Operators and Matrices
Volume 8, Number 4 (2014), 935–955
doi:10.7153/oam-08-52

FREDHOLMNESS AND INDEX OF SIMPLEST SINGULAR INTEGRAL OPERATORS WITH TWO SLOWLY OSCILLATING SHIFTS

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(*Communicated by L. Rodman*)

Abstract. Let α and β be orientation-preserving diffeomorphisms (shifts) of $\mathbb{R}_+ = (0,\infty)$ onto itself with the only fixed points 0 and ∞, where the derivatives α' and β' may have discontinuities of slowly oscillating type at 0 and ∞. For $p \in (1,\infty)$, we consider the weighted shift operators U_α and U_β given on the Lebesgue space $L^p(\mathbb{R}_+)$ by $U_\alpha f = (\alpha')^{1/p}(f\circ\alpha)$ and $U_\beta f = (\beta')^{1/p}(f\circ\beta)$. We apply the theory of Mellin pseudodifferential operators with symbols of limited smoothness to study the simplest singular integral operators with two shifts $A_{ij} = U_\alpha^i P_+ + U_\beta^j P_-$ on the space $L^p(\mathbb{R}_+)$, where $P_\pm = (I \pm S)/2$ are operators associated to the Cauchy singular integral operator S, and $i, j \in \mathbb{Z}$. We prove that all A_{ij} are Fredholm operators on $L^p(\mathbb{R}_+)$ and have zero indices.

1. Introduction

Let $\mathscr{B}(X)$ be the Banach algebra of all bounded linear operators acting on a Banach space X, and let $\mathscr{K}(X)$ be the ideal of all compact operators in $\mathscr{B}(X)$. An operator $A \in \mathscr{B}(X)$ is called *Fredholm* if its image is closed and the spaces $\ker A$ and $\ker A^*$ are finite-dimensional. In that case the number

$$\operatorname{Ind} A = \dim\ker A - \dim\ker A^*$$

is referred to as the *index* of A (see, e.g., [3, Chap. 4]).

A bounded continuous function f on $\mathbb{R}_+ = (0,\infty)$ is called slowly oscillating (at 0 and ∞) if for each (equivalently, for some) $\lambda \in (0,1)$,

$$\lim_{r\to s} \sup_{t,\tau\in[\lambda r, r]} |f(t) - f(\tau)| = 0 \quad \text{for} \quad s \in \{0,\infty\}.$$

The set $SO(\mathbb{R}_+)$ of all slowly oscillating functions forms a C^*-algebra. This algebra properly contains $C(\overline{\mathbb{R}}_+)$, the C^*-algebra of all continuous functions on $\overline{\mathbb{R}}_+ := [0,+\infty]$. Suppose α is an orientation-preserving diffeomorphism of $\mathbb{R}_+$ onto itself, which has

Mathematics subject classification (2010): Primary 45E05; Secondary 47A53, 47B35, 47G10, 47G30.
Keywords and phrases: Slowly oscillating shift, Cauchy singular integral operator, Fredholmness, index, Mellin pseudodifferential operator.
The authors were supported by CEAF/IST/UTL, under the project PEst-OE/MAT/UI4032/2011 (Portugal).
The first author was also supported by CMA/FCT/UNL, under the project PEst-OE/MAT/UI0297/2011 (Portugal).
The second author was also supported by the CONACYT Project No. 168104 (México) and by PROMEP via "Proyecto de Redes" (México).

Paper OaM-08-52

only two fixed points 0 and ∞. We say that α is a slowly oscillating shift if $\log\alpha'$ is bounded and $\alpha' \in SO(\mathbb{R}_+)$. The set of all slowly oscillating shifts is denoted by $SOS(\mathbb{R}_+)$.

We suppose that $1 < p < \infty$. It is easily seen that if $\alpha \in SOS(\mathbb{R}_+)$, then the shift operator W_α defined by $W_\alpha f = f \circ \alpha$ is bounded and invertible on all spaces $L^p(\mathbb{R}_+)$ and its inverse is given by $W_\alpha^{-1} = W_{\alpha_{-1}}$, where α_{-1} is the inverse function to α. Along with W_α we consider the weighted shift operator

$$U_\alpha := (\alpha')^{1/p} W_\alpha$$

being an isometric isomorphism of the Lebesgue space $L^p(\mathbb{R}_+)$ onto itself. It is well known that the Cauchy singular integral operator S given by

$$(Sf)(t) := \lim_{\varepsilon\to 0} \frac{1}{\pi i} \int\limits_{\mathbb{R}_+\setminus(t-\varepsilon,t+\varepsilon)} \frac{f(\tau)}{\tau - t}\, d\tau, \quad t \in \mathbb{R}_+,$$

is bounded on all Lebesgue spaces $L^p(\mathbb{R}_+)$ for $1 < p < \infty$. Put

$$P_\pm := (I \pm S)/2.$$

This paper is in some sense a continuation of our papers [7, 8] where we found a Fredholm criterion for the singular integral operator

$$N = (aI - bW_\alpha)P_+ + (cI - dW_\alpha)P_-$$

with coefficients $a, b, c, d \in SO(\mathbb{R}_+)$ and a shift $\alpha \in SOS(\mathbb{R}_+)$. Here we make the next step towards the completion of the Fredholm theory for the operator N and compute indices of simplest singular integral operators with shifts

$$A_{ij} := U_\alpha^i P_+ + U_\beta^j P_-, \quad i, j \in \mathbb{Z}, \tag{1.1}$$

whose coefficients are pure isometric shift operators U_α^i and/or U_β^j under the mild assumption that $\alpha, \beta \in SOS(\mathbb{R}_+)$. To achieve this goal, we employ the machinery of Mellin pseudodifferential operators with slowly oscillating symbols developed in the series of papers of the second author [10, 11, 12, 13]. The main result of this paper is a necessary piece of our work in progress [9] dedicated to the calculation of the index of N. The techniques of that work are quite heavy and deserve to be illustrated on a simple example. Here we show in detail how the Fredholm theory for Mellin pseudodifferential operators can be used to study operators beyond that class on the example of simplest operators of the form (1.1).

The main result of this paper is the following.

THEOREM 1.1. *Let $\alpha, \beta \in SOS(\mathbb{R}_+)$. For all $i, j \in \mathbb{Z}$, the operator A_{ij} given by (1.1) is Fredholm on the space $L^p(\mathbb{R}_+)$ and* $\operatorname{Ind} A_{ij} = 0$.

The paper is organized as follows. In Section 2 we collect necessary facts on slowly oscillating functions and shifts and also on the algebra $\mathscr{A}$ generated by the operators I and S. In particular, we recall its description in terms of Mellin convolution operators and that it contains the family R_y, $y \in (1,\infty)$, of operators with fixed singularities given by

$$(R_y f)(t) := \frac{1}{\pi i}\int_0^\infty \left(\frac{t}{\tau}\right)^{1/y-1/p} \frac{f(\tau)}{\tau+t}\,d\tau, \quad t \in \mathbb{R}_+. \tag{1.2}$$

The operator $R := R_p$ is of special importance because $S^2 = I + R^2$ (in contrast to the case of the real line, where $S^2 = I$, whence $P_\pm$ are projections). We conclude Section 2 with the important fact that the operators in $\mathscr{A}$ commute with U_α and U_α^{-1} up to compact operators. In Section 3 we gather all needed facts on Mellin pseudodifferential operators with slowly oscillating symbols: the boundedness, compactness, and Fredholmness results. The latter is valid for symbols in the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$. In Section 4 we show that the operators $U_\alpha R_y$ and $U_\alpha^{-1} R_y$ for all $y \in (1,\infty)$ can be realized as Mellin pseudodifferential operators with symbols in the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$ up to compact summands. By using the above mentioned results, in Section 5 we show that the operator $(U_\alpha P_+ + P_-)(U_\alpha^{-1} P_+ + P_-)$ can be realized as a Mellin pseudodifferential operator with a symbol in the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$ up to a compact summand. We prove that the latter pseudodifferenial operator is Fredholm of index zero by using a fact from Section 3. Then we infer that the operators $U_\alpha P_+ + P_-$ and $U_\alpha^{-1} P_+ + P_-$ are both Fredholm and their indices are equal to zero. From this result we almost immediately get Theorem 1.1.

2. Preliminaries

2.1. Slowly oscillating functions and shifts

Repeating the proof of [6, Proposition 3.3] with minor modifications, we obtain the following statement.

LEMMA 2.1. *Suppose* $\varphi \in C^1(\mathbb{R}_+)$ *and put* $\psi(t) := t\varphi'(t)$ *for* $t \in \mathbb{R}_+$. *If* φ, ψ *belong to* $SO(\mathbb{R}_+)$, *then*

$$\lim_{t\to s} \psi(t) = 0 \quad \textit{for} \quad s \in \{0,\infty\}.$$

LEMMA 2.2. ([7, Lemma 2.2]) *An orientation-preserving shift* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to* $SOS(\mathbb{R}_+)$ *if and only if*

$$\alpha(t) = te^{\omega(t)}, \quad t \in \mathbb{R}_+,$$

for some real-valued function $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ *such that the function* $t \mapsto t\omega'(t)$ *also belongs to* $SO(\mathbb{R}_+)$ *and*

$$\inf_{t\in\mathbb{R}_+} \big(1 + t\omega'(t)\big) > 0.$$

LEMMA 2.3. ([7, Lemma 2.4]) *If* $\alpha \in SOS(\mathbb{R}_+)$, *then* $\alpha_{-1} \in SOS(\mathbb{R}_+)$.

LEMMA 2.4. (a) *If* $c \in SO(\mathbb{R}_+)$ *and* $\alpha \in SOS(\mathbb{R}_+)$, *then* $c \circ \alpha \in SO(\mathbb{R}_+)$ *and*

$$\lim_{t \to s}(c(t) - c[\alpha(t)]) = 0 \quad \textit{for} \quad s \in \{0, \infty\}.$$

(b) *If* $\alpha, \beta \in SOS(\mathbb{R}_+)$, *then* $\alpha \circ \beta \in SOS(\mathbb{R}_+)$.

Proof. Part (a) was proved in [7, Lemma 2.3].

(b) Let $\gamma = \alpha \circ \beta$. Since $\alpha, \beta \in SOS(\mathbb{R}_+)$, the logarithms of their derivatives $\log \alpha'$ and $\log \beta'$ are bounded. In view of

$$\log \gamma' = \log(\alpha' \circ \beta) + \log \beta',$$

the logarithm of the derivative $\log \gamma'$ is bounded, too. Further, by definition of slowly oscillating shifts, the functions α' and β' belong to $SO(\mathbb{R}_+)$. Therefore, by part (a), the function $\alpha' \circ \beta$ belongs to $SO(\mathbb{R}_+)$. Taking into account the fact that $SO(\mathbb{R}_+)$ is an algebra, we conclude that $\gamma' = (\alpha' \circ \beta) \cdot \beta' \in SO(\mathbb{R}_+)$. This completes the proof of $\gamma \in SOS(\mathbb{R}_+)$. □

For a shift $\alpha \in SOS(\mathbb{R}_+)$, put $\alpha_0(t) := t$ and $\alpha_i(t) := \alpha[\alpha_{i-1}(t)]$ for $i \in \mathbb{Z}$ and $t \in \mathbb{R}_+$. From Lemmas 2.3 and 2.4(b) we immediately get the following.

COROLLARY 2.5. *If* $\alpha, \beta \in SOS(\mathbb{R}_+)$, *then* $\alpha_i \circ \beta_j \in SOS(\mathbb{R}_+)$ *for all* $i, j \in \mathbb{Z}$.

2.2. Fourier and Mellin convolution operators

Let $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denote the Fourier transform,

$$(\mathscr{F} f)(x) := \int_{\mathbb{R}} f(y) e^{-ixy} dy, \quad x \in \mathbb{R},$$

and let $\mathscr{F}^{-1} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the inverse of $\mathscr{F}$. A function $a \in L^\infty(\mathbb{R})$ is called a Fourier multiplier on $L^p(\mathbb{R})$ if the mapping $f \mapsto \mathscr{F}^{-1} a \mathscr{F} f$ maps $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ onto itself and extends to a bounded operator on $L^p(\mathbb{R})$. The latter operator is then denoted by $W^0(a)$. We let $\mathscr{M}_p(\mathbb{R})$ stand for the set of all Fourier multipliers on $L^p(\mathbb{R})$. One can show that $\mathscr{M}_p(\mathbb{R})$ is a Banach algebra under the norm

$$\|a\|_{\mathscr{M}_p(\mathbb{R})} := \|W^0(a)\|_{\mathscr{B}(L^p(\mathbb{R}))}.$$

Let $d\mu(t) = dt/t$ be the (normalized) invariant measure on $\mathbb{R}_+$. Consider the Fourier transform on $L^2(\mathbb{R}_+, d\mu)$, which is usually referred to as the Mellin transform and is defined by

$$\mathscr{M} : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}), \quad (\mathscr{M} f)(x) = \int_{\mathbb{R}_+} f(t) t^{-ix} \frac{dt}{t}.$$

It is an invertible operator, with inverse given by

$$\mathscr{M}^{-1}: L^2(\mathbb{R}) \to L^2(\mathbb{R}_+, d\mu), \quad (\mathscr{M}^{-1}g)(t) = \frac{1}{2\pi}\int_{\mathbb{R}} g(x)t^{ix}\,dx.$$

Let E be the isometric isomorphism

$$E: L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R}), \quad (Ef)(x) := f(e^x), \quad x \in \mathbb{R}. \tag{2.1}$$

Then the map

$$A \mapsto E^{-1}AE$$

transforms the Fourier convolution operator $W^0(a) = \mathscr{F}^{-1}a\mathscr{F}$ to the Mellin convolution operator

$$\mathrm{Co}(a) := \mathscr{M}^{-1}a\mathscr{M}$$

with the same symbol a. Hence the class of Fourier multipliers on $L^p(\mathbb{R})$ coincides with the class of Mellin multipliers on $L^p(\mathbb{R}_+, d\mu)$.

2.3. Algebra of singular integral operators

Let $\mathscr{A}$ be the smallest closed subalgebra of $\mathscr{B}(L^p(\mathbb{R}_+))$ that contains the operators I and S. Consider the isometric isomorphism

$$\Phi: L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+, d\mu), \quad (\Phi f)(t) := t^{1/p}f(t) \quad (t \in \mathbb{R}_+). \tag{2.2}$$

The following statement is well known (see, e.g., [2], [5, Section 2.1.2] and [15, Sections 4.2.2–4.2.3]).

LEMMA 2.6. *Let* $1 < p < \infty$.
(a) *For every* $y \in (1,\infty)$, *the function* s_y *given by*

$$s_y(x) := \coth[\pi(x + i/y)], \quad x \in \mathbb{R},$$

belongs to $\mathscr{M}_p(\mathbb{R})$ *and* $S = \Phi^{-1}\mathrm{Co}(s_p)\Phi$.
(b) *For every* $y \in (1,\infty)$, *the function* r_y *given by*

$$r_y(x) := 1/\sinh[\pi(x + i/y)], \quad x \in \mathbb{R},$$

belongs to $\mathscr{M}_p(\mathbb{R})$, *the singular integral operator with fixed singularities* R_y *given by* (1.2) *belongs to the algebra* $\mathscr{A}$, *and* $R_y = \Phi^{-1}\mathrm{Co}(r_y)\Phi$.
(c) *The algebra* $\mathscr{A}$ *is commutative and* $S^2 - R_p^2 = I$.

The relation of Lemma 2.6(c) shows the importance of the operator R_p. For simplicity of notation, we will denote R_p by R.

2.4. Compactness of commutators

For bounded linear operators A and B, we will write $A \simeq B$ if $A - B$ is a compact operator.

LEMMA 2.7. *If* $\alpha \in SOS(\mathbb{R}_+)$, $A \in \mathscr{A}$, *then* $U_\alpha A \simeq AU_\alpha$ *and* $U_\alpha^{-1}A \simeq AU_\alpha^{-1}$.

Proof. If $\alpha \in SOS(\mathbb{R}_+)$, then $\alpha' \in SO(\mathbb{R}_+)$. Taking into account that $SO(\mathbb{R}_+)$ is a C^*-algebra and $\alpha' > 0$, we get $(\alpha')^{1/p} \in SO(\mathbb{R}_+)$. Then, by [7, Corollary 6.4],

$$(\alpha')^{1/p}A \simeq A(\alpha')^{1/p}I, \quad W_\alpha A \simeq AW_\alpha$$

for all $A \in \mathscr{A}$. Hence

$$U_\alpha A = (\alpha')^{1/p}W_\alpha A \simeq (\alpha')^{1/p}AW_\alpha \simeq A(\alpha')^{1/p}W_\alpha = AU_\alpha.$$

In view of Lemma 2.3, $\alpha_{-1} \in SOS(\mathbb{R}_+)$. Then, by the relation just proved,

$$U_\alpha^{-1}A = U_{\alpha_{-1}}A \simeq AU_{\alpha_{-1}} = AU_\alpha^{-1},$$

which completes the proof. □

3. Mellin pseudodifferential operators

3.1. Boundedness of Mellin pseudodifferential operators

A handy theory of Fourier pseudodifferential operators with slowly oscillating symbols of limited smoothness was developed in [10]. On the other hand, as we have seen in Subsection 2.3, singular integral operators $P_\pm$ and R on $\mathbb{R}_+$ can be realized as Mellin convolution operators. Hence, for our purposes the Mellin setting is more convenient. In this section we translate necessary results from [10] to the Mellin setting with the aid of the transformation

$$A \mapsto E^{-1}AE,$$

where $A \in \mathscr{B}(L^p(\mathbb{R}))$ and the isometric isomorphism $E : L^p(\mathbb{R}_+, d\mu) \to L^p(\mathbb{R})$ is defined by (2.1).

Let a be an absolutely continuous function of finite total variation

$$V(a) = \int_{\mathbb{R}} |a'(x)|dx$$

on $\mathbb{R}$. The set $V(\mathbb{R})$ of all absolutely continuous functions of finite total variation on $\mathbb{R}$ becomes a Banach algebra equipped with the norm

$$\|a\|_V := \|a\|_{L^\infty(\mathbb{R})} + V(a).$$

Following [10, 11], let $C_b(\mathbb{R}_+,V(\mathbb{R}))$ denote the Banach algebra of all bounded continuous $V(\mathbb{R})$-valued functions on $\mathbb{R}_+$ with the norm

$$\|\mathfrak{a}(\cdot,\cdot)\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} = \sup_{t\in\mathbb{R}_+} \|\mathfrak{a}(t,\cdot)\|_V.$$

As usual, let $C_0^\infty(\mathbb{R}_+)$ be the set of all infinitely differentiable functions of compact support on $\mathbb{R}_+$.

The following boundedness result for Mellin pseudodifferential operators follows from [11, Theorem 6.1] (see also [10, Theorem 3.1]).

THEOREM 3.1. *If* $\mathfrak{a}\in C_b(\mathbb{R}_+,V(\mathbb{R}))$, *then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$, *defined for functions* $f\in C_0^\infty(\mathbb{R}_+)$ *by the iterated integral*

$$[\mathrm{Op}(\mathfrak{a})f](t) = \frac{1}{2\pi}\int_{\mathbb{R}} dx \int_{\mathbb{R}_+} \mathfrak{a}(t,x)\left(\frac{t}{\tau}\right)^{ix} f(\tau)\frac{d\tau}{\tau} \quad \textit{for} \quad t\in\mathbb{R}_+,$$

extends to a bounded linear operator on the space $L^p(\mathbb{R}_+,d\mu)$ *and there is a number* $C_p\in(0,\infty)$ *depending only on* p *such that*

$$\|\mathrm{Op}(\mathfrak{a})\|_{\mathscr{B}(L^p(\mathbb{R}_+,d\mu))} \leqslant C_p\|\mathfrak{a}\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))}.$$

Obviously, if $\mathfrak{a}(t,x)=a(x)$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$, then the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ becomes the Mellin convolution operator

$$\mathrm{Op}(\mathfrak{a}) = \mathrm{Co}(a).$$

3.2. Compactness of Mellin pseudodifferential operators

Let $SO(\mathbb{R}_+,V(\mathbb{R}))$ denote the Banach subalgebra of $C_b(\mathbb{R}_+,V(\mathbb{R}))$ consisting of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ on $\mathbb{R}_+$ that slowly oscillate at 0 and ∞, that is,

$$\lim_{r\to 0}\mathrm{cm}_r^C(\mathfrak{a}) = \lim_{r\to\infty}\mathrm{cm}_r^C(\mathfrak{a}) = 0,$$

where

$$\mathrm{cm}_r^C(\mathfrak{a}) = \max\left\{\left\|\mathfrak{a}(t,\cdot)-\mathfrak{a}(\tau,\cdot)\right\|_{L^\infty(\mathbb{R})} : t,\tau\in[r,2r]\right\}.$$

Let $\mathscr{E}(\mathbb{R}_+,V(\mathbb{R}))$ be the Banach algebra of all $V(\mathbb{R})$-valued functions $\mathfrak{a}$ in the algebra $SO(\mathbb{R}_+,V(\mathbb{R}))$ such that

$$\lim_{|h|\to 0}\sup_{t\in\mathbb{R}_+}\left\|\mathfrak{a}(t,\cdot)-\mathfrak{a}^h(t,\cdot)\right\|_V = 0$$

where $\mathfrak{a}^h(t,x) := \mathfrak{a}(t,x+h)$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$.

Applying the relation

$$\mathrm{Op}(\mathfrak{a}) = E^{-1}a(x,D)E \tag{3.1}$$

between the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ and the Fourier pseudodifferential operator $a(x,D)$ considered in [10], where

$$\mathfrak{a}(t,x) = a(\ln t, x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{3.2}$$

and E is given by (2.1), we infer from [10, Theorem 4.4] the following compactness result.

THEOREM 3.2. *If* $\mathfrak{a} \in \mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$ *and*

$$\lim_{\ln^2 t + x^2 \to \infty} \mathfrak{a}(t,x) = 0,$$

then the Mellin pseudodifferential operator $\mathrm{Op}(\mathfrak{a})$ *is compact on the space* $L^p(\mathbb{R}_+, d\mu)$.

3.3. Products of Mellin pseudodifferential operators

The next result on compactness of semi-commutators of Mellin pseudodifferential operators immediately follows from (2.1), (3.1)–(3.2) and [10, Theorem 8.3].

THEOREM 3.3. *If* $\mathfrak{a}, \mathfrak{b} \in \mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$*, then*

$$\mathrm{Op}(\mathfrak{a})\,\mathrm{Op}(\mathfrak{b}) \simeq \mathrm{Op}(\mathfrak{a}\mathfrak{b}).$$

From (2.1), (3.1)–(3.2), [10, Lemmas 7.1, 7.2], and the proof of [10, Lemma 8.1] we can extract the following.

LEMMA 3.4. *If* $\mathfrak{a}, \mathfrak{b}, \mathfrak{c} \in \mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$ *are such that* $\mathfrak{a}$ *depends only on the first variable and* $\mathfrak{c}$ *depends only on the second variable, then*

$$\mathrm{Op}(\mathfrak{a})\,\mathrm{Op}(\mathfrak{b})\,\mathrm{Op}(\mathfrak{c}) = \mathrm{Op}(\mathfrak{a}\mathfrak{b}\mathfrak{c}).$$

3.4. Fredholmness of Mellin pseudodifferential operators

For a unital commutative Banach algebra $\mathfrak{A}$, let $M(\mathfrak{A})$ denote its maximal ideal space. Identifying the points $t \in \overline{\mathbb{R}}_+$ with the evaluation functionals $t(f) = f(t)$ for $f \in C(\overline{\mathbb{R}}_+)$, we get $M(C(\overline{\mathbb{R}}_+)) = \overline{\mathbb{R}}_+$. Consider the fibers

$$M_s(SO(\mathbb{R}_+)) := \left\{ \xi \in M(SO(\mathbb{R}_+)) : \xi|_{C(\overline{\mathbb{R}}_+)} = s \right\}$$

of the maximal ideal space $M(SO(\mathbb{R}_+))$ over the points $s \in \{0, \infty\}$. By [13, Proposition 2.1], the set

$$\Delta := M_0(SO(\mathbb{R}_+)) \cup M_\infty(SO(\mathbb{R}_+))$$

coincides with $(\mathrm{clos}_{SO^*} \mathbb{R}_+) \setminus \mathbb{R}_+$ where $\mathrm{clos}_{SO^*} \mathbb{R}_+$ is the weak-star closure of $\mathbb{R}_+$ in the dual space of $SO(\mathbb{R}_+)$. Then $M(SO(\mathbb{R}_+)) = \Delta \cup \mathbb{R}_+$.

Let $\mathfrak{a} \in \mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$. For every $t \in \mathbb{R}_+$, the function $\mathfrak{a}(t,\cdot)$ belongs to $V(\mathbb{R})$ and, therefore, has finite limits at $\pm\infty$, which will be denoted by $\mathfrak{a}(t,\pm\infty)$. Now we explain how to extend the function $\mathfrak{a}$ to $\Delta \times \overline{\mathbb{R}}$. By analogy with [10, Lemma 2.7] one can prove the following.

LEMMA 3.5. *Let* $s \in \{0,\infty\}$ *and* $\mathfrak{a} \in \mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$. *For each* $\xi \in M_s(SO(\mathbb{R}_+))$ *there exist a sequence* $\{t_n\} \subset \mathbb{R}_+$ *and a function* $\mathfrak{a}(\xi,\cdot) \in V(\mathbb{R})$ *such that* $t_n \to s$ *as* $n \to \infty$ *and*

$$\mathfrak{a}(\xi,x) = \lim_{n\to\infty} \mathfrak{a}(t_n,x) \quad \textit{for every} \quad x \in \overline{\mathbb{R}}.$$

To study the Fredholmness of Mellin pseudodifferential operators, we need the Banach algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$ consisting of all functions $\mathfrak{a}$ belonging to $\mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$ and such that

$$\lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}\setminus[-m,m]} \left| \frac{\partial \mathfrak{a}(t,x)}{\partial x} \right| dx = 0. \tag{3.3}$$

Below we need the following Fredholm criterion and index formula for Mellin pseudodifferential operators $\mathrm{Op}(\mathfrak{a})$ with symbols $\mathfrak{a} \in \widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$, which were obtained in [13, Theorem 4.3] on the base of [10, Theorems 12.2 and 12.5] and (2.1), (3.1)–(3.2). Note that for infinite differentiable slowly oscillating symbols $\mathfrak{a}$ such result was obtained earlier in [14, Theorem 2.6].

THEOREM 3.6. *If* $\mathfrak{a} \in \widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$, *then the Mellin pseudodifferential operator* $\mathrm{Op}(\mathfrak{a})$ *is Fredholm on the space* $L^p(\mathbb{R}_+, d\mu)$ *if and only if*

$$\mathfrak{a}(t,\pm\infty) \neq 0 \ \textit{for all} \ t \in \mathbb{R}_+, \quad \mathfrak{a}(\xi,x) \neq 0 \ \textit{for all} \ (\xi,x) \in \Delta \times \overline{\mathbb{R}}. \tag{3.4}$$

In the case of Fredholmness

$$\mathrm{Ind}\,\mathrm{Op}(\mathfrak{a}) = \lim_{\tau\to+\infty} \frac{1}{2\pi} \big\{ \arg \mathfrak{a}(t,x) \big\}_{(t,x)\in\partial\Pi_\tau},$$

where $\Pi_\tau = [\tau^{-1}, \tau] \times \overline{\mathbb{R}}$ *and* $\big\{ \arg \mathfrak{a}(t,x) \big\}_{(t,x)\in\partial\Pi_\tau}$ *denotes the increment of* $\arg \mathfrak{a}(t,x)$ *when the point* (t,x) *traces the boundary* $\partial\Pi_\tau$ *of* Π_τ *counter-clockwise.*

4. Applications of Mellin pseudodifferential operators

4.1. Some important functions in the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$

We start with the following obvious auxiliary statement.

LEMMA 4.1. *For every* $p \in (1,\infty)$ *and* $j \in \{0,1\}$, *we have*

$$C_j^\infty(p) := \sup_{x\in\mathbb{R}} |x|^j |r_p(x)| < \infty, \tag{4.1}$$

$$C_j^1(p) := \int_{\mathbb{R}} |x|^j |r_p(x)|\, dx < \infty, \tag{4.2}$$

$$M_0(p) := \sup_{x\in\mathbb{R}} \big|\pi s_p(x)\big| < \infty. \tag{4.3}$$

Consider now simple “bricks” in our construction, functions depending only on one variable.

LEMMA 4.2. *Let $g \in SO(\mathbb{R}_+)$. Then for every $p \in (1,\infty)$ the functions*

$$\mathfrak{g}(t,x) := g(t), \quad \mathfrak{s}_p(t,x) := s_p(x), \quad \mathfrak{r}_p(t,x) := r_p(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R},$$

belong to the Banach algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+,V(\mathbb{R}))$.

Proof. We have already shown in [7, Lemma 7.1] that these functions belong to the algebra $\mathscr{E}(\mathbb{R}_+,V(\mathbb{R}))$. Since $\mathfrak{g}$ does not depend on x, condition (3.3) holds trivially for $\mathfrak{g}$. Thus, $\mathfrak{g} \in \widetilde{\mathscr{E}}(\mathbb{R}_+,V(\mathbb{R}))$.

From $s_p'(x) = -\pi(r_p(x))^2$ and $r_p'(x) = -\pi s_p(x) r_p(x)$, taking into account (4.1) and (4.3), we see that

$$\lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}\setminus[-m,m]} \left| \frac{\partial \mathfrak{s}_p(t,x)}{\partial x} \right| dx\pi \leqslant C_0^\infty(p) \lim_{m\to\infty} \int_{\mathbb{R}\setminus[-m,m]} |r_p(x)|\, dx = 0,$$

$$\lim_{m\to\infty} \sup_{t\in\mathbb{R}_+} \int_{\mathbb{R}\setminus[-m,m]} \left| \frac{\partial \mathfrak{r}_p(t,x)}{\partial x} \right| dx \leqslant M_0(p) \lim_{m\to\infty} \int_{\mathbb{R}\setminus[-m,m]} |r_p(x)|\, dx = 0.$$

Thus, $\mathfrak{s}_p, \mathfrak{r}_p \in \widetilde{\mathscr{E}}(\mathbb{R}_+,V(\mathbb{R}))$. $\square$

The next statement is crucial for our analysis.

LEMMA 4.3. *Suppose $\omega \in SO(\mathbb{R}_+)$ is a real-valued function. Then for every $p \in (1,\infty)$ the function*

$$\mathfrak{b}(t,x) := e^{i\omega(t)x} r_p(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.4}$$

belongs to the Banach algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+,V(\mathbb{R}))$ and there is a constant $C(p) \in (0,\infty)$ depending only on p such that

$$\|\mathfrak{b}\|_{C_b(\mathbb{R}_+,V(\mathbb{R}))} \leqslant C(p) \left(1 + \sup_{t\in\mathbb{R}_+} |\omega(t)| \right). \tag{4.5}$$

Proof. First, by analogy with [7, Lemma 7.3] we will show that $\mathfrak{b} \in C_b(\mathbb{R}_+,V(\mathbb{R}))$. Through the proof we assume that $t,\tau \in \mathbb{R}_+$ and $x \in \mathbb{R}$. From (4.4) and (4.1) it follows that

$$\|\mathfrak{b}(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant C_0^\infty(p). \tag{4.6}$$

Since $\omega \in SO(\mathbb{R}_+)$, we have

$$M_1(\omega) := \sup_{t\in\mathbb{R}_+} |\omega(t)| < \infty. \tag{4.7}$$

From (4.4) it easily follows that

$$\frac{\partial \mathfrak{b}}{\partial x}(t,x) = \big(i\omega(t) - \pi s_p(x)\big)\mathfrak{b}(t,x), \tag{4.8}$$

$$\frac{\partial^2 \mathfrak{b}}{\partial x^2}(t,x) = \big(\pi^2 (r_p(x))^2 + (i\omega(t) - \pi s_p(x))^2\big)\mathfrak{b}(t,x). \tag{4.9}$$

From (4.8), (4.7) and (4.2), (4.3) we obtain

$$V(\mathfrak{b}(t,\cdot)) = \int_{\mathbb{R}} \left| \frac{\partial \mathfrak{b}(t,x)}{\partial x} \right| dx \leqslant (M_1(\omega) + M_0(p)) C_0^1(p). \tag{4.10}$$

Combining (4.6) and (4.10), we arrive at

$$\|\mathfrak{b}(t,\cdot)\|_V \leqslant C_0^\infty(p) + (M_1(\omega) + M_0(p)) C_0^1(p) \leqslant C(p)(1 + M_1(\omega)), \tag{4.11}$$

where $C(p) := \max(C_0^\infty(p) + C_0^1(p) M_0(p), C_0^1(p))$. Further from (4.4), we get

$$\left|\mathfrak{b}(t,x) - \mathfrak{b}(\tau,x)\right| = \left| ix \left(\int_{\omega(\tau)}^{\omega(t)} e^{i\theta x} d\theta \right) r_p(x) \right| \leqslant |\omega(t) - \omega(\tau)|\,|x|\,|r_p(x)|. \tag{4.12}$$

From (4.12) and (4.1) we obtain

$$\|\mathfrak{b}(t,\cdot) - \mathfrak{b}(\tau,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant C_1^\infty(p)|\omega(t) - \omega(\tau)|. \tag{4.13}$$

From (4.8) it follows that

$$\begin{aligned} \frac{\partial \mathfrak{b}(t,x)}{\partial x} - \frac{\partial \mathfrak{b}(\tau,x)}{\partial x} =& i(\omega(t) - \omega(\tau))\mathfrak{b}(t,x) \\ &+ (i\omega(\tau) - \pi s_p(x))(\mathfrak{b}(t,x) - \mathfrak{b}(\tau,x)). \end{aligned} \tag{4.14}$$

Starting with (4.14) and taking into account (4.12), (4.3), and (4.7), we arrive at

$$\left| \frac{\partial \mathfrak{b}(t,x)}{\partial x} - \frac{\partial \mathfrak{b}(\tau,x)}{\partial x} \right| \leqslant |\omega(t) - \omega(\tau)| \big(|r_p(x)| + (M_1(\omega) + M_0(p))|x|\,|r_p(x)| \big).$$

Therefore, taking into account (4.2), we infer from the latter inequality that

$$V(\mathfrak{b}(t,\cdot) - \mathfrak{b}(\tau,\cdot)) = \int_{\mathbb{R}} \left| \frac{\partial \mathfrak{b}(t,x)}{\partial x} - \frac{\partial \mathfrak{b}(\tau,x)}{\partial x} \right| dx \leqslant L_1(\omega,p)|\omega(t) - \omega(\tau)|, \tag{4.15}$$

where $L_1(\omega,p) := C_0^1(p) + (M_1(\omega) + M_0(p)) C_1^1(p)$. Combining (4.13) and (4.15), we arrive at

$$\|\mathfrak{b}(t,\cdot) - \mathfrak{b}(\tau,\cdot)\|_V \leqslant L_2(\omega,p)|\omega(t) - \omega(\tau)| \tag{4.16}$$

for $t,\tau \in \mathbb{R}_+$, where $L_2(\omega,p) := C_1^\infty(p) + L_1(\omega,p)$. From (4.11) and (4.16) it follows that $\mathfrak{b}$ is a bounded and continuous $V(\mathbb{R})$-valued function. Thus, $b \in C_b(\mathbb{R}_+, V(\mathbb{R}))$. Estimate (4.5) follows immediately from (4.11).

Further, we will show by analogy with [7, Lemma 7.4] that $\mathfrak{b}$ belongs to the algebra $\mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$. Estimate (4.13) immediately implies that

$$\mathrm{cm}_r^C(\mathfrak{b}) \leqslant C_1^\infty(p)\,\mathrm{osc}(\omega,[r,2r]), \quad r \in \mathbb{R}_+.$$

Since $\omega \in SO(\mathbb{R}_+)$, from this estimate we obtain

$$\lim_{r\to s} \mathrm{cm}_r^C(\mathfrak{b}) = \lim_{r\to s} \mathrm{osc}(\omega, [r, 2r]) = 0, \quad s \in \{0, \infty\}.$$

Thus, $\mathfrak{b} \in SO(\mathbb{R}_+, V(\mathbb{R}))$.

From (4.8), (4.7) and (4.1), (4.3) we infer that

$$\left| \frac{\partial \mathfrak{b}(t,x)}{\partial x} \right| \leqslant M_2(\omega, p),$$

where $M_2(\omega, p) := (M_1(\omega) + M_0(p)) C_0^\infty(p) < \infty$.

Let $h \in \mathbb{R}$. Then

$$|\mathfrak{b}(t,x) - \mathfrak{b}^h(t,x)| = \left| \int_x^{x+h} \frac{\partial \mathfrak{b}}{\partial y}(t,y)\, dy \right| \leqslant \left| \int_x^{x+h} \left| \frac{\partial \mathfrak{b}}{\partial y}(t,y) \right| dy \right| \leqslant M_2(\omega, p)|h|.$$

Therefore,

$$\sup_{t\in\mathbb{R}_+} \|\mathfrak{b}(t,\cdot) - \mathfrak{b}^h(t,\cdot)\|_{L^\infty(\mathbb{R})} \leqslant M_2(\omega, p)|h|, \quad h \in \mathbb{R}. \tag{4.17}$$

From (4.9), (4.7) and (4.1), (4.3) we obtain

$$\left| \frac{\partial^2 \mathfrak{b}(t,x)}{\partial x^2} \right| \leqslant M_3(\omega, p)|r_p(x)|,$$

where $M_3(\omega, p) := (\pi C_0^\infty(p))^2 + (M_1(\omega) + M_0(p))^2 < \infty$. Fix $h > 0$. Then

$$\begin{aligned} V(\mathfrak{b}(t,\cdot) - \mathfrak{b}^h(t,\cdot)) &= \int_{\mathbb{R}} \left| \frac{\partial \mathfrak{b}}{\partial x}(t, x+h) - \frac{\partial \mathfrak{b}}{\partial x}(t,x) \right| dx = \int_{\mathbb{R}} \left| \int_x^{x+h} \frac{\partial^2 \mathfrak{b}}{\partial y^2}(t,y) dy \right| dx \\ &\leqslant \int_{\mathbb{R}} \int_x^{x+h} \left| \frac{\partial^2 \mathfrak{b}}{\partial y^2}(t,y) \right| dy\, dx \leqslant M_3(\omega, p) \int_{\mathbb{R}} \int_x^{x+h} |r_p(y)| dy\, dx. \end{aligned} \tag{4.18}$$

Changing the order of integration in the integral on the right-hand side of (4.18) and taking into account (4.1), we get for $h \in \mathbb{R}$,

$$\int_{\mathbb{R}} \int_x^{x+h} |r_p(y)| dy\, dx = \int_{\mathbb{R}} \int_{y-h}^{y} |r_p(y)| dx\, dy = h \int_{\mathbb{R}} |r_p(y)| dy = C_0^1(p) h. \tag{4.19}$$

Combining (4.18) and (4.19), we see that

$$V(\mathfrak{b}(t,\cdot) - \mathfrak{b}^h(t,\cdot)) \leqslant M_4(\omega, p) h \quad (h > 0), \tag{4.20}$$

where $M_4(\omega, p) := C_0^1(p) M_3(\omega, p)$. Analogously it can be shown that

$$V(\mathfrak{b}(t,\cdot) - \mathfrak{b}^h(t,\cdot)) \leqslant M_4(\omega, p)(-h) \quad (h < 0). \tag{4.21}$$

From (4.20) and (4.21) we get for $h \in \mathbb{R}$,

$$\sup_{t\in\mathbb{R}_+} V\big(\mathfrak{b}(t,\cdot) - \mathfrak{b}^h(t,\cdot)\big) \leqslant M_4(\omega,p)|h|. \tag{4.22}$$

Combining (4.17) with (4.22), we arrive at

$$\lim_{|h|\to 0}\sup_{t\in\mathbb{R}_+} \|\mathfrak{b}(t,\cdot) - \mathfrak{b}^h(t,\cdot)\|_V \leqslant (M_2(\omega,p)+M_4(\omega,p))\lim_{|h|\to 0}|h| = 0,$$

which implies that $\mathfrak{b} \in SO(\mathbb{R}_+, V(\mathbb{R}))$ actually belongs to $\mathscr{E}(\mathbb{R}_+, V(\mathbb{R}))$.

Finally, from (4.8), (4.7) and (4.3) it follows that

$$\lim_{m\to\infty}\sup_{t\in\mathbb{R}_+}\int_{\mathbb{R}\setminus[-m,m]}\left|\frac{\partial\mathfrak{b}}{\partial x}(t,x)\right| dx \leqslant (M_1(\omega)+M_0(p))\lim_{m\to\infty}\int_{\mathbb{R}\setminus[-m,m]}|r_p(x)|\,dx = 0,$$

which in view of (3.3) implies that $\mathfrak{b} \in \widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$. □

4.2. Product of a shift operator and an operator with fixed singularities

In this subsection we show that the operators $U_\alpha R_y$ and $U_\alpha^{-1} R_y$ can be realized as Mellin pseudodifferential operators with symbols in the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$ for arbitrary $y \in (1,\infty)$.

LEMMA 4.4. *Let $\alpha \in SOS(\mathbb{R}_+)$ and U_α be the associated isometric shift operator on $L^p(\mathbb{R}_+)$. For every $y \in (1,\infty)$, the operator $U_\alpha R_y$ can be realized as the Mellin pseudodifferential operator:*

$$U_\alpha R_y = \Phi^{-1}\operatorname{Op}(\mathfrak{c}_{\alpha,y})\Phi,$$

where the function $\mathfrak{c}_{\alpha,y}$, given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{c}_{\alpha,y}(t,x) := (1+t\omega'(t))^{1/p} e^{i\omega(t)x} r_y(x) \quad \text{with} \quad \omega(t) := \log[\alpha(t)/t], \tag{4.23}$$

belongs to the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. We follow the proof of [7, Lemma 8.3]. By Lemma 2.2, $\alpha(t) = te^{\omega(t)}$, where $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ is a real-valued function. Hence

$$\alpha'(t) = \Omega(t)e^{\omega(t)}, \quad \text{where} \quad \Omega(t) := 1 + t\omega'(t), \quad t \in \mathbb{R}_+. \tag{4.24}$$

Assume that $f \in C_0^\infty(\mathbb{R}_+)$. Taking into account (4.24), we have

$$\begin{aligned}(\Phi U_\alpha R_y \Phi^{-1} f)(t) &= \frac{(\alpha'(t))^{1/p}}{\pi i}\int_{\mathbb{R}_+}\left(\frac{\alpha(t)}{\tau}\right)^{1/y-1/p}\frac{f(\tau)(t/\tau)^{1/p}}{\tau+\alpha(t)}\,d\tau \\ &= \frac{(\Omega(t))^{1/p}e^{\omega(t)/p}}{\pi i}\int_{\mathbb{R}_+}\left(\frac{te^{\omega(t)}}{\tau}\right)^{1/y-1/p}\frac{f(\tau)(t/\tau)^{1/p}}{1+e^{\omega(t)}(t/\tau)}\frac{d\tau}{\tau} \\ &= \frac{(\Omega(t))^{1/p}e^{\omega(t)/y}}{\pi i}\int_{\mathbb{R}_+}\frac{f(\tau)(t/\tau)^{1/y}}{1+e^{\omega(t)}(t/\tau)}\frac{d\tau}{\tau} \\ &= (\Omega(t))^{1/p}(I_y f)(t), \end{aligned} \tag{4.25}$$

where

$$(I_y f)(t) := \frac{e^{\omega(t)/y}}{\pi i}\int_{\mathbb{R}_+}\frac{f(\tau)(t/\tau)^{1/y}}{1+e^{\omega(t)}(t/\tau)}\frac{d\tau}{\tau}, \quad y \in (1,\infty). \tag{4.26}$$

From [4, formula 3.194.4] it follows that for $y \in (1,\infty)$, $k>0$, and $x \in \mathbb{R}$,

$$\frac{1}{\pi i}\int_{\mathbb{R}_+}\frac{t^{1/y}}{1+kt}t^{-ix}\frac{dt}{t} = \frac{1}{k^{1/y-ix}}\cdot\frac{1}{i\sin[\pi(1/y-ix)]} = e^{i(x+i/y)\log k}r_y(x).$$

Taking the inverse Mellin transform, we get

$$\frac{1}{\pi i}\frac{t^{1/y}}{1+kt} = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i(x+i/y)\log k} r_y(x) t^{ix}\,dx. \tag{4.27}$$

Hence, for $y \in (1,\infty)$, we infer from (4.26)–(4.27) that

$$\begin{aligned}(I_y f)(t) &= \frac{e^{\omega(t)/y}}{2\pi}\int_{\mathbb{R}_+}\left(\int_{\mathbb{R}} e^{i\omega(t)(x+i/y)} r_y(x)\left(\frac{t}{\tau}\right)^{ix}dx\right) f(\tau)\frac{d\tau}{\tau} \\ &= \frac{1}{2\pi}\int_{\mathbb{R}}dx\int_{\mathbb{R}_+} e^{i\omega(t)x} r_y(x)\left(\frac{t}{\tau}\right)^{ix} f(\tau)\frac{d\tau}{\tau}. \end{aligned} \tag{4.28}$$

From (4.25) and (4.28) we obtain for $f \in C_0^\infty(\mathbb{R}_+)$,

$$\Phi U_\alpha R_y \Phi^{-1} f = \mathrm{Op}(\mathfrak{c}_{\alpha,y}) f. \tag{4.29}$$

By Lemma 2.2, the function Ω belongs to $SO(\mathbb{R}_+)$. Then $\Omega^{1/p}$ is also in $SO(\mathbb{R}_+)$. Therefore, from Lemmas 4.2 and 4.3 it follows that the function $\mathfrak{c}_{\alpha,y}$ belongs to the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R})) \subset C_b(\mathbb{R}_+, V(\mathbb{R}))$. Then Theorem 3.1 implies that $\mathrm{Op}(\mathfrak{c}_{\alpha,y})$ extends to a bounded operators on $L^p(\mathbb{R}_+, d\mu)$ and, therefore, from (4.29) we obtain $\Phi U_\alpha R_y \Phi^{-1} = \mathrm{Op}(\mathfrak{c}_{\alpha,y})$, which completes the proof. $\square$

Now we show that the symbol in the lemma above can be simplified if we allow equality up to a compact summand instead of the exact equality. Moreover, the operator $U_\alpha^{-1}R_y$ can be similarly treated as well.

LEMMA 4.5. *Let $\alpha \in SOS(\mathbb{R}_+)$ and U_α be the associated isometric shift operator on $L^p(\mathbb{R}_+)$. For every $y \in (1,\infty)$, the operators $U_\alpha R_y$ and $U_\alpha^{-1}R_y$ can be realized as the Mellin pseudodifferential operators up to compact operators:*

$$U_\alpha^{\pm 1}R_y \simeq \Phi^{-1}\operatorname{Op}(\mathfrak{c}^{\pm}_{\alpha,y})\Phi,$$

where the functions $\mathfrak{c}^{\pm}_{\alpha,y}$ given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{c}^{\pm}_{\alpha,y}(t,x) := e^{\pm i\omega(t)x} r_y(x) \quad \textit{with} \quad \omega(t) := \log[\alpha(t)/t],$$

belong to the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$.

Proof. The functions $\mathfrak{c}^{\pm}_{\alpha,y}$ belong to the algebra $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$ due to Lemma 4.3. Let us first prove that

$$U_\alpha R_y \simeq \Phi^{-1}\operatorname{Op}(\mathfrak{c}^{+}_{\alpha,y})\Phi. \tag{4.30}$$

In view of Lemma 4.4 it is sufficient to show that

$$\operatorname{Op}(\mathfrak{c}_{\alpha,y}) \simeq \operatorname{Op}(\mathfrak{c}^{+}_{\alpha,y}), \tag{4.31}$$

where the function $\mathfrak{c}_{\alpha,y}$ is given by (4.23). Let Ω be given by (4.24) and

$$m_\Omega := \inf_{t\in\mathbb{R}_+} \Omega(t), \quad M_\Omega := \sup_{t\in\mathbb{R}_+} \Omega(t).$$

From Lemma 2.2 it follows that $m_\Omega > 0$ and $\omega \in SO(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$. Moreover, $t\omega'(t)$ is also in $SO(\mathbb{R}_+)$. Then $\Omega \in SO(\mathbb{R}_+)$, whence $M_\Omega < \infty$. By Lemma 2.1,

$$\lim_{t\to s}(t\omega'(t)) = 0, \quad s \in \{0,\infty\},$$

whence

$$\lim_{t\to s}(\Omega(t))^{1/p} = \Big(1 + \lim_{t\to s}(t\omega'(t))\Big)^{1/p} = 1 \quad \text{for} \quad s \in \{0,\infty\}. \tag{4.32}$$

On the other hand, obviously,

$$\lim_{|x|\to\infty} r_y(x) = 0, \quad y \in (1,\infty). \tag{4.33}$$

Combining (4.32)–(4.33) with the definitions of $\mathfrak{c}_{\alpha,y}$ and $\mathfrak{c}^{+}_{\alpha,y}$, we arrive at

$$\lim_{\ln^2 t + x^2 \to\infty} (\mathfrak{c}_{\alpha,y}(t,x) - \mathfrak{c}^{+}_{\alpha,y}(t,x)) = 0.$$

From these equalities and Theorem 3.2 we get (4.31).

Now let us prove that

$$U_\alpha^{-1} R_y \simeq \Phi^{-1} \mathrm{Op}(\mathfrak{c}_{\alpha,y}^-)\Phi. \tag{4.34}$$

In view of Lemma 2.3, $\alpha_{-1} \in SOS(\mathbb{R}_+)$ along with α. Then, in view of (4.30) with the shift α_{-1} in place of the shift α, we get

$$U_\alpha^{-1} R_y = U_{\alpha_{-1}} R_y \simeq \Phi^{-1} \mathrm{Op}(\widetilde{\mathfrak{c}}_{\alpha,y}^+)\Phi, \tag{4.35}$$

where the function $\widetilde{\mathfrak{c}}_{\alpha,y}^+$, defined by $\widetilde{\mathfrak{c}}_{\alpha,y}^+(t,x) := e^{i\widetilde{\omega}(t)x} r_y(x)$ for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$, belongs to $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$ and $\widetilde{\omega}(t) := \log[\alpha_{-1}(t)/t]$ is a real-valued function in $SO(\mathbb{R}_+)$ (see also Lemma 2.2). On the other hand, for $t \in \mathbb{R}_+$,

$$\widetilde{\omega}(t) = \log \frac{\alpha_{-1}(t)}{t} = -\log \frac{t}{\alpha_{-1}(t)} = -\log \frac{\alpha[\alpha_{-1}(t)]}{\alpha_{-1}(t)} = -\omega[\alpha_{-1}(t)].$$

Hence, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ we have

$$\begin{aligned} |\mathfrak{c}_{\alpha,y}^-(t,x) - \widetilde{\mathfrak{c}}_{\alpha,y}^+(t,x)| &= \left| e^{-i\omega(t)x} - e^{-i\omega[\alpha_{-1}(t)]x} \right| |r_y(x)| \\ &= \left| ix \left(\int_{-\omega[\alpha_{-1}(t)]}^{-\omega(t)} e^{i\theta x}\, d\theta \right) r_y(x) \right| \\ &\leqslant |\omega(t) - \omega[\alpha_{-1}(t)]|\, |x|\, |r_y(x)|. \end{aligned} \tag{4.36}$$

By Lemma 2.4(a), for $s \in \{0, \infty\}$,

$$\lim_{t \to s} |\omega(t) - \omega[\alpha_{-1}(t)]| = 0. \tag{4.37}$$

On the other hand, obviously,

$$\lim_{|x| \to \infty} |x r_y(x)| = 0. \tag{4.38}$$

Combining (4.36)–(4.38), we arrive at the equality

$$\lim_{\ln^2 t + x^2 \to \infty} (\mathfrak{c}_{\alpha,y}^-(t,x) - \widetilde{\mathfrak{c}}_{\alpha,y}^+(t,x)) = 0. \tag{4.39}$$

From the latter equality and Theorem 3.2 it follows that

$$\mathrm{Op}(\widetilde{\mathfrak{c}}_{\alpha,y}^+) \simeq \mathrm{Op}(\mathfrak{c}_{\alpha,y}^-). \tag{4.40}$$

Finally, from (4.35) and (4.40) we obtain (4.34). $\square$

5. Proof of the main result

5.1. Technical lemma

We start with the following technical lemma.

LEMMA 5.1. *Let $p \in (1,\infty)$, ω be a real-valued function in $SO(\mathbb{R}_+)$, and*

$$\widetilde{\mathfrak{g}}(t,x,\theta) := 1-(e^{i\theta\omega(t)x}+e^{-i\theta\omega(t)x}-2)\frac{(r_p(x))^2}{4}, \quad (t,x,\theta)\in\mathbb{R}_+\times\mathbb{R}\times[0,1], \tag{5.1}$$

with

$$\widetilde{\mathfrak{g}}(t,\pm\infty,\theta) := \lim_{x\to\pm\infty}\widetilde{\mathfrak{g}}(t,x,\theta) = 1 \quad \text{for} \quad (t,\theta)\in\mathbb{R}_+\times[0,1].$$

Then there is a constant $c = c(\omega,p) > 0$ depending only on ω and p such that

$$|\widetilde{\mathfrak{g}}(t,x,\theta)| \geqslant c \quad \text{for all} \quad (t,x,\theta)\in\mathbb{R}_+\times\overline{\mathbb{R}}\times[0,1]. \tag{5.2}$$

Proof. By using elementary transformations of hyperbolic trigonometry (see, e.g., [1, Section 4.5]), $\widetilde{\mathfrak{g}}$ can be rewritten as

$$\begin{aligned}\widetilde{\mathfrak{g}}(t,x,\theta) &= \big(\cosh[2\pi(x+i/p)]-\cosh[i\theta\omega(t)x]\big)\frac{(r_p(x))^2}{2}\\ &= \frac{\sinh[\pi(x+i/p)+i\theta\omega(t)x/2]}{\sinh[\pi(x+i/p)]}\cdot\frac{\sinh[\pi(x+i/p)-i\theta\omega(t)x/2]}{\sinh[\pi(x+i/p)]},\end{aligned}$$

whence, by [1, formula 4.5.54],

$$\begin{aligned}|\widetilde{\mathfrak{g}}(t,x,\theta)| &= \sqrt{\frac{\sinh^2(\pi x)+\sin^2(\pi/p+\theta\omega(t)x/2)}{\sinh^2(\pi x)+\sin^2(\pi/p)}}\\ &\quad\times\sqrt{\frac{\sinh^2(\pi x)+\sin^2(\pi/p-\theta\omega(t)x/2)}{\sinh^2(\pi x)+\sin^2(\pi/p)}}.\end{aligned} \tag{5.3}$$

Let the constant $M_1(\omega)$ be given by (4.7). Assume that $2\leqslant p<\infty$ and $t\in\mathbb{R}_+$, $\theta\in[0,1]$. If $|x|\leqslant\pi/(pM_1(\omega))$, then

$$\left|\frac{\theta\omega(t)x}{2}\right| \leqslant \frac{\theta M_1(\omega)}{2}\cdot\frac{\pi}{pM_1(\omega)} \leqslant \frac{\pi}{2p}.$$

Hence

$$\frac{\pi}{2p} = \frac{\pi}{p}-\frac{\pi}{2p} \leqslant \frac{\pi}{p}\pm\frac{\theta\omega(t)x}{2} \leqslant \frac{\pi}{p}+\frac{\pi}{2p} \leqslant \pi-\frac{\pi}{p}+\frac{\pi}{2p} = \pi-\frac{\pi}{2p}.$$

From here it follows that

$$\sin^2\left(\frac{\pi}{p}\pm\frac{\theta\omega(t)x}{2}\right) \geqslant \sin^2\left(\frac{\pi}{2p}\right) \quad \text{for} \quad |x| \leqslant \frac{\pi}{pM_1(\omega)}. \tag{5.4}$$

If $p \in (1,2)$, then $p' := p/(p-1) \in (2,\infty)$. In this case from (5.4) we get

$$\sin^2\left(\frac{\pi}{p} \pm \frac{\theta\omega(t)x}{2}\right) = \sin^2\left(\pi - \left(\frac{\pi}{p'} \mp \frac{\theta\omega(t)x}{2}\right)\right) = \sin^2\left(\frac{\pi}{p'} \mp \frac{\theta\omega(t)x}{2}\right)$$
$$\geqslant \sin^2\left(\frac{\pi}{2p'}\right) \quad \text{for} \quad |x| \leqslant \frac{\pi}{p'M_1(\omega)}. \tag{5.5}$$

Now let $p \in (1,\infty)$ and put $q := \max(p,p')$. Taking into account that the function $\varphi(x) = \sinh^2(\pi x)$ is even on $\mathbb{R}$ and is increasing and positive on $\mathbb{R}_+$, from (5.3)–(5.5) we obtain for $|x| \leqslant \pi/(qM_1(\omega))$,

$$|\widetilde{\mathfrak{g}}(t,x,\theta)| \geqslant \frac{\sin^2(\pi/(2q))}{\sinh^2(\pi^2/(qM_1(\omega))) + \sin^2(\pi/p)} =: c_1(\omega,p) > 0; \tag{5.6}$$

and for $|x| > \pi/(qM_1(\omega))$,

$$|\widetilde{\mathfrak{g}}(t,x,\theta)| \geqslant \frac{\sinh^2(\pi x)}{\sinh^2(\pi x) + \sin^2(\pi/p)} = \frac{1}{1+\sin^2(\pi/p)\sinh^{-2}(\pi x)}$$
$$\geqslant \frac{1}{1+\sin^2(\pi/p)\sinh^{-2}(\pi^2/(qM_1(\omega)))} =: c_2(\omega,p) > 0. \tag{5.7}$$

Combining (5.6)–(5.7), we arrive at (5.2) with $c := \min(c_1(\omega,p), c_2(\omega,p)) > 0$. □

5.2. Singular integral operators with one shift

Let X be a Banach space and $A \in \mathscr{B}(X)$. Recall that an operator $B_r \in \mathscr{B}(X)$ (resp. $B_l \in \mathscr{B}(X)$) is said to be a right (resp. left) regularizer for A if

$$AB_r - I \in \mathscr{K}(X) \quad (\text{resp.} \quad B_lA - I \in \mathscr{K}(X)).$$

It is well known that the operator A is Fredholm on X if and only if it admits simultaneously a right and a left regularizers. Moreover, each right regularizer differs from each left regularizer by a compact operator (see, e.g., [3, Chap. 4, Section 7]).

Now we will prove Theorem 1.1 for $i = \pm 1$ and $j = 0$.

THEOREM 5.2. *If $\alpha \in SOS(\mathbb{R}_+)$, then the operators*

$$G_\alpha := U_\alpha P_+ + P_-, \quad G_{\alpha_{-1}} := U_\alpha^{-1} P_+ + P_-$$

are Fredholm on the space $L^p(\mathbb{R}_+)$ and $\operatorname{Ind} G_\alpha = \operatorname{Ind} G_{\alpha_{-1}} = 0$.

Proof. We will follow the proof of [12, Theorem 9.4], where this result was proved for weighted L^p spaces, but under stronger conditions on the smoothness of the slowly oscillating shift.

From Lemma 2.7 it follows that

$$G_\alpha G_{\alpha_{-1}} \simeq P_+^2 + U_\alpha P_+ P_- + U_\alpha^{-1} P_- P_+ + P_-^2, \tag{5.8}$$
$$G_{\alpha_{-1}} G_\alpha \simeq P_+^2 + U_\alpha^{-1} P_+ P_- + U_\alpha P_- P_+ + P_-^2. \tag{5.9}$$

From Lemma 2.6(c) we immediately get the following identities:

$$P_+P_- = P_-P_+ = -\frac{R^2}{4}, \quad P_\pm^2 = P_\pm + \frac{R^2}{4}. \tag{5.10}$$

Combining (5.8)–(5.10), we arrive at

$$G_\alpha G_{\alpha_{-1}} \simeq G_{\alpha_{-1}} G_\alpha \simeq I - (U_\alpha + U_\alpha^{-1} - 2I)\frac{R^2}{4}. \tag{5.11}$$

From Lemma 2.6(b) and Lemma 4.2 it follows that

$$R = \Phi^{-1}\,\mathrm{Op}(\mathfrak{r}_p)\Phi, \tag{5.12}$$

where $\mathfrak{r}_p(t,x) = r_p(x)$ belongs to $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$. By Lemma 4.5,

$$(U_\alpha + U_\alpha^{-1} - 2I)R \simeq \Phi^{-1}\,\mathrm{Op}(\mathfrak{f})\Phi, \tag{5.13}$$

where the function $\mathfrak{f}$, given for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$ by

$$\mathfrak{f}(t,x) := (e^{i\omega(t)x} + e^{-i\omega(t)x} - 2)r_p(x) \quad \text{with} \quad \omega(t) = \log[\alpha(t)/t],$$

belongs to $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From (5.12)–(5.13) and Lemma 3.4 we conclude that

$$I - (U_\alpha - U_\alpha^{-1} - 2I)\frac{R^2}{4} \simeq I - \Phi^{-1}\,\mathrm{Op}(\mathfrak{f})\,\mathrm{Op}(\mathfrak{r}_p/4)\Phi = \Phi^{-1}\,\mathrm{Op}(\mathfrak{g})\Phi, \tag{5.14}$$

where

$$\mathfrak{g}(t,x) := 1 - (e^{i\omega(t)x} + e^{-i\omega(t)x} - 2)\frac{(r_p(x))^2}{4}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{5.15}$$

belongs to $\widetilde{\mathscr{E}}(\mathbb{R}_+, V(\mathbb{R}))$. From (5.15) and Lemmas 3.5 and 5.1 it follows that

$$\mathfrak{g}(t, \pm\infty) = 1 \neq 0 \quad \text{for all} \quad t \in \mathbb{R}_+, \quad \mathfrak{g}(\xi, x) \neq 0 \quad \text{for all} \quad (\xi, x) \in \Delta \times \overline{\mathbb{R}}.$$

Thus, by Theorem 3.6, the operator $F := \Phi^{-1}\,\mathrm{Op}(\mathfrak{g})\Phi$ is Fredholm on $L^p(\mathbb{R}_+)$. Let $F_r^{(-1)}$ and $F_l^{(-1)}$ be some of its right and left regularizers. From (5.11) and (5.14) we obtain

$$G_\alpha(G_{\alpha_{-1}} F_r^{(-1)}) \simeq F F_r^{(-1)} \simeq I, \quad (F_l^{(-1)} G_{\alpha_{-1}})G_\alpha \simeq F_l^{(-1)} F \simeq I,$$

whence $G_{\alpha_{-1}} F_r^{(-1)}$ is a right regularizer and $F_l^{(-1)} G_{\alpha_{-1}}$ is a left regularizer for G_α. Thus, G_α is Fredholm. Similarly one can show that $G_{\alpha_{-1}}$ is Fredholm.

For $\tau > 1$, consider $\Pi_\tau := [\tau^{-1}, \tau] \times \overline{\mathbb{R}}$. Since the function $\widetilde{\mathfrak{g}}$ given by (5.1) is continuous and separated from 0 for all $(t,x,\theta) \in \mathbb{R}_+ \times \overline{\mathbb{R}} \times [0,1]$ in view of Lemma 5.1, we conclude that $\{\arg \widetilde{\mathfrak{g}}(t,x,\theta)\}_{(t,x)\in\partial\Pi_\tau}$ does not depend on $\theta \in [0,1]$. Consequently,

$$\{\arg \mathfrak{g}(t,x)\}_{(t,x)\in\partial\Pi_\tau} = \{\arg \widetilde{\mathfrak{g}}(t,x,0)\}_{(t,x)\in\partial\Pi_\tau} = 0. \tag{5.16}$$

By Theorem 3.6 and (5.16),

$$\operatorname{Ind}\Phi^{-1}\operatorname{Op}(\mathfrak{g})\Phi = \lim_{\tau\to\infty}\frac{1}{2\pi}\{\arg\mathfrak{g}(t,x)\}_{(t,x)\in\partial\Pi_\tau} = 0. \tag{5.17}$$

From (5.11) and (5.14) it follows that

$$\operatorname{Ind}G_\alpha + \operatorname{Ind}G_{\alpha_{-1}} = \operatorname{Ind}\Phi^{-1}\operatorname{Op}(\mathfrak{g})\Phi = 0. \tag{5.18}$$

Let $\mathscr{C}$ be the operator of complex conjugation given by $\mathscr{C}f = \overline{f}$. This operator is isometric and anti-linear on $L^p(\mathbb{R}_+)$. It is not difficult to see that $\mathscr{C}U_\alpha\mathscr{C} = U_\alpha$ and $\mathscr{C}P_\pm\mathscr{C} = P_\mp$. Then

$$\mathscr{C}G_\alpha\mathscr{C} = U_\alpha P_- + P_+ = U_\alpha G_{\alpha_{-1}}.$$

Hence $\operatorname{Ind}G_\alpha = \operatorname{Ind}G_{\alpha_{-1}}$, which implies due to (5.18) that $\operatorname{Ind}G_\alpha = \operatorname{Ind}G_{\alpha_{-1}} = 0$. □

5.3. Singular integral operators with two shifts

Proof of Theorem 1.1. The result is trivial for $i = j = 0$. By Corollary 2.5, the shifts α_i, β_j, and $\gamma_{ij} := \alpha_i\circ\beta_{-j}$ belong to $SOS(\mathbb{R}_+)$ for all $i,j\in\mathbb{Z}$. By Theorem 5.2, the operator $A_{i0} = U_{\alpha_i}P_+ + P_-$ is Fredholm and $\operatorname{Ind}A_{i0} = 0$ for all $i\in\mathbb{Z}\setminus\{0\}$. In all remaining cases $A_{ij} = U_\beta^j(U_{\gamma_{ij}}P_+ + P_-)$, where $U_{\gamma_{ij}} = U_{\beta_{-j}}U_{\alpha_i} = U_\beta^{-j}U_\alpha^i$. The operator $U_{\gamma_{ij}}P_+ + P_-$ is Fredholm and its index is equal to zero due to Theorem 5.2. It remains to observe that the operator U_β^j is invertible. Thus, A_{ij} is Fredholm and $\operatorname{Ind}A_{ij} = 0$. □

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, 10th printing, with corrections, National Bureau of Standards, A Wiley-Interscience Publication, John Wiley & Sons, New York etc. (1972).

[2] R. DUDUCHAVA, *On algebras generated by convolutions and discontinuous functions*, Integr. Equat. Oper. Theory, Vol. **10**, 4 (1987), 505–530.

[3] I. GOHBERG AND N. KRUPNIK, *One-Dimensional Linear Singular Integral Equations, I. Introduction*, Operator Theory: Advances and Applications, Vol. **53**. Birkhäuser, Basel (1992).

[4] I. S. GRADSHTEYN AND I. M. RYZHIK, *Table of Integrals, Series, and Products*, 7th edn. Elsevier/Academic Press, Amsterdam (2007).

[5] R. HAGEN, S. ROCH, AND B. SILBERMANN, *Spectral Theory of Approximation Methods for Convolution Equations*, Operator Theory: Advances and Applications, Vol. **74**. Birkhäuser, Basel (1994).

[6] A. YU. KARLOVICH, YU. I. KARLOVICH, AND A. B. LEBRE, *Invertibility of functional operators with slowly oscillating non-Carleman shifts*, Operator Theory: Advances and Applications, Vol. **142** (2003), in: “Singular Integral Operators, Factorization and Applications”, p. 147–174.

[7] A. YU. KARLOVICH, YU. I. KARLOVICH, AND A. B. LEBRE, *Sufficient conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data*, Integr. Equ. Oper. Theory, Vol. **70**, 4 (2011), 451–483.

[8] A. YU. KARLOVICH, YU. I. KARLOVICH, AND A. B. LEBRE, *Necessary conditions for Fredholmness of singular integral operators with shifts and slowly oscillating data*, Integr. Equ. Oper. Theory, Vol. **71**, 1 (2011), 29–53.

[9] A. YU. KARLOVICH, YU. I. KARLOVICH, AND A. B. LEBRE, *The index of singular integral operators with shifts and slowly oscillating data*, work in progress, (2013).

[10] YU. I. KARLOVICH, *An algebra of pseudodifferential operators with slowly oscillating symbols*, Proc. London Math. Soc., Vol. **92**, 3 (2006), 713–761.

[11] YU. I. KARLOVICH, *Pseudodifferential operators with compound slowly oscillating symbols*, Operator Theory: Advances and Applications, Vol. **171** (2006), in: "The Extended Field of Operator Theory", p. 189–224.
[12] YU. I. KARLOVICH, *Algebras of pseudo-differential operators with discontinuous symbols*, Operator Theory: Advances and Applications, Vol. **172** (2007), in: "Modern Trends in Pseudo-Differential Operators", p. 207–233.
[13] YU. I. KARLOVICH, *An algebra of shift-invariant singular integral operators with slowly oscillating data and its application to operators with a Carleman shift*, Operator Theory: Advances and Applications, in: "Analysis, Partial Differential Equations and Applications. The Vladimir Maz'ya Anniversary Volume", Vol. **193** (2009), p. 81–95.
[14] V. S. RABINOVICH, *Mellin pseudodifferential operators techniques in the theory of singular integral operators on some Carleson curves*, Operator Theory: Advances and Applications, Vol. 102 (1998), in "Differential and Integral Operators (Regensburg, 1995)", p. 201–218.
[15] S. ROCH, P. A. SANTOS, AND B. SILBERMANN, *Non-Commutative Gelfand Theories. A Tool-kit for Operator Theorists and Numerical Analysts*, Universitext. Springer-Verlag London, London (2011).

(Received September 2, 2013)

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