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# PROPERTIES OF COMPLEX SYMMETRIC OPERATORS

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*Abstract.* An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be complex symmetric if there exists a conjugation $C$ on $\mathcal{H}$ such that $T = CT^*C$. In this paper, we prove that every complex symmetric operator is biquasitriangular. Also, we show that if a complex symmetric operator $T$ is weakly hypercyclic, then both $T$ and $T^*$ have the single-valued extension property and that if $T$ is a complex symmetric operator which has the property $(\delta)$, then Weyl's theorem holds for $f(T)$ and $f(T)^*$ where $f$ is any analytic function in a neighborhood of $\sigma(T)$. Finally, we establish equivalence relations among Weyl type theorems for complex symmetric operators.

## 1. Introduction

Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators on a separable complex Hilbert space $\mathcal{H}$ and let $\mathcal{K}(\mathcal{H})$ be the ideal of all compact operators on $\mathcal{H}$. If $T \in \mathcal{L}(\mathcal{H})$, we write $\sigma(T)$, $\sigma_{su}(T)$, $\sigma_a(T)$, $\sigma_e(T)$, $\sigma_{le}(T)$, and $\sigma_{re}(T)$ for the spectrum, the surjective spectrum, the approximate point spectrum, the essential spectrum, the left essential spectrum, and the right essential spectrum of $T$, respectively.

A *conjugation* on $\mathcal{H}$ is an antilinear operator $C : \mathcal{H} \to \mathcal{H}$ which satisfies $\langle Cx, Cy\rangle = \langle y, x\rangle$ for all $x, y \in \mathcal{H}$ and $C^2 = I$. For any conjugation $C$, there is an orthonormal basis $\{e_n\}_{n=0}^{\infty}$ for $\mathcal{H}$ such that $Ce_n = e_n$ for all $n$ (see [15] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *complex symmetric* if there exists a conjugation $C$ on $\mathcal{H}$ such that $T = CT^*C$. In this case, we say that $T$ is complex symmetric with conjugation $C$. This concept is due to the fact that $T$ is a complex symmetric operator if and only if it is unitarily equivalent to a symmetric matrix with complex entries, regarded as an operator acting on an $l^2$-space of the appropriate dimension (see [13]). The class of complex symmetric operators is unexpectedly large. This class includes all normal operators, Hankel operators, some truncated Toeplitz operators, and some Volterra integration operators. We refer the reader to [13], [14], [15], [16], [17], [18] for more details, including historical comments and references.

The aim of this paper is to study local spectral and spectral properties of complex symmetric operators. In this paper, we prove that every complex symmetric operator

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is biquasitriangular. Also, we show that if a complex symmetric operator $T$ is weakly hypercyclic, then both $T$ and $T^*$ have the single-valued extension property and that if $T$ is a complex symmetric operator which has the property $(\delta)$, then Weyl's theorem holds for $f(T)$ and $f(T)^*$ where $f$ is any analytic function in a neighborhood of $\sigma(T)$. Finally, we establish equivalence relations among Weyl type theorems for complex symmetric operators.

## 2. Preliminaries

An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *the single-valued extension property* (or SVEP) if for every open subset $G$ of $\mathbb{C}$ and any $\mathcal{H}$-valued analytic function $f$ on $G$ such that $(T-z)f(z) \equiv 0$ on $G$, we have $f(z) \equiv 0$ on $G$. For an operator $T \in \mathcal{L}(\mathcal{H})$ and for $x \in \mathcal{H}$, the *local resolvent set* $\rho_T(x)$ of $T$ at $x$ is defined as the union of every open subset $G$ of $\mathbb{C}$ on which there is an analytic function $f : G \to \mathcal{H}$ such that $(T-z)f(z) \equiv x$ for all $z \in G$. The *local spectrum* of $T$ at $x$ is given by

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

We define the *local spectral subspace* of an operator $T \in \mathcal{L}(\mathcal{H})$ by

$$H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$$

for a subset $F$ of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Dunford's property* $(C)$ if $H_T(F)$ is closed for each closed subset $F$ of $\mathbb{C}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to have *Bishop's property* $(\beta)$ if for every open subset $G$ of $\mathbb{C}$ and every sequence $\{f_n\}$ of $\mathcal{H}$-valued analytic functions on $G$ such that $(T-z)f_n(z)$ converges uniformly to 0 in norm on compact subsets of $G$, we get that $f_n(z)$ converges uniformly to 0 in norm on compact subsets of $G$. We say that $T$ has *the property* $(\delta)$ if for every $x \in \mathcal{H}$ and for every open cover $\{U, V\}$ of $\mathbb{C}$, we can write $x = u + v$ where $u$ and $v$ are vectors in $\mathcal{H}$ such that $(T-z)f(z) \equiv u$ on $\mathbb{C} \setminus \overline{U}$ and $(T-z)g(z) \equiv v$ on $\mathbb{C} \setminus \overline{V}$ for some analytic functions $f : \mathbb{C} \setminus \overline{U} \to \mathcal{H}$ and $g : \mathbb{C} \setminus \overline{V} \to \mathcal{H}$. When $T$ has the single-valued extension property, $T$ has the property $(\delta)$ if and only if for any open cover $\{U, V\}$ of $\mathbb{C}$, the decomposition $\mathcal{H} = H_T(\overline{U}) + H_T(\overline{V})$ holds. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *decomposable* if for every open cover $\{U, V\}$ of $\mathbb{C}$ there are $T$-invariant subspaces $\mathcal{X}$ and $\mathcal{Y}$ such that $\mathcal{H} = \mathcal{X} + \mathcal{Y}$, $\sigma(T|_{\mathcal{X}}) \subset \overline{U}$, and $\sigma(T|_{\mathcal{Y}}) \subset \overline{V}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is decomposable precisely when it has the properties $(\beta)$ and $(\delta)$. It is well known that

$$\text{Bishop's property } (\beta) \Rightarrow \text{Dunford's property } (C) \Rightarrow \text{SVEP}.$$

Any of the converse implications does not hold, in general. In addition, $T$ has the property $(\beta)$ if and only if $T^*$ has the property $(\delta)$ (see [8] and [28] for more details).

An operator $T \in \mathcal{L}(\mathcal{H})$ is called *upper semi-Fredholm* if $T$ has closed range and $dim \ker(T) < \infty$, and $T$ is called *lower semi-Fredholm* if $T$ has closed range and $dim(\mathcal{H}/\mathrm{ran}(T)) < \infty$. When $T$ is upper semi-Fredholm or lower semi-Fredholm, $T$

is said to be *semi-Fredholm*. The *index of a semi-Fredholm operator* $T \in \mathcal{L}(\mathcal{H})$, denoted $ind(T)$, is given by

$$ind(T) = dim\ker(T) - dim(\mathcal{H}/\mathrm{ran}(T))$$

and this value is an integer or $\pm\infty$. Also an operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Fredholm* if it is both upper and lower semi-Fredholm. An operator $T \in \mathcal{L}(\mathcal{H})$ is said to be *Weyl* if it is Fredholm of index zero. If there is a positive integer $m$ such that $\ker(T^m) = \ker(T^{m+1})$, then $T$ is said to have *finite ascent*. If there is a positive integer $n$ satisfying $\mathrm{ran}(T^n) = \mathrm{ran}(T^{n+1})$, then $T$ is said to have *finite descent*. We say that $T \in \mathcal{L}(\mathcal{H})$ is *Browder* if it has finite ascent and finite descent. We define the Weyl spectrum $\sigma_w(T)$ and the Browder spectrum $\sigma_b(T)$ by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Browder}\}.$$

It is evident that

$$\sigma_e(T) \subset \sigma_w(T) \subset \sigma_b(T).$$

We say that *Weyl's theorem holds* for $T$ if

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T), \text{ or equivalently, } \sigma(T) \setminus \sigma_w(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\lambda \in iso(\sigma(T)) : 0 < dim\ker(T - \lambda) < \infty\}$ and $iso(\sigma(T))$ denotes the set of all isolated points of $\sigma(T)$. We say that *Browder's theorem holds* for $T \in \mathcal{L}(\mathcal{H})$ if $\sigma_b(T) = \sigma_w(T)$.

For an operator $T \in \mathcal{L}(\mathcal{H})$, a *hole* in $\sigma_e(T)$ is a bounded component of $\mathbb{C} \setminus \sigma_e(T)$ and thus is an open set. A *pseudohole* in $\sigma_e(T)$ is a component of $\sigma_e(T) \setminus \sigma_{le}(T)$ or $\sigma_e(T) \setminus \sigma_{re}(T)$, which is a subset of $\sigma_e(T)$ and open in $\mathbb{C}$. The *spectral picture* of an operator $T \in \mathcal{L}(\mathcal{H})$ (notation : $SP(T)$) is the structure consisting of the set $\sigma_e(T)$, the collection of holes and pseudoholes in $\sigma_e(T)$, and the indices associated with these holes and pseudoholes (see [33] for more details).

## 3. Main results

In this section, we first show that every complex symmetric operator $T \in \mathcal{L}(\mathcal{H})$ is biquasitriangular. An operator $T$ in $\mathcal{L}(\mathcal{H})$ is called *quasitriangular* if $T$ can be written as sum $T = T_0 + K$, where $T_0$ is a triangular operator (i.e., there exists an orthonormal basis for $\mathcal{H}$ with respect to which the matrix for $T_0$ has upper triangular form) and $K \in \mathcal{K}(\mathcal{H})$. We say that $T$ is *biquasitriangular* if both $T$ and $T^*$ are quasitriangular (see [33] for more details). We denote the Calkin map by $\pi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and put $\Delta^* := \{\overline{\lambda} : \lambda \in \Delta\}$ for any set $\Delta$ in $\mathbb{C}$.

THEOREM 3.1. *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric. Then $SP(T^*)$ consists of holes in $\sigma_e(T)^*$ and pseudoholes in $\sigma_e(T)^* \setminus \sigma_{le}(T)^*$ or $\sigma_e(T)^* \setminus \sigma_{re}(T)^*$ associated with zero indices. Moreover, it is biquasitriangular.*

*Proof.* Since $T$ is complex symmetric, $\sigma_{le}(T)^* = \sigma_{le}(T^*)$ and $\sigma_{re}(T)^* = \sigma_{re}(T^*)$ by [24]. Furthermore, it follows from [24] that $dim\ker(T-\lambda) = dim\ker(T-\lambda)^*$ and that $T-\lambda$ has closed range if and only if $(T-\lambda)^*$ does. Hence $ind(T-\lambda) = 0$ for all $\lambda \notin \sigma_{le}(T) \cap \sigma_{re}(T)$, and so we get the first assertion. Since we know from [23, Theorem 6.15] that an operator $S \in \mathcal{L}(\mathcal{H})$ is biquasitriangular if and only if $ind(S-\lambda) = 0$ for all $\lambda \notin \sigma_{le}(S) \cap \sigma_{re}(S)$, we conclude that $T$ is biquasitriangular. □

Remark that if $T \in \mathcal{L}(\mathcal{H})$ is an operator such that $SP(T)$ or $SP(T^*)$ contains a hole or pseudohole associated with a negative index, then $T$ or $T^*$ is not quasitriangular from [10]. Hence $T$ is not complex symmetric from Theorem 3.1.

COROLLARY 3.2. *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric. If $T$ is invertible, then $T^{-1}$ is biquasitriangular.*

*Proof.* If $T$ is complex symmetric with conjugation $C$, then we have $CT^{-1}C = (CTC)^{-1} = T^{*-1}$, and so $T^{-1}$ is complex symmetric with the same conjugation $C$. Hence the proof follows from Theorem 3.1. □

The following corollary follows from [29].

COROLLARY 3.3. *If $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric, then*
*(i) $T \in \{R \in \mathcal{L}(\mathcal{H}) : \sigma(R)$ is totally disconnected$\}^-$ and*
*(ii) $T \in \{R \in \mathcal{L}(\mathcal{H}) : R$ is similar to $S \in \mathcal{N}(\mathcal{H}) + \mathcal{K}(\mathcal{H})\}^-$*
*where $\mathcal{N}(\mathcal{H})$ denotes the class of all normal operators in $\mathcal{L}(\mathcal{H})$ and the closures are uniform.*

From Theorem 3.1, we observe that the class of complex symmetric operators is contained in that of biquasitriangular operators. Normal operators, 2-normal operators, algebraic operators of order 2, some Volterra integration operators, Hankel operators, and some truncated Toeplitz operators are examples of complex symmetric operators (see [15], [16], and [18]), and hence they are biquasitriangular. On the other hand, the following example, used in the proof of [18, Theorem 2], indicates that there exists a biquasitriangular operator which is not complex symmetric.

EXAMPLE 3.4. Let $T = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ 0 & 0 & 0 \end{pmatrix} \oplus D$ be with $|a| \neq |b|$ where $D$ is a diagonal operator on $\mathbb{C}^{n-3}$ with $n \geqslant 3$. Then $T$ is an algebraic operator of order $n$ and so it is biquasitriangular from [33]. But $T$ is not complex symmetric as stated in the proof of [18, Theorem 2]. Hence the class of complex symmetric operators is properly contained in the class of biquasitriangular operators.

Now we shall introduce some important $T$-invariant subspaces. For an operator $T \in \mathcal{L}(\mathcal{H})$, the *algebraic core* $Alg(T)$ is defined as the greatest subspace $\mathcal{M}$ of $\mathcal{H}$ for which $T\mathcal{M} = \mathcal{M}$. We note that $x \in Alg(T)$ if and only if there exists a sequence $\{u_n\} \subset \mathcal{H}$ such that $x = u_0$ and $Tu_{n+1} = u_n$ for every integer $n \geqslant 0$. The *analytical* *core* $Anal(T)$ of $T$ is the set of all $x \in \mathcal{H}$ with the property that there is a sequence $\{u_n\} \subset \mathcal{H}$ and a constant $\delta > 0$ such that $x = u_0$, $Tu_{n+1} = u_n$, and $\|u_n\| \leqslant \delta^n \|x\|$ for every integer $n \geqslant 0$. In this sense, this subspace is regarded as the analytic counterpart of the algebraic core $Alg(T)$ (see [1] for more details). An operator $T \in \mathcal{L}(\mathcal{H})$ is *irreducible* if it has no nontrivial reducing subspaces. For an operator $T \in \mathcal{L}(\mathcal{H})$, we say that a $T$-invariant subspace $\mathcal{M}$ is a *spectral maximal subspace* of $T$ provided that it contains any $T$-invariant subspace $\mathcal{N}$ with $\sigma(T|_{\mathcal{N}}) \subset \sigma(T|_{\mathcal{M}})$. In the following theorem, we show that if $T$ is complex symmetric with conjugation $C$, then the mapping $C$ induces some one-to-one correspondences.

THEOREM 3.5. *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric with conjugation $C$. Then the following assertions hold.*

*(i) $Alg(T^*) = C(Alg(T))$ and $Anal(T^*) = CAnal(T)$.*

*(ii) $\mathcal{M}$ is a spectral maximal subspace of $T$ if and only if $C\mathcal{M}$ is a spectral maximal subspace of $T^*$. Moreover, $\sigma(T|_{\mathcal{M}})^* = \sigma(T^*|_{C\mathcal{M}})$ for any $T$-invariant subspace $\mathcal{M}$ of $\mathcal{H}$.*

*(iii) $\mathcal{M}$ is a hyperinvariant subspace for $T$ if and only if $C\mathcal{M}$ is a hyperinvariant subspace for $T^*$.*

*(iv) $T$ is irreducible if and only if $T^*$ is irreducible.*

*Proof.* (i) Since $T(Alg(T)) = Alg(T)$, we have

$$T^*C(Alg(T)) = CT(Alg(T)) = C(Alg(T)).$$

In addition, if $\mathcal{M}$ is a subspace of $\mathcal{H}$ such that $T^*\mathcal{M} = \mathcal{M}$, then it holds that $TC\mathcal{M} = CT^*\mathcal{M} = C\mathcal{M}$ and so $C\mathcal{M} \subset Alg(T)$, i.e., $\mathcal{M} \subset C(Alg(T))$. Hence $Alg(T^*) = C(Alg(T))$.

For the second identity, let $x \in Anal(T)$. Then there exists a sequence $\{u_n\} \subset \mathcal{H}$ and a constant $\delta > 0$ such that $x = u_0$, $Tu_{n+1} = u_n$, and $\|u_n\| \leqslant \delta^n \|x\|$ for every integer $n \geqslant 0$. This implies that $Cx = Cu_0$, $T^*Cu_{n+1} = CTu_{n+1} = Cu_n$, and $\|Cu_n\| = \|u_n\| \leqslant \delta^n\|x\| = \delta^n\|Cx\|$ for every integer $n \geqslant 0$. Thus $Cx \in Anal(T^*)$. Hence $CAnal(T) \subset Anal(T^*)$. Since $T^*$ is also complex symmetric, we obtain that $CAnal(T^*) \subset Anal(T)$. Since $C^2 = I$, we have $Anal(T^*) = CAnal(T)$.

(ii) Let $\mathcal{M}$ be a $T$-invariant subspace and let $\lambda \in \rho(T|_{\mathcal{M}})$ be arbitrary. We first note that $C\mathcal{M}$ is a $T^*$-invariant subspace by [25]. If $(T^*|_{C\mathcal{M}} - \overline{\lambda})x = 0$ for some $x \in C\mathcal{M}$, then $0 = C(T^* - \overline{\lambda})x = (T - \lambda)Cx$ and $Cx \in \mathcal{M}$. Hence $Cx = 0$. Since $C^2 = I$, we have $x = 0$, and so $T^*|_{C\mathcal{M}} - \overline{\lambda}$ is one-to-one. If $y \in C\mathcal{M}$, then there is $x \in \mathcal{M}$ such that $(T - \lambda)x = Cy$. This implies that $y = C(T - \lambda)x = (T^* - \overline{\lambda})Cx$ and $Cx \in C\mathcal{M}$, which ensures that $T^*|_{C\mathcal{M}} - \overline{\lambda}$ is onto. Thus $\overline{\lambda} \in \rho(T^*|_{C\mathcal{M}})$. Therefore we get that $\sigma(T|_{\mathcal{M}})^* \supset \sigma(T^*|_{C\mathcal{M}})$. The opposite inclusion holds similarly, and so $\sigma(T|_{\mathcal{M}})^* = \sigma(T^*|_{C\mathcal{M}})$.

We now suppose that $\mathcal{M}$ is a spectral maximal subspace of $T$. Then $C\mathcal{M}$ is a $T^*$-invariant subspace. If $\mathcal{N}$ is any $T^*$-invariant subspace with $\sigma(T^*|_{\mathcal{N}}) \subset \sigma(T^*|_{C\mathcal{M}})$, then we have

$$\sigma(T|_{C\mathcal{N}}) = \sigma(T^*|_{\mathcal{N}})^* \subset \sigma(T^*|_{C\mathcal{M}})^* = \sigma(T|_{\mathcal{M}}).$$

Hence $C\mathscr{N} \subset \mathscr{M}$, which means that $\mathscr{N} \subset C\mathscr{M}$. Thus $C\mathscr{M}$ is a spectral maximal subspace of $T^*$. The converse argument holds by a similar way.

(iii) It suffices to show one direction. Let $\mathscr{M}$ be a hyperinvariant subspace for $T$, and let $S \in \{T^*\}'$. We first note that $C\mathscr{M}$ is a closed subspace of $\mathscr{H}$ by [25]. Moreover, since $CSC \in \{T\}'$, we have $S(C\mathscr{M}) = C(CSC\mathscr{M}) \subset C\mathscr{M}$. Hence $C\mathscr{M}$ is a hyperinvariant subspace for $T^*$.

(iv) If $\mathscr{M}$ is a reducing subspace for $T$, then $\mathscr{M}$ is an invariant subspace for $T$ and $T^*$. Thus $C\mathscr{M}$ is an invariant subspace for $T^*$ and $T$ by the proof of (iii) (or see [25]). Since $\mathscr{M}$ is nontrivial if and only if $C\mathscr{M}$ is, we complete the proof. $\square$

COROLLARY 3.6. *Let $T \in \mathscr{L}(\mathscr{H})$ be complex symmetric with conjugation $C$. Then the following assertions hold.*

*(i) $CAnal(T^*) \subset Alg(T)$. In particular, if $Alg(T)$ is closed, then the identity $CAnal(T^*) = Alg(T)$ holds and $\mathscr{M} \subset CAnal(T^*)$ for any subspace $\mathscr{M}$ of $\mathscr{H}$ with $T\mathscr{M} = \mathscr{M}$.*

*(ii) $Alg(T) = T^\infty(\mathscr{H})$ if and only if $Alg(T^*) = T^{*\infty}(\mathscr{H})$, where $T^\infty(\mathscr{H}) = \cap_{n=0}^{\infty} T^n\mathscr{H}$.*

*(iii) If $\sigma \subset \sigma(T)$ is a spectral set (i.e., open and closed in $\sigma(T)$) and $P_\sigma$ is the spectral projection associated with $\sigma$, then $CP_\sigma(\mathscr{H})$ is a spectral maximal subspace of $T^*$.*

*(iv) If $T$ has the single-valued extension property and $\mathscr{M}$ is a spectral maximal subspace of $T$, then $\sigma_{T^*}(x) = \sigma_{T^*|_{C\mathscr{M}}}(x)$ for every $x \in C\mathscr{M}$.*

*(v) If $T$ has Dunford's property $(C)$, then $CH_T(F)$ is a spectral maximal subspace of $T^*$ and $\sigma(T^*|_{CH_T(F)}) \subset F^* \cap \sigma(T^*)$ for any closed set $F$ in $\mathbb{C}$.*

*Proof.* (i) Since $Anal(T) = CAnal(T^*)$ by Theorem 3.5, the proof follows from [1].

(ii) If $Alg(T) = T^\infty(\mathscr{H})$, then $CAlg(T^*) = T^\infty(\mathscr{H})$ by Theorem 3.5. This implies that

$$Alg(T^*) = CT^\infty(\mathscr{H}) = \cap_{n=0}^{\infty} CT^n\mathscr{H} = \cap_{n=0}^{\infty} T^{*n}C\mathscr{H} = \cap_{n=0}^{\infty} T^{*n}\mathscr{H} = T^{*\infty}(\mathscr{H}).$$

The converse holds similarly.

(iii) Since $P_\sigma(\mathscr{H})$ is a spectral maximal subspace of $T$ by [1], we complete the proof from Theorem 3.5.

(iv) Since $T^*$ has the single-valued extension property by [24] and $C\mathscr{M}$ is a spectral maximal subspace of $T^*$ by Theorem 3.5, the proof follows from [1] or [8].

(v) If $T$ has Dunford's property $(C)$, then $H_T(F)$ is a spectral maximal subspace of $T$ and $\sigma(T|_{H_T(F)}) \subset F \cap \sigma(T)$ for any closed set $F$ in $\mathbb{C}$ (see [8] or [28]), and so the proof follows from Theorem 3.5. $\square$

For an operator $T \in \mathscr{L}(\mathscr{H})$, the *quasinilpotent part* of $T$ is defined by

$$H_0(T) := \{x \in \mathscr{H} : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

Then $H_0(T)$ is a linear (not necessarily closed) subspace of $\mathscr{H}$. We remark from [2] that if $T$ has the single-valued extension property, then

$$H_0(T-\lambda)=\{x\in\mathscr{H}:\lim_{n\to\infty}\|(T-\lambda)^n x\|^{\frac{1}{n}}=0\}=H_T(\{\lambda\})$$

for all $\lambda\in\mathbb{C}$. It is well known from [1] and [2] that if $H_0(T-\lambda)=\{0\}$ for all $\lambda\in\mathbb{C}$, then $T$ has the single-valued extension property. Next we give some applications of $H_0(T)$. We denote by $\mathscr{H}(\sigma(T))$ the space of functions analytic in an open neighborhood of $\sigma(T)$.

THEOREM 3.7. *Let $T\in\mathscr{L}(\mathscr{H})$ be complex symmetric. If $T$ has the property $(\delta)$, then Weyl's theorem holds for $f(T)$ and $f(T)^*$, where $f\in\mathscr{H}(\sigma(T))$. In particular, Weyl's theorem holds for $T$ and $T^*$.*

*Proof.* If $T$ has the property $(\delta)$, then $T$ is subscalar by [25]. Hence [1] implies that for each $\lambda\in\mathbb{C}$, there exists $m_\lambda\in\mathbb{N}$ such that

$$H_0(T-\lambda)=\ker(T-\lambda)^{m_\lambda}. \tag{1}$$

Let $CTC=T^*$ for some conjugation $C$ and set $n_\lambda=m_{\overline{\lambda}}$ for $\lambda\in\mathbb{C}$. Since $H_0(T^*-\lambda)\supset\ker(T^*-\lambda)^{n_\lambda}$ is trivial for any $\lambda\in\mathbb{C}$, it suffices to show the reverse inclusion. Let $x\in H_0(T^*-\lambda)$. Then we obtain that

$$\|(T-\overline{\lambda})^n Cx\|^{\frac{1}{n}}=\|C(T^*-\lambda)^n x\|^{\frac{1}{n}}=\|(T^*-\lambda)^n x\|^{\frac{1}{n}}\to 0 \text{ as } n\to\infty. \tag{2}$$

Since $Cx\in H_0(T-\overline{\lambda})=\ker(T-\overline{\lambda})^{n_\lambda}$, we have $(T^*-\lambda)^{n_\lambda}x=C(T-\overline{\lambda})^{n_\lambda}Cx=0$, and so $H_0(T^*-\lambda)\subset\ker(T^*-\lambda)^{n_\lambda}$. Consequently, for each $\lambda\in\mathbb{C}$

$$H_0(T^*-\lambda)=\ker(T^*-\lambda)^{n_\lambda}. \tag{3}$$

Hence, combining (1) and (3) with [1], we get that Weyl's theorem holds for $f(T)$ and $f(T)^*$, where $f\in\mathscr{H}(\sigma(T))$. $\square$

COROLLARY 3.8. *Let $T\in\mathscr{L}(\mathscr{H})$ be complex symmetric. Suppose that for each $\lambda\in\mathbb{C}$, there exists a positive integer $m_\lambda$ such that $H_0(T-\lambda)=\ker(T-\lambda)^{m_\lambda}$. If $\mathscr{M}$ is any $T^*$-invariant subspace, then $H_0(T^*|_{\mathscr{M}}-\lambda)=\ker(T^*|_{\mathscr{M}}-\lambda)^{n_\lambda}$ for all $\lambda\in\mathbb{C}$, where $n_\lambda=m_{\overline{\lambda}}$. Furthermore, Weyl's theorem holds for $T^*|_{\mathscr{M}}$.*

*Proof.* Suppose that $T$ is complex symmetric and $\mathscr{M}$ is any $T^*$-invariant subspace. From the proof of Theorem 3.7, we have $H_0(T^*-\lambda)=\ker(T^*-\lambda)^{n_\lambda}$ for all $\lambda\in\mathbb{C}$, where $n_\lambda=m_{\overline{\lambda}}$. Hence we get that

$$H_0(T^*|_{\mathscr{M}}-\lambda)\subset\ker(T^*-\lambda)^{n_\lambda}\cap\mathscr{M}=\ker(T^*|_{\mathscr{M}}-\lambda)^{n_\lambda}$$

for all $\lambda\in\mathbb{C}$. Since the opposite inclusion holds obviously, it follows that

$$H_0(T^*|_{\mathscr{M}}-\lambda)=\ker(T^*|_{\mathscr{M}}-\lambda)^{n_\lambda}$$

for all $\lambda\in\mathbb{C}$. Therefore, we conclude from [1] that Weyl's theorem holds for $T^*|_{\mathscr{M}}$. $\square$

COROLLARY 3.9. *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric. If $H_0(T-\lambda)$ is closed for all $\lambda \in \mathbb{C}$, then $H_{T^*}(\{\lambda\})$ is closed for all $\lambda \in \mathbb{C}$, $T^*$ has the single-valued extension property, and Browder's theorem holds for $T^*$.*

*Proof.* Choose a conjugation $C$ with $CTC = T^*$. Suppose that $H_0(T-\lambda)$ is closed for every $\lambda \in \mathbb{C}$. Then $T$ has the single-valued extension property from [1], and so does $T^*$ by [25]. Therefore, Browder's theorem holds for $T^*$ by [1] and $H_{T^*}(\{\lambda\}) = H_0(T^*-\lambda)$ for all $\lambda \in \mathbb{C}$ by [2]. It follows from (2) that

$$H_{T^*}(\{\lambda\}) = H_0(T^*-\lambda) = CH_0(T-\overline{\lambda})$$

for all $\lambda \in \mathbb{C}$. Since it is easy to see that $C$ maps any closed subspace onto a closed one, we obtain that $H_{T^*}(\{\lambda\})$ is closed for all $\lambda \in \mathbb{C}$. $\square$

Let $\{T\}' = \{S \in \mathcal{L}(\mathcal{H}) : ST = TS\}$ denote the commutant of an operator $T \in \mathcal{L}(\mathcal{H})$. Recall that the local spectral subspace of $T \in \mathcal{L}(\mathcal{H})$ is defined by $H_T(F) = \{x \in \mathcal{H} : \sigma_T(x) \subset F\}$, where $F$ is a subset of $\mathbb{C}$. By [8] or [28], if $T$ has Dunford's property $(C)$, then $H_T(F)$ is a spectral maximal subspace of $T$ for any closed set $F$ in $\mathbb{C}$. In the following theorem, we give some properties of local spectral subspaces for complex symmetric operators.

THEOREM 3.10. *Let $\Omega$ be any closed subset of $\mathbb{C}$, and let $T$ be complex symmetric with conjugation $C$. For any $\varepsilon > 0$, put $\Omega_\varepsilon = \{\lambda \in \mathbb{C} : dist(\lambda, \Omega) < \varepsilon\}$.*

*If $T$ has the property $(\beta)$ and $S \in \{T\}'$, then the following relations hold.*

*(i) $\overline{ran(CSC)} \subset H_{T^*}(\Omega)$ if and only if $H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) \subset \ker(CSC)$ for any $\varepsilon > 0$.*

*(ii) $\overline{ran(CSC)} \subset CH_T(\Omega^*)$ if and only if $CH_T(\mathbb{C} \setminus \Omega_\varepsilon^*) \subset \ker(CSC)$ for any $\varepsilon > 0$.*

*Proof.* (i) Assume that $\overline{\mathrm{ran}(CSC)} \subset H_{T^*}(\Omega)$ and $x \in H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon)$ is arbitrary. Since $CSC$ commutes with $T^*$, it follows that $\sigma_{T^*}(CSCx) \subset \sigma_{T^*}(x) \subset \mathbb{C} \setminus \Omega_\varepsilon$. Hence $\sigma_{T^*}(CSCx) \subset \Omega \cap (\mathbb{C} \setminus \Omega_\varepsilon) = \emptyset$ and so $CSCx \in H_{T^*}(\emptyset)$. Since $T$ has the property $(\beta)$ and it is complex symmetric, $T^*$ has the single-valued extension property from [24], and thus $CSCx = 0$. Hence $H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) \subset \ker(CSC)$.

On the other hand, suppose that $H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) \subset \ker(CSC)$ for any $\varepsilon > 0$. Let $\varepsilon > 0$ be given and choose $\delta$ with $0 < \delta < \varepsilon$. Since $T$ has the property $(\beta)$, it follows from [25] and [28] that $T^*$ has both the single-valued extension property and the property $(\delta)$. Since $\mathbb{C} = \Omega_\varepsilon \cup (\mathbb{C} \setminus \overline{\Omega_\delta})$, we get that

$$\mathcal{H} = H_{T^*}(\overline{\Omega_\varepsilon}) + H_{T^*}(\mathbb{C} \setminus \Omega_\delta) \subset H_{T^*}(\overline{\Omega_\varepsilon}) + \ker(CSC).$$

For any $x \in \mathcal{H}$, set $x = u + v$ where $u \in H_{T^*}(\overline{\Omega_\varepsilon})$ and $v \in \ker(CSC)$. Then it holds that

$$CSCx = CSCu \in H_{T^*}(\overline{\Omega_\varepsilon}).$$

Since $\varepsilon$ is an arbitrary positive number and $\Omega$ is closed, $CSCx \in H_{T^*}(\Omega)$. Therefore $\mathrm{ran}(CSC) \subset H_{T^*}(\Omega)$. Since $T$ has Dunford's property $(C)$, so does $T^*$ by [24], and so we conclude that $\overline{\mathrm{ran}(CSC)} \subset H_{T^*}(\Omega)$.

(ii) Since we obtain from [24] that $H_{T^*}(F) = CH_T(F^*)$ for any set $F$ in $\mathbb{C}$, the proof follows from (i). $\square$

COROLLARY 3.11. *Assume that $T \in \mathscr{L}(\mathscr{H})$ is complex symmetric with conjugation $C$ and has the property $(\beta)$. If there exists an operator $S \in \{T\}'$ such that $\overline{ran(CSC)} \subset H_{T^*}(\Omega)$ and $\sigma_p(T)^* \setminus \Omega_\varepsilon \neq \emptyset$ for some closed set $\Omega$ in $\mathbb{C}$ and some $\varepsilon > 0$, where $\Omega_\varepsilon$ denotes the open $\varepsilon$-neighborhood of $\Omega$, then* $\ker(CSC) \neq \{0\}$ *and so $\sigma_p(CSC) \neq \emptyset$.*

*Proof.* Since $\overline{\operatorname{ran}(CSC)} \subset H_{T^*}(\Omega)$, Theorem 3.10 implies that

$$H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) \subset \ker(CSC).$$

Thus it is enough to show that $H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) \neq \{0\}$. Suppose that $H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) = \{0\}$. Choose any point $\lambda_0 \in \sigma_p(T)^* \setminus \Omega_\varepsilon$. Since $T$ is complex symmetric, $\lambda_0 \in \sigma_p(T^*)$ from [24] and then there exists a nonzero vector $x \in \ker(T^* - \lambda_0)$. By [28], we get that $x \in H_{T^*}(\{\lambda_0\})$. Since $\lambda_0 \notin \Omega_\varepsilon$, we have $x \in H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) = \{0\}$, which is a contradiction. So we obtain that $H_{T^*}(\mathbb{C} \setminus \Omega_\varepsilon) \neq \{0\}$. $\square$

We say that an operator $T \in \mathscr{L}(\mathscr{H})$ has *the property* $(E)$ (respectively, $(E_s)$) if there exist sequences $\{B_n\} \subset \{T\}'$ and $\{K_n\} \subset \mathscr{K}(\mathscr{H})$ (respectively, $\{E_n\} \subset \mathscr{L}(\mathscr{H})$) such that $\|B_n - T\| \to 0$, $K_nB_n = B_nK_n$ (respectively, $E_nB_n = B_nE_n$) for each $n \in \mathbb{N}$ and $\{K_n\}$ is a nontrivial sequence of compact operators (respectively, $\{E_n\}$ is a sequence of finite-rank projections weakly convergent to the identity operator $I$ on $\mathscr{H}$). An operator $T$ in $\mathscr{L}(\mathscr{H})$ will be said to have *the property* $(PS)$ if there exist sequences $\{S_n\} \subset \{T\}'$ and $\{K_n\} \subset \mathscr{K}(\mathscr{H})$ such that $\|S_n - K_n\| \to 0$ and $\{K_n\}$ is a nontrivial sequence of compact operators. An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *the property* $(A)$ provided that for every (not necessarily strict) contraction $S$, every operator $X$ with dense range such that $TX = XS$, and every vector $x \in \mathscr{H}$, there exists a nonzero polynomial $p(z)$ such that $p(T)x$ belongs to $\operatorname{ran}(X)$ (see [5] for more details). An operator $T \in \mathscr{L}(\mathscr{H})$ is said to have *the property* $(K)$ if for every $\lambda \in \sigma(T)$ and for every $\varepsilon > 0$, there exists a unit vector $x_{\lambda,\varepsilon}$ in $\mathscr{H}$ such that $\limsup_{n\to\infty} \|(T-\lambda)^n x_{\lambda,\varepsilon}\|^{\frac{1}{n}} < \varepsilon$. An operator $T \in \mathscr{L}(\mathscr{H})$ is called *power regular* if $\lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$ exists for every $x \in \mathscr{H}$. Preserver problems concern characterizations of maps on matrices, operators, or other algebraic objects with special properties. In the following proposition, we show that a complex symmetric operator and its adjoint preserve some properties each other.

PROPOSITION 3.12. *Let $T \in \mathscr{L}(\mathscr{H})$ be complex symmetric. Then the following relations hold.*

*(i) $T$ has the property $P$ if and only if $T^*$ does, where $P$ is $(E)$, $(PS)$, $(A)$ or $(K)$.*

*(ii) $T$ is power regular if and only if $T^*$ is.*

*Proof.* Assume that $T$ is complex symmetric with conjugation $C$.

(i) If $T$ has the property $(E)$, then there exist sequences $\{B_n\} \subset \{T\}'$ and $\{K_n\} \subset \mathscr{K}(\mathscr{H})$ such that $\|B_n - T\| \to 0$, $K_nB_n = B_nK_n$ for each $n \in \mathbb{N}$, and $\{K_n\}$ is a nontrivial sequence of compact operators. It is obvious that $\{CB_nC\} \subset \{T^*\}'$, $(CK_nC)(CB_nC)$

$=(CB_nC)(CK_nC)$ for each $n \in \mathbb{N}$, and $\{CK_nC\}$ is a nontrivial sequence of compact operators. Furthermore, we have

$$\|CB_nC - T^*\| = \|C(B_n - T)C\| = \|B_n - T\| \to 0.$$

Hence $T^*$ has the property $(E)$. The converse holds similarly.

If $T$ has the property $(PS)$, we can choose sequences $\{S_n\} \subset \{T\}'$ and $\{K_n\} \subset \mathcal{K}(\mathcal{H})$ such that $\|S_n - K_n\| \to 0$ and $\{K_n\}$ is a nontrivial sequence of compact operators. Clearly, $\{CS_nC\} \subset \{T^*\}'$ and $\{CK_nC\}$ is a nontrivial sequence of compact operators. Since $\|CS_nC - CK_nC\| = \|C(S_n - K_n)C\| = \|S_n - K_n\| \to 0$, we get that $T^*$ has the property $(PS)$. The converse statement holds similarly.

We suppose that $T$ has the property $(A)$. Let $S$ be a contraction, $X$ be an operator with dense range such that $T^*X = XS$, and let $x \in \mathcal{H}$. It is trivial that $T(CXC) = (CXC)(CSC)$ and $\|CSC\| = \|S\| \leqslant 1$. Moreover, there is a sequence $\{x_n\}$ with $\|Xx_n - Cx\| \to 0$ as $n \to \infty$, which implies that

$$\|(CXC)(Cx_n) - x\| = \|C(Xx_n - Cx)\| = \|Xx_n - Cx\| \to 0 \text{ as } n \to \infty.$$

Hence $CXC$ has dense range. Since $T$ has the property $(A)$, it follows that there exists a nonzero polynomial $p(z) = \sum_{j=0}^m a_j z^j$, with $a_j \in \mathbb{C}$ and $m \in \mathbb{N}$, such that $p(T)Cx = CXCy$ for some $y \in \mathcal{H}$. Setting the nonzero polynomial $q(z) = \overline{p(\overline{z})}$, we get that

$$q(T^*)x = \sum_{j=0}^m \overline{a_j} T^{*j} x = C\Big(\sum_{j=0}^m a_j T^j Cx\Big) = Cp(T)Cx = XCy \in \operatorname{ran}(X).$$

Thus $T^*$ has the property $(A)$. By replacing $T$ with $T^*$, the converse statement also holds.

Suppose that $T$ has the property $(K)$. Then for every $\lambda \in \sigma(T)$ and every $\varepsilon > 0$, there exists a unit vector $x_{\lambda,\varepsilon}$ in $\mathcal{H}$ such that $\limsup_{n\to\infty} \|(T-\lambda)^n x_{\lambda,\varepsilon}\|^{\frac{1}{n}} < \varepsilon$. Let $\mu \in \sigma(T^*)$ and $\varepsilon > 0$ be given. Since $\sigma(T^*) = \sigma(T)^*$ by [24], we have $\overline{\mu} \in \sigma(T)$ and so

$$\|(T^* - \mu)^n Cx_{\overline{\mu},\varepsilon}\| = \|(T - \overline{\mu})^n x_{\overline{\mu},\varepsilon}\|.$$

Moreover, $Cx_{\overline{\mu},\varepsilon}$ is a unit vector. Hence $T^*$ has the property $(K)$. The converse statement holds similarly.

(ii) Suppose that $T$ is power regular. Since

$$\|T^{*n}x\| = \|CT^nCx\| = \|T^nCx\| \tag{4}$$

for all $n$ and $T$ is power regular, $\lim_{n\to\infty} \|T^{*n}x\|^{\frac{1}{n}}$ exists for every $x \in \mathcal{H}$, i.e., $T^*$ is power regular. The converse statement holds in a similar way. □

EXAMPLE 3.13. If $T^2 = T$, then $T$ is quasidiagonal from [11], and hence $T$ has the property $(E_s)$ by [26]. Therefore, $T$ has the property $(E)$. Since $T$ is complex symmetric from [18], Proposition 3.12 implies that $T^*$ also has the property $(E)$.

COROLLARY 3.14. *Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator that is not a scalar multiple of the identity operator on $\mathcal{H}$. Then the following assertions hold:*

*(i) If $T$ has the property $(A)$, then for every contraction $S$ and every operator $X$ with dense range such that $T^*X = XS$, there exists a nonzero polynomial $p(z)$ and an operator $Z$ such that $p(T^*) = XZ$. In addition, if $X$ is one-to-one, then $p(S) = ZX$.*

*(ii) If $T$ has the property $(\beta)$, then $T$ and $T^*$ are power regular.*

*Proof.* (i) If $T$ has the property $(A)$, then so does $T^*$ by Proposition 3.12, and hence the proof follows from [5].

(ii) It is known that every operator with the property $(\beta)$ is power regular. Hence $T$ is power regular, and so is $T^*$ by Proposition 3.12. □

For an operator $T \in \mathcal{L}(\mathcal{H})$, a vector $x \in \mathcal{H}$ is said to be *cyclic* if the linear span of the orbit $O(x,T) = \{T^n x : n = 0,1,2,\cdots\}$ is norm dense in $\mathcal{H}$, i.e., $\bigvee O(x,T) = \mathcal{H}$. If there is a cyclic vector $x$ for $T$, then we say that $T$ is a *cyclic* operator. If $O(x,T)$ is norm dense in $\mathcal{H}$ for some $x \in \mathcal{H}$, then $T$ is called *hypercyclic*. An operator $T \in \mathcal{L}(\mathcal{H})$ having a vector $x \in \mathcal{H}$ whose orbit is weakly dense in $\mathcal{H}$ is said to be *weakly hypercyclic*. A *net* in $\mathcal{H}$ is a pair $\big((J,\leqslant),x\big)$ where $x$ is a function from $J$ to $\mathcal{H}$ and $(J,\leqslant)$ is a directed set, i.e., a partially ordered set such that if $i_1, i_2 \in J$, then there is $i_3 \in J$ such that $i_3 \geqslant i_1$ and $i_3 \geqslant i_2$. Usually, we will write $x_i$ instead of $x(i)$, and $\{x_i\}$ will be called a net in $\mathcal{H}$. We say that *a net $\{x_i\}$ in $\mathcal{H}$ converges* (*weakly*) *to $x_0$* if for every (weakly) open subset $\mathcal{U}$ of $\mathcal{H}$ with $x_0 \in \mathcal{U}$, there is $i_0 = i_0(\mathcal{U})$ such that $x_i \in \mathcal{U}$ for all $i \geqslant i_0$. It is well known that a net $\{x_i\}$ converges weakly to $x_0$ if and only if $\langle x_i, y\rangle \to \langle x_0, y\rangle$ for any $y \in \mathcal{H}$. In the following theorem, we consider a complex symmetric operator which is weakly hypercyclic.

THEOREM 3.15. *Let $T \in \mathcal{L}(\mathcal{H})$ be complex symmetric. If $T$ is weakly hypercyclic, then both $T$ and $T^*$ have the single-valued extension property.*

*Proof.* Suppose that $T$ is a complex symmetric operator that is weakly hypercyclic. We will first show that $T^*$ is also weakly hypercyclic. Since

$$\begin{aligned} CO(x,T) &= \{CT^n x : n = 0,1,2,\cdots\} \\ &= \{T^{*n} Cx : n = 0,1,2,\cdots\} = O(Cx, T^*) \end{aligned}$$

for any $x \in \mathcal{H}$, it suffices to prove the following claim.

*Claim.* If $\mathcal{M}$ is a weakly dense set in $\mathcal{H}$, then so is $C\mathcal{M}$.

Let $\mathcal{M}$ be a weakly dense set in $\mathcal{H}$ and let $x_0 \in \mathcal{H}$ be given. Since $Cx_0$ belongs to the weak closure of $\mathcal{M}$, there exists a net $\{x_i\}$ in $\mathcal{M}$ converging weakly to $Cx_0$. Then we get that

$$\langle x_0 - Cx_i, y\rangle = \langle Cy, Cx_0 - x_i\rangle \to 0$$

for all $y \in \mathcal{H}$. That is, $\{Cx_i\}$ is a net in $C\mathcal{M}$ which converges weakly to $x_0$, and so $C\mathcal{M}$ is weakly dense in $\mathcal{H}$.

From the claim above, $T^*$ is weakly hypercyclic. Hence $\sigma_p(T) = \sigma_p(T^*) = \emptyset$ from [32], and these identities imply that $T$ and $T^*$ have the single-valued extension property by [28]. $\square$

COROLLARY 3.16. *If a complex symmetric operator $T$ is weakly hypercyclic, then the following statements hold.*

*(i) $\sigma(T) = \sigma_a(T) = \sigma_{su}(T) = \cup\{\sigma_T(x) : x \in \mathscr{H}\}$.*

*(ii) $H_T(\{\lambda\}) = H_0(T-\lambda)$ and $H_{T^*}(\{\lambda\}) = H_0(T^*-\lambda)$ for all $\lambda \in \mathbb{C}$.*

*Proof.* Since both $T$ and $T^*$ have the single-valued extension property by Theorem 3.15, the proof follows from [8] or [28]. $\square$

For an operator $T \in \mathscr{L}(\mathscr{H})$ and a vector $x \in \mathscr{H}$, we define the *local spectral radius* of $T$ at $x$ by $r_T(x) = \limsup_{n\to\infty} \|T^n x\|^{\frac{1}{n}}$.

PROPOSITION 3.17. *If $T \in \mathscr{L}(\mathscr{H})$ is complex symmetric with conjugation $C$, then the following statements hold.*

*(i) $\sigma_T(x) = \sigma(T)$ for every cyclic vector $x \in \mathscr{H}$ of $T$ if and only if $\sigma_{T^*}(y) = \sigma(T^*)$ for every cyclic vector $y \in \mathscr{H}$ of $T^*$.*

*(ii) The equality $r_T(x) = r_{T^*}(Cx)$ holds for every $x \in \mathscr{H}$. Moreover, $r_T(x) = r(T)$ for every cyclic vector $x \in \mathscr{H}$ of $T$ if and only if $r_{T^*}(y) = r(T)$ for every cyclic vector $y \in \mathscr{H}$ of $T^*$.*

*Proof.* (i) It suffices to show one direction. Suppose that $\sigma_T(x) = \sigma(T)$ for every cyclic vector $x \in \mathscr{H}$ of $T$. If $y \in \mathscr{H}$ is a cyclic vector for $T^*$, then

$$\mathscr{H} = C\mathscr{H} = \bigvee_{n=0}^{\infty} \{CT^{*n}y\} = \bigvee_{n=0}^{\infty} \{T^n Cy\}.$$

This means that $Cy$ is cyclic for $T$, and so $\sigma_T(Cy) = \sigma(T)$. Hence, we obtain from [24] that

$$\sigma_{T^*}(y) = \sigma_T(Cy)^* = \sigma(T)^* = \sigma(T^*).$$

(ii) Note that $r_T(x) = r_{T^*}(Cx)$ holds for every $x \in \mathscr{H}$ from the equalities (4). Assume that $r_T(x) = r(T)$ for every cyclic vector $x \in \mathscr{H}$ of $T$ and $y \in \mathscr{H}$ is a cyclic vector for $T^*$. Since $Cy$ is cyclic for $T$ as in (i), we get that $r(T) = r_T(Cy) = r_{T^*}(y)$. The converse statement is true by a similar method. $\square$

COROLLARY 3.18. *If $T \in \mathscr{L}(\mathscr{H})$ is complex symmetric, then the following properties hold.*

*(i) If $T$ has Bishop's property $(\beta)$, then $\sigma_{T^*}(x) = \sigma(T^*)$ and $r_{T^*}(x) = r(T)$ for any cyclic vector $x$ in $\mathscr{H}$ of $T^*$. Moreover, $r_{T^*}(x) = \lim_{n\to\infty} \|T^{*n}x\|^{\frac{1}{n}}$ for all $x \in \mathscr{H}$.*

*(ii) For all $x \in \mathscr{H}$, $\lim_{n\to\infty} \|T^n x\| = 0$ if and only if $\lim_{n\to\infty} \|T^{*n}x\| = 0$.*

*Proof.* Suppose that $T$ is complex symmetric with conjugation $C$.

(i) If $T$ has Bishop's property $(\beta)$, then $\sigma_T(x)=\sigma(T)$ and $r_T(x)=r(T)$ for every cyclic vector $x\in\mathscr{H}$ of $T$ by [28]. Therefore the first statement follows from Proposition 3.17. In addition, $r_T(x)=\lim_{n\to\infty}\|T^nx\|^{\frac{1}{n}}$ for any $x\in\mathscr{H}$ by [28]. Hence we obtain from Proposition 3.17 that

$$r_{T^*}(x)=r_T(Cx)=\lim_{n\to\infty}\|T^nCx\|^{\frac{1}{n}}=\lim_{n\to\infty}\|T^{*n}x\|^{\frac{1}{n}}$$

for any $x\in\mathscr{H}$.

(ii) If $\lim_{n\to\infty}\|T^nx\|=0$ for all $x\in\mathscr{H}$, then from equalities (4) we get $\lim_{n\to\infty}\|T^{*n}x\|=\lim_{n\to\infty}\|T^nCx\|=0$ for all $x\in\mathscr{H}$. The converse implication holds in a similar way. □

## 4. Weyl type Theorem

In this section, we deal with Weyl type theorems for complex symmetric operators. We recall the definitions of some spectra;

$$\sigma_{ea}(T):=\cap\{\sigma_a(T+K):K\in\mathscr{K}(\mathscr{H})\}$$

is the essential approximate point spectrum, and

$$\sigma_{ab}(T):=\cap\{\sigma_a(T+K):TK=KT\text{ and }K\in\mathscr{K}(\mathscr{H})\}$$

is the Browder essential approximate point spectrum. We put

$$\pi_{00}(T):=\{\lambda\in\text{iso }\sigma(T):0<dim\ker(T-\lambda)<\infty\}$$

and

$$\pi_{00}^a(T):=\{\lambda\in\text{iso }\sigma_a(T):0<dim\ker(T-\lambda)<\infty\}.$$

Let $T\in\mathscr{L}(\mathscr{H})$. We say that
(i) *$a$-Browder's theorem holds for $T$* if $\sigma_{ea}(T)=\sigma_{ab}(T)$;
(ii) *$a$-Weyl's theorem holds for $T$* if $\sigma_a(T)\setminus\sigma_{ea}(T)=\pi_{00}^a(T)$;
(iii) $T$ has *the property* $(w)$ if $\sigma_a(T)\setminus\sigma_{ea}(T)=\pi_{00}(T)$.
It is known that

$$\begin{array}{ccc}\text{Property }(w) & \Longrightarrow & a\text{-Browder's theorem}\\ \Downarrow & & \Uparrow\\ \text{Weyl's theorem} & \Longleftarrow & a\text{-Weyl's theorem.}\end{array}$$

We refer the reader to [1], [20], [21] for more details.

Let $T_n=T|_{\text{ran}(T^n)}$ for each nonnegative integer $n$; in particular, $T_0=T$. If $T_n$ is upper semi-Fredholm for some nonnegative integer $n$, then $T$ is called a *upper semi-B-Fredholm* operator. In this case, by [6], $T_m$ is a upper semi-Fredholm operator and

$ind(T_m) = ind(T_n)$ for each $m \geqslant n$. Thus, one can consider the *index* of $T$, denoted by $ind_B(T)$, as the index of the semi-Fredholm operator $T_n$. Similarly, we define lower semi-B-Fredholm operators. We say that $T \in \mathscr{L}(\mathscr{H})$ is *B-Fredholm* if it is both upper and lower semi-B-Fredholm. In [6], Berkani proved that $T \in \mathscr{L}(\mathscr{H})$ is B-Fredholm if and only if $T = T_1 \oplus T_2$ where $T_1$ is Fredholm and $T_2$ is nilpotent. Let $SBF_+^-(\mathscr{H})$ be the class of all upper semi-$B$-Fredholm operators such that $ind_B(T) \leqslant 0$, and let

$$\sigma_{SBF_+^-}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathscr{H})\}.$$

An operator $T \in \mathscr{L}(\mathscr{H})$ is called *B-Weyl* if it is B-Fredholm of index zero. The *B-Weyl spectrum* $\sigma_{BW}(T)$ of $T$ is defined by

$$\sigma_{BW}(T) := \{\lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator }\}.$$

We say that $\lambda \in \sigma_a(T)$ is a *left pole* of $T$ if it has finite ascent, i.e., $a(T) < \infty$ and $\operatorname{ran}(T^{a(T)+1})$ is closed where $a(T) = dim\ker(T)$. The notation $p_0(T)$ (respectively, $p_0^a(T)$) denotes the set of all poles (respectively, left poles) of $T$, while $\pi_0(T)$ (respectively, $\pi_0^a(T)$) is the set of all eigenvalues of $T$ which is an isolated point in $\sigma(T)$ (respectively, $\sigma_a(T)$).

Let $T \in \mathscr{L}(\mathscr{H})$. We say that
(i) $T$ satisfies *generalized Browder's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus p_0(T)$;
(ii) $T$ satisfies *generalized $a$-Browder's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus p_0^a(T)$;
(iii) $T$ satisfies *generalized Weyl's theorem* if $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$;
(iv) $T$ satisfies *generalized $a$-Weyl's theorem* if $\sigma_{SBF_+^-}(T) = \sigma_a(T) \setminus \pi_0^a(T)$.
It is known that

$$\begin{array}{ccc} \text{generalized } a\text{-Weyl's theorem} & \Longrightarrow & \text{generalized Weyl's theorem} \\ \Downarrow & & \Downarrow \\ \text{generalized } a\text{-Browder's theorem} & \Longrightarrow & \text{generalized Browder's theorem.} \end{array}$$

We now establish equivalence relations among Weyl type theorems for complex symmetric operators. We begin with the following lemma.

LEMMA 4.1. *If $T \in \mathscr{L}(\mathscr{H})$ is complex symmetric, the following statements hold:*
*(i)* $\sigma(T) = \sigma_a(T)$.
*(ii)* $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T) = \sigma_{ea}(T) = \sigma_w(T)$.
*(iii) If* $\sigma_p(T) = \emptyset$, *then*

$$\sigma(T) = \sigma_a(T) = \sigma_e(T) = \sigma_{le}(T) = \sigma_{re}(T) = \sigma_{ea}(T) = \sigma_w(T).$$

*Proof.* (i) Since $T$ is a complex symmetric operator, it follows from [24] that $\sigma_a(T)^* = \sigma_a(T^*)$. Hence we have $\sigma(T) = \sigma_a(T) \cup \sigma_a(T^*)^* = \sigma_a(T)$ by [19].

(ii) Note that $\sigma_{re}(S)^* = \sigma_{le}(S^*)$ and $\sigma_e(S) = \sigma_{le}(S) \cup \sigma_{re}(S)$ for any $S \in \mathscr{L}(\mathscr{H})$. Since $T$ is complex symmetric, it follows from [24] that $\sigma_{le}(T)^* = \sigma_{le}(T^*)$. Thus we get that $\sigma_{le}(T) = \sigma_{re}(T)$ and so $\sigma_{le}(T) = \sigma_{re}(T) = \sigma_e(T)$. In addition, we note that

$T-\lambda$ is Weyl for any $\lambda \notin \sigma_{le}(T) \cap \sigma_{re}(T)$ as in the proof of Theorem 3.1 and it is known that $\lambda \notin \sigma_{ea}(T)$ if and only if $T-\lambda$ is semi-Fredholm with $ind(T-\lambda) \leqslant 0$ (see [1]). Thus we obtain that $\sigma_e(T) = \sigma_w(T) = \sigma_{ea}(T)$.

(iii) Assume that $T$ is a complex symmetric operator with $\sigma_p(T) = \emptyset$. For all $S \in \mathcal{L}(\mathcal{H})$, $\sigma(S) = \sigma_e(S) \cup \sigma_p(S) \cup \sigma_p(S^*)^*$. Since $T$ is complex symmetric, $\sigma_p(T^*)^* = \sigma_p(T) = \emptyset$ by [25] and thus $\sigma(T) = \sigma_e(T)$. Therefore we complete the proof from (i) and (ii). □

THEOREM 4.2. *If $T \in \mathcal{L}(\mathcal{H})$ is complex symmetric, then the following statements are equivalent:*

*(i) $a$-Weyl's theorem holds for $T$.*

*(ii) Weyl's theorem holds for $T$.*

*(iii) $T$ has the property $(w)$.*

*Proof.* It is obvious that (i) $\Rightarrow$ (ii). Assume that $T$ satisfies Weyl's theorem. Since $T$ is complex symmetric, it follows from Lemma 4.1 that $\sigma_a(T) = \sigma(T)$ and $\sigma_w(T) = \sigma_{ea}(T)$, which yields that

$$\pi_{00}^a(T) = \pi_{00}(T) = \sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{ea}(T).$$

Hence $a$-Weyl's theorem holds for $T$, and so we have (ii) $\Rightarrow$ (i). Similarly, since $\pi_{00}^a(T) = \pi_{00}(T)$, we can show that (i) $\Leftrightarrow$ (iii). So we complete the proof. □

COROLLARY 4.3. *Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator. Then the following statements holds.*

*(i) $T$ satisfies $a$-Weyl's theorem if and only if $T^*$ does.*

*(ii) $T$ has the property $(w)$ if and only if $T^*$ does.*

*Proof.* (i) If $T$ satisfies $a$-Weyl's theorem, then Weyl's theorem holds for $T$, and so Weyl's theorem holds for $T^*$ by [24]. Since $T^*$ is also complex symmetric, it satisfies $a$-Weyl's theorem by Theorem 4.2. The converse statement holds by replacing $T$ with $T^*$.

(ii) Since Theorem 4.2 gives the equivalent relation that $a$-Weyl's theorem holds for $T$ (respectively, $T^*$) if and only if $T$ (respectively, $T^*$) has the property $(w)$, the proof follows from (i). □

THEOREM 4.4. *If $T \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator with the single-valued extension property, then the following statements are equivalent.*

*(i) $T$ satisfies generalized $a$-Weyl's theorem.*

*(ii) $T$ satisfies generalized Weyl's theorem.*

*Proof.* Since (i) $\Rightarrow$ (ii) follows from [7, Theorem 3.7], it suffices to show that (ii) $\Rightarrow$ (i). Suppose that $T$ satisfies generalized Weyl's theorem. Then we have $\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T)$. Since $T$ is complex symmetric, it follows from Lemma 4.1 that $\sigma_a(T) = \sigma(T)$ and so

$$\sigma_{BW}(T) = \sigma(T) \setminus \pi_0(T) = \sigma_a(T) \setminus \pi_0^a(T).$$

Hence it suffices to show that $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$. If $\lambda \notin \sigma_{SBF_+^-}(T)$, then $T-\lambda$ is semi-B-Fredholm and $ind_B(T-\lambda) \leqslant 0$. Since $T$ is a complex symmetric operator with the single-valued extension property, it follows from [25] that $T^*$ has the single-valued extension property. Therefore, we obtain from [1] that $ind_B(T-\lambda) \geqslant 0$ for every $\lambda \notin \sigma_{SBF_+^-}(T)$. Thus we have $ind_B(T-\lambda) = 0$ for every $\lambda \notin \sigma_{SBF_+^-}(T)$. This means that $\sigma_{SBF_+^-}(T) \supset \sigma_{BW}(T)$. Since $\sigma_{SBF_+^-}(T) \subset \sigma_{BW}(T)$ is clear, we obtain that

$$\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_a(T) \setminus \pi_{00}^a(T),$$

that is, generalized $a$-Weyl's theorem holds for $T$. □

COROLLARY 4.5. *If $T \in \mathcal{L}(\mathcal{H})$ is a complex symmetric operator which satisfies that for every $\lambda \in \mathbb{C}$, there is a positive integer $m_\lambda$ such that $H_0(T-\lambda) = \ker(T-\lambda)^{m_\lambda}$, then $f(T)$ and $f(T)^*$ obey generalized $a$-Weyl's theorem where $f$ is any analytic function in a neighborhood of $\sigma(T)$.*

*Proof.* As the proof of Theorem 3.7, it holds for all $\lambda \in \mathbb{C}$ that $H_0(T^*-\lambda) = \ker(T^*-\lambda)^{n_\lambda}$ where $n_\lambda = m_{\overline{\lambda}}$. Since $\sigma(T) = \sigma_a(T)$ and $\sigma(T^*) = \sigma_a(T^*)$ by Lemma 4.1, we complete the proof from [30, Proposition 1.9]. □

In the following theorem, we consider Browder type theorems for complex symmetric operators.

THEOREM 4.6. *Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator. Then the following arguments are equivalent.*

*(i) $T$ satisfies Browder's theorem.*

*(ii) $T$ satisfies $a$-Browder's theorem.*

*(iii) $T$ satisfies the generalized Browder's theorem.*

*(iv) $T$ satisfies the generalized $a$-Browder's theorem.*

*Proof.* Since $\sigma(T) = \sigma_a(T)$ from Lemma 4.1, we have $p_0(T) = p_0^a(T)$. Moreover, $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T)$ as in the proof of Theorem 4.4. Using these results, we get that (iii) ⇔ (iv). Since it is well known that (i) ⇔ (iii) and (ii) ⇔ (iv) from [3, Theorem 2.1 and Theorem 2.2], we complete the proof. □

REMARK. A similar equivalent statement to Corollary 4.3 is satisfied for $a$-Browder's theorem of complex symmetric operators. Let $T \in \mathcal{L}(\mathcal{H})$ be a complex symmetric operator. We obtain from Lemma 4.1 that

$$\sigma_{ea}(T)^* = \sigma_e(T)^* = \sigma_e(T^*) = \sigma_{ea}(T^*).$$

In addition, since we know from [1, Corollary 3.45 and Theorem 3.65] that

$$\sigma_{ab}(S) = \sigma_{ea}(S) \cup acc(\sigma_a(S))$$

for any operator $S \in \mathscr{L}(\mathscr{H})$, where $acc(\sigma(S))$ denotes the set of all accumulation points of $\sigma(S)$, it follows that

$$\begin{aligned}\sigma_{ab}(T^*) &= \sigma_{ea}(T^*) \cup acc(\sigma_a(T^*)) = \sigma_{ea}(T^*) \cup acc(\sigma_a(T)^*) \\ &= [\sigma_{ea}(T) \cup acc(\sigma_a(T))]^* = \sigma_{ab}(T)^*.\end{aligned}$$

Hence we conclude that $T$ satisfies $a$-Browder's theorem if and only if $T^*$ does.

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