{T}_n(S))$ such that $g_1(x) = \lambda^{-1}x + [x, A]$ and $g_2(x) = \lambda^2 g_1(x)$ for all $x \in \mathscr{T}_n(S)$, where $A \in \mathscr{T}_n(S)$ with $A[\mathscr{T}_n(S), \mathscr{T}_n(S)] = 0 = [\mathscr{T}_n(S), \mathscr{T}_n(S)]A$. It suffices to show that $A = 0$. Set

$$A = \sum_{\substack{i,j=1 \\ i \leqslant j}}^{n} a_{ij} e_{ij},$$

where $a_{ij} \in S$ and e_{ij} denotes the standard matrix unit of $\mathscr{T}_n(S)$. We get from the property $A[\mathscr{T}_n(S), \mathscr{T}_n(S)] = 0$ that in particular, $A[S,S] = 0$ and then $a_{ij}[S,S] = 0$ for every a_{ij} in A. Since S is a noncommutative prime ring we easily check that each $a_{ij} = 0$. Hence $A = 0$. $\square$

A *nest* $\mathscr{N}$ is a totally ordered set of closed subspaces of a Hilbert space H such that $\{0\}$, $H \in \mathscr{N}$, and $\mathscr{N}$ is closed under the taking of arbitrary intersections and closed linear spans of its elements. The *nest algebra* associated to $\mathscr{N}$ is the set $\mathscr{T}(\mathscr{N}) = \{T \in \mathscr{B}(H) \mid TN \subseteq N \text{ for all } N \in \mathscr{N}\}$.

A nest algebra $\mathscr{T}(\mathscr{N})$ is called *trivial* if $\mathscr{N} = \{0, H\}$. A nontrivial nest algebra can be viewed as a triangular algebra. Namely, if $N \in \mathscr{N} \setminus \{0, H\}$ and E is the orthonormal projection onto N, then $\mathscr{N}_1 = E(\mathscr{N})$ and $\mathscr{N}_2 = (1-E)(\mathscr{N})$ are nests of N and $N^{\perp}$, respectively. Moreover, $\mathscr{T}(\mathscr{N}_1) = ET(\mathscr{N})E$, $\mathscr{T}(\mathscr{N}_2) = (1-E)\mathscr{T}(\mathscr{N})(1-E)$ are nest algebras. Thus

$$\mathscr{T}(\mathscr{N}) = \begin{pmatrix} \mathscr{T}(\mathscr{N}_1) & E\mathscr{T}(\mathscr{N})(1-E) \\ & \mathscr{T}(\mathscr{N}_2) \end{pmatrix}$$

is a triangular ring. We refer the reader to [9] for the general theory of nest algebras.

COROLLARY 3.4. *Let $\mathscr{N}$ be a nest of a complex Hilbert space H with $\dim(H) > 2$. If g_1, g_2 are a pair of generalized derivations of $\mathscr{T}(\mathscr{N})$ such that*

$$[g_1(x), g_2(y)] = [x, y]$$

for all $x, y \in \mathscr{T}(\mathscr{N})$, then there exist $\lambda \in C$ and $A \in \mathscr{T}(\mathscr{N})$ with the property

$$A[\mathscr{T}(\mathscr{N}), \mathscr{T}(\mathscr{N})] = 0 = [\mathscr{T}(\mathscr{N}), \mathscr{T}(\mathscr{N})]A$$

such that $g_1(x) = \lambda^{-1}x + [x, A]$ and $g_2(x) = \lambda^2 g_1(x)$ for all $x \in \mathscr{T}(\mathscr{N})$.

Proof. If $\mathscr{N}$ is a trivial nest, then $\mathscr{T}(\mathscr{N}) = \mathscr{B}(H)$ is a prime ring and hence the conclusion follows from [20, Corollary 2.12]. Thus, we may assume that $\mathscr{N}$ is a nontrivial nest. Since $\dim(H) > 2$ it follows that either $\dim(\mathscr{T}(\mathscr{N}_1)) > 1$ or $\dim(\mathscr{T}(\mathscr{N}_2)) > 1$. If $\dim(\mathscr{T}(\mathscr{N})_1) > 1$, then either $\mathscr{T}(\mathscr{N}_1) = \mathscr{B}(\mathscr{N})$ is a noncommutative prime ring or $\mathscr{T}(\mathscr{N}_1)$ is a triangular algebra. Similarly, if $\dim(\mathscr{N}_2) > 1$, then either $\mathscr{T}(\mathscr{N}_2) = \mathscr{B}(\mathscr{N}_2)$ is a noncommutative prime ring or $\mathscr{T}(\mathscr{N}_2)$ is a triangular algebra. Consequently, either $\mathscr{T}(\mathscr{N}_1)$ or $\mathscr{T}(\mathscr{N}_2)$ has no nonzero central ideals (see [3, Lemma 2.6]). Thus, the result follows from Theorem 2.1. □

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