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THE BLOCK NUMERICAL RANGE OF ANALYTIC OPERATOR FUNCTIONS

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Abstract. We introduce the block numerical range $W^n(\mathscr{L})$ of an operator function $\mathscr{L}$ with respect to a decomposition $H = H_1 \oplus \ldots \oplus H_n$ of the underlying Hilbert space. Our main results include the spectral inclusion property and estimates of the norm of the resolvent for analytic $\mathscr{L}$. They generalise, and improve, the corresponding results for the numerical range (which is the case $n = 1$) since the block numerical range is contained in, and may be much smaller than, the usual numerical range. We show that refinements of the decomposition entail inclusions between the corresponding block numerical ranges and that the block numerical range of the operator matrix function $\mathscr{L}$ contains those of its principal subminors. For the special case of operator polynomials, we investigate the boundedness of $W^n(\mathscr{L})$ and we prove a Perron-Frobenius type result for the block numerical radius of monic operator polynomials with coefficients that are positive in Hilbert lattice sense.

1. Introduction

The numerical range of a linear operator is a useful tool to localise the spectrum and, for nonselfadjoint operators, to keep control of the resolvent norm. There are various generalisations of the concept of numerical range, on the one hand the numerical range of analytic operator functions (see e.g. [18]) and on the other hand the block numerical range of operators that admit a matrix representation (see e.g. [31]). Both generalisations share the spectral inclusion property and resolvent estimates with the classical numerical range, but not the convexity and even connectivity. However, although convexity may appear to be an advantage, it is rather an obstacle for tight spectral enclosures and finer resolvent estimates.

Non-linear operator functions appear in a wide range of applications such as vibrating systems, signal processing, quantum mechanics, fluid dynamics, and many more; the most common non-linearities are polynomial (especially quadratic, see [30]) and rational. Striking evidence of the importance of corresponding spectral problems has been given in the recent paper [1] (see also the corresponding MATLAB toolbox).

In this paper we introduce an analytic tool to study the spectral properties of operator functions, the so-called *block numerical range*. This new notion includes both

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generalisations mentioned above as special cases and provides even tighter spectral enclosures and resolvent estimates. If $\mathscr{L} = \big(L_{ij}\big)_{i,j=1}^{n} : \Omega \to L(H)$ is an analytic operator function on a domain $\Omega \subset \mathbb{C}$ whose values are bounded linear operators in a Hilbert space $H = H_1 \oplus \ldots \oplus H_n$, then the block numerical range is defined as

$$W^n(\mathscr{L}) := W^n_{H_1\oplus\ldots\oplus H_n}(\mathscr{L}) := \bigcup\nolimits_{x_i\in H_i, \|x_i\|=1} \sigma_p(\mathscr{L}_{(x_1,\ldots,x_n)})$$

where $\mathscr{L}_{(x_1,\ldots,x_n)} : \Omega \to M_n(\mathbb{C})$ is the $n\times n$ matrix function given by

$$\mathscr{L}_{(x_1,\ldots,x_n)}(z) := \big((L_{ij}(z)x_j, x_i)\big)_{i,j=1}^{n}, \quad z \in \Omega.$$

In the special case $\mathscr{L}(z) = \mathscr{A} - z$, $z \in \mathbb{C}$, the block numerical range of the operator function $\mathscr{L}$ coincides with the block numerical range of the operator $\mathscr{A}$ which was introduced in [15], and further studied in [11], for the case $n = 2$ and in [33] for $n \geqslant 2$; the case $n = 1$ is the classical numerical range.

The first crucial property of the block numerical range is that, while being contained in the classical numerical range, it still exhibits the *spectral inclusion property*. More precisely, we always have $\sigma_p(\mathscr{L}) \subset W^n(\mathscr{L}) \subset W(\mathscr{L})$, and

$$\sigma(\mathscr{L}) \subset \overline{W^n(\mathscr{L})} \subset \overline{W(\mathscr{L})} \tag{1.1}$$

provided that

$$\exists s \in \mathbb{N}_0\ \exists z_0 \in \Omega:\ 0 \notin \overline{\{(\det \mathscr{L}_{(x_1,\ldots,x_n)})^{(s)}(z_0) : x_i \in H_i, \|x_i\| = 1\}}. \tag{1.2}$$

This theorem improves an earlier result in [32] for the quadratic numerical range, i.e. the case $n = 2$; there it was assumed that $0 \notin \overline{W^2(\mathscr{L}(z_0))}$ for some $z_0 \in \Omega$ which corresponds to the special case $s = 0$ in (1.2).

The second key property of the block numerical range is that it provides an estimate for the norm of the resolvent. Since $W^n(\mathscr{L}) \subset W(\mathscr{L})$, this estimate applies in more points of the complex plane and gives tighter bounds. More precisely, if $C \subset \Omega$ is a bounded connected component of $\overline{W^n(\mathscr{L})}$, then there is a $\gamma > 0$ such that

$$\big\|\mathscr{L}^{-1}(z)\big\| \leqslant \frac{\gamma}{\operatorname{dist}(z,C)^{\nu_C}}, \quad z \in \overline{U} \setminus \overline{C}, \tag{1.3}$$

where ν_C is the number of zeros of $\det \mathscr{L}_{(x_1,\ldots,x_n)}$ in C counted with multiplicities (which is independent of $(x_1,\ldots,x_n)$, $\|x_i\| = 1$) and $U \subset \Omega$ is a bounded domain such that $\overline{U} \supset \overline{C}$ separates C from $\overline{W^n(\mathscr{L})} \setminus C$. As a consequence of the estimate (1.3), we obtain that the length of a Jordan chain of an eigenvalue on the boundary of C cannot exceed ν_C.

Since the block numerical range may be considerably smaller than the numerical range, the above results may provide tighter spectral enclosures and resolvent esimtates; in fact, it may even happen that the block numerical range is bounded, while the numerical range is unbounded. More generally, if a decomposition $H = H_1 \oplus \ldots \oplus H_n$ is further refined to a decomposition $H = \widetilde{H}_1 \oplus \ldots \oplus \widetilde{H}_m$ with $m > n$ and $H_i = \widetilde{H}_{j_i} \oplus \ldots \oplus \widetilde{H}_{j_{i+1}-1}$, then

$$W^m(\mathscr{L}) \subset W^n(\mathscr{L}) \quad \big(\text{more precisely, } W^m_{\widetilde{H}_1\oplus\ldots\oplus\widetilde{H}_m}(\mathscr{L}) \subset W^n_{H_1\oplus\ldots\oplus H_n}(\mathscr{L})\big).$$

Other interesting features of the block numerical range include various inclusions of (other block) numerical ranges and, for operator polynomials, criteria for boundedness and a Perron-Frobenius type result for the block numerical radius.

This paper is organised as follows. In Section 2 we introduce the block numerical range and present some of its basic properties. In Section 3 we derive various inclusions between the block numerical ranges of $\mathscr{L}$ and other (block) numerical ranges; we prove that a refinement of the decomposition of H entails inclusion between the respective block numerical ranges (Proposition 3.1) and that, under certain dimension restrictions, $W^n(\mathscr{L})$ contains the block numerical ranges of the principal subminors of $\mathscr{L}$ and, in particular, the numerical ranges $W(L_{ii})$ of the diagonal elements L_{ii} of $\mathscr{L}$ (Proposition 3.3). In Section 4 we prove the spectral inclusion (1.1) using tools from complex analysis such as the theorems of Hurwitz and Montel (Theorem 4.1). In Section 5 we establish the estimate (1.3) for the norm of the resolvent $\mathscr{L}(z)^{-1}$ in points $z\in\Omega\setminus\overline{W^n(\mathscr{L})}$ in terms of the distance of z to a bounded component of $W^n(\mathscr{L})$ (Theorem 5.1). In Section 6 we consider the special case of operator polynomials and find criteria when the block numerical range is bounded (Theorem 6.4). In Section 7 we prove a Perron-Frobenius type result for monic operator polynomials $\mathscr{P}$ of the form $\mathscr{P}(z)=z^k-A_{k-1}z^{k-1}-\ldots-A_1z-A_0$ on a Hilbert lattice H with positive coefficients A_i in lattice sense; it shows that the block numerical radius $w^n(\mathscr{P})$ of $\mathscr{P}$ is contained in the closure of the block numerical range (Theorem 7.3) and the vector $x\in H$ at which it is attained may be chosen to be positive in lattice sense.

Part of our results were obtained in the PhD thesis [36] of the third author, M. Wagenhofer. The illustrating plots of block numerical ranges for the six examples, which include a quadratic 8×8 matrix polynomial of a gyroscopic system with 8 degrees of freedom, were produced with a C++-code due to M. Wagenhofer (see [35]).

2. Definition and preliminaries

Throughout this paper, H denotes a complex Hilbert space with scalar product $(\cdot,\cdot)$, $L(H)$ is the set of bounded linear operators on H, $\Omega\subset\mathbb{C}$ is a domain, and $\mathscr{L}:\Omega\to L(H)$ is an operator function which is assumed to be analytic in the main results. The *resolvent set*, *spectrum*, and *point spectrum* of $\mathscr{L}$ are defined as (see [18, § 11.2])

$$\begin{aligned}\rho(\mathscr{L}) &:= \{z\in\Omega : \mathscr{L}(z) \text{ is bijective}\},\\ \sigma(\mathscr{L}) &:= \Omega\setminus\rho(\mathscr{L}),\\ \sigma_p(\mathscr{L}) &:= \{z\in\Omega : \mathscr{L}(z) \text{ is not injective}\};\end{aligned}$$

the operator function $\mathscr{L}^{-1}$ is defined as

$$\mathscr{L}^{-1}:\rho(\mathscr{L})\to L(H),\quad \mathscr{L}^{-1}(z):=(\mathscr{L}(z))^{-1};$$

note that, if $\mathscr{L}$ is analytic, then so is $\mathscr{L}^{-1}$. The *numerical range* of $\mathscr{L}$ is the set (see [18, § 26.3])

$$W(\mathscr{L}):=\{\lambda\in\Omega:\exists x\in H, x\neq 0, (\mathscr{L}(\lambda)x,x)=0\}.$$

In the following, a decomposition of the Hilbert space H as an orthogonal direct sum of closed subspaces $H_1,\ldots,H_n$ of H is given,

$$H = H_1 \oplus \ldots \oplus H_n. \tag{2.1}$$

If H is given as a product of $H = H_1 \times \ldots \times H_n$, of Hilbert spaces $H_1,\ldots,H_n$, then H_i will be tacitly identified with the subspace $\{0\}\times\ldots\times\{0\}\times H_i\times\{0\}\times\ldots\times\{0\}$ in (2.1).

With respect to the decomposition (2.1) of H, we can write $\mathscr{L}$ as an operator matrix function

$$\mathscr{L}(z) = \big(L_{ij}(z)\big)_{i,j=1}^{n} \tag{2.2}$$

with operator functions

$$L_{ij} : \Omega \to L(H_j, H_i), \quad i,j = 1,\ldots,n,$$

the values of which are bounded linear operators from H_j to H_i; note that if $\mathscr{L}$ is analytic, then so are L_{ij}.

The following definition generalises various earlier notions: the numerical and the quadratic numerical range of an analytic operator function (see [18, § 26.3] and [32]) as well as the quadratic numerical range and block numerical range of a linear operator (see [11] and [33]).

DEFINITION 2.1. Let $\mathscr{L}(\cdot) = (L_{ij}(\cdot))_{i,j=1}^{n} : \Omega \to L(H)$ be an operator function. For $x = (x_1,\ldots,x_n) \in H_1\times\ldots\times H_n$ define the $n\times n$-matrix function $\mathscr{L}_x : \Omega \to M_n(\mathbb{C})$ by

$$\mathscr{L}_x(z) := \mathscr{L}(z)_x := \big((L_{ij}(z)x_j, x_i)\big)_{i,j=1}^{n}, \quad z \in \Omega,$$

and denote the product of the unit spheres of H_i, $i = 1,\ldots,n$, by

$$\mathscr{S}^n := \{(x_1,\ldots,x_n) : x_i \in H_i, \|x_i\| = 1, i = 1,\ldots,n\}.$$

Then the *block numerical range* of $\mathscr{L}$ is defined as the union of all eigenvalues of the matrix functions $\mathscr{L}_x$, $x \in \mathscr{S}^n$,

$$W^n(\mathscr{L}) := \bigcup_{x\in\mathscr{S}^n} \sigma(\mathscr{L}_x).$$

Clearly, the block numerical range depends on the decomposition (2.1) of H since so does the matrix representation (2.2) of $\mathscr{L}$; if necessary, we specify this dependence by writing $W^n_{H_1\oplus\ldots\oplus H_n}(\mathscr{L})$ instead of $W^n(\mathscr{L})$. For $n = 3$, the block numerical range is also called *cubic numerical range*.

The following properties of the block numerical range are easy to check.

REMARK 2.2.

i) For an $n\times n$ matrix function $\mathscr{L} : \Omega \to L(\mathbb{C}^n)$ (where $H = \mathbb{C}^n$),

$$W^n(\mathscr{L}) = \sigma_p(\mathscr{L}) = \sigma(\mathscr{L}).$$

ii) If we set $H_i^* := H_i \setminus \{0\}$ and $H^* := H_1^* \oplus \ldots \oplus H_n^*$, then

$$W^n(\mathscr{L}) := \bigcup_{x \in H^*} \sigma(\mathscr{L}_x)$$

since, for $x = (x_1, \ldots, x_n) \in H^*$ and $\widehat{x} = (\widehat{x}_1, \ldots, \widehat{x}_n) \in \mathscr{S}^n$ with $\widehat{x}_i := \dfrac{x_i}{\|x_i\|}$,

$$\det \mathscr{L}(z)_x = 0 \iff \det \mathscr{L}(z)_{\widehat{x}} = 0.$$

iii) For $n = 1$, the block numerical range is the usual numerical range, i.e.

$$W^1(\mathscr{L}) = W(\mathscr{L}).$$

iv) For a linear operator function $\mathscr{L} : \mathbb{C} \to L(H)$, $\mathscr{L}(z) := \mathscr{A} - z$, $z \in \mathbb{C}$, with $\mathscr{A} \in L(H)$, the block numerical range of $\mathscr{L}$ and of $\mathscr{A}$ coincide, i.e.

$$W^n(\mathscr{L}) = W^n(\mathscr{A}).$$

v) With the identification iii), the block numerical range can also be written as

$$W^n(\mathscr{L}) = \{z \in \Omega : 0 \in W^n(\mathscr{L}(z))\}. \tag{2.3}$$

PROPOSITION 2.3. *If* $\mathscr{L} : \Omega \to L(H)$ *is an operator function,* $\Omega^* := \{z \in \mathbb{C} : \overline{z} \in \Omega\}$*, and*

$$\mathscr{L}^* : \Omega^* \to L(H), \quad \mathscr{L}^*(z) := \mathscr{L}(\overline{z})^*,$$

is its adjoint operator function*, then,*

$$W^n(\mathscr{L}^*) = W^n(\mathscr{L})^* := \{\lambda \in \mathbb{C} : \overline{\lambda} \in W^n(\mathscr{L})\}.$$

In particular, if $\mathscr{L}$ *is* self-adjoint*, i.e.* $\Omega^* = \Omega$ *and* $\mathscr{L}^* = \mathscr{L}$*, then* $W^n(\mathscr{L})$ *is symmetric with respect to the real axis.*

Proof. Let $z \in \Omega$. By [33, Rem. 2.3] we have $W^n(\mathscr{L}(\overline{z})^*) = W^n(\mathscr{L}(\overline{z}))^*$. Hence, by (2.3),

$$\begin{aligned} z \in W^n(\mathscr{L}^*) &\iff 0 \in W^n(\mathscr{L}^*(z)) \iff 0 \in W^n(\mathscr{L}(\overline{z})^*) \iff 0 \in W^n(\mathscr{L}(\overline{z}))^* \\ &\iff 0 \in W^n(\mathscr{L}(\overline{z})) \iff \overline{z} \in W^n(\mathscr{L}) \iff z \in W^n(\mathscr{L})^*. \quad \square \end{aligned}$$

3. Inclusions between block numerical ranges and classical numerical ranges

In this section we show that the block numerical ranges become smaller if the decomposition is refined; in particular, the block numerical range of an operator function is always contained in the numerical range. Moreover, we show that the numerical ranges of all principal submatrices, in particular, of all diagonal entries L_{ii} are contained in the block numerical range if certain dimension conditions hold.

Both results are straightforward generalisations of respective theorems [33, Thm. 3.5, Thm. 3.1] for the operator case.

PROPOSITION 3.1. *Let $\mathscr{L}:\Omega\to L(H)$ be an operator function and let*

$$H=\underbrace{H_1^1\oplus\cdots\oplus H_1^{m_1}}_{=H_1}\oplus\cdots\oplus\underbrace{H_n^1\oplus\cdots\oplus H_n^{m_n}}_{=H_n}$$

be a refinement of the decomposition $H=H_1\oplus\cdots\oplus H_n$. Then, with $m:=\sum_{i=1}^n m_i$,

$$W^m_{H_1^1\oplus\cdots\oplus H_1^{m_1}\oplus\cdots\oplus H_n^1\oplus\cdots\oplus H_n^{m_n}}\subset W^n_{H_1\oplus\cdots\oplus H_n},$$

or, briefly,

$$W^m(\mathscr{L})\subset W^n(\mathscr{L}). \tag{3.1}$$

Proof. Let $\lambda\in W^m(\mathscr{L})$. Then $0\in W^m(\mathscr{L}(\lambda))\subset W^n(\mathscr{L}(\lambda))$ by [33, Thm. 3.5], and thus $\lambda\in W^n(\mathscr{L})$. $\square$

EXAMPLE 3.2. Consider the quadratic 6×6 matrix polynomial

$$\mathscr{P}(z):=\begin{pmatrix}1&0&0&0&0&0\\0&1&1&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&1&1&0\\0&0&0&0&0&1\end{pmatrix}z^2+\begin{pmatrix}-1&0&0&0&0&0\\0&0&0&0&1&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\0&-1&0&0&0&0\\0&0&0&0&0&-1\end{pmatrix}z+\begin{pmatrix}1&0&1&0&0&0\\0&1&0&0&0&0\\1&0&1&0&0&0\\0&0&0&1&0&1\\0&0&0&0&1&0\\0&0&0&1&0&1\end{pmatrix},$$
$$z\in\mathbb{C},$$

with respect to the refined decompositions $\mathbb{C}^6=\mathbb{C}^2\times\mathbb{C}^4=\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2$. Proposition 3.1 yields that $W^3_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{P})\subset W^2_{\mathbb{C}^4\times\mathbb{C}^2}(\mathscr{P})\subset W(\mathscr{P})$; the corresponding block numerical ranges are displayed in Figure 1. Note that in this case, only the cubic numerical range is considerably smaller than the numerical range.

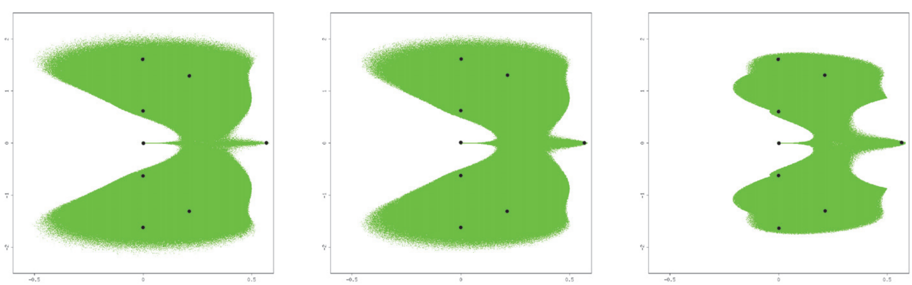

Figure 1: $W(\mathscr{P})$, $W^2_{\mathbb{C}^4\times\mathbb{C}^2}(\mathscr{P})$, $W^3_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{P})$ *and eigenvalues of the quadratic* 6×6 *matrix polynomial* $\mathscr{P}$ *in Ex.* 3.2.

The following results show that, if all H_i are infinite dimensional, then the block numerical ranges of all principal submatrix operator functions of $\mathscr{L}$ are contained in $W^n(\mathscr{L})$ and, in particular, $W(L_{ii})\subset W^n(\mathscr{L})$. If some of the components are finite dimensional, these inclusions only hold under some restrictions on their dimensions.

PROPOSITION 3.3. *Let* $\mathscr{L} = (L_{ij})_{i,j=1}^n : \Omega \to L(H)$ *be an operator function,* $l \in \mathbb{N}$, $I := \{i_1,\ldots,i_l\} \subset \{1,\ldots,n\}$, *and let*

$$\mathscr{L}_I := (L_{i_j i_k})_{j,k=1}^l : \Omega \to L(H_{i_1} \oplus \ldots \oplus H_{i_l})$$

be the operator function arising from $\mathscr{L}$ *by selecting the rows and columns* $i_1,\ldots, i_l$. *If there exists an enumeration of* $\{i'_1,\ldots,i'_{n-l}\} := \{1,\ldots,n\} \setminus \{i_1,\ldots,i_l\}$ *such that* $\dim H_{i'_j} > n-j$, $j = 1,\ldots,n-l$, *then*

$$W^l(\mathscr{L}_I) \subset W^n(\mathscr{L}); \tag{3.2}$$

in particular, the inclusion (3.2) *holds if* $\dim H_{i'_j} = \infty$, $j = 1,\ldots,n-l$.

Proof. The claim follows from [33, Thm. 3.1] in a straightforward way if we use the characterization (2.3) of the block numerical range. □

COROLLARY 3.4. *Let* $\mathscr{L} = (L_{ij})_{i,j=1}^n : \Omega \to L(H)$ *be an operator function and* $i \in \{1,\ldots,n\}$. *If there exists an enumeration of* $\{i'_1,\ldots,i'_{n-1}\} := \{1,\ldots,i-1, i+1,\ldots,n\}$ *with* $\dim H_{i'_j} > n-j$, $j = 1,\ldots,n-1$, *then*

$$W(L_{ii}) \subset W^n(\mathscr{L}); \tag{3.3}$$

in particular, if $\dim H_j > n$, $j = 1,\ldots,n$, *then* $W(L_{ii}) \subset W^n(\mathscr{L})$ *for all* $i = 1,\ldots,n$.

REMARK 3.5. If $\dim H_j = \infty$, $j = 1,\ldots,n$, then

$$W^{|I|}(\mathscr{L}_I) \subset W^n(\mathscr{L}) \quad \text{for all } I \subset \{1,\ldots,n\};$$

in particular, $W(L_{ii}) \subset W^n(\mathscr{L})$ for all $i = 1,\ldots,n$.

In the special case $n = 2$ of the quadratic numerical range, the dimension conditions in Proposition 3.3 and Corollary 3.4 reduce to $\dim H_2 > 1$ for $W(L_{11}) \subset W^n(\mathscr{L})$ and $\dim H_1 > 1$ for $W(L_{22}) \subset W^n(\mathscr{L})$ (see [32, Proposition 4.1].

If one H_i has finite dimension, there is another criterion for the inclusion in (3.3).

PROPOSITION 3.6. *Let* $\mathscr{L} = (L_{ij})_{i,j=1}^n : \Omega \to L(H)$ *be an operator function and* $i \in \{1,\ldots,n\}$ *such that* $\dim H_i < \infty$ *and* $\dim H_i < \dim H_j$ *for all* $j \in \{1,\ldots,n\}$, $j \neq i$. *Then*

$$W(L_{ii}) \subset W^n(\mathscr{L}).$$

Proof. Without loss of generality, we may assume that $i = 1$. Let $\lambda \in W(L_{11})$. By the assumption on the dimensions, there exist $x_j \in H_j$, $\|x_j\| = 1$, with $L_{1j}(\lambda)x_j = 0$,

$j = 2, \ldots, n$. Since $\lambda \in W(L_{11})$, there exists $x_1 \in H_1$, $\|x_1\| = 1$, such that $(L_{11}(\lambda)x_1, x_1) = 0$. Then $x = (x_1, \ldots, x_n) \in \mathscr{S}^n$,

$$\mathscr{L}_x(\lambda) = \begin{pmatrix} 0 & \cdots & 0 \\ (L_{21}(\lambda)x_1, x_2) & \cdots & (L_{2n}(\lambda)x_n, x_2) \\ \vdots & & \vdots \\ (L_{n1}(\lambda)x_1, x_n) & \cdots & (L_{nn}(\lambda)x_n, x_n) \end{pmatrix},$$

and thus $0 \in W^n(\mathscr{L}(\lambda))$ or, equivalently, $\lambda \in W^n(\mathscr{L})$. $\square$

REMARK 3.7. If $\dim H_i < \infty$ for some $i \in \{1, \ldots, n\}$ and $\dim H_i \geqslant n-1$, then the dimension condition in Corollary 3.4 is weaker than the one in Proposition 3.6; if $\dim H_i = 1$, then the dimension condition in Proposition 3.6 is weaker than the one in Corollary 3.4; if $1 < \dim H_i < n-1$, the conditions are not comparable.

We remark that, unlike the operator case (comp. [33, Cor. 3.3]), one cannot conclude, for any $n \geqslant 2$, that the numerical ranges $W(L_{ii})$ of the diagonal operator functions L_{ii} are contained in a connected component of $W^n(\mathscr{L})$. The reason for this is that the numerical range of an operator function is not connected in general; the simplest example is a strongly damped quadratic operator polynomial whose numerical range consists of two disjoint real intervals (see [10] and [18, § 31] for higher order so-called hyperbolic operator polynomials).

4. Spectral inclusion

In this section we prove the spectral inclusion property of the block numerical range of an analytic operator function. It generalises all earlier spectral inclusion results for the numerical range and the block numerical range of bounded linear operators (see [4, Thm. 1.2-1], [33, Thm. 2.5]) as well as those for the numerical range and the quadratical numerical range of analytic operator functions (see [18, Thm. 26.6], [32, Thm. 3.5]).

Like the case of the numerical range ($n = 1$) and quadratic numerical range ($n = 2$) of an operator function $\mathscr{L}$, spectral inclusion holds only if an additional condition is satisfied: there exists a $z_0 \in \Omega$ such that $0 \notin \overline{W^n(\mathscr{L}(z_0))}$ for $n = 1, 2$. Note that this condition is automatically satisfied if $\mathscr{L}(z) = \mathscr{A} - z$ with a bounded linear operator $\mathscr{A}$.

In the following spectral inclusion theorem for the block numerical range ($n \in \mathbb{N}$) this condition is weakened. Unlike the results in the previous section, the claim cannot easily be deduced from the operator case; the crucial step is to prove the equivalence (4.2) below.

THEOREM 4.1. *Let $\mathscr{L} : \Omega \to L(H)$ be an operator function. Then*

$$\sigma_p(\mathscr{L}) \subset W^n(\mathscr{L}).$$

If, additionally, $\mathscr{L}$ is analytic and

$$\exists s \in \mathbb{N}_0 \; \exists z_0 \in \Omega: \; 0 \notin \overline{\{(\det \mathscr{L}_x)^{(s)}(z_0) : x \in \mathscr{S}^n\}}, \tag{4.1}$$

then

$$\sigma(\mathscr{L}) \subset \overline{W^n(\mathscr{L})}.$$

Figure 1 illustrates the inclusion of the eigenvalues for the quadratic matrix polynomial $\mathscr{P}$ in Example 3.2; due to Proposition 3.1 higher order block numerical ranges may yield tighter and tighter spectral inclusions (see also Figures 2–6 below).

Proof of Theorem 4.1. Let $\lambda_0 \in \sigma_p(\mathscr{L})$, i.e. $0 \in \sigma_p(\mathscr{L}(\lambda_0))$. Then the spectral inclusion [33, Thm. 2.5] for the block numerical range of operators yields $\sigma_p(\mathscr{L}(\lambda_0)) \subset W^n(\mathscr{L}(\lambda_0))$, thus, $\lambda_0 \in W^n(\mathscr{L})$.

Now suppose that $\mathscr{L}$ is analytic and $\lambda_0 \in \sigma(\mathscr{L})$, i.e. $0 \in \sigma(\mathscr{L}(\lambda_0))$. Then, again by [33, Thm. 2.5], it follows that $\sigma(\mathscr{L}(\lambda_0)) \subset \overline{W^n(\mathscr{L}(\lambda_0))}$. The main part of the proof consists in showing that, if (4.1) holds, then the equivalence

$$0 \in \overline{W^n(\mathscr{L}(\lambda))} \quad \Longleftrightarrow \quad \lambda \in \overline{W^n(\mathscr{L})} \tag{4.2}$$

prevails for $\lambda \in \Omega$ (see Lemma 4.5 below). This proves $\lambda_0 \in \overline{W^n(\mathscr{L})}$, as required. □

The proof of the equivalence (4.2) is divided into several lemmas.

LEMMA 4.2. *For a subset $\mathscr{M} \subset M_n(\mathbb{C})$ define*

$$\sigma(\mathscr{M}) := \bigcup_{A \in \mathscr{M}} \sigma(A).$$

If $\mathscr{M}$ is bounded with respect to the norm topology, then $\sigma\left(\overline{\mathscr{M}}\right) = \overline{\sigma(\mathscr{M})}$.

Proof. "$\supset$" The set $\overline{\mathscr{M}} \subset M_n(\mathbb{C})$ is compact being a closed and bounded set in a finite dimensional space. By [6, Cor. 4.2.2] the set $\sigma(\overline{\mathscr{M}})$ is compact as well and hence

$$\sigma(\overline{\mathscr{M}}) = \overline{\sigma(\overline{\mathscr{M}})} \supset \overline{\sigma(\mathscr{M})}.$$

"$\subset$" For a metric space (X,d) we equip the set of all compact subsets

$$\mathscr{K}(X) := \{K \subset X : K \text{ compact}, K \neq \emptyset\}$$

with the *Hausdorff metric* $d_H : \mathscr{K}(X) \times \mathscr{K}(X) \to [0,\infty)$, defined as

$$d_H(K_1,K_2) := \max\left\{\max_{x_1 \in K_1} \operatorname{dist}(x_1,K_2), \max_{x_2 \in K_2} \operatorname{dist}(x_2,K_1)\right\},$$

where $\operatorname{dist}(x,K) := \min\{d(x,y) : y \in K\}$ for $K \in \mathscr{K}(X)$ and $x \in X$. Moreover, by [6, Cor. 4.2.2], the mapping

$$\sigma : (\mathscr{K}(M_n(\mathbb{C})), d_H) \to (\mathscr{K}(\mathbb{C}), d_H), \quad K \mapsto \sigma(K),$$

is continuous.

Now let $\lambda_0 \in \sigma(\overline{\mathscr{M}})$, $\lambda_0 \in \sigma(A)$ with $A \in \overline{\mathscr{M}}$, and $\varepsilon > 0$. Since σ is continuous, there exists a $\delta > 0$ such that $d_H(\sigma(M), \sigma(A)) < \varepsilon$ for $M \in M_n(\mathbb{C})$ with $d_H(\{M\}, \{A\}) < \delta$; observe that $d_H(\{M\}, \{A\}) = d(M, A)$ where d denotes the metric on $M_n(\mathbb{C})$ induced by the norm on $M_n(\mathbb{C})$. Since $A \in \overline{\mathscr{M}}$, there exists a $B \in \mathscr{M}$ such that $d(A, B) < \delta$ and thus $d_H(\sigma(A), \sigma(B)) < \varepsilon$. In particular, there is a $\lambda \in \sigma(B)$ such that $|\lambda - \lambda_0| < \varepsilon$. As $\sigma(B) \subset \underline{\sigma(\mathscr{M})}$, we have $B_\varepsilon(\lambda_0) \cap \sigma(\mathscr{M}) \neq \emptyset$. Because ε was arbitrary, this implies that $\lambda_0 \in \overline{\sigma(\mathscr{M})}$. $\square$

LEMMA 4.3. *For* $\mathscr{A} \in L(H)$ *define the* block determinant set

$$D^n(\mathscr{A}) := \{\det \mathscr{A}_x : x \in \mathscr{S}^n\}, \tag{4.3}$$

Then, for $z \in \mathbb{C}$,

$$z \in \overline{W^n(\mathscr{A})} \quad \Longleftrightarrow \quad 0 \in \overline{D^n(\mathscr{A} - z)};$$

in particular,

$$0 \in \overline{W^n(\mathscr{A})} \quad \Longleftrightarrow \quad 0 \in \overline{D^n(\mathscr{A})}.$$

REMARK 4.4. It is an open problem whether even the equivalence $z \in \partial W^n(\mathscr{A}) \Longleftrightarrow 0 \in \partial D^n(\mathscr{A} - z)$ holds.

Proof. "$\Longrightarrow$" If $z \in \overline{W^n(\mathscr{A})}$, there exist sequences $(x_k)_{k\in\mathbb{N}} \subset \mathscr{S}^n$ and $(z_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $\det(\mathscr{A}_{x_k} - z_k) = 0$, $k \in \mathbb{N}$, and $z_k \to z$, $k \to \infty$. Since

$$(\mathscr{A}_{x_k} - z_k) - (\mathscr{A}_{x_k} - z) = z - z_k \to 0, \quad k \to \infty,$$

and since $\det(\cdot)$ is uniformly continuous on compact subsets of $M_n(\mathbb{C})$, we conclude

$$\big|\det(\mathscr{A}_{x_k} - z)\big| = \big|\det(\mathscr{A}_{x_k} - z) - \det(\mathscr{A}_{x_k} - z_k)\big| \to 0, \quad k \to \infty.$$

Therefore, $0 \in \overline{D^n(\mathscr{A} - z)}$.

"$\Longleftarrow$" Consider the mapping

$$\chi : \mathscr{S}^n \to M_n(\mathbb{C}), \quad x \mapsto \mathscr{A}_x.$$

As $\|\mathscr{A}_x\| \leqslant \|\mathscr{A}\|$ for all $x \in \mathscr{S}^n$ by [33, Rem. 2.3], the image $\mathscr{M} := \chi(\mathscr{S}^n)$ is bounded and hence $\sigma(\overline{\mathscr{M}}) = \overline{\sigma(\mathscr{M})}$ by Lemma 4.2. For $z \in \mathbb{C} \setminus \overline{W^n(\mathscr{A})}$ define

$$\Delta_z : \overline{\mathscr{M}} \to \mathbb{C}, \quad \Delta_z(B) := \det(B - z).$$

If $\Delta_z(B) = 0$, then $z \in \sigma(B) \subset \sigma(\overline{\mathscr{M}}) = \overline{\sigma(\mathscr{M})} = \overline{W^n(\mathscr{A})}$, in contradiction to the assumption $z \in \mathbb{C} \setminus \overline{W^n(\mathscr{A})}$. As Δ_z is continuous and $\overline{\mathscr{M}}$ is compact, this leads to

$$0 < \min_{B \in \overline{\mathscr{M}}} |\Delta_z(B)| = \inf_{B \in \mathscr{M}} |\det(B - z)| = \inf_{x \in \mathscr{S}^n} |\det(\mathscr{A}_x - z)|$$

for all $z \in \mathbb{C} \setminus \overline{W^n(\mathscr{A})}$. Thus, $0 \notin \overline{D^n(\mathscr{A} - z)}$. $\square$

LEMMA 4.5. *Let $\mathscr{L}:\Omega\to L(H)$ be a continuous operator function and $z\in\Omega$. Then*

$$z\in\overline{W^n(\mathscr{L})}\quad\Longrightarrow\quad 0\in\overline{W^n(\mathscr{L}(z))}.$$

If, in addition, $\mathscr{L}$ is analytic and (4.1) is fulfilled, then

$$z\in\overline{W^n(\mathscr{L})}\quad\Longleftrightarrow\quad 0\in\overline{W^n(\mathscr{L}(z))}. \tag{4.4}$$

Proof. "$\Longrightarrow$" Assume that $z\in\overline{W^n(\mathscr{L})}\cap\Omega$. Then there exist sequences $(z_k)_{k\in\mathbb{N}}\subset\Omega$ and $(x_k)_{k\in\mathbb{N}}\subset\mathscr{S}^n$ such that $\det\mathscr{L}_{x_k}(z_k)=0$, $k\in\mathbb{N}$, and $z_k\to z$, $k\to\infty$. By [33, Rem. 2.3] we have $\|\mathscr{A}_x\|\leqslant\|\mathscr{A}\|$ for every $\mathscr{A}\in L(H)$, $x\in\mathscr{S}^n$. Thus

$$\left\|\mathscr{L}_{x_k}(z_k)-\mathscr{L}_{x_k}(z)\right\|=\left\|(\mathscr{L}(z_k)-\mathscr{L}(z))_{x_k}\right\|\leqslant\|\mathscr{L}(z_k)-\mathscr{L}(z)\|\to 0,\quad k\to\infty,$$

as $\mathscr{L}$ is continuous at z. Since $\det(\cdot)$ is uniformly continuous on compact subsets of $M_n(\mathbb{C})$, we obtain

$$\left|\det(\mathscr{L}_{x_k}(z))\right|=\left|\det(\mathscr{L}_{x_k}(z))-\det(\mathscr{L}_{x_k}(z_k))\right|\to 0,\quad k\to\infty.$$

This means that $0\in\overline{D^n(\mathscr{L}(z))}$ and hence $0\in\overline{W^n(\mathscr{L}(z))}$ by Lemma 4.3.

"$\Longleftarrow$" Suppose that (4.1) is satisfied. Let $z\in\Omega$ be such that $0\in\overline{W^n(\mathscr{L}(z))}$ and thus $0\in\overline{D^n(\mathscr{L}(z))}$ by Lemma 4.3. Then there exists a sequence $(x_k)_{k\in\mathbb{N}}\subset\mathscr{S}^n$ such that

$$\lim_{k\to\infty}\det\mathscr{L}_{x_k}(z)=0.$$

The sequence $(\varphi_k)_{k\in\mathbb{N}}$ of analytic functions defined by

$$\varphi_k:\Omega\to\mathbb{C},\quad \varphi_k(\lambda):=\det\mathscr{L}_{x_k}(\lambda),$$

is uniformly bounded on compact subsets of Ω. Hence, by Montel's Theorem (see [17, Thm. I.17.17 (p. 415_1)]) we may assume that $(\varphi_k)_{k\in\mathbb{N}}$ converges uniformly to an analytic function $\varphi:\Omega\to\mathbb{C}$ on every compact subset of Ω. By assumption,

$$\varphi(z)=\lim_{k\to\infty}\det\mathscr{L}_{x_k}(z)=0.$$

To show that $\varphi\not\equiv 0$, assume that $\varphi\equiv 0$. Then, for every $s\in\mathbb{N}_0$ and $\lambda\in\Omega$, using that $\varphi^{(s)}$ is the pointwise limit of the sequence $(\varphi_k^{(s)})_{k\in\mathbb{N}}$, we conclude that

$$0=\varphi^{(s)}(\lambda)=\lim_{k\to\infty}\varphi_k^{(s)}(\lambda)=\lim_{k\to\infty}(\det\mathscr{L}_{x_k})^{(s)}(\lambda),$$

which yields that $0\in\overline{\{(\det\mathscr{L}_x)^{(s)}(\lambda):x\in\mathscr{S}^n\}}$, a contradiction to assumption (4.1). Hence $\varphi\not\equiv 0$. As a consequence of Hurwitz's Theorem (see [17, Thm. II.2.5 (p. 49_2)]), there exists a sequence $(z_k)_{k\in\mathbb{N}}\subset\Omega$ such that $z_k\to z, k\to\infty$, and $\varphi_k(z_k)=0$ for all $k\in\mathbb{N}$. This implies that $0\in W^n(\mathscr{L}(z_k))$ for all $k\in\mathbb{N}$, i.e. $z_k\in W^n(\mathscr{L})$ and therefore $z\in\overline{W^n(\mathscr{L})}$. $\square$

COROLLARY 4.6. *If H is finite dimensional and $\mathscr{L}:\Omega\to L(H)$ is continuous, then $W^n(\mathscr{L})$ is closed in Ω.*

Proof. If $z\in\overline{W^n(\mathscr{L})}\cap\Omega$, then $0\in\overline{W^n(\mathscr{L}(z))}$ by the first part of Lemma 4.5. Since H is finite dimensional, the set $W^n(\mathscr{L}(z))$ is compact, see [33, Rem. 2.3]. Thus, $\overline{W^n(\mathscr{L}(z))}=W^n(\mathscr{L}(z))$ and hence $0\in W^n(\mathscr{L}(z))$ which means $z\in W^n(\mathscr{L})$. □

The following example illustrates the spectral inclusion in Theorem 4.1 for a matrix function which is not an operator polynomial and for which it seems to be hardly possible to obtain analytic information about the eigenvalues.

EXAMPLE 4.7. ($\sigma(\mathscr{L})=\sigma_p(\mathscr{L})\subset W^n(\mathscr{L})$) The spectrum of the 4×4 matrix function $G:\mathbb{C}\to L(\mathbb{C}^4)$ given by

$$G(z)=\begin{pmatrix}2-z & \mathrm{i} & 1 & -\sin z\\ \mathrm{i} & 2-z & 3+\mathrm{i} & 1\\ 1 & 3+\mathrm{i} & -2-z & \mathrm{i}\\ 3+\mathrm{i} & 1 & \mathrm{i} & -2-z\end{pmatrix},\quad z\in\mathbb{C},$$

consists only of eigenvalues $\lambda\in\sigma_p(G)$, given as the zeros of the characteristic determinant $\det G(\lambda)=0$, where

$$\det G(z)=z^4-(16+6\mathrm{i})z^2+(6-18\mathrm{i})z+68+24\mathrm{i}+\sin(z)(1+\mathrm{i})((2-\mathrm{i})z^2+(1+\mathrm{i})z-30).$$

Figure 2 illustrates that the cubic numerical range of G with respect to $\mathbb{C}^4=\mathbb{C}\times\mathbb{C}\times\mathbb{C}^2$ gives a tighter inclusion of the eigenvalues than the quadratic numerical range with respect to $\mathbb{C}^4=\mathbb{C}^2\times\mathbb{C}^2$.

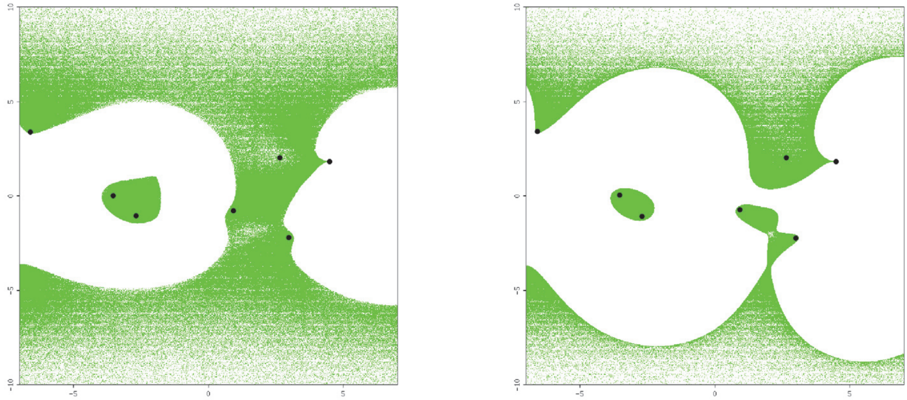

Figure 2: $W^2_{\mathbb{C}^2\times\mathbb{C}^2}(G)$, $W^3_{\mathbb{C}\times\mathbb{C}\times\mathbb{C}^2}(G)$ *and eigenvalues of the matrix function G in Ex. 4.7 in the rectangle* $[-7,7]\times[-10,10]$.

The following example shows that condition (4.1) is necessary for the inclusion of the spectrum in the closure of the block numerical range of an operator function.

EXAMPLE 4.8. Let $f : \Omega \to \mathbb{C}$ be analytic, $f \not\equiv 0, \mathscr{A} \in L(H)$, and

$$\mathscr{L} : \Omega \to L(H), \quad \mathscr{L}(z) := f(z)\mathscr{A}.$$

Then, if $N_\Omega(f)$ denotes the set of zeros of f in Ω, it is not difficult to see that

$$\sigma(\mathscr{L}) = \begin{cases} \Omega, & 0 \in \sigma(\mathscr{A}), \\ N_\Omega(f), & 0 \notin \sigma(\mathscr{A}), \end{cases} \qquad W^n(\mathscr{L}) = \begin{cases} \Omega, & 0 \in W^n(\mathscr{A}), \\ N_\Omega(f), & 0 \notin W^n(\mathscr{A}). \end{cases}$$

If we choose $\mathscr{A}$ such that

$$0 \in \sigma(\mathscr{A}) \text{ and } 0 \in \overline{W^n(\mathscr{A})} \setminus W^n(\mathscr{A}), \tag{4.5}$$

then

$$\Omega = \sigma(\mathscr{L}) \not\subset W^n(\mathscr{L}) = N_\Omega(f).$$

In fact, condition (4.1) is not satisfied for $\mathscr{L}$ since

$$\forall s \in \mathbb{N}_0 \; \forall z_0 \in \Omega \;\; 0 \in \overline{\{f^{(s)}(z_0) \det \mathscr{A}_x : x \in \mathscr{S}^n\}};$$

the latter holds because $0 \in \overline{W^n(\mathscr{A})}$, i.e. $0 \in \overline{\{\det \mathscr{A}_x : x \in \mathscr{S}^n\}}$.

For example, we can choose $\mathscr{A} := \mathrm{Id} - L$ where L is the left shift on $\ell^2(\mathbb{N})$,

$$L : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}), \quad (x_n)_{n\in\mathbb{N}} \mapsto (x_{n+1})_{n\in\mathbb{N}}.$$

It is well-known that $\sigma_p(L) = \{z \in \mathbb{C} : |z| < 1\}$, $\sigma(L) = \overline{\sigma_p(L)}$, and $W(L) = \{z \in \mathbb{C} : |z| < 1\}$ (see [5, Problem 82] and [4, Chapt. 1, Ex. 2]). Since $1 \in \overline{\sigma(L)}$ and, for an arbitrary decomposition of $\ell^2(\mathbb{N})$, $1 \notin W(L) \supset W^n(L)$, we conclude

$$0 \in \sigma(\mathscr{A}) = \{1 - \lambda : \lambda \in \sigma(L)\},$$
$$0 \notin W^n(\mathscr{A}) = \{1 - \lambda : \lambda \in W^n(L)\}, \quad 0 \in \sigma(\mathscr{A}) \subset \overline{W^n(\mathscr{A})}.$$

Thus, (4.5) is satisfied.

5. Resolvent estimates

In this section we establish upper bounds for the norm of the resolvent $\mathscr{L}(\lambda)^{-1}$ of an operator function in terms of the distance of λ to the block numerical range of $\mathscr{L}$. This result generalises corresponding estimates for the operator case (see [33, Thm. 4.2]) as well as for operator functions in terms of the numerical range (see [19, Thm. 1]) and the quadratic numerical range (see [32, Thm. 5.2]).

Due to the inclusions $W^n(\mathscr{L}) \subset W^2(\mathscr{L}) \subset W(\mathscr{L})$ for $n \geqslant 2$, the block numerical range provides tighter bounds and bounds even in points $\lambda \in \overline{W(\mathscr{L})} \setminus W^n(\mathscr{L})$ and $\lambda \in \overline{W^2(\mathscr{L})} \setminus W^n(\mathscr{L})$, respectively.

THEOREM 5.1. *Let $\mathscr{L} : \Omega \to L(H)$ be an analytic operator function such that (4.1) holds. Let $C \subset W^n(\mathscr{L})$ be a bounded connected component of $W^n(\mathscr{L})$ with*

$$\overline{C} \subset \Omega, \quad \overline{C} \cap \overline{W^n(\mathscr{L}) \setminus C} = \emptyset. \tag{5.1}$$

Then

i) *the number of zeros* $\nu_C(\det \mathscr{L}_x)$ *of* $\det \mathscr{L}_x$ *in* C (*counted with multiplicities*) *is finite and independent of* $x \in \mathscr{S}^n$,

$$\nu_C := \nu_C(\det \mathscr{L}_x), \quad x \in \mathscr{S}^n;$$

ii) *if* $U \subset \Omega$ *is a bounded domain such that*

$$\overline{C} \subset U, \quad \overline{U} \cap \overline{W^n(\mathscr{L}) \setminus C} = \emptyset,$$

then there exists a constant $\gamma > 0$ *such that*

$$\|\mathscr{L}^{-1}(z)\| \leqslant \frac{\gamma}{\operatorname{dist}(z,C)^{\nu_C}}, \quad z \in \overline{U} \setminus \overline{C}. \tag{5.2}$$

For the proof we need the following proposition. Here, and in the following, we use the suggestive notation $\inf|D^n(\mathscr{L}(z))| = \inf\{|\lambda| : \lambda \in D^n(\mathscr{L}(z))\}$.

PROPOSITION 5.2. *Let* $\mathscr{L} : \Omega \to L(H)$ *be an operator function. If* $0 \notin \overline{D^n(\mathscr{L}(z))}$ *for some* $z \in \mathbb{C}$, *then* $\mathscr{L}(z)$ *is invertible and*

$$\|\mathscr{L}^{-1}(z)\| \leqslant \frac{\|\mathscr{L}(z)\|^{n-1}}{\inf|D^n(\mathscr{L}(z))|}. \tag{5.3}$$

Proof. If $0 \notin \overline{D^n(\mathscr{L}(z))}$, then $0 \notin \overline{W^n(\mathscr{L}(z))}$ by Lemma 4.3, and thus $0 \notin \sigma(\mathscr{L}(z))$ by the spectral inclusion [33, Thm. 2.5], i.e. $\mathscr{L}(z)$ is (boundedly) invertible and $\mathscr{L}_x(z) = \mathscr{L}(z)_x$ is invertible for all $x \in \mathscr{S}^n$. Moreover, it follows from [7, Chapter I, (4.12)] and [33, Rem. 2.3] that

$$\|\mathscr{L}_x(z)^{-1}\| \leqslant \frac{\|\mathscr{L}_x(z)\|^{n-1}}{|\det \mathscr{L}_x(z)|} \leqslant \frac{\|\mathscr{L}(z)\|^{n-1}}{\inf|D^n(\mathscr{L}(z))|}, \quad x \in \mathscr{S}^n. \tag{5.4}$$

By [33, Lemma 4.1], if $\mathscr{A} \in L(H)$ is (boundedly) invertible with $\mathscr{A}_x$ invertible and $\|\mathscr{A}_x^{-1}\| \leqslant \gamma$ for all $x \in \mathscr{S}^n$, then also $\|\mathscr{A}^{-1}\| \leqslant \gamma$. Thus the estimate (5.3) follows from (5.4). $\square$

Proof of Theorem 5.1. The proof follows the ideas of the proof of [19, Thm. 1].

Assumption (5.1) implies, by [18, Lemma 28.5], that there exist domains $U \subset \Omega$ as in (ii) and $V \subset \Omega$ such that $\Gamma := \partial V$ consists of finitely many piecewise smooth Jordan curves not intersecting each other and

$$\overline{U} \subset V, \quad \overline{V} \cap \overline{W^n(\mathscr{L}) \setminus C} \neq \emptyset.$$

i) For a continuous function f on Γ with $f(t) \neq 0$, $t \in \Gamma$, the index $\operatorname{ind}_\Gamma f$ of f on Γ is defined as the change of the argument of $f(t)$ when t varies on Γ; if f is holomorphic in the interior of Γ, then $\operatorname{ind}_\Gamma f$ is equal to the number of zeros of f inside Γ by the argument principle (see [18, § 25.1]).

Since $\nu_C(\det\mathscr{L}_x) = \nu_{\overline{V}}(\det\mathscr{L}_x) = \operatorname{ind}_\Gamma(\det\mathscr{L}_x)$, $x \in H^*$, the claim in i) now follows immediately from the connectedness of $H^* = (H_1 \setminus \{0\}) \oplus \cdots \oplus (H_n \setminus \{0\})$ and from the continuity of the index function $\operatorname{ind}_\Gamma$.

ii) Let $x \in \mathscr{S}^n$ and denote by $\lambda_1(x), \ldots, \lambda_{\nu_C}(x) \in C$ the zeros of $\det\mathscr{L}_x$ in C counted with multiplicities. Then the functions $g_x : \Omega \to \mathbb{C}$, $x \in \mathscr{S}^n$, defined by

$$g_x(z) := \frac{\det\mathscr{L}_x(z)}{(z-\lambda_1(x))\cdots(z-\lambda_{\nu_C}(x))}, \quad z \in \Omega \setminus \{\lambda_1(x), \ldots, \lambda_{\nu_C}(x)\},$$

are analytic and do not have any zeros in $\overline{V}$. Moreover, we have

$$|\det\mathscr{L}_x(z)| = |g_x(z)|\,|z-\lambda_1(x)|\cdots\left|z-\lambda_{\nu_C}(x)\right| \geqslant |g_x(z)|\operatorname{dist}(z,C)^{\nu_C}, \quad z \in \Omega. \tag{5.5}$$

Assume that we have shown that condition (4.1) implies that

$$\inf\{|g_x(z)| : x \in \mathscr{S}^n, z \in \overline{U}\} =: d > 0. \tag{5.6}$$

Then, by (5.5),

$$\inf|D^n(\mathscr{L}(z))| \geqslant d\operatorname{dist}(z,C)^{\nu_C} > 0, \quad z \in \overline{U} \setminus \overline{C}. \tag{5.7}$$

Setting $\gamma := d^{-1}\max\{\|\mathscr{L}(z)\|^{n-1} : z \in \overline{U}\}$, we obtain from (5.3)

$$\left\|\mathscr{L}^{-1}(z)\right\| \leqslant \frac{\|\mathscr{L}(z)\|^{n-1}}{\inf|D^n(\mathscr{L}(z))|} \leqslant \frac{\|\mathscr{L}(z)\|^{n-1}}{d\operatorname{dist}(z,C)^{\nu_C}} \leqslant \frac{\gamma}{\operatorname{dist}(z,C)^{\nu_C}}, \quad z \in \overline{U} \setminus \overline{C},$$

as required. To prove (5.6), suppose to the contrary that there are sequences $(z_k)_{k\in\mathbb{N}} \subset \overline{U}$ and $(x_k)_{k\in\mathbb{N}} \subset \mathscr{S}^n$ such that $g_{x_k}(z_k) \to 0$, $k \to \infty$; as $\overline{U}$ is compact, we may assume that $z_k \to z_0 \in \overline{U} \subset V$, $k \to \infty$. The inequality (5.5) implies that

$$|g_x(z)| \leqslant \frac{|\det\mathscr{L}_x(z)|}{\operatorname{dist}(z,C)^{\nu_C}} \leqslant \frac{M}{\operatorname{dist}(\Gamma,C)^n} =: N, \quad z \in \Gamma = \partial V,\ x \in \mathscr{S}^n,$$

where $M := \sup\{|\det\mathscr{L}_x(z)| : x \in \mathscr{S}^n, z \in \Gamma\} < \infty$. Since g is analytic in V, the maximum modulus principle (see [17, Thm. I.17.5]) yields that $|g_x(z)| \leqslant N$, $x \in \mathscr{S}^n$, $z \in V$. Thus, by Montel's Theorem (see [17, Thm. I.17.7]), we may assume that $(g_{x_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of V to an analytic function $g : V \to \mathbb{C}$. From $g_{x_k}(z_k) \to 0, k \to \infty$, and the uniform convergence of the sequence $(g_{x_k})_{k\in\mathbb{N}}$ in a neighbourhood of z_0 it readily follows that $g(z_0) = 0$. In order to see that $g \not\equiv 0$, we note that for $z \in V \setminus \overline{C}$ we have $z \notin \overline{W^n(\mathscr{L})}$. Due to assumption (4.1), we can apply (4.4) in Lemma 4.5 which yields that $0 \notin \overline{W^n(\mathscr{L}(z))}$. By Lemma 4.3 this implies that $0 \notin \overline{D^n(\mathscr{L}(z))}$. Hence, in particular, $\inf\{\left|g_{x_k}(z)\right| : k \in \mathbb{N}\} > 0$ for every $z \in V \setminus \overline{C}$ and thus $g \not\equiv 0$. According to Hurwitz's Theorem there is a sequence $(\mu_k)_{k\in\mathbb{N}} \subset U$ such that $\mu_k \to z_0$, $k \to \infty$, and $g_{x_k}(\mu_k) = 0$, $k \in \mathbb{N}$. This is a contradiction to $g_{x_k}(z) \neq 0$, $z \in U$, $k \in \mathbb{N}$. $\square$

Resolvent estimates provide upper bounds for the lengths of Jordan chains in boundary points of the numerical range or block numerical range. Recall that for

$\lambda_0 \in \sigma_p(\mathscr{L})$ a vector $x_0 \in H \setminus \{0\}$ such that $\mathscr{L}(\lambda_0)x_0 = 0$ is called an *eigenvector of $\mathscr{L}$ in λ_0* and $x_0, x_1, \ldots, x_{m-1} \in H \setminus \{0\}$ are called a *Jordan chain of $\mathscr{L}$ in λ_0 of length m* if

$$\sum_{j=0}^{k} \frac{1}{k!} \mathscr{L}^{(j)}(\lambda_0) x_{k-j} = 0, \quad k = 0, \ldots, m-1.$$

Since the numerical and block numerical range of an operator function is not convex, in general, an additional property of the boundary point is required.

COROLLARY 5.3. *Let $\mathscr{L} : \Omega \to L(H)$ be analytic such that (4.1) holds, let $C \subset W^n(\mathscr{L})$ be a bounded connected component of $W^n(\mathscr{L})$ fulfilling (5.1), and let ν_C be as in Theorem 5.1. Let $\lambda_0 \in \partial C$ be an eigenvalue of $\mathscr{L}$ with the* exterior cone property, *i.e. there exists a closed cone K with vertex λ_0 and positive aperture and an $r > 0$ such that*

$$K \cap \overline{B_r(\lambda)} \cap \overline{W} = \{\lambda_0\}.$$

Then the lengths of all Jordan chains of $\mathscr{L}$ in λ_0 are at most ν_C.

Proof. The proof is identical to that of [19, Thm. 2] and hence omitted. □

6. Operator polynomials

In this section we consider a special class of operator functions, namely operator polynomials $\mathscr{P}$. We establish a sufficient condition for the spectral inclusion, improve the resolvent estimates, and prove criteria for the block numerical range $W^n(\mathscr{P})$ to be bounded or unbounded. Finally, we show that $W^n(\mathscr{P})$ is contained in some block numerical range of the linear companion pencil associated with $\mathscr{P}$.

An operator polynomial $\mathscr{P}$ of degree $d \in \mathbb{N}$ is of the form

$$\mathscr{P} : \mathbb{C} \to L(H), \quad \mathscr{P}(z) = A^{[d]} z^d + A^{[d-1]} z^{d-1} + \cdots + A^{[1]} z + A^{[0]} \tag{6.1}$$

with $A^{[l]} \in L(H)$, $l = 0, \ldots, d$, and $A^{[d]} \neq 0$; if $A^{[d]} = \mathrm{Id}_H$, then $\mathscr{P}$ is called *monic*.

6.1. Spectral inclusion

For operator polynomials there is a simple sufficient condition in terms of the leading coefficient for assumption (4.1) in the spectral inclusion Theorem 4.1. This condition is easy to check but far from being necessary; below we show that it implies that the block numerical range is bounded (see Theorem 6.4 i)).

THEOREM 6.1. *Let $\mathscr{P} : \mathbb{C} \to L(H)$ be an operator polynomial of degree d. Then*

$$\sigma_p(\mathscr{P}) \subset W^n(\mathscr{P}).$$

If, in addition, $0 \notin \overline{W^n(A^{[d]})}$, then

$$\sigma(\mathscr{P}) \subset \overline{W^n(\mathscr{P})}; \tag{6.2}$$

in particular, (6.2) always holds if $\mathscr{P}$ is monic.

For the proof of Theorem 6.1, we use the following lemmas; the proof of the first one is immediate.

LEMMA 6.2. *If $\mathscr{P} : \mathbb{C} \to L(H)$ is an operator polynomial of degree d, then for $z \in \mathbb{C}$ and $x \in H_1 \times \cdots \times H_n$*

$$\det \mathscr{P}_x(z) = \sum_{l=0}^{nd} \delta_x^{[l]} z^l = \det A_x^{[d]} z^{nd} + \sum_{l=1}^{nd-1} \delta_x^{[l]} z^l + \det A_x^{[0]} \tag{6.3}$$

where

$$\delta_x^{[l]} = \sum_{\substack{0 \leqslant l_1,\dots,l_n \leqslant d \\ l_1+\cdots+l_n = l}} \det(A_{ij}^{[l_i]} x_j, x_i)_{i,j=1}^n, \quad l = 0,\dots,nd. \tag{6.4}$$

LEMMA 6.3. *If $\mathscr{P} : \mathbb{C} \to L(H)$ is an operator polynomial of degree d and if $0 \notin \overline{W^n(A^{[d]})}$, then $0 \notin \overline{W^n(\mathscr{P}(z))}$ for large values of $|z|$. In particular, $\mathscr{P}$ satisfies (4.1) and hence, for $z \in \mathbb{C}$,*

$$z \in \overline{W^n(\mathscr{P})} \quad \Longleftrightarrow \quad 0 \in \overline{W^n(\mathscr{P}(z))}. \tag{6.5}$$

Proof. By the assumption $0 \notin \overline{W^n(A^{[d]})}$ and Lemma 4.3 it follows that $0 \notin \overline{D^n(A^{[d]})}$, i.e. $\delta := \inf_{x\in\mathscr{S}^n} \left|\det A_x^{[d]}\right| > 0$. Let $m := nd$ and define $C := \sup\{\delta_x^{[l]} : x \in \mathscr{S}^n, l = 0,\dots,m-1\}$ with $\delta_x^{[l]}$ as in (6.4). Then, for arbitrary $x \in \mathscr{S}^n$,

$$|\det \mathscr{P}_x(z)| = \left| \det A_x^{[d]} z^m + \sum_{l=0}^{m-1} \delta_x^{[l]} z^l \right| \geqslant \delta\, |z|^m - C \sum_{l=0}^{m-1} |z|^l \to \infty, \quad |z| \to \infty,$$

independently of $x \in \mathscr{S}^n$ and hence $\inf_{x\in\mathscr{S}^n} |\det \mathscr{P}_x(z)| > 0$ for large values of $|z|$. The latter means that $0 \notin \overline{D^n(\mathscr{P}(z))}$, which is equivalent to $0 \notin \overline{W^n(\mathscr{P}(z))}$ by Lemma 4.3, and it implies that condition (4.1) holds (with $s = 0$). Now the equivalence in (6.5) is immediate from Lemma 4.5. □

Proof of Theorem 6.1. By Lemma 6.3, the additional assumption on $\mathscr{P}$ guarantees that $\mathscr{P}$ satisfies the condition (4.1) in the spectral inclusion Theorem 4.1 for analytic operator functions. Therefore the latter implies both claims. □

6.2. Boundedness and connected components of $W^n(\mathscr{P})$

The boundedness of $W^n(\mathscr{P})$ is closely related to the position of the point 0 with respect to (the closure of) the block numerical range $W^n(A^{[d]})$ of the leading coefficient of $\mathscr{P}$. In the finite dimensional case it was proved in [13, Thm. 2.3] that the numerical range $W(\mathscr{P})$ is bounded if and only if $0 \notin W(A^{[d]})$ $(= \overline{W(A^{[d]})})$.

The following theorem shows that, in infinite dimensions, not only for the numerical range but also for the block numerical range, we only have

$$0 \notin \overline{W^n(A^{[d]})} \implies W^n(\mathscr{P}) \text{ is bounded.}$$

The converse, which was conjectured for $n=1$ in [13, Rem. after Ex. 3, p. 1260], is not true in general; this can be seen e.g. from Example 4.8 where $\mathscr{L}(z)=p(z)\mathscr{A}$ with $\mathscr{A}=\mathrm{Id}-L$ and L the left shift on $\ell^2(\mathbb{N})$ and a polynomial p. Here the (block) numerical range of the constant coefficient also plays a role.

THEOREM 6.4. *Let $\mathscr{P}:\mathbb{C}\to L(H)$ be an operator polynomial of degree d.*

i) *If $0\notin\overline{W^n(A^{[d]})}$, then $W^n(\mathscr{P})$ is bounded.*

ii) *If there exists $x\in\mathscr{S}^n$ with $\det A_x^{[d]}\neq 0$ and if $0\in W^n(A^{[d]})$, then $W^n(\mathscr{P})$ is unbounded.*

iii) *If there exists $x\in\mathscr{S}^n$ with $\det A_x^{[d]}\neq 0$ and if $0\in\overline{W^n(A^{[d]})}$ but $0\notin\overline{W^n(A^{[0]})}$, then $W^n(\mathscr{P})$ is unbounded.*

For the proof we use the following lemma.

LEMMA 6.5. *Let $p_k(z)=a_k^{[m]}z^m+\cdots+a_k^{[1]}z+a_k^{[0]}, z\in\mathbb{C}$, be complex polynomials of degree m, i.e. $a_k^{[m]}\neq 0$, $k\in\mathbb{N}$, such that $a_k^{[l]}\to a^{[l]}$, $k\to\infty$, $l=0,\dots,m$. If $a^{[m]}=0$ and $a^{[s]}\neq 0$ for some $s\in\{0,\dots,m-1\}$, then there exists a sequence $(\lambda_k)_{k\in\mathbb{N}}\subset\mathbb{C}$ such that $|\lambda_k|\to\infty$, and $p_k(\lambda_k)=0$, $k\in\mathbb{N}$.*

Proof. Let $p_k(z)=:a_k^{[m]}(z-\mu_{k1})\cdots(z-\mu_{km})$, $p(z):=a^{[m]}z^m+\ldots+a^{[1]}z+a^{[0]}$. Suppose the claim is false. Then there exists a constant $C>0$ such that $|\mu_{kl}|\leqslant C$, $k\in\mathbb{N}$, $l=0,\dots,m$; thus

$$|p(z)|=\lim_{k\to\infty}|p_k(z)|\leqslant\lim_{k\to\infty}\left|a_k^{[m]}\right|(|z|+C)^m=0,\quad z\in\mathbb{C},$$

i.e. $p\equiv 0$, a contradiction to the assumption that $a^{[s]}\neq 0$ for some s. □

Proof of Theorem 6.4. i) By Lemma 6.3 we have $0\notin\overline{W^n(\mathscr{P}(z))}$ and thus, by Lemma 4.5, $z\notin\overline{W^n(\mathscr{P})}$ for large values of $|z|$, which shows that $W^n(\mathscr{P})$ is bounded.

ii) Let $m:=nd$ and

$$N:=\{x\in\mathscr{S}^n:\ \det A_x^{[d]}=0\}.$$

Then $\emptyset\neq N\neq\mathscr{S}^n$ by the assumptions. Let $x_0\in\partial N\subset N$ and $(x_k)_{k\in\mathbb{N}}\subset\mathscr{S}^n\setminus N$ be a sequence converging to x_0. For the sequences $\left(\delta_{x_k}^{[l]}\right)_{k\in\mathbb{N}}$ of the corresponding coefficients of the polynomials $\det\mathscr{P}_{x_k}$ (see (6.3), (6.4)) we have

$$\lim_{k\to\infty}\delta_{x_k}^{[l]}=\delta_{x_0}^{[l]},\quad l=0,\dots,m.$$

Now consider two cases: If $\delta_{x_0}^{[l]}=0$ for all $l=0,\dots,m$, then

$$\det\mathscr{P}_{x_0}(z)=\lim_{k\to\infty}\det\mathscr{P}_{x_k}(z)=\lim_{k\to\infty}\sum_{l=0}^{m}\delta_{x_k}^{[l]}z^l=0,\quad z\in\mathbb{C},$$

that is, $z \in W^n(\mathscr{P})$ for every $z \in \mathbb{C}$, and thus $W^n(\mathscr{P}) = \mathbb{C}$ is unbounded. If, on the other hand, there is an $s \in \{0,\dots,m-1\}$ such that $\delta_{x_0}^{[s]} \neq 0$, then Lemma 6.5 applies to the sequence of polynomials $\left(\det \mathscr{P}_{x_k}\right)_1^\infty$; note that $\delta_{x_0}^{[m]} = \det A_{x_0}^{[d]} = 0$ and $\delta_{x_k}^{[m]} \neq 0$, as $x_k \notin N$, $k \in \mathbb{N}$. It yields a sequence $(\lambda_k)_{k\in\mathbb{N}} \subset \mathbb{C}$ such that $|\lambda_k| \to \infty$, $k \to \infty$, and $\det \mathscr{P}_{x_k}(\lambda_k) = 0$, $k \in \mathbb{N}$. In particular, $\lambda_k \in W^n(\mathscr{P})$, $k \in \mathbb{N}$, and hence $W^n(\mathscr{P})$ is unbounded.

iii) We may assume that $0 \in \overline{W^n(A^{[d]})} \setminus W^n(A^{[d]})$; otherwise the claim already follows from ii). Then there exists a sequence $(x_k)_{k\in\mathbb{N}} \subset \mathscr{S}^n$ such that $\det A_{x_k}^{[d]} \to 0$, $k \to \infty$, and $\det A_{x_k}^{[d]} \neq 0$, $k \in \mathbb{N}$. By passing to appropriate subsequences, we may assume that the sequences $\left(\delta_{x_k}^{[l]}\right)_{k\in\mathbb{N}}$, defined as in the proof of ii), converge to numbers $\delta^{[l]}, l = 0,\dots,m$. Then $\delta^{[m]} = 0$, and $\delta^{[0]} \neq 0$ according to the assumption $0 \notin \overline{W^n(A^{[0]})}$ (which is equivalent to $0 \notin \overline{D^n(A^{[0]})}$ by Lemma 4.3). In particular, Lemma 6.5 again applies to the sequence of polynomials $\left(\det \mathscr{P}_{x_k}\right)_{k\in\mathbb{N}}$; thus, the claim follows in the same way as in the second case in the proof of ii). $\square$

The following examples illustrate various phenomena that may occur for the block numerical range of operator polynomials. The first example shows that, unlike the numerical range, the block numerical range of a non-constant operator polynomial may be empty.

EXAMPLE 6.6. ($W^2(\mathscr{P}) = \emptyset$) For a non-constant operator polynomial $\mathscr{P}$ the numerical range $W(\mathscr{P})$ is never empty. This is no longer true for the block numerical range. An example is the linear operator polynomial

$$\mathscr{P}(z) := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{C},$$

in $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$. Then $\det \mathscr{P}_x(z) = 1 \neq 0$ for every $x \in \mathscr{S}^n$ and thus $W^2(\mathscr{P}) = \emptyset$.

The second example shows that the block numerical range may be bounded and non-empty, even though the numerical range is unbounded.

EXAMPLE 6.7. ($W(\mathscr{P})$ *unbounded, but* $W^2(\mathscr{P})$ *bounded and* $W^2(\mathscr{P}) \neq \emptyset$) It is not difficult to see that $W(\mathscr{P}) = \mathbb{C}$ if there exists a common non-zero isotropic vector $x \in H$, i.e. $(A^{[d]}x,x) = \dots = (A^{[0]}x,x) = 0$. For the quadratic polynomial

$$\mathscr{P}(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z^2 + \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z^2 + z & 1 \\ z & 0 \end{pmatrix}, \quad z \in \mathbb{C},$$

$x = (0,1)^{\mathrm{t}}$ is a common isotropic vector for all coefficient matrices and therefore $W(\mathscr{P}) = \mathbb{C}$.

On the other hand, with respect to the decomposition $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$, we have $W^2(\mathscr{P}) = \{0\}$ since $\det \mathscr{P}_x(z) = -z = 0$ if and only if $z = 0$; note that $0 \in W^2(A^{[2]})$

here. This example shows that a block-analogue of the existence of an isotropic vector cannot be as simple as the existence of some non-zero $x \in \mathscr{S}^n$ with

$$\det A_x^{[d]} = \cdots = \det A_x^{[0]} = 0.$$

EXAMPLE 6.8. ($W^n(\mathscr{P})$ *bounded*) For the operator polynomial $\mathscr{P}$ in Example 3.2, it is easy to see that $W(A^{[2]}) = \{\lambda \in \mathbb{C} : |\lambda - 1| \leqslant 1/2\}$, thus $0 \notin W(A^{[2]}) \supset W^n(A^{[2]})$ for all $n \in \mathbb{N}$. Hence it follows from (3.1) and Theorem 6.4 i) that all block numerical ranges $W^n(\mathscr{P})$ of $\mathscr{P}$ are bounded. The usual, quadratic and cubic numerical range of $\mathscr{P}$ for the two decompositions $\mathbb{C}^6 = \mathbb{C}^4 \times \mathbb{C}^2 = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ are illustrated in Figure 1.

The last example illustrates Theorem 6.4 ii); at the same time, it shows that, in general, it is difficult to obtain any analytical information about the shape of the numerical and block numerical range even in the case of operator polynomials.

EXAMPLE 6.9. ($W^n(\mathscr{P})$ *unbounded, but* $W^n(\mathscr{P}) \neq \mathbb{C}$) For the quadratic 3×3 matrix polynomial

$$P(z) = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) z^2 + \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 1 & 0 \end{array}\right) z + \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \hline 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} z^2 & 0 & 0 \\ 0 & z & 1 \\ \hline 0 & z & z^2 \end{array}\right), \; z \in \mathbb{C},$$

in $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}^1$, we have

$$\det A_x^{[2]} = \det \begin{pmatrix} (x_1, x_1) & 0 \\ 0 & x_3\overline{x_3} \end{pmatrix} = \|x_1\|^2 |x_3|^3, \quad x = (x_1\, x_2 \,|\, x_3)^{\mathrm{t}},$$

and so $\det A_x^{[2]} \neq 0$ e.g. for

$$x = (1\;0\,|\,1) \in \mathscr{S}^2 = \{x = (x_1\, x_2 \,|\, x_3)^{\mathrm{t}} \in \mathbb{C}^2 \times \mathbb{C} : |x_1|^2 + |x_2|^2 = 1, |x_3| = 1\};$$

moreover, $\det A_x^{[2]} = 0$ e.g. for $x = (0\;1\,|\,1) \in \mathscr{S}^2$ and thus $0 \in W^2(A^{[2]})$. Hence $\mathscr{P}$ satisfies the assumptions of Theorem 6.4 ii) and so $W^2(\mathscr{P})$ is unbounded; since $W^2(\mathscr{P}) \subset W(\mathscr{P})$, so must be $W(\mathscr{P})$.

To calculate $W^2(\mathscr{P})$, let $x = (x_1\, x_2 \,|\, x_3)^{\mathrm{t}} \in \mathscr{S}^2$ be arbitrary, $x = (t\;w\,|\,1)^t$ with $t \in [0,1]$ and $|w|^2 = 1 - t^2$. Then

$$\det \mathscr{P}_x(z) = \det \begin{pmatrix} t^2 z^2 & \overline{w} \\ wz & z^2 \end{pmatrix} = z(t^2 z^3 + (1 - t^2)(z^2 - 1)), \quad z \in \mathbb{C},$$

i.e. $W^2(\mathscr{P})$ consists of 0 and all zeros of the polynomials

$$p_t(z) = t^2 z^3 + (1 - t^2)(z^2 - 1), \quad t \in [0,1].$$

For $t = 0$ we obtain $-1,\ 1 \in W^2(\mathscr{P})$; for $t = 1$, we obtain $0 \in W^2(\mathscr{P})$ again. For $t \in (0,1)$ and $z \in \mathbb{C}$, $z \notin \{-1,1\}$, we have

$$p_t(z) = 0 \quad \Longleftrightarrow \quad \frac{z^3}{z^2 - 1} = \frac{t^2 - 1}{t^2} < 0,$$

and thus,

$$W^2(\mathscr{P}) \setminus \{-1,0,1\} = \left\{ z \in \mathbb{C} : \frac{z^3}{z^2-1} \in (-\infty,0) \right\}.$$

In particular, we have $W^2(\mathscr{P}) \cap \mathbb{R} = (-\infty,-1] \cup [0,1]$. Thus, $W^2(\mathscr{P})$ is unbounded, and hence also $W(\mathscr{P})$, but $W^2(\mathscr{P}) \neq \mathbb{C}$ (see Figure 3).

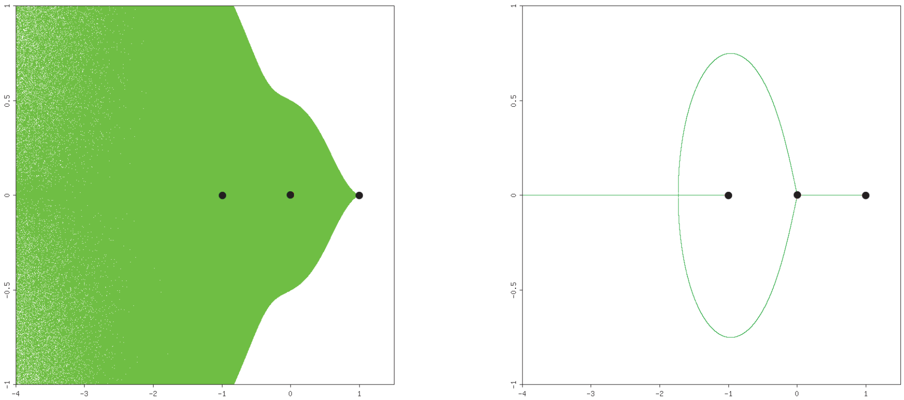

Figure 3: $W(\mathscr{P})$, $W^2_{\mathbb{C}^2\times\mathbb{C}}(\mathscr{P})$ *and eigenvalues of the quadratic* 3×3 *matrix polynomial* $\mathscr{P}$ *in Ex.* 6.9 *in the rectangle* $[-4,1.5]\times[-1,1]$.

In fact, we can prove analytically that the imaginary part of $W^2(\mathscr{P})$ is bounded, whereas that of $W(\mathscr{P})$ is not. To prove the former, we note that

$$\frac{z^3}{z^2-1} \in (-\infty,0) \iff \frac{z^2-1}{z^3} \in (-\infty,0) \iff |z|^4\overline{z} - \overline{z}^3 \in (-\infty,0),$$

where we have multiplied numerator and denominator by $\overline{z}^3$ for the last equivalence. Writing $z = a + \mathrm{i}b$, $a,b \in \mathbb{R}$, the latter holds if and only if the two conditions

$$\begin{aligned} a^5 + 2a^3b^2 + ab^4 - a^3 + 3ab^2 &< 0, \\ -a^4b - 2a^2b^3 - b^5 + 3a^2b - b^3 &= 0 \end{aligned}$$

hold. The second condition implies that

$$b = 0 \quad \text{or} \quad a^2 = -\tfrac{1}{2}\left((2b^2-3) \pm \sqrt{9-16b^2}\right);$$

Since a^2 is real, the discriminant must be non-negative, i.e. $9-16b^2 \geqslant 0$ and hence $b \in [-\frac{3}{4},\frac{3}{4}]$. This proves that $\operatorname{Im} W^2(\mathscr{P}) \in [-\frac{3}{4},\frac{3}{4}]$; note that Figure 3 suggests that this estimate is sharp.

To see that the imaginary part of the numerical range is unbounded, we note that

$$(\mathscr{P}(z)x,x) = z^2(|x_1|^2 + |x_3|^2) + z(|x_2|^2 + x_2\overline{x_3}) + x_3\overline{x_2}$$

for $x=(x_1,x_2,x_3)\in\mathbb{C}^3$, $\|x\|=1$. If we let $\varepsilon\in(0,\frac{1}{\sqrt{2}})$ and choose $x_1=x_3=\varepsilon$, $x_2=\sqrt{1-2\varepsilon^2}\,\mathrm{i}$, it is not difficult to check that the equation $(\mathscr{P}(z)x,x)=0$ has the solutions

$$\begin{aligned} z_\pm(\varepsilon)&=\frac{1}{4\varepsilon^2}\Big(2\varepsilon^2-1+\varepsilon\sqrt{1-2\varepsilon^2}\,\mathrm{i}\pm\sqrt{(1-2\varepsilon^2+\sqrt{1-2\varepsilon^2}\varepsilon\,\mathrm{i})^2+8\sqrt{1-2\varepsilon^2}\varepsilon^3\,\mathrm{i}}\Big)\in W(\mathscr{P})\\ &=: a(\varepsilon)+b(\varepsilon)\,\mathrm{i}\pm\sqrt{c(\varepsilon)+d(\varepsilon)\,\mathrm{i}} \end{aligned}$$

where $\sqrt{\cdot}$ denotes the principal branch of the square root. Clearly, $\lim_{\varepsilon\to 0}b(\varepsilon)=\lim_{\varepsilon\to 0}\frac{\varepsilon\sqrt{1-2\varepsilon^2}}{4\varepsilon^2}=\infty$. If we set $\zeta(\varepsilon):=z_+(\varepsilon)$ if $\operatorname{Im}\sqrt{c(\varepsilon)+d(\varepsilon)\,\mathrm{i}}>0$ and $\zeta(\varepsilon):=z_-(\varepsilon)$ if $\operatorname{Im}\sqrt{c(\varepsilon)+d(\varepsilon)\,\mathrm{i}}<0$, we obtain that $\zeta(\varepsilon)\in W(\mathscr{P})$ and $\lim_{\varepsilon\to 0}\operatorname{Im}\zeta(\varepsilon)\geqslant\lim_{\varepsilon\to 0}b(\varepsilon)=\infty$. This proves that $\operatorname{Im}W(\mathscr{P})$ is unbounded, as one could guess from Figure 3.

We close this subsection by giving an upper bound for the number of connected components of the block numerical range of an operator polynomial $\mathscr{P}$. Note that the assumption $0\notin W^n(A^{[d]})$ below allows for unbounded components (unlike Theorem 5.1 i)).

PROPOSITION 6.10. *Let $\mathscr{P}:\mathbb{C}\to L(H)$ be an operator polynomial of degree d. If $0\notin W^n(A^{[d]})$, then $W^n(\mathscr{P})$ consists of at most nd connected components.*

For each connected component C of $W^n(\mathscr{P})$ the number $\nu_C(\det\mathscr{P}_x(\cdot))$ of zeros of the polynomial $\det\mathscr{P}_x(\cdot)$ in C counting multiplicities does not depend on $x\in\mathscr{S}^n$.

Proof. Let $0\notin W^n(A^{[d]})$, i.e. $\det A_x^{[d]}\neq 0$ for all $x\in\mathscr{S}^n$. Hence $\det\mathscr{P}_x(z)=0$ if and only if

$$0=z^{nd}+\sum_{l=0}^{nd-1}\frac{\delta_x^{[l]}}{\det A_x^{[d]}}z^l=:z^{nd}+\sum_{l=0}^{nd-1}\tilde{\delta}_x^{[l]}z^l=:p_x(z),$$

where $\delta_x^{[l]}$ is defined as in (6.4). The mapping

$$B:\mathscr{S}^n\to M_{nd}(\mathbb{C}),\quad B(x):=\begin{pmatrix} 0 & 1 & & 0\\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & 1\\ -\tilde{\delta}_x^{[0]} & -\tilde{\delta}_x^{[1]} & \cdots & -\tilde{\delta}_x^{[nd-1]}\end{pmatrix},$$

is continuous, and for every $x\in\mathscr{S}^n$ the zeros of the polynomial p_x coincide with the eigenvalues of $B(x)$ counting multiplicities. Therefore,

$$W^n(\mathscr{P})=\bigcup_{x\in\mathscr{S}^n}\sigma_p(B(x))=\sigma_p(B(\mathscr{S}^n)).$$

As $B(\mathscr{S}^n)\subset M_{nd}(\mathbb{C})$ is connected, it follows from [34, Appendix B] that $\sigma_p(B(\mathscr{S}^n))$ consists of at most nd connected components and $\nu_C(\det\mathscr{P}_x)=\nu_C(B(x))$ for a connected component C of $W^n(\mathscr{P})$ does not depend on $x\in\mathscr{S}^n$. $\square$

6.3. Resolvent estimates

For an operator polynomial $\mathscr{P}$ of degree d with $0 \notin \overline{W^n(A^{[d]})}$, Lemma 6.3 shows that $W^n(\mathscr{P})$ is bounded and that there exists $z_0 \in \mathbb{C}$ such that $0 \notin \overline{W^n(\mathscr{L}(z_0))}$; in particular, every connected component of $W^n(\mathscr{P})$ is bounded. Thus the resolvent estimate in Theorem 5.1 applies to each connected component C of $W^n(\mathscr{P})$ for which $\overline{C}$ is a connected component of $\overline{W^n(\mathscr{P})}$; note that condition (5.1) holds for such C.

The following estimate improves the resolvent bound in (5.2) for the special case of operator polynomials.

THEOREM 6.11. *Let $\mathscr{P} : \mathbb{C} \to L(H)$ be an operator polynomial of degree d. If $0 \notin \overline{W^n(A^{[d]})}$, then*

$$\left\|\mathscr{P}^{-1}(z)\right\| \leqslant \frac{\|\mathscr{P}(z)\|^{n-1}}{\inf\left|D^n(A^{[d]})\right| \cdot \operatorname{dist}(z, W^n(\mathscr{P}))^{nd}}, \quad z \in \mathbb{C} \setminus \overline{W^n(\mathscr{P})}, \tag{6.6}$$

where $D^n(A^{[d]})$ is the block determinant set of $A^{[d]}$ (see (4.3)) and $\inf\left|D^n(A^{[d]})\right| = \inf\{|z| : z \in D^n(A^{[d]})\}$. More exactly, if $C_1, \ldots, C_s$ are the connected components of $W^n(\mathscr{P})$ and $\nu_j := \nu_{C_j}(\det \mathscr{P}_x(\cdot))$, $x \in \mathscr{S}^n$, is the number of zeros of $\det \mathscr{P}_x(\cdot)$ in C_j, $j = 1, \ldots, s$, then

$$\left\|\mathscr{P}^{-1}(z)\right\| \leqslant \frac{\|\mathscr{P}(z)\|^{n-1}}{\inf\left|D^n(A^{[d]})\right| \cdot \Pi_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j}}, \quad z \in \mathbb{C} \setminus \overline{W^n(\mathscr{P})}. \tag{6.7}$$

Proof. The proof follows the lines of the proof of [33, Thm. 4.2] in the operator case. Note that from $0 \notin \overline{W^n(A^{[d]})}$ and Theorem 6.1 it follows that $\mathbb{C} \setminus \overline{W^n(\mathscr{P})} \subset \rho(\mathscr{P})$ and $\inf\left|D^n(A^{[d]})\right| > 0$ (the latter by Lemma 4.3). If $\lambda_1^{[j]}(x), \ldots, \lambda_{\nu_j}^{[j]}(x)$ are the zeros of $\det \mathscr{P}_x$ on C_j, $j = 1, \ldots, s$, counted with multiplicities, we have

$$|\det \mathscr{P}_x(z)| = \left|\det A_x^{[d]}\right| \cdot \prod_{j=1}^{s} \prod_{i=1}^{\nu_j} \left|z - \lambda_i^{[j]}\right| \geqslant \inf\left|D^n(A^{[d]})\right| \cdot \prod_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j}$$

for $x \in \mathscr{S}^n$ and $z \in \mathbb{C}$. Hence,

$$\inf|D^n(\mathscr{P}(z))| \geqslant \inf\left|D^n(A^{[d]})\right| \cdot \prod_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j} > 0,$$

for all $z \in \mathbb{C} \setminus \overline{W^n(\mathscr{P})}$. Now the estimate (6.6) follows from Proposition 5.2. $\square$

COROLLARY 6.12. *If $\mathscr{P}$ in Theorem 6.11 is monic, then*

$$\left\|\mathscr{P}^{-1}(z)\right\| \leqslant \frac{\|\mathscr{P}(z)\|^{n-1}}{\Pi_{j=1}^{s} \operatorname{dist}(z, C_j)^{\nu_j}} \leqslant \frac{\|\mathscr{P}(z)\|^{n-1}}{\operatorname{dist}(z, W^n(\mathscr{P}))^{nd}}, \quad z \in \mathbb{C} \setminus \overline{W^n(\mathscr{P})}.$$

Since for an operator polynomial of degree d, the number of connected components of $W^n(\mathscr{P})$ is always bounded by nd, we obtain the following a priori upper bound on the lengths of Jordan chains at eigenvalues on the boundary of $W^n(\mathscr{P})$, complementing Corollary 5.3.

COROLLARY 6.13. *Let $\mathscr{P}$ be an operator polynomial of degree d such that $0 \notin \overline{W^n(A^{[d]})}$, let C be a connected component of $W^n(\mathscr{P})$ and $\nu_C := \nu_C(\det \mathscr{P}_x(\cdot))$, $x \in \mathscr{S}^n$.*

If $\lambda_0 \in \partial C$ is an eigenvalue of $\mathscr{P}$ with the exterior cone property, then the length m_0 of a Jordan chain of $\mathscr{P}$ in λ_0 is bounded by $m_0 \leqslant \nu_C \leqslant nd$. In particular, if $W^n(\mathscr{P})$ consists of the maximal number nd of connected components, then there are no associated vectors at λ_0.

6.4. The linear companion pencil

For an operator polynomial $\mathscr{P}$ of degree $d > 0$ the *linear companion pencil* $\mathscr{C}^{\mathscr{P}}$: $\mathbb{C} \to \mathscr{L}(H^d)$ is defined as

$$\mathscr{C}^{\mathscr{P}}(z) := \begin{pmatrix} \mathrm{Id}_H & & 0 \\ & \ddots & \\ & & \mathrm{Id}_H \\ 0 & & A^{[d]} \end{pmatrix} z + \begin{pmatrix} 0 & -\mathrm{Id}_H & & 0 \\ & 0 & \ddots & \\ & & \ddots & -\mathrm{Id}_H \\ A^{[0]} & A^{[1]} & \dots & A^{[d-1]} \end{pmatrix} =: \mathscr{A}^{[1]} z + \mathscr{A}^{[0]}.$$

Note that $\mathscr{C}^{\mathscr{P}} = \mathscr{P}$ if $\mathscr{P}$ is already linear (i.e. $d = 1$). If $\mathscr{P}$ is monic, then $\mathscr{C}^{\mathscr{P}}$ is monic as well and $-\mathscr{A}^{[0]}$ is called the *companion operator of* $\mathscr{P}$.

REMARK 6.14. The linear companion pencil $\mathscr{C}^{\mathscr{P}}$ is a linearisation of the operator polynomial $\mathscr{P}$ (see e.g. [18, § 12.2 and Lemma 12.2]). In particular, the spectra and point spectra of $\mathscr{P}$ and $\mathscr{C}^{\mathscr{P}}$ coincide, i.e. $\sigma(\mathscr{P}) = \sigma(\mathscr{C}^{\mathscr{P}})$ and $\sigma_p(\mathscr{P}) = \sigma_p(\mathscr{C}^{\mathscr{P}})$, and for finite dimensional H we have the equivalence

$$\det \mathscr{P}(z) = 0 \quad \Longleftrightarrow \quad \det \mathscr{C}^{\mathscr{P}}(z) = 0, \quad z \in \mathbb{C}. \tag{6.8}$$

If $\mathscr{P}$ is monic, then the spectra and point spectra of $\mathscr{P}$ and of its companion operator $-\mathscr{A}^{[0]}$ coincide.

Now let $H_1, \dots, H_n$ be closed subspaces of H such that $H = H_1 \oplus \dots \oplus H_n$. In the next proposition we study the block numerical range of the linear companion pencil with respect to the decomposition

$$H^d \cong (H_1 \oplus \dots \oplus H_n) \oplus \dots \oplus (H_1 \oplus \dots \oplus H_n) \tag{6.9}$$

induced by the given decomposition of H. The special case of a monic polynomial and the trivial decomposition of $H = H_1$ (i.e. $n = 1$) was considered in [33, Thm. 5.1].

PROPOSITION 6.15. *Let $\mathscr{P}:\mathbb{C}\to L(H)$ be an operator polynomial of degree d. Then, with respect to the decomposition $H = H_1 \oplus \cdots \oplus H_n$ of H and (6.9) of H^d,*

$$W^n(\mathscr{P}) \subset W^{nd}(\mathscr{C}^{\mathscr{P}}). \tag{6.10}$$

Proof. Let $x = (x_1,\dots,x_n) \in \mathscr{S}^n$. Then $x^d := (x_1,\dots,x_n,\dots,x_1,\dots,x_n) \in \mathscr{S}^{nd}$, and it follows that

$$\mathscr{C}^{\mathscr{P}}_{x^d}(z) = \begin{pmatrix} \mathrm{Id}_{\mathbb{C}^n} & & & 0 \\ & \ddots & & \\ & & \mathrm{Id}_{\mathbb{C}^n} & \\ 0 & & & A_x^{[d]} \end{pmatrix} z + \begin{pmatrix} 0 & -\mathrm{Id}_{\mathbb{C}^n} & & 0 \\ & 0 & \ddots & \\ & & \ddots & -\mathrm{Id}_{\mathbb{C}^n} \\ A_x^{[0]} & A_x^{[1]} & \dots & A_x^{[d-1]} \end{pmatrix}, \quad z\in\mathbb{C}.$$

In particular, if $\lambda \in W^n(\mathscr{P})$, i.e., $\det \mathscr{P}_x(\lambda) = 0$ for some $x \in \mathscr{S}$, it follows from (6.8) applied to the matrix polynomial $\mathscr{P}_x$ that

$$0 = \det \mathscr{C}^{\mathscr{P}_x}(\lambda) = \det \mathscr{C}^{\mathscr{P}}_{x^d}(\lambda),$$

and therefore $\lambda \in W^{nd}(\mathscr{C}^{\mathscr{P}})$. $\square$

EXAMPLE 6.16. The cubic numerical range $W^3(\mathscr{P})$ of the quadratic 6×6 matrix polynomial

$$P(z) = z^2 + \left(\begin{array}{cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right) z + \left(\begin{array}{cc|cc|cc} -3 & 0 & 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ \hline -1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array}\right), \quad z \in \mathbb{C},$$

with respect to the decomposition $\mathbb{C}^6 = \mathbb{C}^2 \times \mathbb{C}^2 \times \mathbb{C}^2$ and the block numerical range $W^6(\mathscr{C}^{\mathscr{P}})$ of its linear companion pencil $\mathscr{C}^{\mathscr{P}}$ with respect to the corresponding decomposition $\mathbb{C}^{12} = (\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2)\times(\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2)$ are shown in Figure 4 to illustrate the inclusion (6.10).

The eigenvalues of $\mathscr{P}$ are the zeros of the characteristic determinant

$$\det \mathscr{P}(z) = z^3(z-1)(z+2)\Big(z-\frac{1}{2}-\frac{1}{2}\sqrt{7}\mathrm{i}\Big)\Big(z-\frac{1}{2}+\frac{1}{2}\sqrt{7}\mathrm{i}\Big)(z^5-8z^3+17z-13);$$

in particular, the smallest real eigenvalue of $\mathscr{P}$ is equal to -2 and the largest real eigenvalue of $\mathscr{P}$ is the largest (and only) real zero of $z^5-8z^3+17z-13=0$ which is approximately 2.46336583493925.

REMARK 6.17. Note that the leftmost real point of $W^3_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{P})$ is a corner, but no eigenvalue of $\mathscr{P}$ and the rightmost real point of $W^6_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{C}^{\mathscr{P}})$ is a corner, but no eigenvalue of $\mathscr{C}^{\mathscr{P}}$ (i.e. of $\mathscr{P}$).

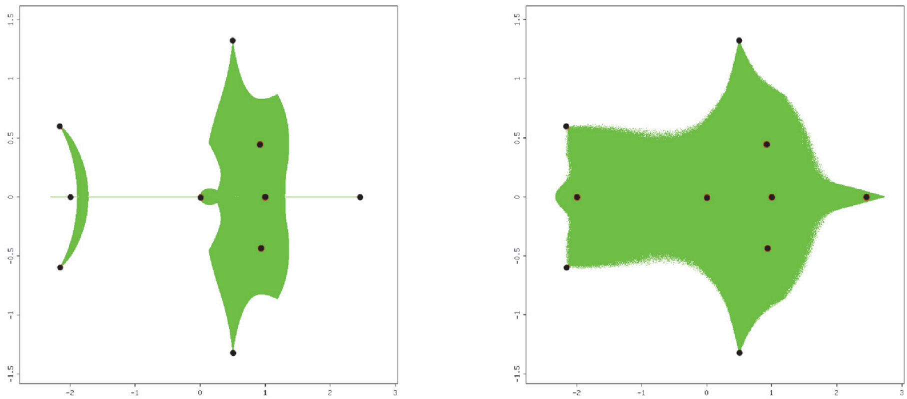

Figure 4: $W^3_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{P})$ *and eigenvalues of the quadratic* 6×6 *matrix polynomial* $\mathscr{P}$ *in Ex.* 6.16 *and* $W^6_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{C}^{\mathscr{P}})$ *of its linear companion pencil* $\mathscr{C}^{\mathscr{P}}$.

This shows that the result that corners of the numerical range belong to the spectrum does *not* hold for the block numerical range of linear operators or operator functions (see [36, Thm. 5.10] and the forthcoming paper [25]), and not even for the quadratic numerical range of linear operators (see [14, Thm. 3.1]).

In analogy to the latter result, the left real corner of $W^3_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{P})$ is an eigenvalue of the principal minor of the operator polynomial $\mathscr{P}$ which arises if we delete the last block row and block column (i.e. the last two rows and columns), approximately given by -2.23169667970097.

Similarly, the right real corner of $W^6_{\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^2}(\mathscr{C}^{\mathscr{P}})$ is an eigenvalue of the principal minor of the companion operator $\mathscr{C}^{\mathscr{P}}$ which arises if we delete the third block row and block column (i.e. the fifth and sixth row and column), approximately given by 2.735865482.

7. Monic operator polynomials with positive coefficients

In this section we consider operator polynomials on Hilbert lattices the coefficients of which are positive operators. The classical result of Perron and Frobenius shows that the spectral radius of a positive matrix is always contained in its spectrum (see [23], [3]). Infinite dimensional analogues were proved by Kreĭn/Rutman, Bonsall, and Schaefer in different settings (see [8], [9], [2], [27, Prop. V.4.1]). A corresponding result for the spectral radius and spectrum of monic operator polynomials $\mathscr{P}$ with positive coefficients, and even more general cases, was established by Maibaum (see [16][1]), using that the corresponding companion operator is positive and has the same spectrum as $\mathscr{P}$.

[1] We thank the reviewer for bringing this reference to our attention.

Here we prove the analogous theorem that the numerical and block numerical radius are contained in the numerical and block numerical range, respectively, of monic operator polynomials $\mathscr{P}$ on a Hilbert lattice H with positive coefficients. The numerical range analogue in the finite dimensional case was proved in [22]. The numerical and block numerical range analogue for positive operators $\mathscr{A}$ proved in [24, Prop. 2.5 (ii), Prop. 3.3 (ii)] is the special case of a monic linear operator pencil $\mathscr{P}(z) = z - \mathscr{A}$ of the general result below.

DEFINITION 7.1.

i) An orderd vector space $(H_{\mathbb{R}}, \leqslant)$ over $\mathbb{R}$ is called a *vector lattice* if $\sup\{x,y\}$ and $\inf\{x,y\}$ exist for all $x,y \in H_{\mathbb{R}}$; in this case we set $|x| := \sup\{x,0\} + \sup\{-x,0\}$ for $x \in H_{\mathbb{R}}$.

ii) If there exists an inner product $(\cdot,\cdot) : H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathbb{R}$ such that $(H_{\mathbb{R}}, (\cdot,\cdot))$ is a Hilbert space and $|x| \leqslant |y|$ implies $(x,x) \leqslant (y,y)$, $x,y \in H_{\mathbb{R}}$, then $(H_{\mathbb{R}}, (\cdot,\cdot), \leqslant)$ is called a (*real*) *Hilbert lattice*.

iii) If $H = H_{\mathbb{R}} \oplus \mathrm{i} H_{\mathbb{R}}$ with $(\alpha + \mathrm{i}\beta)(x + \mathrm{i}y) = \alpha x - \beta y + \mathrm{i}(\beta x + \alpha y)$ for $\alpha, \beta \in \mathbb{R}$, $x,y \in H_{\mathbb{R}}$, is the complexification of $H_{\mathbb{R}}$, then $(H, (\cdot,\cdot), \leqslant)$ is called a *complex Hilbert lattice*.

iv) The positive cone of H is defined as $H_+ := \{x \in H_{\mathbb{R}} : x \geqslant 0\}$; for $z = x + \mathrm{i}y \in H$, $x,y \in H_{\mathbb{R}}$, we define
$$|z| := \sup_{0 \leqslant \theta < 2\pi} |(\cos\theta)x + (\sin\theta)y| \in H_+.$$

v) An operator $T \in L(H)$ is called *positive* ($T \geqslant 0$) if $TH_+ \subset H_+$.

vi) Let $T \in L(H)$ and $T_1, T_2 \in L(H_{\mathbb{R}})$ such that $T = T_1 \oplus \mathrm{i} T_2$. Then
$$|T| := \sup_{0 \leqslant \theta < 2\pi} |(\cos\theta)T_1 + (\sin\theta)T_2|$$
if the supremum exists in the canonical order of $L(H_{\mathbb{R}})$.

An extensive treatment of positive operators may be found in [27] and [20]. Note that the above concept of a *positive operator* should not be mixed up with positive (semi-)definite operators on Hilbert spaces for which the numerical range lies in $(0,\infty)$ ($[0,\infty)$, respectively).

DEFINITION 7.2. Let $\mathscr{A} \in L(H)$ and let $\mathscr{L}(\cdot) = (L_{ij}(\cdot))_{i,j=1}^n : \Omega \to L(H)$ be an operator function. Then the *block numerical radius of* $\mathscr{A}$ is defined as
$$w^n(\mathscr{A}) := \sup\{\, |\lambda| : \lambda \in W^n(\mathscr{A})\},$$
and the *block numerical radius of* $\mathscr{L}$ is defined as
$$w^n(\mathscr{L}) := \sup\{\, |\lambda| : \lambda \in W^n(\mathscr{L})\}. \tag{7.1}$$

Note that for the special case $\mathscr{L}(z) = \mathscr{A} - z$ with $\mathscr{A} \in L(H)$, the two definitions coincide and that for $n = 1$ the block numerical radius is the usual numerical radius $w(\mathscr{A})$ and $w(\mathscr{L})$, respectively.

In order to comply with classical Perron-Frobenius theory where $\mathscr{P}(z) = z - \mathscr{A}$, we follow the same sign convention for higher order polynomials (unlike Section 6, comp. (6.1)).

THEOREM 7.3. *Let $H = H_1 \oplus \ldots \oplus H_n$ be a Hilbert lattice such that $H_1, \ldots, H_n$ are closed lattice ideals in H. Let $\mathscr{P} : \mathbb{C} \to L(H)$ be a monic operator polynomial of degree $d \in \mathbb{N}$ of the form*

$$\mathscr{P}(z) = z^d - A^{[d-1]} z^{d-1} - \ldots - A^{[1]} z - A^{[0]}, \quad z \in \mathbb{C},$$

with positive operators $A^{[i]} \in L(H)$, $i = 0, \ldots, d-1$. Then the following hold.

i) *If $w^n(\mathscr{P}) > 0$, then $w^n(\mathscr{P}) \in \overline{W^n(\mathscr{P})}$.*

ii) *If $0 < w^n(\mathscr{P}) \in W^n(\mathscr{P})$, then there exists $x \in \mathscr{S}^n$, $x \geqslant 0$, with $0 \in \sigma\big(\mathscr{P}_x(w^n(\mathscr{P}))\big)$.*

For the proof of Theorem 7.3 we use two lemmas; the first one applies to arbitrary operator polynomials, without positivity assumptions on the coefficients.

LEMMA 7.4. *Let H be a Hilbert space, let $\mathscr{P} : \mathbb{C} \to L(H)$ be a monic operator polynomial of degree $d \in \mathbb{N}$, and define the operator function $\mathscr{Q}$ by*

$$\mathscr{Q}(z) := A^{[d-1]} + A^{[d-2]} \tfrac{1}{z} + \cdots + A^{[0]} \tfrac{1}{z^{d-1}}, \quad z \in \mathbb{C} \setminus \{0\}. \tag{7.2}$$

Then, for every $\lambda \in \mathbb{C} \setminus \{0\}$,

$$\lambda \in W^n(\mathscr{P}) \iff \lambda \in W^n(\mathscr{Q}(\lambda)), \tag{7.3}$$

$$\lambda \in \overline{W^n(\mathscr{P})} \iff \lambda \in \overline{W^n(\mathscr{Q}(\lambda))}. \tag{7.4}$$

Proof. Let $\lambda \in \mathbb{C} \setminus \{0\}$. The relation $\mathscr{P}(\lambda) = \lambda^{d-1}(\lambda\,\mathrm{Id}_H - \mathscr{Q}(\lambda))$ implies that

$$W^n(\mathscr{P}(\lambda)) = W^n(\lambda\,\mathrm{Id}_H - \mathscr{Q}(\lambda)) = (-1)^n W^n(\mathscr{Q}(\lambda) - \lambda\,\mathrm{Id}_H) = (-1)^n (W^n(\mathscr{Q}(\lambda)) - \lambda)$$

and hence, by applying Lemma 4.5,

$$\lambda \in \overline{W^n(\mathscr{P})} \iff 0 \in \overline{W^n(\mathscr{P}(\lambda))} \iff 0 \in \overline{(W^n(\mathscr{Q}(\lambda)) - \lambda)} \iff \lambda \in \overline{W^n(\mathscr{Q}(\lambda))},$$

and analogously without closures. $\square$

The next lemma shows that the block numerical radius of an operator function $\mathscr{Q}$ of the form (7.2) with positive coefficients defines a monotonically decreasing function on the positive real axis having a fixed point.

LEMMA 7.5. *Under the assumptions of Theorem 7.3, the function*

$$q_n : (0, \infty) \to [0, \infty), \quad t \mapsto w_n(\mathscr{Q}(t)),$$

is continuous and monotonically decreasing. If $w^n(\mathscr{P}) > 0$, then $q_n(w^n(\mathscr{P})) = w^n(\mathscr{P})$.

Proof. The proof is similar to the proof of [26, Prop. 2.1] of the analogous assertion for the spectral radius. Let $0<\lambda\leqslant\mu$. Then, as all the coefficients $A^{[i]}$ of $\mathscr{Q}$ are positive, $\mathscr{Q}(z)$ is an operator polynomial of degree $\leqslant d-1$ in $\frac{1}{z}$ and hence $z^{d-1}\mathscr{Q}(z)$ is an operator polynomial, it follows that

$$0\leqslant\mathscr{Q}(\mu)\leqslant\mathscr{Q}(\lambda),\quad 0\leqslant\lambda^{d-1}\mathscr{Q}(\lambda)\leqslant\mu^{d-1}\mathscr{Q}(\mu).$$

Hence

$$0\leqslant\frac{\lambda^{d-1}}{\mu^{d-1}}\mathscr{Q}(\lambda)\leqslant\mathscr{Q}(\mu)\leqslant\mathscr{Q}(\lambda)\leqslant\frac{\mu^{d-1}}{\lambda^{d-1}}\mathscr{Q}(\mu).$$

Since the block numerical range is homogeneous and the block numerical radius is monotonically increasing (see [24, Prop. 3.3 (v)]), this implies that

$$0\leqslant\frac{\lambda^{d-1}}{\mu^{d-1}}q_n(\lambda)\leqslant q_n(\mu)\leqslant q_n(\lambda)\leqslant\frac{\mu^{d-1}}{\lambda^{d-1}}q_n(\mu),$$

which proves both the continuity and monotonicity of q_n.

If $w^n(\mathscr{P})>0$, then there exists $0\neq\lambda\in W^n(\mathscr{P})$. Hence $\lambda\in\mathscr{Q}(\lambda)$ by (7.3) and thus $w_n(\mathscr{Q}(\lambda))>0$. Since $|\mathscr{Q}(\lambda)|\leqslant\mathscr{Q}(|\lambda|)$, the monotonicity of the block numerical radius ([24, Prop. 3.3]) implies that $0<w_n(\mathscr{Q}(\lambda))\leqslant w_n(\mathscr{Q}(|\lambda|))$, and so $q_n\not\equiv 0$. Thus, since q_n is monotonically decreasing, q_n has a fixed point $t_0\in(0,\infty)$.

Using [24, Prop. 3.3 (ii)] we conclude that

$$t_0=q_n(t_0)=w_n(\mathscr{Q}(t_0))\in\overline{W^n(\mathscr{Q}(t_0))}.$$

By Lemma 7.4 this implies $t_0\in\overline{W^n(\mathscr{P})}$, and thus $t_0\leqslant w^n(\mathscr{P})$. Since q_n is monotonically decreasing by Lemma 7.5, we obtain

$$q_n(w^n(\mathscr{P}))\leqslant q_n(t_0)=t_0\leqslant w^n(\mathscr{P}).\tag{7.5}$$

By the definition of $w^n(\mathscr{P})$ in (7.1), there exists $\xi\in\mathbb{C}$, $|\xi|=1$, with $\xi w^n(\mathscr{P})\in\overline{W^n(\mathscr{P})}$. Again by Lemma 7.4 this yields $\xi w^n(\mathscr{P})\in\overline{W^n(\mathscr{Q}(\xi w^n(\mathscr{P})))}$, and thus

$$|\xi w^n(\mathscr{P})|\leqslant w_n(\mathscr{Q}(\xi w^n(\mathscr{P}))).\tag{7.6}$$

Since $|\mathscr{Q}(\xi w^n(\mathscr{P}))|\leqslant\mathscr{Q}(w^n(\mathscr{P}))$, the monotonicity of the block numerical radius yields

$$w_n(\mathscr{Q}(\xi w^n(\mathscr{P})))\leqslant w_n(\mathscr{Q}(w^n(\mathscr{P}))).\tag{7.7}$$

Then, using (7.6), (7.7), and (7.5), we find that

$$|\xi w^n(\mathscr{P})|=w^n(\mathscr{P})\leqslant w_n(\mathscr{Q}(w^n(\mathscr{P})))=q_n(w^n(\mathscr{P}))\leqslant t_0\leqslant w^n(\mathscr{P}),\tag{7.8}$$

and hence $t_0=w^n(\mathscr{P})$. $\square$

Proof of Theorem 7.3. i) If $w^n(\mathscr{P})>0$, then Lemma 7.5 yields that

$$w^n(\mathscr{P})=w_n(\mathscr{Q}(w^n(\mathscr{P}))).$$

As $\mathscr{Q}(w^n(\mathscr{P})) \geqslant 0$ we know that $w^n(\mathscr{P}) \in \overline{W^n(\mathscr{Q}(w^n(\mathscr{P})))}$ by [24, Prop. 3.3]. The assertion then follows from Lemma 7.4.

ii) If $w^n(\mathscr{P}) \in W^n(\mathscr{P})$, then $w^n(\mathscr{P}) \in W^n(\mathscr{Q}(w^n(\mathscr{P})))$. As $\mathscr{Q}(w^n(\mathscr{P})) \geqslant 0$ and $w^n(\mathscr{P}) = w_n(\mathscr{Q}(w^n(\mathscr{P})))$ by Lemma 7.5, there exists $x \in \mathscr{S}^n$, $x \geqslant 0$, with $w^n(\mathscr{P}) \in \sigma(\mathscr{Q}(w^n(\mathscr{P}))_x)$ by [24, Prop. 3.3 (iv)]. Using that $\mathscr{P}(\lambda) = \lambda^{d-1}(\lambda\,\mathrm{Id}_H - \mathscr{Q}(\lambda))$ as in the proof of Lemma 7.4, we see that $0 \in \sigma\big(\mathscr{P}(w^n(\mathscr{P}))_x\big)$. $\square$

EXAMPLE 7.6. The cubic 4×4 matrix polynomial

$$\mathscr{P}(z) = z^3 - \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix} z^2 - \begin{pmatrix} 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1\\ 1&1&1&1 \end{pmatrix} z - \begin{pmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 1&0&0&0 \end{pmatrix}, \quad z \in \mathbb{C},$$

satisfies the positivity assumptions in Theorem 7.3. The eigenvalues of $\mathscr{P}$ are the zeros of the polynomial

$$\det(P(z)) = z^{12} - 4z^{11} - 4z^{10} - 4z^8 - 4z^7 - 4z^5 - 4z^4 - 4z^2 - 4z - 1, \quad z \in \mathbb{C}, \tag{7.9}$$

of degree 12. It is not difficult to check that -1 and $\pm\mathrm{i}$ are zeros of $\det(P(\cdot))$; then the first and last term in (7.9) cancel and the middle terms cancel pairwise. Since the eigenvalues belong to all block numerical ranges $W^n(\mathscr{P})$, we have $w^n(\mathscr{P}) \geqslant 1 > 0$ for $n = 1,2,3,4$. Hence Theorem 7.3 yields that $w^n(\mathscr{P}) \in \overline{W^n(\mathscr{P})} = W^n(\mathscr{P})$ for $n = 1,2,3,4$.

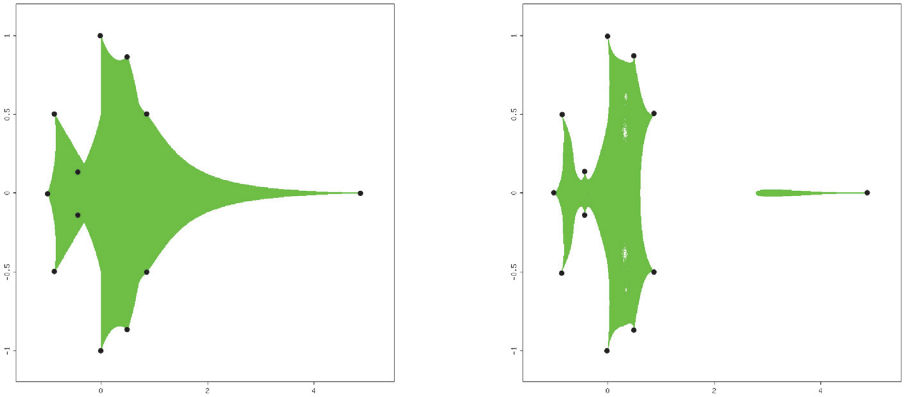

Figure 5: *$W(\mathscr{P})$, $W^3_{\mathbb{C}\times\mathbb{C}^2\times\mathbb{C}}(\mathscr{P})$ and eigenvalues of the cubic 4×4 matrix polynomial $\mathscr{P}$ in Ex. 7.6.*

In fact, all 12 zeros of $\det(P(\cdot))$ can be found explicitly e.g. using Maple; they are all different, i.e. all eigenvalues of $\mathscr{P}$ are simple. The eigenvalue $\lambda_{\max}$ with largest modulus is real and given by

$$\lambda_{\max} = w^n(\mathscr{P}) = \frac{1}{6}(1196 + 12\sqrt{177})^{1/3} + \frac{56}{3(1196+12\sqrt{177})^{1/3}} + \frac{4}{3}. \tag{7.10}$$

Figure 5, which shows the numerical and cubic numerical range together with the eigenvalues of $\mathscr{P}$, suggests that the numerical radius $w^1(\mathscr{P}) \in W(\mathscr{P})$ and the block numerical radius $w^3(\mathscr{P}) \in W^3(\mathscr{P})$ coincide with the eigenvalue $\lambda_{\max} \sim 4.864536513$ of largest modulus of $\mathscr{P}$, and thus with the spectral radius of $\mathscr{P}$.

8. An application to gyroscopic systems

Gyroscopic systems are non-proportionally damped systems which are known to exhibit instabilities that are not well understood (see e.g. [30, Section 2.2] and references therein). Separating time in the corresponding equations of motion, one arrives at an eigenvalue problem for a quadratic operator polynomial

$$\mathscr{Q}(\mu) := \mu^2 M + \mu G + K, \quad \mu \in \mathbb{C},$$

where M, G, and K are certain linear (differential) operators representing mass, gyroscopic terms, and stiffness; after a discretised finite element analysis, the corresponding mass, gyroscopic and stiffness matrix are of the form

$$M = \begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix}, \quad G = \begin{pmatrix} 0 & -G_1 \\ G_1 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} K_1 & 0 \\ 0 & K_1 \end{pmatrix}.$$

As an example, we consider an eight degrees of freedom approximation to a mechanical system considered in [12, Ex. 7] where M, G, and K are 8×8 matrices with

$$M_1 = \begin{pmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & \frac{1}{9} \end{pmatrix}, \quad G_1 = 150 \begin{pmatrix} 0.4 & 0 & 0 & 0 \\ 0 & 1.6 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & \frac{7}{36} \end{pmatrix},$$

$$K_1 = \begin{pmatrix} -2800 & -1200 & 0 & -1200 \\ -1200 & -15600 & -1200 & 0 \\ 0 & -1200 & -2800 & 1200 \\ -1200 & 0 & 1200 & 561.48 \end{pmatrix}.$$

This example arises in the study of elastically supported rotors (see [21, Section 2.3.3.6] and also [29]) if one chooses e.g.

$$\begin{aligned} &k_0 = k_u = 500\,\frac{\mathrm{kg}}{\mathrm{s}^2}, \ k_h = k_w = 1200\,\frac{\mathrm{kg}}{\mathrm{s}^2}, \ m_0 = m_u = 0.2\,\mathrm{kg}, \ m_R = 0.8\,\mathrm{kg}, \\ &c = d = 15\,\mathrm{cm}, \quad N = 666\,\mathrm{kg}\frac{\mathrm{cm}}{\mathrm{s}^2}, \quad K = 118.8\,\mathrm{kg}\frac{\mathrm{cm}}{\mathrm{s}^2}, \\ &\Omega = 150\,\mathrm{Hz}, \quad C = 6.25\,\mathrm{kg\,cm}^2, \quad A = 25\,\mathrm{kg\,cm}^2. \end{aligned}$$

In [12, Ex. 7 and Fig. 7] it was stated that the eigenvalues of the associated operator polynomial $\mathscr{P}(\lambda) = \mathscr{Q}(-\mathrm{i}\lambda)$, $\lambda \in \mathbb{C}$, are all real and distinct, and a sketch of the numerical range $W(\mathscr{P})$ was given. It was conjectured that $W(\mathscr{P})$ consists of the real segment $[-300, 300]$ on which the eigenvalues lie and an elliptical region symmetric to the origin and to the real axis.

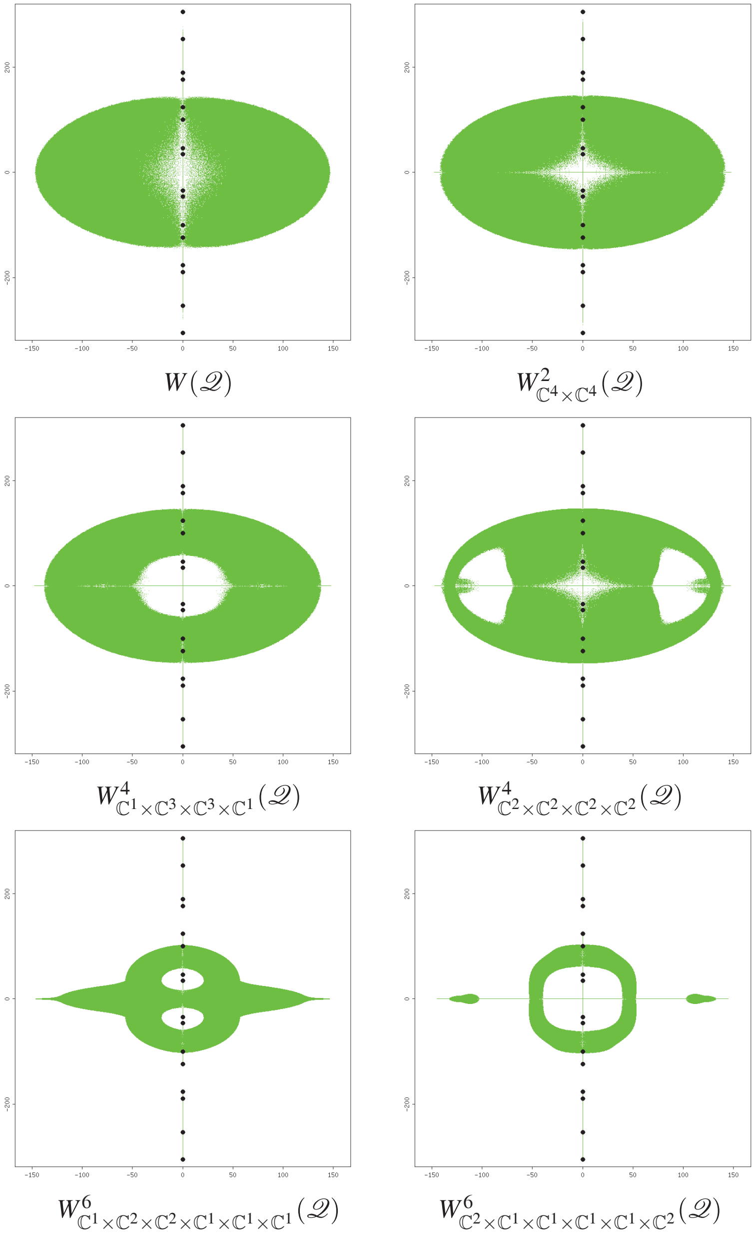

Figure 6: *Numerical range and various block numerical ranges of the quadratic* 8×8 *matrix polynomial* $\mathscr{Q}$ *for the gyroscopic system in Sect.* 8.

We computed a series of numerical and block numerical ranges for the original matrix polynomial $\mathscr{Q}$ (related to those of $\mathscr{P}$ by a rotation of $90°$). The block numerical ranges exhibit more interesting structures than the numerical range. Some of them develop holes, others split into three connected components thus separating the eigenvalues into three groups (see Figure 6).

REMARK 8.1. Note that each of the five block numerical ranges of $\mathscr{Q}$ in Figure 6 has two real corners that are no eigenvalues of $\mathscr{Q}$. This is another example that the corner result for the numerical range of linear operators fails for block numerical ranges (see Remark 6.17 on Example 6.16).

These two points, given by ± 147.668334804856 precisely, are eigenvalues of certain principal minors of $\mathscr{Q}$, namely of the principal minor consisting of the first three rows and columns for $W^6_{\mathbb{C}^1\times\mathbb{C}^2\times\mathbb{C}^2\times\mathbb{C}^1\times\mathbb{C}^1\times\mathbb{C}^1}(\mathscr{Q})$ and of the principal minor consisting of the last four rows and columns for all other block numerical ranges.

REFERENCES

[1] T. BETCKE, N. J. HIGHAM, V. MEHRMANN, C. SCHRÖDER, F. TISSEUR, *NLEVP: a collection of nonlinear eigenvalue problems*, ACM Trans. Math. Software, **39** (2): Art. 7, 28, 2013.

[2] F. F. BONSALL, *Endomorphisms of partially ordered vector spaces; Endomorphisms of a partially ordered vector space without order unit*, J. London Math. Soc., **30**: 144–153, 1955.

[3] G. FROBENIUS, *Über Matrizen aus nicht negativen Elementen*, Sitzungsber. Königl. Preuss. Akad. Wiss., Berlin, 456–477, 1912.

[4] K. E. GUSTAFSON AND D. K. M. RAO, *Numerical Range*, Universitext, Springer-Verlag, New York, 1997, The field of values of linear operators and matrices.

[5] P. R. HALMOS, *A Hilbert Space Problem Book*, volume 19 of Graduate Texts in Mathematics, Springer-Verlag, New York, second edition, 1982, Encyclopedia od Mathematics and its Applications, 17.

[6] D. HINRICHSEN AND A. J. PRITCHARD, *Mathematical systems theory, I*, volume 48 of Texts in Applied Mathematics, Springer-Verlag, Berlin, 2005, Modelling, state space analysis, stability and robustness.

[7] T. KATO, *Perturbation theory for linear operators*, Classics in Mathematics, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[8] M. KREĬN, *Sur les opérations linéaires transformant un certain ensemble conique en lui-même*, C. R. (Doklady) Acad. Sci. URSS (N.S.), **23**: 749–752, 1939.

[9] M. G. KREĬN, M. A. RUTMAN, *Linear operators leaving invariant a cone in a Banach space*, Uspehi Matem. Nauk (N. S.), Vol. 3, **1** (23): 3–95, 1948.

[10] H. LANGER, *Über stark gedämpfte Scharen im Hilbertraum*, J. Math. Mech., **17**: 685–705, 1967/1968.

[11] H. LANGER, A. S. MARKUS, V. I. MATSAEV, AND C. TRETTER, *A new concept for block operator matrices: the quadratic numerical range*, Linear Algebra Appl., **330** (1-3): 89–112, 2001.

[12] P. LANCASTER AND P. J. PSARRAKOS, *The numerical range of self-adjoint quadratic matrix polynomials*, SIAM J. Matrix Anal. Appl., **23** (3): 615–631 (electronic), 2001/02.

[13] C.-K. LI AND L. RODMAN, *Numerical range of matrix polynomials*, SIAM J. Matrix Anal. Appl., **15** (4): 1256–1265, 1994.

[14] H. LANGER, A. S. MARKUS, C. TRETTER, *Corners of numerical ranges*, Oper. Theory Adv. Appl., **124**: 385–400, Birkhäuser, Basel, 2001.

[15] H. LANGER AND C. TRETTER, *Spectral decomposition of some nonselfadjoint block operator matrices*, J. Operator Theory, **39** (2), 339–359, 1998.

[16] G. MAIBAUM, *Über Scharen positiver Operatoren*, Math. Ann., **184**: 238–256, 1969/1970.

[17] A. I. MARKUSHEVICH, *Theory of functions of a complex variable*, Vol. I, II, III, Chelsea Publishing Co., New York, english edition, 1977. Translated and edited by Richard A. Silverman.

[18] A. S. MARKUS, *Introduction to the spectral theory of polynomial operator pencils*, volume 71 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1988.

[19] A. S. MARKUS AND V. I. MATSAEV, *Some estimates for the resolvent and for the lengths of Jordan chains of an analytic operator function*, In recent advances in operator theory (Groningen, 1998), volume 124 of Oper. Theory Adv. Appl., pages 473–479, Birkhäuser, Basel, 2001.

[20] P. MEYER-NIEBERG, *Banach Lattices*, Universitext, Springer-Verlag, Berlin, 1991.
[21] P. C. MÜLLER, *Stabilität und Matrizen*, Springer-Verlag, Berlin, 1977. Matrizenverfahren in der Stabilitätstheorie linearer dynamischer Systeme, Ingenieurwissenschaftliche Bibliothek/Engineering Science Library.
[22] P. J. PSARRAKOS AND M. J. TSATSOMEROS, *A primer of Perron-Frobenius theory for matrix polynomials*, Linear Algebra Appl., **393**: 333–351, 2004.
[23] O. PERRON, *Zur Theorie der Matrices*, Math. Ann., **64** (2): 248–263, 1907.
[24] A. RADL, *The block numerical range of positive operators on Hilbert lattices*, Integral Equ. Oper. Theory, **75**: 459–472, 2013 DOI:10.1007/s00020-013-2032-x.
[25] A. RADL, C. TRETTER, M. WAGENHOFER, *Corners of the block numerical range of analytic operator functions*, Preprint, 16 pp., 2013.
[26] R. T. RAU, *On the peripheral spectrum of monic operator polynomials with positive coefficients*, Integral Equ. Oper. Theory, **15** (3): 479–495, 1992.
[27] H. H. SCHAEFER, *Banach Lattices and Positive Operators*, Springer-Verlag, New York, 1974, Die Grundlehren der mathematischen Wissenschaften, Band 215.
[28] H. H. SCHAEFER, *Some spectral properties of positive linear operators*, Pacific J. Math., **10**: 1009–1019, 1960.
[29] G. SCHWEITZER, W. SCHICHTEN, P. C. MÜLLER, W. HÜBNER, J. LÜCKEL, G. SANDWEG AND R. LAUTENSCHLAGER, *Kreiselverhalten eines elastisch gelagerten Rotors*, Ingenieur Archiv, **41** (2): 110–140, 1972.
[30] F. TISSEUR AND K. MEERBERGEN, *The quadratic eigenvalue problem*, SIAM Rev., **43** (2): 235–286, 2001.
[31] C. TRETTER, *Spectral Theory of Block Operator Matrices and Applications*, Imperial College Press, London, 2008.
[32] C. TRETTER, *The quadratic numerical range of an analytic operator function*, Complex Anal. Oper. Theory, **4** (2): 449–469, 2010.
[33] C. TRETTER AND M. WAGENHOFER, *The block numerical range of an $n \times n$ block operator matrix*, SIAM J. Matrix Anal. Appl., **24** (4): 1003–1017 (electronic), 2003.
[34] M. WAGENHOFER, *Der blocknumerische Wertebereich*, 2000. Diploma Thesis – Universität Regensburg.
[35] M. WAGENHOFER, *C++-code for calculating numerical ranges, quadratic numerical ranges, and block numerical ranges of analytic operator functions*, Private communication, 2007.
[36] M. WAGENHOFER, *The block numerical range of an $n \times n$ block operator matrix*, Thesis (Ph.D.), 88 pp. Universität Bremen, 2007.

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