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THE BEHAVIOR OF THE ORBITS OF POWER BOUNDED OPERATORS

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Abstract. Let T be a power bounded operator on a Banach space X and let $\sigma_T(x)$ be the local spectrum of T at $x \in X$. In this paper, we study the asymptotic behavior of the orbits $\{T^n x : n \geqslant 0\}$ in terms of the local spectrum of T at x.

1. Introduction

Let X be a complex Banach space and let $B(X)$ be the algebra of all bounded, linear operators on X. For $T \in B(X)$, we denote by $\sigma(T)$, the spectrum of T and by $R_z(T) := (zI - T)^{-1}$ $(z \notin \sigma(T))$ the resolvent of T. The unit circle in the complex plane will be denoted by Γ, whereas D indicates the open unit disc. The set $\sigma(T) \cap \Gamma$ will be called the *unitary spectrum* of T.

Recall that $T \in B(X)$ is called *stable* if $\lim_{n\to\infty} \|T^n x\| = 0$ for all $x \in X$. Generally speaking, the asymptotic behavior of the orbits $\{T^n x : n \geqslant 0\}$ is frequently related to unitary spectrum of underlying operator. This is well illustrated by the following classical result of Nagy-Foias [16, Proposition II. 6.7]. If T is a completely non-unitary contraction on a Hilbert space and if the unitary spectrum of T is of Lebesgue measure zero, then T is stable.

For arbitrary $T \in B(X)$ and $x \in X$, we define $\rho_T(x)$ to be the set of all $\lambda \in \mathbb{C}$ for which there exists a neighborhood O_λ of λ with $u(z)$ analytic on O_λ having values in X such that $(zI - T)u(z) = x$, $\forall z \in O_\lambda$. This set is open and contains the resolvent set $\rho(T)$ of T. By definition, the *local spectrum* of T at x, denoted by $\sigma_T(x)$ is the complement of $\rho_T(x)$, so it is a closed subset of $\sigma(T)$. This object is most tractable if the operator T has the *single-valued extension property* (SVEP) i.e. for every open set U in $\mathbb{C}$, the only analytic function $f : U \to X$ for which the equation $(zI - T)f(z) = 0$ holds, is the constant function $f \equiv 0$. In that case, for every $x \in X$ there exists a maximal analytic extension of $R_z(T)x$ to $\rho_T(x)$. It follows that if T has the SVEP, then $\sigma_T(x) \neq \emptyset$, whenever $x \neq 0$. It is easy to see that an operator $T \in B(X)$ having spectrum without interior points has the SVEP.

Note that the local spectrum of T may be "very small" with respect to its usual spectrum. To see this, let σ be a "small" part of $\sigma(T)$ such that both σ and $\sigma(T) \setminus \sigma$ are closed sets. Let P_σ be the spectral projection associated with σ and let $X_\sigma := P_\sigma X$.

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Then, X_σ is a closed T-invariant subspace of X and $\sigma(T|_{X_\sigma}) = \sigma$. It is easy to check that $\sigma_T(x) \subset \sigma$, for every $x \in X_\sigma$.

An operator T acting on a Banach space is called *power bounded* if

$$\sup_{n \geqslant 0} \|T^n\| < \infty$$

(by changing to an equivalent norm it can be made contractive). If T is power bounded, then $\sigma(T) \subset \overline{D}$ and $\sigma_T(x) \cap \Gamma$, the *local unitary spectrum* of $x \in X$ consists of all $\xi \in \Gamma$ such that the function $R_z(T)x$ $(|z| > 1)$ has no analytic extension to a neighborhood of ξ. Clearly,

$$\sigma(T) \cap \Gamma = \bigcup_{x \in X} (\sigma_T(x) \cap \Gamma).$$

An operator $T \in B(X)$ is called *stable* at $x \in X$ if $\lim_{n\to\infty} \|T^n x\| = 0$. Local version of the Nagy-Foias Theorem was proved in [9]: If T is a completely non-unitary contraction on a Hilbert space and if $\sigma_T(x) \cap \Gamma$ is of Lebesgue measure zero, then T is stable at $x \in X$.

Let T be a power bounded operator on a Banach space. Assume that the unitary spectrum of T is countable. Discrete version of Arendt-Batty-Lyubich-Phong (ABLP) theorem asserts that if T^* has no unitary eigenvalues, then T is stable (see, [2] and [17, Chapter 5]).

In this paper, for the stability of T at $x \in X$, some spectral conditions are found on T and on x.

2. Preliminaries

This section deals with some preliminaries that will be used later.

If E is an invariant subspace of $T \in B(X)$, we denote by T_E or by $T|_E$ the restriction of T to E. We will need the following.

LEMMA 2.1. *Let T be a power bounded operator on a Banach space X and let E be a (closed) T-invariant subspace of X. Then, for every $x \in E$, we have*

$$\sigma_{T_E}(x) \cap \Gamma = \sigma_T(x) \cap \Gamma.$$

Proof. Let $x \in E$. Clearly, $\rho_{T_E}(x) \subset \rho_T(x)$ and so

$$\sigma_T(x) \cap \Gamma \subset \sigma_{T_E}(x) \cap \Gamma.$$

For the reverse inclusion, let $\xi \in \rho_T(x) \cap \Gamma$ and let $\pi : X \to X/E$ be the canonical mapping. Then, there exists a neighborhood O_ξ of ξ with $u(z)$ analytic on O_ξ having values in X such that $(zI - T)u(z) = x$ on O_ξ. Notice that

$$u(z) = R_z(T)x = \sum_{n=0}^{\infty} z^{-n-1} T^n x \in E,$$

for all $z \in O_\xi$ with $|z|>1$. Therefore, we have $\pi u(z)=0$, for all $z \in O_\xi$ with $|z|>1$. By uniqueness theorem, $\pi u(z)=0$, for all $z \in O_\xi$. Hence, we obtain that $u(z) \in E$, for all $z \in O_\xi$. Consequently, we can write

$$(zI - T_E)\,u(z) = x,\ \forall z \in O_\xi.$$

This shows that $\xi \in \rho_{T_E}(x) \cap \Gamma$. $\square$

As an illustration of Lemma 2.1, consider the following example. Let K be a Hilbert space and let $H^2(K)$ be the Hardy space of K-valued analytic functions on D. By S_K, we denote the forward shift operator on $H^2(K)$;

$$(S_K f)(z) = zf(z).$$

Its adjoint, the backward shift, is given by

$$(S_K^* f)(z) = \frac{f(z) - f(0)}{z},\ f \in H^2(K).$$

It is easy to verify that for every $f \in H^2(K)$ and $\lambda \in \mathbb{C}$ with $|\lambda|>1$,

$$(\lambda I - S_K^*)^{-1} f(z) = \frac{\lambda^{-1} f\left(\lambda^{-1}\right) - zf(z)}{1 - \lambda z}.$$

Hence, $\sigma_{S_K^*}(f) \cap \Gamma$ consists of all $\xi \in \Gamma$ for which the function f has no analytic extension to a neighborhood of ξ. Now, let T be a stable contraction on a Hilbert space H i.e.

$$\lim_{n\to\infty} \|T^n x\| = 0, \forall x \in H.$$

Let $\mathscr{D} := (I - T^*T)^{\frac{1}{2}}$ and $K := \overline{\mathscr{D}H}$. By well-known Model Theorem of Nagy-Foias [16, Chapter VI], there exists S_K^*-invariant subspace E of $H^2(K)$ and a unitary operator $U : H \to E$ such that

$$T = U^{-1}(S_K^*|_E)U,$$

where

$$(Ux)(z) = \sum_{n=0}^{\infty} z^n \mathscr{D} T^n x\ (x \in H).$$

It follows from Lemma 2.1 that if $x \in H$, then

$$\sigma_T(x) \cap \Gamma = \sigma_{S_K^*|_E}(Ux) \cap \Gamma = \sigma_{S_K^*}(Ux) \cap \Gamma.$$

Hence, $\sigma_T(x) \cap \Gamma$ consists of all $\xi \in \Gamma$ such that the function $z \mapsto (Ux)(z)$ has no analytic extension to a neighborhood of ξ.

Let V be an isometry on a Banach space. It is well known that if $\sigma(V) \neq \overline{D}$, then V is invertible. Recall also that $x \in X$ is a *cyclic vector* of $T \in B(X)$ if

$$\overline{\operatorname{span}}\{T^n x : n \geqslant 0\} = X.$$

The following result was proved in [9, Lemma 1.3].

LEMMA 2.2. *Let V be an isometry on a Banach space X. If $x \in X$ is a cyclic vector of V, then*

$$\sigma(V) \cap \Gamma = \sigma_V(x) \cap \Gamma.$$

By l.i.m.$_n a_n$ we will denote Banach limit of the bounded sequence $\{a_n\}$.

The following result is well known (see for instance, [9, 11] and [17, Chapter 5]).

LEMMA 2.3. *If T is a power bounded operator on a Banach space X, then there exist a Banach space Y, a bounded linear operator $J : X \to Y$ with dense range, and an isometry V on Y with the following properties:*

(a) $VJ = JT$.

(b) $\|Jx\| = l.i.m._n \|T^n x\|$, $\forall x \in X$.

(c) $\sigma(V) \subset \sigma(T)$.

If X is assumed to be a Hilbert space, then Y is a Hilbert space, also.

The triple (Y, J, V) will be called the *limit isometry associated with T*. Notice that $Jx = 0$ if and only if $\lim_{n\to\infty} \|T^n x\| = 0$. Notice also that if $x \in X$ is a cyclic vector of T, then Jx is a cyclic vector of V.

LEMMA 2.4. *Let T be a power bounded operator on a Banach space X and let (Y, J, V) be the limit isometry associated with T. Then we have*

$$\sigma_V(Jx) \subset \sigma_T(x), \ \forall x \in X.$$

Proof. If $\lambda \in \rho_T(x)$, then there exists a neighborhood U_λ of λ with $u(z)$ analytic on U_λ having values in X such that $(zI - T)u(z) = x$, $\forall z \in U_\lambda$. It follows that $(zJ - JT)u(z) = Jx$. Since $JT = VJ$, we have $(zI - V)Ju(z) = Jx$, $\forall z \in U_\lambda$. This shows that $\lambda \in \rho_V(Jx)$. $\square$

The following lemma was proved in [14, Lemma 3].

LEMMA 2.5. *Let V be an invertible isometry on a Banach space X with countable spectrum. For arbitrary $\varphi \in X^*$, there exist a Hilbert space H_φ, a bounded linear operator $J_\varphi : X \to H_\varphi$ with dense range, and a unitary operator U_φ on H_φ with the following properties:*

(a) $U_\varphi J_\varphi = J_\varphi V$.

(b) $\sigma(U_\varphi) \subset \sigma(V)$.

(c) $\bigcap_{\varphi \in X^*} \ker J_\varphi = \{0\}$.

The triple $(H_\varphi, J_\varphi, U_\varphi)$ will be called the *unitary operator associated with the pair* (V, φ). As in the proof of Lemma 2.4, we can see that for every $\varphi \in X^*$ and $x \in X$,

$$\sigma_{U_\varphi}(J_\varphi x) \subset \sigma_V(x). \tag{2.1}$$

3. Hilbert space operators

In this section, we consider stability problem for operators on Hilbert space with ”thin” spectra.

We denote by $\mathscr{A}$ the set of all continuous functions on Γ having an absolutely convergent Fourier series. $\mathscr{A}$ is a commutative Banach algebra under the norm

$$\|f\|_1 := \sum_{n\in\mathbb{Z}} \left|\widehat{f}(n)\right|,$$

where $\widehat{f}(n)$ is the nth Fourier coefficient of $f \in \mathscr{A}$.

Recall [19, Chapter 5] that a closed set S in Γ is a *Helson set* if for every continuous function g on S there corresponds a function $f \in \mathscr{A}$ such that $f(s) = g(s)$, for all $s \in S$.

Let $M(\Gamma)$ be the space of regular complex Borel measures on Γ. The nth Fourier coefficient of $\mu \in M(\Gamma)$ is defined by

$$\widehat{\mu}(n) = \int_0^{2\pi} e^{-int} d\mu(t) \quad (n \in \mathbb{Z}).$$

It is well known that if $\widehat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$, then $\mu = 0$.

The Helson Theorem [19, Theorem 5.6.10] asserts the following.

THEOREM 3.1. *Let $S \subset \Gamma$ be a Helson set and let $\mu \in M(\Gamma)$ be given such that $supp\, \mu \subset S$. If $\lim_{|n|\to\infty} |\widehat{\mu}(n)| = 0$, then $\mu = 0$.*

As an application, we have the following.

THEOREM 3.2. *Let T be a power bounded operator on a Hilbert space H and let $x \in H$. Assume that*

(i) $\sigma_T(x) \cap \Gamma$ is contained in a Helson set,

(ii) $T^n x \to 0$ weakly as $n \to \infty$.

Then,

$$\lim_{n\to\infty} \|T^n x\| = 0.$$

Proof. Let L be the closed linear span of $\{T^n x : n \geqslant 0\}$. Then, L is a T-invariant subspace of H. Let (K, J, V) be the limit isometry associated with T_L. By Lemma 2.4, $\sigma_V(Jx) \subset \sigma_{T_L}(x)$. Consequently, we have

$$\sigma_V(Jx) \cap \Gamma \subset \sigma_{T_L}(x) \cap \Gamma.$$

Taking into account Lemma 2.1, we can write

$$\sigma_V(Jx) \cap \Gamma \subset \sigma_T(x) \cap \Gamma.$$

Further, since Jx is a cyclic vector of V by Lemma 2.2, we obtain

$$\sigma(V)\cap\Gamma=\sigma_V(Jx)\cap\Gamma\subset\sigma_T(x)\cap\Gamma.$$

Consequently, V is a unitary operator and $\sigma(V)$ is contained in a Helson set.

Let $E(\cdot)$ be the spectral measure of V and let μ_{Jx} be the scalar measure defined on the Borel subsets of Γ by

$$\mu_{Jx}(\Delta)=\langle E(\Delta)Jx,Jx\rangle=\|E(\Delta)Jx\|^2.$$

From the spectral decomposition of V, we can write

$$\begin{aligned}\widehat{\mu_{Jx}}(n) &= \int_0^{2\pi} e^{-int}\,d\mu_{Jx}(t)\\ &= \int_0^{2\pi} e^{-int}\,d\langle E_t Jx,Jx\rangle=\langle V^{*n}Jx,Jx\rangle\ \ (n\in\mathbb{Z}).\end{aligned}$$

On the other hand, from Lemma 2.3 (a), we have $J^*V^{*n}=T_L^{*}J^*\ \ (n\in\mathbb{N})$ which implies

$$\begin{aligned}\langle V^{*n}Jx,Jx\rangle &= \langle J^*V^{*n}Jx,x\rangle=\langle T_L^{*n}J^*Jx,x\rangle\\ \langle J^*Jx,T^nx\rangle &= \overline{\langle T^nx,J^*Jx\rangle}\to 0\ \ (n\to\infty).\end{aligned}$$

Similarly,

$$\begin{aligned}\widehat{\mu_{Jx}}(-n) &= \langle V^nJx,Jx\rangle\\ &= \langle JT^nx,Jx\rangle=\langle T^nx,J^*Jx\rangle\to 0\,(n\to\infty).\end{aligned}$$

Thus, we have

$$\lim_{|n|\to\infty}\left|\widehat{\mu_{Jx}}(n)\right|=0.$$

Since $\operatorname{supp}\mu_{Jx}$ is contained in a Helson set, by Theorem 3.1, $\mu_{Jx}=0$. Consequently, $E(\Delta)Jx=0$ for every Borel subset Δ of Γ. Therefore, we have $VJx=0$. It follows that $Jx=0$. This means that $\lim_{n\to\infty}\|T^nx\|=0$. $\square$

If Λ is a subset of $\mathbb{Z}$, we denote by C_Λ the space of all continuous functions f on Γ such that $\widehat{f}(n)=0$ if $n\notin\Lambda$. A subset Λ of $\mathbb{Z}$ is called a *Sidon set* if for every trigonometric polynomial $f\in C_\Lambda$, there exists a constant $C>0$ such that

$$\sum\left|\widehat{f}(n)\right|\leqslant C\|f\|_\infty.$$

We need the following result [20].

THEOREM 3.3. *Suppose that Λ is a Sidon set in $\mathbb{Z}_+$. If $\mu\in M(\Gamma)$ is such that $\widehat{\mu}(n)=0$ for each $n\in\mathbb{Z}_+\setminus\Lambda$, then μ is absolutely continuous with respect to Lebesgue measure on Γ.*

As an application, we have the following.

THEOREM 3.4. *Let T be a power bounded operator on a Hilbert space H and let $x \in H$. Let Λ be a Sidon set in $\mathbb{Z}_+$. Assume that*
(i) The Lebesgue measure of $\sigma_T(x) \cap \Gamma$ is zero,
(ii) $\lim_{k\to\infty}\langle T^{k+n}x, T^k x\rangle = 0,\ \forall n \in \mathbb{Z}_+ \setminus \Lambda$.
Then,

$$\lim_{n\to\infty} \|T^n x\| = 0.$$

Proof. Let L be the closed linear span of $\{T^n x : n \geqslant 0\}$ and let (K, J, V) be the limit isometry associated with T_L. As in the proof of Theorem 3.2, we have

$$\sigma(V) \cap \Gamma \subset \sigma_T(x) \cap \Gamma.$$

It follows that V is unitary and

$$\sigma(V) \subset \sigma_T(x) \cap \Gamma.$$

Consequently, the Lebesgue measure of $\sigma(V)$ is zero.
We can write

$$\begin{aligned}\langle V^n Jx, Jx\rangle &= \langle JT^n x, Jx\rangle \\ &= \text{l.i.m.}_k \langle T^{k+n}x, T^k x\rangle = 0,\ \forall n \in \mathbb{Z}_+ \setminus \Lambda.\end{aligned}$$

Let $E(\cdot)$ be the spectral measure of V and let μ_{Jx} be the scalar measure defined on the Borel subsets of Γ by

$$\mu_{Jx}(\Delta) = \langle E(\Delta) Jx, Jx\rangle = \|E(\Delta) Jx\|^2.$$

We have

$$\widehat{\mu_{Jx}}(n) = \langle V^n Jx, Jx\rangle = 0, \forall n \in \mathbb{Z}_+ \setminus \Lambda.$$

By the preceding theorem, μ_{Jx} is absolutely continuous with respect to Lebesgue measure. Consequently, $E(\Delta) Jx = 0$ for every Borel subset Δ of $\sigma(V)$. Therefore, we have $VJx = 0$. It follows that $Jx = 0$. This means that $\lim_{n\to\infty} \|T^n x\| = 0$. □

Recall that $\mathscr{A}$ is a commutative regular semisimple Banach algebra. The elements of $\mathscr{A}^*$ are called *pseudomeasures*. We will write $\varphi = \{\widehat{\varphi}(n)\}_{n\in\mathbb{Z}}$, where

$$\widehat{\varphi}(n) := \langle \varphi, e^{int}\rangle \ (n \in \mathbb{Z})$$

is the Fourier coefficients of a pseudomeasure φ. If $f \in \mathscr{A}$, then the duality being implemented by the formula

$$\langle \varphi, f\rangle = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(n) \widehat{f}(n).$$

The $\operatorname{hull}(I)$ of an ideal $I \subset \mathscr{A}$ is defined as

$$\operatorname{hull}(I) = \{\xi \in \Gamma : f(\xi) = 0,\ \forall f \in I\}.$$

If φ is a pseudomeasure, then

$$I_{\varphi} := \{f \in \mathscr{A} : \varphi \cdot f = 0\}$$

is a closed ideal in $\mathscr{A}$, where $\varphi \cdot f$ is a pseudomeasure defined by

$$\langle \varphi \cdot f, g \rangle = \langle \varphi, fg \rangle, \; g \in \mathscr{A}.$$

Recall that the *support* of a pseudomeasure φ is defined as follows. For $\xi \in \Gamma$, we let $\xi \notin \operatorname{supp}\varphi$ iff there is a neighborhood O_{ξ} of ξ such that $\langle \varphi, f \rangle = 0$ for all $f \in \mathscr{A}$ with $\operatorname{supp} f \subset O_{\xi}$. An equivalent definition for $\operatorname{supp}\varphi$ is that $\xi \in \operatorname{supp}\varphi$ iff $\varphi \cdot f = 0$ implies $f(\xi) = 0$. Consequently, for every pseudomeasure φ, we have

$$\operatorname{supp}\varphi = \operatorname{hull}\left(I_{\varphi}\right).$$

The well-known Loomis Theorem [13] states that if the support of a pseudomeasure φ is at most countable, then φ is almost periodic.

If $\mu \in M(\Gamma)$, then

$$\varphi_{\mu} := \{\widehat{\mu}(n)\}_{n \in \mathbb{Z}}$$

is a pseudomeasure. Notice that $\operatorname{supp}\varphi_{\mu}$ and $\operatorname{supp}\mu$ in the usual sense are the same. Notice also that if φ_{μ} is an almost periodic pseudomeasure, then

$$C_{\xi}\left(\varphi_{\mu}\right) = \mu\{\xi\},$$

where $C_{\xi}\left(\varphi_{\mu}\right)$ is the Fourier-Bohr coefficients of φ_{μ}. It follows from the uniqueness theorem that if φ_{μ} is a nonzero almost periodic pseudomeasure, then the corresponding measure μ has a nontrivial discrete part.

Next, we have the following.

THEOREM 3.5. *Let T be a power bounded operator on a Hilbert space H which has no unitary eigenvalues. Assume that there exists a vector $x \in H$ such that*

(i) $\inf_{n \geqslant 0} \|T^n x\| > 0$,

(ii) $\sigma_T(x) \cap \Gamma$ *is countable.*

Then, there exists a nonzero vector $y \in H$ such that

$$\lim_{n \to \infty} \|T^n y\| = 0.$$

Proof. Let L be the closed linear span of $\{T^n x : n \geqslant 0\}$ and let (K, J, V) be the limit isometry associated with T_L. As in the proof of Theorem 3.2, we can see that V is unitary and

$$\sigma(V) \subset \sigma_T(x) \cap \Gamma.$$

Consequently, $\sigma(V)$ is countable.

Let $E(\cdot)$ be the spectral measure of V and let μ_{Jx} be the scalar measure defined on the Borel subsets of Γ by

$$\mu_{Jx}(\Delta) = \langle E(\Delta) Jx, Jx \rangle = \|E(\Delta) Jx\|^2.$$

We have

$$\langle V^n Jx, Jx\rangle = \widehat{\mu_{Jx}}(n) \quad (n \in \mathbb{Z})$$

and $\operatorname{supp}\mu_{Jx} \subset \sigma(V)$. Consequently, $\operatorname{supp}\mu_{Jx}$ is countable. By Loomis theorem,

$$\varphi_{\mu_{Jx}} := \left\{\widehat{\mu_{Jx}}(n)\right\}_{n\in\mathbb{Z}}$$

is an almost periodic pseudomeasure and

$$\widehat{\mu_{Jx}}(0) = \|Jx\|^2 = \lim_{n\to\infty} \|T^n x\|^2 > 0.$$

It follows that the measure μ_{Jx} has a nontrivial discrete part. Therefore, $\mu_{Jx}\{\xi_0\} \neq 0$ for some $\xi_0 \in \Gamma$. Consequently, we have $E\{\xi_0\}Jx \neq 0$.

Let us show that $E\{\xi_0\}Jx = Ju$ for some $u \in L$. For this purpose, consider the function

$$f(z) := \frac{1+\overline{\xi_0}z}{2}.$$

Then, $f(\xi_0) = 1$ and $|f(z)| < 1$ for all $z \in \overline{D}\setminus\{\xi_0\}$. We claim that the operator

$$f(T) := \frac{1+\overline{\xi_0}T}{2}$$

is power bounded. Indeed, we have

$$\begin{aligned}\|f(T)^n\| &= \frac{1}{2^n}\left\|\left(1+\overline{\xi_0}T\right)^n\right\| = \frac{1}{2^n}\left\|\sum_{k=0}^{n}\binom{n}{k}\overline{\xi_0}^k T^k\right\| \\ &\leqslant \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left\|T^k\right\| \leqslant \sup_k \left\|T^k\right\|.\end{aligned}$$

Taking a subsequence if necessary we can assume that $\{f(T)^n x\}_{n\in\mathbb{N}}$ is weakly convergent to some $u \in L$. It follows that $Jf(T)^n x \to Ju$ weakly. Let arbitrary $v \in L$ be given. In view of Lemma 2.3 (a), we can write

$$f(V)^n Jx = Jf(T)^n x \ (n \in \mathbb{N}).$$

Consequently, we have

$$\begin{aligned}\langle Ju, v\rangle &= \lim_{n\to\infty}\langle Jf(T)^n x, v\rangle \\ &= \lim_{n\to\infty}\langle f(V)^n Jx, v\rangle = \lim_{n\to\infty}\int_{\Gamma} f^n(\xi)\, d\langle E(\xi)Jx, v\rangle \\ &= \langle E(\xi_0)Jx, v\rangle + \lim_{n\to\infty}\int_{\Gamma\setminus\{\xi_0\}} f^n(\xi)\, d\langle E(\xi)Jx, v\rangle \\ &= \langle E(\xi_0)Jx, v\rangle.\end{aligned}$$

Thus, we obtain that $E\{\xi_0\}Jx = Ju$. As $E\{\xi_0\}Jx \neq 0$, we have $u \neq 0$.

Notice that $E\{\xi_0\}Jx$ is an eigenvector of V corresponding to the eigenvalue ξ_0. Therefore, Ju is an eigenvector of V corresponding to the eigenvalue ξ_0;

$$VJu = \xi_0 Ju.$$

Since $VJu = JTu$, we have $JTu = \xi_0 Ju$. By Lemma 2.3 (b), this means that

$$\lim_{n\to\infty} \|T^n (Tu - \xi_0 u)\| = 0.$$

Let $y := Tu - \xi_0 u$. Since T has no unitary eigenvalues, we have that $y \neq 0$ and $\lim_{n\to\infty} \|T^n y\| = 0$. $\square$

Recall that the subspace E of X is *hyperinvariant* for $T \in B(X)$ if $SE \subset E$ for every $S \in B(X)$ which commutes with T.

COROLLARY 3.6. *Let T be a power bounded operator on a Hilbert space H which is not a multiple of the identity. Assume that there exists $x \in H$ such that:*
(i) $\inf_{n\geqslant 0} \|T^n x\| > 0$;
(ii) $\sigma_T(x) \cap \Gamma$ *is countable.*
Then, T has a nontrivial hyperinvariant subspace.

4. Banach space operators

In this section, we present local version of a theorem of Gelfand [6] on doubly power bounded operators, and another of Katznelson and Tzafriri [8] on power bounded operators ones.

An invertible operator T on a Banach space is called *doubly power bounded* if

$$\sup_{n\in\mathbb{Z}} \|T^n\| < \infty.$$

Now, let T be a doubly power bounded operator on a Banach space X. Then, $\sigma(T) \subset \Gamma$ and therefore T has the SVEP. For a given $f \in \mathscr{A}$, we can define $f(T) \in B(X)$ by

$$f(T) = \sum_{n\in\mathbb{Z}} \widehat{f}(n) T^n.$$

Then, $h : f \to f(T)$ is a continuous algebra homomorphism with the norm

$$\|h\| = \sup_{n\in\mathbb{Z}} \|T^n\|.$$

It is easy to check that $\sigma(T) = \mathrm{hull}(\ker T)$.

Recall that the *Carleman transform* $\Phi(z)$ of a pseudomeasure $\varphi = \{\widehat{\varphi}(n)\}_{n\in\mathbb{Z}}$ is defined by the relation

$$\Phi(z) = \begin{cases} \sum_{n=0}^{\infty} \dfrac{\widehat{\varphi}(n)}{z^n}, & |z| > 1; \\ -\sum_{n=1}^{\infty} \widehat{\varphi}(-n) z^n, & |z| < 1. \end{cases}$$

We know [4, Chapter 3] that $\Phi(z)$ is a function analytic on $\mathbb{C}\setminus \operatorname{supp}\varphi$.

For a given $\varphi \in X^*$ and $x \in X$, let φ_x be a pseudomeasure defined by

$$\langle \varphi_x, f\rangle = \langle \varphi, f(T)x\rangle,\ f \in \mathscr{A}.$$

Since $\widehat{\varphi_x}(n) = \varphi(T^n x)\ \ (n \in \mathbb{Z})$, from the identity

$$R_z(T)x = \begin{cases} \sum\limits_{n=0}^{\infty} \dfrac{T^n x}{z^{n+1}}, & |z| > 1; \\ -\sum\limits_{n=1}^{\infty} z^{n-1} T^{-n} x, & |z| < 1, \end{cases}$$

we have

$$z\langle \varphi, R_z(T)x\rangle = \begin{cases} \sum\limits_{n=0}^{\infty} \dfrac{\widehat{\varphi_x}(n)}{z^n}, & |z| > 1; \\ -\sum\limits_{n=1}^{\infty} z^n \widehat{\varphi_x}(-n), & |z| < 1. \end{cases}$$

This shows that $z\langle \varphi, R_z(T)x\rangle\ \ (|z| \neq 1)$ is the Carleman transform of φ_x. It follows that

$$\sigma_T(x) = \overline{\bigcup_{\varphi \in X^*} \operatorname{supp}\varphi_x},$$

for every $x \in X$.

If $x \in X$, then

$$I_x := \{f \in \mathscr{A} : f(T)x = 0\}$$

is a closed ideal of $\mathscr{A}$ and

$$I_x = \bigcap_{\varphi \in X^*} I_{\varphi_x}.$$

Recall that

$$I_{\varphi_x} = \{f \in \mathscr{A} : \varphi_x \cdot f = 0\}.$$

Since

$$\operatorname{hull}\left(I_{\varphi_x}\right) = \operatorname{supp}\varphi_x,$$

it follows from the general theory of Banach algebras that

$$\operatorname{hull}(I_x) = \overline{\bigcup_{\varphi \in X^*} \operatorname{hull}\left(I_{\varphi_x}\right)} = \overline{\bigcup_{\varphi \in X^*} \operatorname{supp}\varphi_x} = \sigma_T(x).$$

Hence, we have the following.

PROPOSITION 4.1. *If T is a doubly power bounded operator on a Banach space X, then for every $x \in X$, we have*

$$\sigma_T(x) = hull(I_x).$$

From the preceding proposition, it easily follows that for every $f \in \mathscr{A}$ and $x \in X$, the following relations hold:

$$\sigma_T(f(T)x) \subset \sigma_T(x) \cap \operatorname{supp} f, \tag{4.1}$$

$$\sigma_T(x) \cap \{\xi \in \Gamma : f(\xi) \neq 0\} \subset \sigma_T(f(T)x). \tag{4.2}$$

Let T be an invertible operator on X. Recall that $x \in X$ is a *doubly cyclic vector* of T if

$$\overline{\operatorname{span}}\{T^n x : n \in \mathbb{Z}\} = X.$$

COROLLARY 4.2. *Let T be a doubly power bounded operator on a Banach space X. If $x \in X$ is a doubly cyclic vector of T, then*

$$\sigma_T(x) = \sigma(T).$$

REMARK 4.3. An invertible operator T on X is called *nonquasianalytic* [3, Chapter XII] if

$$\sum_{n \in \mathbb{Z}} \frac{\log \|T^n\|}{1+n^2} < \infty.$$

The assertion of the preceding proposition remains valid for nonquasianalytic operators, too.

Given a closed subset S of Γ, there are two distinguished closed ideals of $\mathscr{A}$ with hull equal to S, namely

$$J_S = \overline{\{f \in \mathscr{A} : \operatorname{supp} f \cap S = \emptyset\}}$$

and

$$I_S = \{f \in \mathscr{A} : f(\xi) = 0, \ \forall \xi \in S\}.$$

The set S is called a *set of synthesis* if $J_S = I_S$ ([10, Chapter 8]).

Well-known Gelfand's theorem [6] states that if T is a doubly power bounded operator with $\sigma(T) = \{1\}$, then $T = I$.

We include here the following result which seems to be unnoticed.

PROPOSITION 4.4. *Let T be a doubly power bounded operator on a Banach space X and let $x \in X$. If $\sigma_T(x) = \{\xi_1, ..., \xi_n\}$ $(\xi_i \neq \xi_j,\ i \neq j)$, then*

$$x \in \ker(T - \xi_1 I) \oplus \cdots \oplus \ker(T - \xi_n I).$$

Proof. Let $U_1, ..., U_n$ be a disjoint neighborhoods of $\xi_1, ..., \xi_n$, respectively. Let V_k be a neighborhood of ξ_k such that $\overline{V_k} \subset U_k$ $(k = 1, ..., n)$. Then, there exist functions $f_1, ..., f_n$ in $\mathscr{A}$ such that $f_k = 1$ on V_k and $f_k = 0$ outside U_k $(k = 1, ..., n)$. Put $f = f_1 + ... + f_n$. Since $1 - f$ vanishes in a neighborhood of $\sigma_T(x)$, the function $1 - f$ belongs to the smallest ideal of $\mathscr{A}$ whose hull is $\sigma_T(x)$. It follows from Proposition 4.1

that $1-f\in I_x$, so that $f(T)x=x$. Hence, we have $x=x_1+\ldots+x_n$, where $x_k=f_k(T)x$ $(k=1,\ldots,n)$. Further, it follows from the relations (4.1) and (4.2) that

$$\{\xi_k\}\subset\sigma_T(x_k)\subset\sigma_T(x)\cap\operatorname{supp}f_k=\{\xi_k\}.$$

Hence, we obtain $\sigma_T(x_k)=\{\xi_k\}$. It remains to show that if $y\in X$ with $\sigma_T(y)=\{\xi\}$, then $Ty=\xi y$. By Proposition 4.1, hull$(I_y)=\{\xi\}$. Since $\{\xi\}$ is a set of synthesis [10, Chapter 8], we have $I_y=I_{\{\xi\}}$, so that

$$\{f\in\mathscr{A}:f(T)y=0\}=\{f\in\mathscr{A}:f(\xi)=0\}.$$

If we put in the last identity $f=\zeta-\xi\ \ (\zeta\in\Gamma)$, then we have $Ty=\xi y$. □

REMARK 4.5. Let T be an invertible operator on a Banach space. Assume that there exists $0\leqslant\alpha<1$ such that

$$\|T^n\|\leqslant\text{const}(1+|n|)^{\alpha},\ \forall n\in\mathbb{Z}.$$

In this case, the assertion of the preceding proposition remains valid.

We denote by $\mathscr{A}_+$ the set of all functions

$$f(z)=\sum_{n=0}^{\infty}\widehat{f}(n)z^n$$

analytic on D and satisfying

$$\|f\|_1:=\sum_{n=0}^{\infty}\left|\widehat{f}(n)\right|<\infty.$$

(whence f is a continuous function on $\overline{D}$). $\mathscr{A}_+$ is a commutative Banach algebra under this norm. Let $\varphi\in\mathscr{A}_+^*$ and $\widehat{\varphi}(n):=\langle\varphi,z^n\rangle$ $(n\geqslant 0)$. If $f\in\mathscr{A}_+$, then the duality being implemented by the formula

$$\langle\varphi,f\rangle=\sum_{n=0}^{\infty}\widehat{\varphi}(n)\widehat{f}(n).$$

If T is a power bounded operator on a Banach space X, then for a given $f\in\mathscr{A}_+$, we can define $f(T)\in B(X)$ by

$$f(T)=\sum_{n=0}^{\infty}\widehat{f}(n)T^n.$$

Then, $h:f\to f(T)$ is a continuous algebra homomorphism with the norm

$$\|h\|=\sup_{n\geqslant 0}\|T^n\|.$$

It follows that if f is a power bounded element of $\mathscr{A}_+$ (in particular, if $\|f\|_1 \leqslant 1$), then $f(T)$ is power bounded. Standard Banach algebra techniques shows that the spectral mapping property $\sigma(f(T)) = f(\sigma(T))$ $(f \in \mathscr{A}_+)$ holds.

If $x \in X$, then

$$I_x^+ := \{f \in \mathscr{A}_+ : f(T)x = 0\}$$

is a closed ideal of $\mathscr{A}_+$.

We have the following.

PROPOSITION 4.6. *If T is a power bounded operator on a Banach space X, then for every $x \in X$, we have*

$$\sigma_T(x) \subset hull\left(I_x^+\right).$$

For the proof, we need some preliminary results. For a given $\varphi \in \mathscr{A}_+^*$ and $f \in \mathscr{A}_+$, define

$$\varphi^+(z) := \sum_{n=0}^{\infty} \frac{\widehat{\varphi}(n)}{z^n} \ (|z| > 1), \tag{4.3}$$

$$\widehat{\varphi}(-n) := \sum_{k=0}^{\infty} \widehat{\varphi}(k) \widehat{f}(k+n) \ (n = 1, 2 \ldots),$$

and

$$\psi(z) := \sum_{n=1}^{\infty} \widehat{\varphi}(-n) z^n \ (|z| < 1). \tag{4.4}$$

The following result is contained in [18, Chapter 4, Theorem 10].

LEMMA 4.7. *Let $\varphi \in \mathscr{A}_+^*$ and $f \in \mathscr{A}_+$. Assume that the functions $\varphi^+(z)$ and $\psi(z)$ are defined as in (4.3) and (4.4), respectively. If*

$$\sum_{k=0}^{\infty} \widehat{\varphi}(k+n) \widehat{f}(k) = 0 \ (\forall n \geqslant 0),$$

then

$$\Phi(z) := \begin{cases} \varphi^+(z), & |z| > 1; \\ \frac{\psi(z)}{f(z)}, & |z| < 1 \end{cases}$$

is an analytic function on the complex plane possible expectation of zero set of f.

Proof of Proposition 4.6. Assume that $\lambda \in \overline{D} \backslash \mathrm{hull}(I_x^+)$. Then, there exists a function $f \in \mathscr{A}_+$ such that $f(T)x = 0$ but $f(\lambda) \neq 0$. For a given $\varphi \in X^*$, define $\varphi_x \in \mathscr{A}_+^*$ by

$$\langle \varphi_x, f \rangle = \langle \varphi, f(T)x \rangle, \ f \in \mathscr{A}_+.$$

Since $\widehat{\varphi_x}(n) = \varphi(T^n x)$ and

$$R_z(T)x = \sum_{n=0}^{\infty} \frac{T^n x}{z^{n+1}} \ (|z| > 1),$$

we have

$$\varphi_x^+(z) = \sum_{n=0}^{\infty} \frac{\widehat{\varphi_x}(n)}{z^n} = \sum_{n=0}^{\infty} \frac{\varphi(T^n x)}{z^n} = z\langle \varphi, R_z(T)x\rangle \ \ (|z|>1).$$

On the other hand, as $f(T)x=0,$ we have $f(T)T^k x=0 \ (k\geqslant 0)$ which implies

$$0 = \sum_{n=0}^{\infty} \widehat{f}(n)\,\varphi\left(T^{n+k}x\right) = \sum_{n=0}^{\infty} \widehat{f}(n)\,\widehat{\varphi_x}(n+k).$$

By the preceding lemma, the function $z\mapsto \langle \varphi, R_z(T)x\rangle$ can be analytically extended to a neighborhood of λ for every $\varphi\in X^*$. It follows that $\lambda\in\rho_T(x)$. □

Katznelson and Tzafriri [8] obtained the following generalization of Gelfand's theorem. If T is a power bounded operator on a Banach space, then

$$\lim_{n\to\infty}\left\|T^{n+1}-T^n\right\| = 0$$

if and only if $\sigma(T)\cap\Gamma\subset\{1\}$.

We denote by $\mathscr{A}_+^1$ the set of all $f\in\mathscr{A}_+$ such that $\|f\|_1\leqslant 1,$ $f(1)=1,$ and $|f(z)|<1$ for all $z\in\overline{D}\setminus\{1\}$. For example, if $\{a_n\}_{n=0}^{\infty}$ is a sequence such that $0< a_n<1$ $(n=0,1,\ldots)$ and $\sum_{n=0}^{\infty}a_n=1,$ then the function $f(z)=\sum_{n=0}^{\infty}a_n z^n$ is in $\mathscr{A}_+^1$. Notice that if $f\in\mathscr{A}_+^1,$ then $f(T)$ is power bounded and by the spectral mapping property, $\sigma(f(T))\cap\Gamma\subset\{1\}$. Consequently, for every $f\in\mathscr{A}_+,$ we have that

$$\lim_{n\to\infty}\left\|f(T)^{n+1}-f(T)^n\right\| = 0.$$

Below, we present local quantitative version of Katznelson-Tzafriri theorem (see also [1]).

An entire function f is said to be of *order* ρ if

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M(r)}{\log r},$$

where $M(r)=\sup\{|f(z)| : |z|\leqslant r\}$. An entire function of finite order ρ is said to be of *type* σ if

$$\sigma = \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^\rho}.$$

If the entire function f is of order less than 1 or f is of order 1 and type less than or equal to $\sigma,$ we say f is of *exponential type* σ [5, p. 8].

For a given $\sigma>0,$ we denote by B_σ the set of all bounded on the real line entire functions f of exponential type $\leqslant\sigma,$ i.e., for every $\varepsilon>0,$ there exists a constant $C_\varepsilon>0$ such that

$$|f(z)|\leqslant C_\varepsilon e^{(\sigma+\varepsilon)|z|},\ \forall z\in\mathbb{C}.$$

It follows from the Phragmen-Lindelöf theorem that if $f\in B_\sigma$ and

$$C_f := \sup_{t\in\mathbb{R}}|f(t)|,$$

then

$$|f(z)| \leqslant C_f e^{\sigma|\operatorname{Im} z|}.$$

Notice that B_σ is a Banach space under the norm given by

$$\|f\|_\sigma := \sup_{z\in\mathbb{C}} \left[e^{-\sigma|\operatorname{Im} z|}|f(z)|\right].$$

In fact,

$$\|f\|_\sigma = \sup_{t\in\mathbb{R}} |f(t)|.$$

The following inequality of Bernstein type is well known [7]. If $f \in B_\sigma$, where $0 \leqslant \sigma h \leqslant \frac{\pi}{2}$, then

$$\sup_{t\in\mathbb{R}} |f(t+h) - f(t-h)| \leqslant 2\sin\sigma h \|f\|_\sigma.$$

It follows that for every $f \in B_\sigma$,

$$|f(1) - f(0)| \leqslant 2\sin\frac{\sigma}{2}\|f\|_\sigma \ \ (\sigma \leqslant \pi),$$

$$|f(1) - f(-1)| \leqslant 2\sin\sigma\|f\|_\sigma \ \ \left(\sigma \leqslant \frac{\pi}{2}\right).$$

On the other hand, by Cartwright theorem (see, [5, Chapter 10] and [7]), the inequality

$$\|f\|_\sigma \leqslant \sec\frac{\sigma}{2} \sup_{n\in\mathbb{Z}} |f(n)|$$

holds for every $f \in B_\sigma$ $(\sigma < \pi)$. So, we have

$$|f(1) - f(0)| \leqslant 2\tan\frac{\sigma}{2}\left(\sup_{n\in\mathbb{Z}}|f(n)|\right), \ \forall f \in B_\sigma \ (\sigma < \pi), \tag{4.5}$$

$$|f(1) - f(-1)| \leqslant 2\sin\frac{\sigma}{2}\left(\sup_{n\in\mathbb{Z}}|f(n)|\right), \ \forall f \in B_\sigma \ \left(\sigma \leqslant \frac{\pi}{2}\right). \tag{4.6}$$

Let V be an invertible isometry on a Banach space X. Notice that if $\sigma(V) = \Gamma$, then $\|V - I\| = 2$. Now, assume that $\sigma(V)$ is contained in the arc

$$\Lambda_\sigma := \left\{e^{i\theta} \in \Gamma : |\theta| \leqslant \sigma\right\},$$

where $0 \leqslant \sigma < \pi$ (any proper closed subset of Γ can be rotated so as to lie inside some such Λ_σ). Then $V = e^{iS}$ for some $S \in B(X)$, where $\sigma(S) \subseteq [-\sigma, \sigma]$. For a given $\varphi \in B(X)^*$ with norm one, consider the entire function $f(z) := \varphi\left(e^{izS}\right)$. Since $\left\|e^{inS}\right\| = 1$ for all $n \in \mathbb{Z}$, we have $|f(t)| \leqslant e^{\|S\|}$ for all $t \in \mathbb{R}$. On the other hand, the inequality

$$|f(z)| \leqslant e^{|z|\|S\|}$$

gives us that the order of f is less than or equal to 1. Notice also that the nth derivative of f at zero is $\varphi(i^n S^n)$. Thus, by Levin's theorem [12, p. 84], the type of f is less than or equal to

$$\lim_{n\to\infty} \|S^n\|^{\frac{1}{n}}.$$

On the other hand, the last expression is less than or equal to σ. Consequently, $f \in B_\sigma$. Now, applying the inequalities (4.5) and (4.6) to f, we obtain the following inequalities

$$\|V - I\| \leqslant 2\tan\frac{\sigma}{2} \ (\sigma < \pi), \tag{4.7}$$

$$\left\|V^2 - I\right\| = \left\|V - V^{-1}\right\| \leqslant 2\sin\frac{\sigma}{2} \left(\sigma \leqslant \frac{\pi}{2}\right). \tag{4.8}$$

PROPOSITION 4.8. *Let T be a contraction on a Banach space X and let $x \in X$.*
(a) If $\sigma_T(x) \cap \Gamma \subset \Lambda_\sigma$ $(\sigma < \pi)$, then

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| \leqslant 2\tan\frac{\sigma}{2}\|x\|.$$

(b) If $\sigma_T(x) \cap \Gamma \subset \Lambda_\sigma$ $\left(\sigma \leqslant \frac{\pi}{2}\right)$, then

$$\lim_{n\to\infty} \left\|T^{n+2}x - T^n x\right\| \leqslant 2\sin\frac{\sigma}{2}\|x\|.$$

Proof. Let L be the closed linear span of $\{T^n x : n \geqslant 0\}$ and let (Y, J, V) be the limit isometry associated with T_L. As in the proof of Theorem 3.2, we can see that

$$\sigma(V) \cap \Gamma \subset \sigma_T(x) \cap \Gamma \subset \Lambda_\sigma.$$

Hence, V is an invertible isometry and $\sigma(V) \subset \Lambda_\sigma$. Now, from the identities

$$(V - I)Jx = J(Tx - x), \ \left(V^2 - I\right)Jx = J\left(T^2x - x\right)$$

and from the inequalities (4.7) and (4.8), we can write

$$\begin{aligned}\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| &= \|J(Tx - x)\| = \|(V - I)Jx\| \\ &\leqslant \|V - I\|\,\|x\| \leqslant 2\tan\frac{\sigma}{2}\|x\|,\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty} \left\|T^{n+2}x - T^n x\right\| &= \left\|J\left(T^2x - x\right)\right\| = \left\|\left(V^2 - I\right)Jx\right\| \\ &\leqslant \left\|V^2 - I\right\|\,\|x\| \leqslant 2\sin\frac{\sigma}{2}\|x\|. \quad \square\end{aligned}$$

It follows from the preceding proposition that if T is power bounded and if $x \in X$ with $\sigma_T(x) \cap \Gamma \subset \{1\}$, then

$$\lim_{n\to\infty} \left\|T^{n+1}x - T^n x\right\| = 0.$$

Note that the converse of this fact is not true in general. To see this, let S be the forward shift on the Hardy space H^2. As $\lim_{n\to\infty}\|S^{*n}f\|=0$, we have

$$\lim_{n\to\infty}\left\|S^{*n+1}f-S^{*n}f\right\|=0,\ \forall f\in H^2.$$

Let μ be a positive singular measure on Γ such that $\operatorname{supp}\mu\nsubseteq\{1\}$. Consider the inner function

$$f(z)=\exp\left(-\int_{\Gamma}\frac{\zeta+z}{\zeta-z}d\mu_{\zeta}\right).$$

We know (see, [16, Theorem III.5.1]) that $\operatorname{supp}\mu$ consists of all $\xi\in\Gamma$ for which the function f has no analytic extension to a neighborhood of ξ. Now, as $\sigma_{S^*}(f)=\operatorname{supp}\mu$, we have $\sigma_{S^*}(f)\cap\Gamma\nsubseteq\{1\}$.

PROPOSITION 4.9. *Let T be a power bounded operator on a Banach space X and let $x\in X$. Assume that*

$$\text{l.i.m.}_n\left\|T^{n+1}x-T^nx\right\|=0.$$

If

$$\frac{Tx+\ldots+T^nx}{n}\to 0 \text{ weakly as } n\to\infty,$$

then

$$\lim_{n\to\infty}\|T^nx\|=0.$$

Proof. Let L be the closed linear span of $\{T^nx:n\geqslant 0\}$ and let (Y,J,V) be the limit isometry associated with T_L. From the identity

$$VJx-Jx=J(Tx-x),$$

we have

$$\|VJx-Jx\|=\text{l.i.m.}_n\left\|T^{n+1}x-T^nx\right\|=0,$$

so that $VJx=Jx$. Since Jx is a cyclic vector of V, we have $V=I$. From the identities $Jx=JT^nx\ \ (n\in\mathbb{N})$, we can write

$$Jx=J\frac{Tx+\ldots+T^nx}{n}.$$

Let $y^*\in Y^*$ be given. Then, we have

$$\langle y^*,Jx\rangle=\left\langle J^*y^*,\frac{Tx+\ldots+T^nx}{n}\right\rangle\to 0.$$

Hence, $Jx=0$. This means that $\lim_{n\to\infty}\|T^nx\|=0$. $\square$

REMARK 4.10. If T is a power-bounded operator on X and if $x \in X$, then

$$\frac{1}{n}\sum_{k=1}^{\infty} T^k x \to 0$$

weakly $(n \to \infty)$, implies that $x \in \overline{\operatorname{Ran}(T-I)}$. Consequently, $\frac{1}{n}\sum_{k=1}^{\infty} T^k x \to 0$ strongly as $n \to \infty$.

5. Ergodic conditions

In this section, for the stability of T at $x \in X$, some ergodic spectral conditions are found on T and on x.

The C_0-semigroup version of the following theorem was proved in [17, Theorem 5.1.11].

THEOREM 5.1. *Let T be a power bounded operator on a Banach X and let $x \in X$. Assume that*

(i) $\sigma_T(x) \cap \Gamma$ is countable,

(ii) $\frac{1}{n}\sum_{k=1}^{n} \xi^{-k} T^k x \to 0$ weakly $(n \to \infty)$, $\forall \xi \in \sigma_T(x) \cap \Gamma$.

Then,

$$\lim_{n\to\infty} \|T^n x\| = 0.$$

For the proof of Theorem 5.1 we need the following lemma.

LEMMA 5.2. *Let V be an invertible isometry on a Banach space X and let $x \in X$. Assume that*

(i) $\sigma_V(x)$ is countable,

(ii) $\frac{1}{n}\sum_{k=1}^{n} \xi^{-k} V^k x \to 0$ weakly $(n \to \infty)$, $\forall \xi \in \sigma_V(x)$.

Then, $x = 0$.

Proof. Let $\varphi \in X^*$ and let $\left(H_\varphi, J_\varphi, U_\varphi\right)$ be the unitary operator associated with the pair (V, φ). By (2.1), we have $\sigma_{U_\varphi}\left(J_\varphi x\right) \subset \sigma_V(x)$ and consequently, $\sigma_{U_\varphi}\left(J_\varphi x\right)$ is countable. In view of Lemma 2.5 (a), we can write

$$\langle U_\varphi^k J_\varphi x, J_\varphi x\rangle = \langle J_\varphi V^k x, J_\varphi x\rangle = \langle V^k x, J_\varphi^* J_\varphi x\rangle \ (k \in \mathbb{N}).$$

It follows that for every $\xi \in \sigma_{U_\varphi}\left(J_\varphi x\right)$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} \xi^{-k}\langle U_\varphi^k J_\varphi x, J_\varphi x\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} \xi^{-k}\langle V^k x, J_\varphi^* J_\varphi x\rangle = 0.$$

By Lemma 2.5 (c), it suffices to show that $J_\varphi x = 0$.

To simplify the notation, we put $U := U_\varphi$ and $y := J_\varphi x$. Let $E(\cdot)$ be the spectral measure of U and let μ_y be the scalar measure defined on the Borel subsets of Γ by

$$\mu_y(\Delta) = \langle E(\Delta)y, y\rangle = \|E(\Delta)y\|^2 .$$

Then, for every $\xi \in \operatorname{supp}\mu_y = \sigma_U(y)$, we can write

$$0 = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \xi^{-k}\langle U^k y, y\rangle = \lim_{n\to\infty}\int_\Gamma \left(\frac{1}{n}\sum_{k=1}^{n}\xi^{-k}\zeta^k\right) d\mu_y(\zeta) = \mu_y\{\xi\}.$$

This shows that μ_y is a continuous measure. As is well known, there is no nonzero continuous measure supported by countable set. Consequently, $\mu_y = 0$. This clearly implies that $y = 0$. □

Proof of Theorem 5.1. Let L be the closed linear span of $\{T^n x : n \geqslant 0\}$ and let (Y, J, V) be the limit isometry associated with T_L. As in the proof of Theorem 3.2, we have

$$\sigma(V)\cap\Gamma \subset \sigma_T(x)\cap\Gamma.$$

Consequently, V is an invertible isometry, $\sigma(V) \subset \sigma_T(x)\cap\Gamma$, and $\sigma(V)$ is countable. Since

$$\langle V^k Jx, Jx\rangle = \langle J T_L^k x, Jx\rangle = \langle T^k x, J^* Jx\rangle \ (k\in\mathbb{N}),$$

we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\xi^{-k}\langle V^k Jx, Jx\rangle = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\xi^{-k}\langle T^k x, J^* Jx\rangle = 0,.$$

for every $\xi \in \sigma(V)$. It follows from the preceding lemma that $Jx = 0$. Hence, $\lim_{n\to\infty}\|T^n x\| = 0$. □

Let T be a power bounded operator on a Banach space X and let $x \in X$. Assume that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\langle\varphi, T^k x\rangle\right| = 0,\ \forall\varphi\in X^*.$$

It follows that $x \in \overline{(\xi - T)X}$, for every $\xi\in\Gamma$. Consequently, we have

$$\lim_{n\to\infty}\frac{1}{n}\left\|\sum_{k=1}^{n}\xi^{-k}T^k x\right\| = 0,\ \forall\xi\in\Gamma.$$

Hence, we have the following.

COROLLARY 5.3. *Let T be a power bounded operator on a Banach space X and let $x \in X$. Assume that*

(*i*) $\sigma_T(x)\cap\Gamma$ *is at most countable,*

(*ii*) $\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\langle\varphi, T^k x\rangle\right| = 0,\ \forall\varphi\in X^*.$

Then, $\lim_{n\to\infty}\|T^n x\| = 0.$

REMARK 5.4. Note that the condition (ii) in the preceding corollary can be replaced by the condition

$$\exists \alpha > 0,\ \forall \varphi \in X^*,\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left|\langle \varphi, T^k x\rangle\right|^{\alpha} = 0.$$

For this, it is enough to show that the above condition implies the following.

$$\forall \beta > 0,\ \forall \varphi \in X^*,\ \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \left|\langle \varphi, T^k x\rangle\right|^{\beta} = 0.$$

This follows from the following simple fact. If $\{a_n\}$ is a bounded positive sequence and if $\frac{a_1^\alpha+\ldots+a_n^\alpha}{n} \to 0$, for some $\alpha > 0$, then $\frac{a_1^\beta+\ldots+a_n^\beta}{n} \to 0$, for every $\beta > 0$. To see this, assume on the contrary that $\frac{a_1^\beta+\ldots+a_n^\beta}{n} \nrightarrow 0$. Then, $\frac{a_1^\beta+\ldots+a_{n_i}^\beta}{n_i} \geqslant \delta > 0$ for some subsequence $\{n_i\}$. As the sequence $\{a_{n_i}\}$ is bounded, $a_{n_{i_j}} \to a$ for some subsequence $\{n_{i_j}\}$. Since $a_{n_{i_j}}^\alpha \to a^\alpha$, we have $\frac{a_1^\alpha+\ldots+a_{n_{i_j}}^\alpha}{n_{i_j}} \to a^\alpha$, so that $a=0$. Thus we have $a_{n_{i_j}} \to 0$ and so $a_{n_{i_j}}^\beta \to 0$. Consequently, $\frac{a_1^\beta+\ldots+a_{n_{i_j}}^\beta}{n_{i_j}} \to 0$. This is a contradiction.

Below, we present some applications of Theorem 5.1.

If $T \in B(X)$, we let A_T denote the closure in the uniform operator topology of all polynomials in T. Note that A_T is a commutative unital Banach algebra. The Gelfand space of A_T can be identified with $\sigma_{A_T}(T)$, the spectrum of T with respect to the algebra A_T. It follows from the Shilov's Theorem [10, Theorem 2.3.1] that if T is power bounded, then $\sigma_{A_T}(T)\cap\Gamma = \sigma(T)\cap\Gamma$. Since $\sigma(T)$ is a (closed) subset of $\sigma_{A_T}(T)$, for every $\lambda \in \sigma(T)$, there exists a multiplicative functional ϕ_λ on A_T such that $\phi_\lambda(T) = \lambda$. By $\widehat{S}$, we will denote the Gelfand transform of $S \in A_T$. Here, instead of $\widehat{S}(\phi_\lambda)(=\phi_\lambda(S))$, where $\lambda \in \sigma(T)$, we will use the notation $\widehat{S}(\lambda)$. Notice that $\lambda \mapsto \widehat{S}(\lambda)$ is a continuous function on $\sigma(T)$.

Let T be a power bounded operator on a Hilbert space H and let $Q \in \{T\}'$, the commutant of T. In [11], it was proved that if

$$\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{k=1}^{n} \xi^{-k} T^k Q\right\| = 0$$

holds for every $\xi \in \sigma(T)\cap\Gamma$, then $\lim_{n\to\infty}\|T^n Q\| = 0$.

For a given $T \in B(X)$, we denote by L_T, the left multiplication operator on $B(X)$; $L_T Q = TQ$. We know that $\sigma(L_T) = \sigma(T)$. Now, applying Theorem 5.1 to the operator L_T on the space $B(X)$, we have the following.

COROLLARY 5.5. *Let T be a power bounded operator on a Banach space X with countable unitary spectrum. Then, the following statements are equivalent for $Q \in B(X)$.*

(i) $\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{k=1}^{n} \xi^{-k} T^{k} Q\right\| = 0,\ \forall \xi \in \sigma(T) \cap \Gamma.$

(ii) $\lim_{n\to\infty} \|T^n Q\| = 0.$

COROLLARY 5.6. *Let T be a power bounded operator on a Banach space X with countable unitary spectrum. The following statements are equivalent for compact operator K on X.*

(i) $\frac{1}{n}\sum_{k=1}^{n} \xi^{-k} T^{k} K \to 0$ $(n\to\infty)$ *in the weak operator topology,* $\forall \xi \in \sigma(T) \cap \Gamma.$

(ii) $\lim_{n\to\infty} \|T^n K\| = 0.$

Proof. For every $x \in X$ and $\xi \in \sigma(T) \cap \Gamma$, we have

$$\frac{1}{n}\sum_{k=1}^{n} \xi^{-k} T^{k} K x \to 0 \text{ weakly},$$

By Theorem 5.1,

$$\lim_{n\to\infty} \|T^n K x\| = 0,\ \forall x \in X.$$

Since the set $\{Kx : \|x\| \leqslant 1\}$ is relatively compact, for a given $\varepsilon > 0$, it has a finite ε-mesh, say $\{Kx_1, ..., Kx_m\}$, where $\|x_i\| \leqslant 1$ $(i = 1, ..., m)$. So, we have

$$\|T^n K\| \leqslant \max_{i} \{\|T^n K x_i\|\} + \varepsilon \sup_{n\geqslant 0} \|T^n\|,\ (n \in \mathbb{N}).$$

It follows that $\lim_{n\to\infty} \|T^n K\| = 0.$ $\square$

An operator T acting on a Banach space is called *polynomially bounded* if there exists a constant $C > 0$ such that

$$\|P(T)\| \leqslant C \|P\|_{\infty},$$

for all polynomials P. By the von Neumann inequality, every Hilbert space contraction is polynomially bounded with constant $C = 1$. Notice also that every polynomially bounded operator is power bounded. In [15] it was proved that if T is a polynomially bounded operator with constant C, then for every $Q \in A_T$,

$$\lim_{n\to\infty} \|T^n Q\| \leqslant C \sup_{\xi\in\sigma(T)\cap\Gamma} \left|\widehat{Q}(\xi)\right|. \tag{5.1}$$

We finish the paper with the following.

PROPOSITION 5.7. *If T is a polynomially bounded operator on a Banach space, then the following statements are equivalent for $Q \in A_T$.*

(i) $\lim_{n\to\infty} \frac{1}{n}\left\|\sum_{k=1}^{n} \xi^{-k} T^{k} Q\right\| = 0,\ \forall \xi \in \sigma(T) \cap \Gamma.$

(ii) $\lim_{n\to\infty} \|T^n Q\| = 0.$

Proof. For every $\xi \in \sigma(T) \cap \Gamma$, there exists a multiplicative functional ϕ_ξ on A_T such that $\phi_\xi(T) = \xi$. Then, we have

$$\left|\widehat{Q}(\xi)\right| = \frac{1}{n}\left|\langle \phi_\xi, \sum_{k=1}^{n} \xi^{-k} T^k Q \rangle\right| \leqslant \frac{1}{n}\left\|\sum_{k=1}^{n} \xi^{-k} T^k Q\right\| \to 0 \ (n \to \infty).$$

So, $\widehat{Q}$ vanishes on $\sigma(T) \cap \Gamma$. It follows from (5.1) that $\lim_{n\to\infty} \|T^n Q\| = 0$. □

REFERENCES

[1] G. R. ALLAN AND T. J. RANSFORD, *Power-dominated elements in a Banach algebra*, Studia Math., **94** (1989), 63–79.

[2] C. J. K. BATTY, J. VAN NEERVEN AND F. RÄBIGER, *Local spectra and individual stability of uniformly bounded C_0-semigroups*, Trans. Amer. Math. Soc., **350** (1998), 2071–2085.

[3] B. BEAUZAMY, *Introduction to Operator Theory and Invariant Subspaces*, North-Holland, Amsterdam, 1988.

[4] J. BENEDETTO, *Harmonic Analysis on Totally Disconnected Sets*, Lecture Notes in Math., **202**, Springer-Verlag, 1971.

[5] R. P. BOAS, *Entire Functions*, Academic Press, New York, 1954.

[6] I. M. GELFAND, *Zur theorie der charactere der abelschen topologischen gruppen*, Rec. Math. N. S. (Mat. Sb), **51** (1941), 49–50.

[7] E. A. GORIN, *Bernstein's inequality from the point of view of operator theory*, Selecta Math. Soviet, **7** (1988), 191–209 (transl. from Vestnik Kharkov Univ. **45** (1980), 77–105).

[8] Y. KATZNELSON AND L. TZAFRIRI, *On power bounded operators*, J. Funct. Anal., **68** (1986), 313–328.

[9] L. KÉRCHY AND J. VAN NEERVEN, *Polynomially bounded operators whose spectrum on the unit circle has measure zero*, Acta Sci. Math. (Szeged), **63** (1997), 551–562.

[10] R. LARSEN, *Banach Algebras*, Marcel-Dekker Inc., New York, 1973.

[11] Z. LEKA, *A Katznelson-Tzafriri type theorem in Hilbert spaces*, Proc. Amer. Math. Soc., **137** (2009), 3763–3768.

[12] B. YA. LEVIN, *Distributions of Zeros of Entire Functions*, Amer. Math. Soc. Providence, 1964.

[13] L. H. LOOMIS, *The spectral characterization of a class of almost periodic functions*, Ann. Math., **72** (1960), 362–368.

[14] H. S. MUSTAFAYEV, *The Banach algebra generated by a contraction*, Proc. Amer. Math. Soc., **134** (2006), 2677–2683.

[15] H. S. MUSTAFAYEV, *Asymptotic behavior of polynomially bounded operators*, C. R. Acad. Sci. Paris, Ser. I, **348** (2010), 517–520.

[16] B. SZ.-NAGY AND C. FOIAS, *Harmonic Analysis of Operators on Hilbert Space*, (Russian), Mir, Moscow, 1970.

[17] J. VAN NEERVEN, *The Asymptotic Behavior of Semigroups of Linear Operators*, Oper. Theory Adv. Appl., **88**, Birkhäuser Verlag, Basel, 1996.

[18] B. NYMAN, *On the one-dimensional translation group and semi-group in certain function spaces*, Inaugural Dissertation, Uppsala, 1950.

[19] W. RUDIN, *Fourier Analysis on Groups*, Wiley-Interscience Publication, New-York, 1962.

[20] W. RUDIN, *Trigonometric series with gaps*, J. Math. Mech., **9** (1960), 203–228.

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