Operators
and
Matrices
Volume 8, Number 4 (2014), 999–1011
doi:10.7153/oam-08-55

ON THE EXTENDED HOLOMORPHIC CURVES ON C^*–ALGEBRAS

YINGLI HOU AND KUI JI*

(*Communicated by N.-C. Wong*)

Abstract. For $\Omega \subseteq \mathbb{C}$ a connected open set, and $\mathscr{U}$ a unital C^*-algebra, let $\mathscr{P}(\mathscr{U})$ denote the sets of all projections in $\mathscr{U}$. If $P : \Omega \to \mathscr{P}(\mathscr{U})$ is a holomorphic $\mathscr{U}$-valued map, then P is called an extended holomorphic curve on $\mathscr{P}(\mathscr{U})$. In this note, we focus on discussing the unitary classification problem of extended holomorphic curves. By considering the metric of related determinant jet bundles, we give a necessary and sufficient condition for some extended holomorphic curves on C^*-algebras to be unitary equivalent.

1. Introduction

In this note, we will discuss the unitary equivalence problem of holomorphic maps in Grassmann manifolds in a C^*-algebraic setting. Let $\mathscr{U}$ be a unital C^*-algebra, then $p \in \mathscr{U}$ is called a projection in $\mathscr{U}$ whenever $p^2 = p = p^*$, and $\mathscr{P}(\mathscr{U})$ denote the set of all projections in $\mathscr{U}$ which is called Grassmann manifold of $\mathscr{U}$. Let $\Omega \subseteq \mathbb{C}$ be a connected open set. If $P : \Omega \to \mathscr{P}(\mathscr{U})$ is a holomorphic $\mathscr{U}$-valued map, then it is called an extended holomorphic curve on $\mathscr{P}(\mathscr{U})$ (in order to discriminate ordinary holomorphic curve).

Let $P, Q : \Omega \to P(\mathscr{U})$ be two extended holomorphic curves. We say that P and Q are unitary equivalent (denoted by $P \overset{u}{\sim} Q$) if there exists a unitary $U \in \mathscr{U}$ such that $P(\lambda) = UQ(\lambda)U^*$, $\forall \lambda \in \Omega$ (cf. [6]).

The unitary equivalent problem of extended holomorphic curves originates from the systematic researches of holomorphic curves initiated by M. J. Cowen and R. G. Douglas. Let $\mathscr{H}$ be a complex separable Hilbert space and $Gr(n, \mathscr{H})$ denote n-dimensional Grassmann manifold, the set of all n-dimensional subspaces of $\mathscr{H}$. A map $p : \Omega \to Gr(n, \mathscr{H})$ is called as a holomorphic curve, if there exist n holomorphic $\mathscr{H}$-valued functions $e_1, e_2, \ldots, e_n$ on Ω such that $p(\lambda) = \bigvee\{e_1(\lambda), \ldots, e_n(\lambda)\}$ for each $\lambda \in \Omega$, where symbol "$\bigvee$" denotes the closure of linear span (cf. [2]). In the paper [2], M. J. Cowen and R. G. Douglas introduced a class of operators related to complex geometry now referred to as Cowen-Douglas operators [cf. example 1.2]. There exists a natural connection between holomorphic curves and this class of operators.

Mathematics subject classification (2010): Primary 47C15, 47B37; Secondary 47B48, 47L40.
Keywords and phrases: C^*-algebras, unitary equivalence, curvatures, extended holomorphic curves.
The authors are supported by Chinese NSFC (Grant No. 10901046) and Foundation for the Author of National Excellent Doctoral Dissertation of China (Grant No. 201116).

* The corresponding author.

Paper OaM-08-55

M. J. Cowen and R. G. Douglas obtained a unitary equivalence classification of holomorphic curves in [2]. They proved that a kind of curvature function is a unitary invariant of the holomorphic curves and Cowen-Doulgas operators by means of complex hermitian geometry techniques. Subsequently, the curvature function turns into an important object of the research of classification of Cowen-Douglas operators (cf. [4], [11], [12], [15]). In 1981, a C^*-algebra approach to Cowen-Dougals theory was given by C. Apostol and M. Martin (cf. [1]). And M. Martin and N. Salinas did a series work of holomorphic curves on extended flag manifolds and extended Grassmann manifolds (cf. [6], [7], [8], [11], [14]). In the spirit of the above work, we want to characterize unitary equivalence problem of extended holomorphic curves with some geometry concepts.

The paper is organized as follows. In §1 some notations and known results will be introduced. In §2, We define a special class of extended holomorphic curves analogous to Bott projection in C^*-algebras. By considering the metric of a related determinant jet bundle, we also give a necessary and sufficient condition for two extended holomorphic curves in this class to be unitary equivalent.

We will introduce some notations and results first, and all the notations are adopted from [1], [2], [3] and [6].

To simplify the notation, we use the symbol "$\overline{\partial}^J\partial^I$" denotes partial derivative "$\frac{\partial^{J+I}}{\partial^J\overline{\lambda}\partial^I\lambda}$", where I,J are non-negative integers. And for any I and J,

(1) symbol $\overline{\partial}^J$ stands for $\overline{\partial}^J\partial^0$ and ∂^I stands for $\overline{\partial}^0\partial^I$,

(2) symbol ∂ stands for ∂^1, and $\overline{\partial}$ stands for $\overline{\partial}^1$,

(3) $\overline{\partial}^J\partial^I P = P$, when $J = I = 0$.

Firstly, we need a criterion for determining the holomorphic map from Ω to $\mathscr{P}(\mathscr{U})$.

1.1 [6] Let $\mathscr{U}$ be a unital C^*-algebra. Let $P:\Omega\to\mathscr{P}(\mathscr{U})$ be a $\mathscr{U}$-valued infinitely differentiable map. Then P is called holomorphic if and only if

$$\overline{\partial}P(\lambda) = P(\lambda)\overline{\partial}P(\lambda),\ \forall\lambda\in\Omega. \tag{1.1}$$

Since $P(\lambda)$ is a projection, for any $\lambda\in\Omega$, we can get that

$$\overline{\partial}P(\lambda) = [\overline{\partial}P(\lambda)]P(\lambda) + P(\lambda)[\overline{\partial}P(\lambda)].$$

So (1.1) is equivalent to say that

$$[\overline{\partial}P(\lambda)]P(\lambda) = 0 \Longleftrightarrow \partial P(\lambda) = [\partial P(\lambda)]P(\lambda) \Longleftrightarrow P(\lambda)\partial P(\lambda) = 0,\ \forall\lambda\in\Omega.$$

EXAMPLE 1.2. A class of Cowen-Douglas operator with index n: $B_n(\Omega)$ is defined as follows [2]:

$$\begin{aligned} B_n(\Omega) =: \{T\in\mathscr{L}(\mathscr{H}) : &(i)\ \Omega\subset\sigma(T) =: \{\lambda\in\mathbb{C} : T-\lambda I \text{ is not invertible}\},\\ &(ii)\ \textstyle\bigvee_{\lambda\in\Omega}\mathrm{Ker}(T-\lambda) = \mathscr{H},\\ &(iii)\ \mathrm{Ran}(T-\lambda) = \mathscr{H},\\ &(iv)\ \dim\mathrm{Ker}(T-\lambda) = n,\ \forall\lambda\in\Omega.\}\end{aligned}$$

Let $T \in \mathscr{L}(\mathscr{H})$ be a Cowen-Douglas operator. For any $\lambda \in \Omega$, if $P(\lambda)$ is the projection from $\mathscr{H}$ to $\mathrm{Ker}(T-\lambda)$, then $P : \Omega \to \mathscr{P}(\mathscr{L}(\mathscr{H}))$ is an extended holomorphic curve.

1.3 [6] Let $\mathscr{U}$ be a unital C^*-algebra, and $P : \Omega \to \mathscr{P}(\mathscr{U})$ be an extended holomorphic curve. For each $\lambda \in \Omega$ and every $\alpha \in \mathbb{Z}_+ \cup \{\infty\}$, set

$$\mathscr{B}_\lambda^\alpha = \{\overline{\partial}^J P(\lambda)\partial^I P(\lambda) : I,J \in \mathbb{Z}_+, I,J \leqslant \alpha\}.$$

Let $\mathscr{U}_\lambda^\alpha$ be the closure of $*$-subalgebra of $\mathscr{U}$ generated by $\mathscr{B}_\lambda^\alpha$ with the following property:

$$\mathscr{U}_\lambda^0 \subseteq \mathscr{U}_\lambda^1 \subseteq \cdots \subseteq \mathscr{U}_\lambda^\infty.$$

By using notations mentioned above, M. Martin and N. Salinas defined a substitute in C^*-algebra for Cowen-Douglas class $B_n(\Omega)$:

DEFINITION 1.4. [6] Let $k \geqslant 1$ be an integer. If the following conditions are satisfied, then extended holomorphic curve $P : \Omega \to P(\mathscr{U})$ is said to be in the class $\mathscr{A}_k(\Omega, \mathscr{U})$:

(1) For each $\lambda \in \Omega$, $\mathscr{U}_\lambda^\infty$ is a finite-dimensional C^*-algebra.

(2) If k_λ denotes the cardinal of any maximal collection of mutually orthogonal minimal projections in $\mathscr{U}_\lambda^\infty$, then

$$k_\lambda \leqslant k.$$

(3) If $a \in \mathscr{U}$ and $aP(\lambda) = 0$ for every $\lambda \in \Omega$, then $a = 0$.

DEFINITION 1.5. [6] Let $\lambda \in \Omega$ and $\alpha \in \mathbb{Z}_+$ be a fixed integer. We say that P and Q *have order of contact* α *at* λ if there exists a unitary v such that

$$v\overline{\partial}^J P(\lambda)\partial^I P(\lambda)v^* = \overline{\partial}^J Q(\lambda)\partial^I Q(\lambda), \ \forall 0 \leqslant I,J \leqslant \alpha, \tag{1.2}$$

We say $\mathfrak{G} \subset \mathscr{U}$ is a *separating subset* of $\mathscr{U}$, if $\{a \in \mathscr{U} : as = 0, s \in \mathfrak{G}\} = \{0\}$. Assume $\mathfrak{G}$, $\mathfrak{T}$ are two separating subsets of $\mathscr{U}$, $\theta : \mathfrak{G} \to \mathfrak{T}$ is a given bijection. We say θ is *inner (or semi-inner)*, if there exists a unitary $u \in \mathscr{U}$ (or a unitary $v \in \mathscr{U}$) such that

$$usu^* = \theta(s), \ \forall s \in \mathfrak{G}, \ \ (\text{or } \ vt^*sv^* = \theta(t)\theta(s), \ \forall s,t \in \mathfrak{G}).$$

$\mathscr{U}$ is said to be inner if each semi-inner bijection between two separating subsets of $\mathscr{U}$ is inner.

M. Martin and N. Salinas proved the following related rigidity theorem for $\mathscr{A}_k(\Omega, \mathscr{U})$ class on C^*-algebra.

LEMMA 1.6. [Theorem 4.5, 6] *Suppose that extended holomorphic curves P, Q : $\Omega \to \mathscr{P}(\mathscr{U})$ belong to the class $\mathscr{A}_k(\Omega, \mathscr{U})$. If $\mathscr{U}$ is an inner C*-algebra, then the following two statements are equivalent:*

(1) *P and Q are unitarily equivalent;*

(2) *P and Q have order of contact α at each $\lambda \in \Omega$.*

DEFINITION 1.7. [2] Let $p : \Omega \to Gr(n, \mathscr{H})$ be a holomorphic curve and $p(\lambda) = \bigvee\{e_1(\lambda), \dots, e_n(\lambda)\}$ for any $\lambda \in \Omega$, then the curvature according to p is defined as

$$K_p = -\frac{\partial}{\partial \overline{\lambda}}\Big(h^{-1}\frac{\partial h}{\partial \lambda}\Big),$$

where metric function h is defined as the following:

$$h(\lambda) = (\langle e_i(\lambda), e_j(\lambda)\rangle)_{n\times n},\ \forall \lambda \in \Omega.$$

2. The Bott projection and extended holomorphic curve

2.1 Let $C^\infty(\Omega, \mathscr{U})$ denote the $*-$algebra of all the $\mathscr{U}$-valued infinitely differentiable functions defined on Ω. In this section, we assume the following conditions about our extended holomorphic curve $P : \Omega \to \mathscr{P}(\mathscr{U})$. There exist $f \in C^\infty(\Omega)$ and $a \in C^\infty(\Omega, \mathscr{U})$ satisfying the following three conditions:

(1) $P(\lambda) = f(|\lambda|)a(\lambda)a^*(\lambda),\ \forall \lambda \in \Omega$;
(2) $\partial P(\lambda) = f(|\lambda|)a^*(\lambda), \overline{\partial} P(\lambda) = f(|\lambda|)a(\lambda),\ \forall \lambda \in \Omega$;
(3) $\frac{\partial}{\partial \lambda}a^*(\lambda) = \frac{\partial}{\partial \overline{\lambda}}a(\lambda) = 0,\ \forall \lambda \in \Omega$.

In the following, we will give some examples on extended holomorphic curves with these properties.

EXAMPLE 2.2. Let $\mathscr{U}$ be $M_2(\mathbb{C})$ and $\Omega \subseteq \mathbb{C}$ be a connected open set and closed with respect to conjugation around X-axis. Let $P : \Omega \to M_2(\mathbb{C})$ defined by

$$P(\lambda) = \begin{pmatrix} \frac{1}{1+|\lambda|^2} & \frac{\overline{\lambda}}{1+|\lambda|^2} \\ \frac{\lambda}{1+|\lambda|^2} & \frac{|\lambda|^2}{1+|\lambda|^2} \end{pmatrix} = \frac{1}{1+|\lambda|^2}\begin{pmatrix} 1 & \overline{\lambda} \\ \lambda & |\lambda|^2 \end{pmatrix},\ \forall \lambda \in \Omega.$$

Then P is an extended holomorphic curve on Ω. Since for any $\lambda \in \Omega$,

$$\partial P(\lambda) = \frac{\partial}{\partial \lambda}\left(\frac{1}{1+|\lambda|^2}\begin{pmatrix} 1 & \overline{\lambda} \\ \lambda & |\lambda|^2 \end{pmatrix}\right) = \frac{1}{(1+|\lambda|^2)^2}\begin{pmatrix} -\overline{\lambda} & -\overline{\lambda}^2 \\ 1 & \overline{\lambda} \end{pmatrix},$$

and

$$\overline{\partial} P(\lambda) = \frac{1}{(1+|\lambda|^2)^2}\begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix},$$

then we have

$$\begin{aligned}\overline{\partial} P(\lambda)P(\lambda) &= \frac{\partial}{\partial \overline{\lambda}}\begin{pmatrix} \frac{1}{1+|\lambda|^2} & \frac{\overline{\lambda}}{1+|\lambda|^2} \\ \frac{\lambda}{1+|\lambda|^2} & \frac{|\lambda|^2}{1+|\lambda|^2} \end{pmatrix}\begin{pmatrix} \frac{1}{1+|\lambda|^2} & \frac{\overline{\lambda}}{1+|\lambda|^2} \\ \frac{\lambda}{1+|\lambda|^2} & \frac{|\lambda|^2}{1+|\lambda|^2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{-\lambda}{(1+|\lambda|^2)^2} & \frac{1}{(1+|\lambda|^2)^2} \\ \frac{-\lambda^2}{(1+|\lambda|^2)^2} & \frac{\lambda}{(1+|\lambda|^2)^2} \end{pmatrix}\begin{pmatrix} \frac{1}{1+|\lambda|^2} & \frac{\overline{\lambda}}{1+|\lambda|^2} \\ \frac{\lambda}{1+|\lambda|^2} & \frac{|\lambda|^2}{1+|\lambda|^2} \end{pmatrix} = 0,\end{aligned}$$

where $f(|\lambda|) = \frac{1}{(1+|\lambda|^2)^2}, a(\lambda) = \begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix}$ (See in Definition 2.1).

Notice that for any $\lambda \in \Omega$,

$$\begin{aligned}\overline{\partial}P(\lambda)\partial P(\lambda) &= \frac{1}{(1+|\lambda|^2)^4}\begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix}\begin{pmatrix} -\overline{\lambda} & -\overline{\lambda}^2 \\ 1 & \overline{\lambda} \end{pmatrix} \\ &= \frac{1}{(1+|\lambda|^2)^3}\begin{pmatrix} 1 & \overline{\lambda} \\ \lambda & |\lambda|^2 \end{pmatrix} = \frac{1}{(1+|\lambda|^2)^2}P(\lambda).\end{aligned}$$

And

$$\begin{aligned}\overline{\partial}^2 P(\lambda) &= \frac{\partial}{\partial\overline{\lambda}}\left(\frac{1}{(1+|\lambda|^2)^2}\begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix}\right) = \frac{\partial}{\partial\overline{\lambda}}\left(\frac{1}{(1+|\lambda|^2)^2}\right)\begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix} \\ &= \frac{-2\overline{\lambda}}{1+|\lambda|^2}\overline{\partial}P(\lambda).\end{aligned}$$

So

$$\begin{aligned}\overline{\partial}^2 P(\lambda)\partial^2 P(\lambda) &= \frac{\partial}{\partial\overline{\lambda}}\left(\frac{1}{(1+|\lambda|^2)^2}\right)\frac{\partial}{\partial\lambda}\left(\frac{1}{(1+|\lambda|^2)^2}\right)\begin{pmatrix} -\lambda & 1 \\ -\lambda^2 & \lambda \end{pmatrix}\begin{pmatrix} -\overline{\lambda} & -\overline{\lambda}^2 \\ 1 & \overline{\lambda} \end{pmatrix}, \\ &= f^{2,2}(\lambda)P(\lambda)\end{aligned}$$

where $f^{2,2}(\lambda) = \frac{\partial}{\partial\overline{\lambda}}\left(\frac{1}{(1+|\lambda|^2)^2}\right)\frac{\partial}{\partial\lambda}\left(\frac{1}{(1+|\lambda|^2)^2}\right)(1+|\lambda|^2)$. By mathematical induction, for any non-negative integers I,J, we can find a function $f^{J,I} \in C^\infty(\Omega)$, such that

$$\overline{\partial}^J P(\lambda)\partial^I P(\lambda) = f^{J,I}(\lambda)P(\lambda),\ \forall J,I \in \mathbb{N}.$$

REMARK 2.3. The extended holomorphic curve in example 2.2 is just Bott projection for dimension two in algebraic K-theory [12]. By the calculation above, we know that $P \in \mathscr{A}_1(\Omega, M_2(\mathbb{C}))$.

EXAMPLE 2.4. Let $k \in C^\infty(\Omega)$, $k(\overline{\lambda}) = \overline{k(\lambda)}$, and $\overline{\partial}k(\lambda) = 0$, $\forall\lambda \in \Omega$. Then the following formulae

$$P(\lambda) =: \frac{1}{1+|k(\lambda)|^2}\begin{pmatrix} 1 & k(\overline{\lambda}) \\ k(\lambda) & |k(\lambda)|^2 \end{pmatrix},\ \forall\lambda \in \Omega,$$

defines an extended holomorphic curve on Ω. Similar to the calculation progress in example 2.2, we have that

$$P(\lambda) = \frac{1}{(1+|k(\lambda)|^2)^2}\begin{pmatrix} -k(\lambda) & 1 \\ -k(\lambda)^2 & k(\lambda) \end{pmatrix}\begin{pmatrix} -k(\overline{\lambda}) & -k(\overline{\lambda})^2 \\ 1 & k(\overline{\lambda}) \end{pmatrix},$$

and

$$\partial P(\lambda) = \frac{\frac{\partial}{\partial\lambda}k(\lambda)}{(1+|k(\lambda)|^2)^2}\begin{pmatrix} -k(\overline{\lambda}) & -k(\overline{\lambda})^2 \\ 1 & k(\overline{\lambda}) \end{pmatrix},\ \forall\lambda \in \Omega.$$

$$\overline{\partial}P(\lambda)=\frac{\frac{\partial}{\partial\overline{\lambda}}k(\overline{\lambda})}{(1+|k(\lambda)|^2)^2}\begin{pmatrix}-k(\lambda) & 1\\ -k(\lambda)^2 & k(\lambda)\end{pmatrix},\ \forall\lambda\in\Omega.$$

So P also satisfies assumptions in Definition 2.1. $\square$

By the conditions (2) and (3), for any I and J, we have

$$\partial^I P(\lambda)=\frac{\partial^I}{\partial\lambda}f(|\lambda|)a^*(\lambda),\ \overline{\partial}^J P(\lambda)=\frac{\partial^J}{\partial\overline{\lambda}}f(|\lambda|)a(\lambda),\ \forall\lambda\in\Omega.$$

and there exists $f^{JI}\in C^\infty(\Omega)$ such that

$$\overline{\partial}^J P(\lambda)\partial^I P(\lambda)=\frac{\frac{\partial^I}{\partial\lambda}f(|\lambda|)\frac{\partial^J}{\partial\overline{\lambda}}f(|\lambda|)}{f(|\lambda|)}P(\lambda)=f^{JI}(\lambda)P(\lambda),\ \forall\lambda\in\Omega.$$

Now, we define the function f^{JI} as the following:

$$f^{JI}(\lambda)=:\frac{\frac{\partial^I}{\partial\lambda}f(|\lambda|)\frac{\partial^J}{\partial\overline{\lambda}}f(|\lambda|)}{f(|\lambda|)},\ \forall\lambda\in\Omega,0\leqslant J,I. \tag{2.1}$$

DEFINITION 2.5. [2] Let $(E,\pi),\pi:E\to\Omega$ be a Hermitian holomorphic vector bundle with rank n. Let

$$\wedge^r(E):=\bigcup_{\lambda\in\Omega}\wedge^r(\pi^{-1}(\lambda)),1\leqslant r\leqslant n,$$

where $\wedge^r(\pi^{-1}(\lambda))$ is the exterior power space of fiber $\pi^{-1}(\lambda)$ for any $\lambda\in\Omega$. By giving a proper holomorphic Hermitian structure, the vector space $\wedge^r(\pi^{-1}(E))$ becomes a Hermitian holomorphic vector bundle. When $r=n$, $\wedge^n(E)$ is called the determinant bundle, denoted by $\det E$.

Let $\{s_1,s_2,\cdots,s_n\}$ be a frame for the vector bundle E on some open set $U\subset\Omega$. Then wedge $s_1\wedge s_2\wedge\cdots\wedge s_n$ is the frame of $\det E$ over U. And it's metric is the determinant of the metric of E, i.e.

$$h_{\det E}=\det h_E.$$

If $\sigma=\{e\}$ is a holomorphic frame of E on Ω, then the 1-jet bundle $\mathscr{J}_1(E)$ has an associate frame

$$\mathscr{J}_1(\sigma)=\{e,\partial e\}.$$

And the metric $h(\lambda):=\langle e(\lambda),e(\lambda)\rangle$ induces the metric $\mathscr{J}_1(h)$ for $\mathscr{J}_1(E)$ as the following:

$$\mathscr{J}_1(h)(\lambda):=\begin{pmatrix}\langle e(\lambda),e(\lambda)\rangle & \partial\langle e(\lambda),e(\lambda)\rangle\\ \overline{\partial}\langle e(\lambda),e(\lambda)\rangle & \overline{\partial}\partial\langle e(\lambda),e(\lambda)\rangle\end{pmatrix},\ \forall\lambda\in\Omega.$$

DEFINITION 2.6. Let $P:\Omega\to\mathscr{P}(\mathscr{U})$ be an extended holomorphic curve which satisfies conditions (1),(2) and (3) in 2.1. Then there exists a natural C^*-bundle E_P over Ω induced by P with the fiber defined as

$$E_P(\lambda):=\mathrm{Span}\{\partial^I P(\lambda),I\in\mathbb{N}\},\ \forall\lambda\in\Omega.$$

And the Hermitian inner product be defined as following:

$$\langle \partial^I P(\lambda), \partial^J P(\lambda)\rangle = f^{JI}(\lambda),\ \forall J,I \in \mathbb{Z}^+, \text{ and } f^{11}(\lambda) = f(|\lambda|),$$

where f^{JI} is the function defined as formulae (2.1).

If $P:\Omega \to \mathscr{P}(\mathscr{U})$ is an extended holomorphic curves satisfying 2.1, and there exists a holomorphic curve $p:\Omega \to Gr(1,\mathscr{H})$ with the frame $e_p(\lambda)$ satisfying the following property:

$$\langle e_p(\lambda), e_p(\lambda)\rangle = f(|\lambda|),\ \forall I,J \in \mathbb{N},$$

then we call p is *a supporting holomorphic curve* to P.

Let $Q:\Omega \to \mathscr{P}(\mathscr{U})$ be another extended holomorphic curve, and q be a supporting holomorphic curve to Q. We call $p \overset{u}{\sim} q$ if and only if there exists a unitary $u \in \mathscr{L}(\mathscr{H})$ such that $p(\lambda) = uq(\lambda)$, $\forall \lambda \in \Omega$. And we call p and q *have order of contact* α, if there exists a unitary $U \in \mathscr{L}(\mathscr{H})$ such that $U\partial^I e_p(\lambda) = \partial^I e_q(\lambda)$, $\forall I \in \mathbb{N}$, $\forall \lambda \in \Omega$ (See in [2]).

With same form to the curvature of Hermitian holomorphic bundles in [2], an *curvature function* with $h \in C^\infty(\Omega)$ is defined as

$$K_h(\lambda) = -\frac{\partial}{\partial \overline{\lambda}}\left(h^{-1}\frac{\partial h}{\partial \lambda}\right)(\lambda),\ \forall \lambda \in \Omega.$$

Then $K_f(\lambda) = -\frac{\partial}{\partial \overline{\lambda}}\left(f^{-1}\frac{\partial f}{\partial \lambda}\right)(\lambda)$, $\forall \lambda \in \Omega$, where f is given in Definition 2.1 and 2.6.

By Definition 2.5, we know that *the metric of determinate 1-jet bundle* $\det \mathscr{J}_1(E_p)$ is just equal to f^2K_f i.e.

$$h_{\det \mathscr{J}_1(E_p)} = \det \mathscr{J}_1(h)(E_p) = f^2K_f.$$

By using this metric, we will give our main theorem of this note as the following:

THEOREM 2.7. *Let P and Q be extended holomorphic curves from Ω to $\mathscr{P}(\mathscr{U})$ in 2.1. Let p and q be the supporting holomorphic curves to P and Q respectively. And Let $||e_p(\lambda)||^2 = f(|\lambda|), ||e_q(\lambda)||^2 = g(|\lambda|)$, $\forall \lambda \in \Omega$. Then for any given $\lambda_0 \in \Omega$, we have the following statement:*

If curvature functions $K_{\overline{\partial^J}\partial^I f(\lambda_0)} = K_{\overline{\partial^J}\partial^I g(\lambda_0)} = 0$, $\forall J,I \in \mathbb{N}$, p and q have order of contact α and $p(\lambda_0) \overset{u}{\sim} q(\lambda_0)$, then extended holomorphic curves P and Q have order of contact α at λ_0 if and only if

$$\overline{\partial}^J \partial^I h_{det \mathscr{J}_1(E_p)}(\lambda_0) = \overline{\partial}^J \partial^I h_{det \mathscr{J}_1(E_q)}(\lambda_0),\ \forall J,I \leqslant \alpha - 1.$$

Proof. Firstly, recall that

$$f^{JI}(\lambda) =: \frac{\frac{\partial^I}{\partial \lambda} f(|\lambda|)\frac{\partial^J}{\partial \overline{\lambda}} f(|\lambda|)}{f(|\lambda|)},\ \forall \lambda \in \Omega, 0 \leqslant J,I.$$

We want to prove that for any given $\lambda_0 \in \Omega$ and $0 \leqslant J, I$, $f^{JI}(|\lambda_0|)$ can be expressed in terms of $f(\lambda_0)$ and $\overline{\partial}^j \partial^i h_{\det \mathscr{J}_1(E_p)}(\lambda_0), j, i \in \mathbb{N}$.

Since

$$K_f(\lambda) = -\frac{\partial}{\partial \overline{\lambda}}\Big(\frac{1}{f(|\lambda|)}\frac{\partial f}{\partial \lambda}\Big) = \frac{\frac{\partial}{\partial \lambda} f(|\lambda|) \frac{\partial}{\partial \overline{\lambda}} f(|\lambda|) - f(|\lambda|) \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} f(|\lambda|)}{f^2(|\lambda|)},$$

and

$$f^2 K_f = \det \mathscr{J}_1(E_p),$$

then we have that

$$\frac{\partial}{\partial \overline{\lambda}} f \frac{\partial}{\partial \lambda} f = f^2 K_f + f \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} f = h_{\det \mathscr{J}_{E_p}} + f \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} f.$$

By

$$\frac{\partial}{\partial \lambda}\Big(\frac{\partial}{\partial \overline{\lambda}} f \frac{\partial}{\partial \lambda} f\Big) = \frac{\partial}{\partial \lambda} h_{\det \mathscr{J}_1(E_p)} + \frac{\partial}{\partial \lambda} f \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} f + f \frac{\partial^3}{\partial \overline{\lambda} \partial \lambda^2} f,$$

it follows that

$$\frac{\partial^2}{\partial \overline{\lambda}^2} f \frac{\partial}{\partial \lambda} f = \frac{\partial}{\partial \lambda} h_{\det \mathscr{J}_1(E_p)} + f \frac{\partial^3}{\partial \overline{\lambda} \partial \lambda^2} f.$$

In the same way,

$$\frac{\partial}{\partial \overline{\lambda}} f \frac{\partial^2}{\partial \lambda^2} f = \frac{\partial}{\partial \overline{\lambda}} h_{\det \mathscr{J}_1(E_p)} + f \frac{\partial^3}{\partial \lambda \partial \overline{\lambda}^2} f,$$

$$\frac{\partial^2}{\partial \overline{\lambda}^2} f \frac{\partial^2}{\partial \lambda^2} f = \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} h_{\det \mathscr{J}_1(E_p)} + f \frac{\partial^4}{\partial \lambda^2 \partial \overline{\lambda}^2} f.$$

So when $\max\{I, J\} \leqslant 2$, we have

$$\langle \partial^J P(\lambda), \partial^I P(\lambda) \rangle = f^{JI}(\lambda) = \frac{\overline{\partial}^{J-1} \partial^{I-1} h_{\det \mathscr{J}_1(E_p)(\lambda)}}{f(\lambda)} + \overline{\partial}^J \partial^I f(\lambda).$$

When $\max\{I, J\} \leqslant 3$, we have

$$K_{\partial f} = -\frac{\partial}{\partial \overline{\lambda}}\Big(\partial f^{-1} \frac{\partial}{\partial \lambda}(\partial f)\Big) = -\frac{\frac{\partial^2 f}{\partial \lambda^2} \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} f - \frac{\partial}{\partial \lambda} f \frac{\partial^2}{\partial \lambda^2 \partial \overline{\lambda}} f}{\Big(\frac{\partial}{\partial \lambda} f\Big)^2},$$

$$K_{\overline{\partial} f} = -\frac{\partial}{\partial \overline{\lambda}}\Big(\overline{\partial} f^{-1} \frac{\partial}{\partial \lambda}(\overline{\partial} f)\Big) = -\frac{\frac{\partial^2 f}{\partial \overline{\lambda}^2} \frac{\partial^2}{\partial \overline{\lambda} \partial \lambda} f - \frac{\partial}{\partial \overline{\lambda}} f \frac{\partial^2}{\partial \overline{\lambda}^2 \partial \lambda} f}{\Big(\frac{\partial}{\partial \overline{\lambda}} f\Big)^2}.$$

Since $K_{\overline{\partial}^J\partial^I f}(\lambda)=K_{\overline{\partial}^J\partial^I g}(\lambda)=0$, then

$$\frac{\partial^2 f}{\partial\lambda^2}\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f-\frac{\partial}{\partial\lambda}f\frac{\partial^2}{\partial\lambda^2\partial\overline{\lambda}}f=0, \tag{2.2}$$

$$\frac{\partial^2 f}{\partial\overline{\lambda}^2}\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f-\frac{\partial}{\partial\overline{\lambda}}f\frac{\partial^2}{\partial\overline{\lambda}^2\partial\lambda}f=0. \tag{2.3}$$

And

$$\begin{aligned}\frac{\partial^3}{\partial\overline{\lambda}^3}f\frac{\partial}{\partial\lambda}f &= \frac{\partial}{\partial\overline{\lambda}}\Big(\frac{\partial^2}{\partial\overline{\lambda}^2}f\frac{\partial}{\partial\lambda}f\Big)-\frac{\partial^2}{\partial\overline{\lambda}^2}f\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f\\
&=\frac{\partial}{\partial\overline{\lambda}}\Big(\frac{\partial}{\partial\overline{\lambda}}h_{\det\mathscr{J}_1(E_p)}+f\tfrac{\partial^3}{\partial\lambda\partial\overline{\lambda}^2}f\Big)-\frac{\partial^2}{\partial\overline{\lambda}^2}f\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f\\
&=\frac{\partial^2}{\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^4}{\partial\lambda\partial\overline{\lambda}^3}f-\frac{\partial^2 f}{\partial\overline{\lambda}^2}\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f+\frac{\partial}{\partial\overline{\lambda}}f\frac{\partial^2}{\partial\overline{\lambda}^2\partial\lambda}f\\
&=\frac{\partial^2}{\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^4}{\partial\lambda\partial\overline{\lambda}^3}f;\end{aligned}$$

$$\begin{aligned}\frac{\partial^3}{\partial\overline{\lambda}^3}f\frac{\partial^2}{\partial\lambda^2}f &= \frac{\partial}{\partial\lambda}\Big(\frac{\partial^3}{\partial\overline{\lambda}^3}f\frac{\partial}{\partial\lambda}f\Big)-\frac{\partial}{\partial\lambda}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f\\
&=\frac{\partial}{\partial\lambda}\Big(\frac{\partial^2}{\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f\Big)-\frac{\partial}{\partial\lambda}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f\\
&=\frac{\partial^3}{\partial\lambda\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^5}{\partial\overline{\lambda}^3\partial\lambda^2}f+\frac{\partial}{\partial\lambda}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f-\frac{\partial}{\partial\lambda}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f\\
&=\frac{\partial^3}{\partial\lambda\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^5}{\partial\overline{\lambda}^3\partial\lambda^2}f.\end{aligned}$$

Notice that (2.2) and $K_{\overline{\partial}\partial f}(\lambda)=0$, we have

$$K_{\overline{\partial}\partial f}(\lambda)=-\frac{\partial}{\partial\overline{\lambda}}\Big(\overline{\partial}\partial f^{-1}\frac{\partial}{\partial\lambda}(\overline{\partial}\partial f)\Big)=-\frac{\frac{\partial^3}{\partial\lambda\partial\overline{\lambda}^2}f\frac{\partial^3}{\partial\lambda^2\partial\overline{\lambda}}f-\frac{\partial^2}{\partial\lambda\partial\overline{\lambda}}f\frac{\partial^4}{\partial\lambda^2\partial\overline{\lambda}^2}f}{\Big(\frac{\partial^2}{\partial\lambda\partial\overline{\lambda}f}\Big)^2}=0. \tag{2.4}$$

By (2.2),

$$\frac{\partial^2}{\partial\overline{\lambda}^2}\Big(\frac{\partial^2 f}{\partial\lambda^2}\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f-\frac{\partial}{\partial\lambda}f\frac{\partial^2}{\partial\lambda^2\partial\overline{\lambda}}f\Big)=0,$$

that is

$$\begin{aligned}\frac{\partial}{\partial\overline{\lambda}}\Big(\frac{\partial^2}{\partial\lambda^2}f\frac{\partial^3}{\partial\lambda\partial\overline{\lambda}^2}f-\frac{\partial}{\partial\lambda}f\frac{\partial^4}{\partial\lambda^2\partial\overline{\lambda}^2}f\Big) &= \frac{\partial^3}{\partial\overline{\lambda}\partial\lambda^2}f\frac{\partial^3}{\partial\overline{\lambda}^2\partial\lambda}f+\frac{\partial^2}{\partial\lambda^2}f\frac{\partial^4}{\partial\lambda\partial\overline{\lambda}^3}f\\
&\quad-\frac{\partial^2}{\partial\overline{\lambda}\partial\lambda}f\frac{\partial^4}{\partial\lambda^2\partial\overline{\lambda}^2}f-\frac{\partial}{\partial\lambda}f\frac{\partial^5}{\partial\overline{\lambda}^3\partial\lambda^2}f\\
&=0.\end{aligned}$$

By (2.4), we have

$$\frac{\partial}{\partial\lambda}f\frac{\partial^5}{\partial\overline{\lambda}^3\partial\lambda^2}f-\frac{\partial^2}{\partial\overline{\lambda}^2}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f=0.$$

So

$$\begin{aligned}\frac{\partial^3}{\partial\overline{\lambda}^3}f\frac{\partial^3}{\partial\lambda^3}f&=\frac{\partial}{\partial\lambda}\Big(\frac{\partial^3}{\partial\overline{\lambda}^3}f\frac{\partial^2}{\partial\lambda^2}f\Big)-\frac{\partial^2}{\partial\lambda^2}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f\\&=\frac{\partial}{\partial\lambda}\Big(\frac{\partial^3}{\partial\lambda\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^5}{\partial\overline{\lambda}^3\partial\lambda^2}f\Big)-\frac{\partial^2}{\partial\overline{\lambda}^2}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f\\&=\frac{\partial^4}{\partial\lambda^2\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^6}{\partial\lambda^3\partial\overline{\lambda}^3}f+\frac{\partial}{\partial\lambda}f\frac{\partial^5}{\partial\overline{\lambda}^3\partial\lambda^2}f-\frac{\partial^2}{\partial\overline{\lambda}^2}f\frac{\partial^4}{\partial\overline{\lambda}^3\partial\lambda}f;\\&=\frac{\partial^4}{\partial\lambda^2\partial\overline{\lambda}^2}h_{\det\mathscr{J}_1(E_p)}+f\frac{\partial^6}{\partial\lambda^3\partial\overline{\lambda}^3}f.\end{aligned}$$

So when $\max\{I,J\}\leqslant 3$, we also have that

$$\langle\partial^J P(\lambda),\partial^I P(\lambda)\rangle=f^{JI}(\lambda)=\frac{\overline{\partial}^{J-1}\partial^{I-1}h_{\det\mathscr{J}_1(E_p)(\lambda)}}{f(\lambda)}+\overline{\partial}^J\partial^I f(\lambda).$$

For the general case, by the conditions, we have

$$K_{\overline{\partial}^J f}=0\Leftrightarrow\overline{\partial}^{J+1}f\partial\overline{\partial}^J f-\partial\overline{\partial}^{J+1}f\overline{\partial}^J f=0,\ \forall J\in\mathbb{N};\tag{2.5}$$

$$K_{\partial^I f}=0\Leftrightarrow\partial^{I+1}f\overline{\partial}\partial^I f-\overline{\partial}\partial^{I+1}f\partial^I f=0,\ \forall I\in\mathbb{N};\tag{2.6}$$

$$K_{\overline{\partial}^J\partial^I f}=0\Leftrightarrow\overline{\partial}^{J+1}f\partial^I f\cdot\overline{\partial}^J\partial^{I+1}f-\overline{\partial}^J\partial^I f\cdot\overline{\partial}^{J+1}\partial^I f=0,\ \forall J,I\in\mathbb{N}.\tag{2.7}$$

Claim. For any $J,I\in\mathbb{N}$, we have

$$\langle\partial^J P(\lambda),\partial^I P(\lambda)\rangle=f^{JI}(\lambda)=\frac{\overline{\partial}^{J-1}\partial^{I-1}h_{\det\mathscr{J}_1(E_p)(\lambda)}}{f(\lambda)}+\overline{\partial}^J\partial^I f(\lambda).$$

In the following, by three steps, we will prove the claim.

Step 1. Firstly, we will prove that when $I=1,J\in\mathbb{N}$, the claim is true. That is $\overline{\partial}^J f\partial f=\overline{\partial}^{J-1}h_{\det\mathscr{J}_1(E_p)}+\overline{\partial}^J\partial f,\ \forall J\in\mathbb{N}$.

Since

$$\begin{aligned}\overline{\partial}^{J+1}f\partial f&=\overline{\partial}(\overline{\partial}^J f\cdot\partial f)-\overline{\partial}^J f\cdot\partial\overline{\partial}f\\&=\overline{\partial}(\overline{\partial}^{J-1}h_{\det\mathscr{J}_1(E_p)}+f\cdot\overline{\partial}^J\partial f)-\overline{\partial}^J f\cdot\partial\overline{\partial}f\\&=\overline{\partial}^J h_{\det\mathscr{J}_1(E_p)}+f\cdot\overline{\partial}^{J+1}\partial f+\overline{\partial}f\cdot\overline{\partial}^J\partial f-\overline{\partial}^J f\cdot\partial\overline{\partial}f,\ \forall J\in\mathbb{N}.\end{aligned}$$

We only need to prove that

$$\overline{\partial} f \cdot \overline{\partial}^J \partial f - \overline{\partial}^J f \cdot \partial \overline{\partial} f = 0,\ \forall J \in \mathbb{N}. \tag{2.8}$$

By (2.5), we have that

$$\overline{\partial}^{J+k} f \cdot \overline{\partial}^J \partial f = \overline{\partial}^J f \cdot \overline{\partial}^{J+k} \partial f,\ \forall k, J \in \mathbb{N}. \tag{2.9}$$

Since

$$\overline{\partial}^2 f \cdot \overline{\partial} \partial f = \overline{\partial} f \cdot \overline{\partial}^2 \partial f,$$

by taking the partial derivation of $\overline{\lambda}$ on the two sides of the formulation above, we have that

$$\overline{\partial}^3 f \cdot \overline{\partial} \partial f = \overline{\partial} f \cdot \overline{\partial}^3 \partial f.$$

Set $J = 2, k = 1$ in (2.9) and take the partial derivation of $\overline{\lambda}$. Then we have that

$$\overline{\partial}^4 f \cdot \overline{\partial} \partial f = \overline{\partial} f \cdot \overline{\partial}^4 \partial f.$$

Repeating the procedure above and a routine computation, we have that for any $J \in \mathbb{N}$, (2.8) holds.

Step 2. Secondly, we will prove that

$$\overline{\partial} f \cdot \overline{\partial}^J \partial^I f - \overline{\partial}^J f \cdot \overline{\partial} \partial^I f = 0,\ \forall J, I \in \mathbb{N}. \tag{2.10}$$

and

$$\partial f \cdot \overline{\partial}^J \partial^I f - \partial^J f \cdot \partial \overline{\partial}^I f = 0,\ \forall J, I \in \mathbb{N}. \tag{2.11}$$

By the proof above, we can assume that (2.10) and (2.11) hold for any $\widetilde{f} \in C^\infty(\Omega)$ which satisfies conditions (2.5), (2.6) and (2.7) when $J, I \leqslant k$. In the following, we will prove they also hold when $J, I \leqslant k+1$.

Now we use the mathematical induction. By step 1, we know that (2.10) holds for any $J \in \mathbb{N}$ when $I = 1$. So if we already have that

$$\overline{\partial} f \cdot \overline{\partial}^J \partial^{I-1} f - \overline{\partial}^J f \cdot \overline{\partial} \partial^{I-1} f = 0,\ \forall J, I \leqslant k+1.$$

Then by taking partial derivation of $\overline{\lambda}$, it follows that

$$\partial \overline{\partial} f \cdot \overline{\partial}^J \partial^{I-1} f + \overline{\partial} f \cdot \overline{\partial}^J \partial^I f = \partial \overline{\partial}^J f \cdot \overline{\partial} \partial^{I-1} f + \overline{\partial}^J f \cdot \overline{\partial} \partial^I f = 0.$$

In order to prove that

$$\overline{\partial} f \cdot \overline{\partial}^J \partial^I f - \overline{\partial}^J f \cdot \overline{\partial} \partial^I f = 0,\ \forall J, I \leqslant k+1,$$

we only need to prove

$$\partial \overline{\partial} f \cdot \overline{\partial}^J \partial^{I-1} f = \partial \overline{\partial}^J f \cdot \overline{\partial} \partial^{I-1} f,\ \forall J, I \leqslant k+1.$$

Let $\widetilde{f} = \overline{\partial} f$, then it is to prove

$$\partial \widetilde{f} \cdot \overline{\partial}^{J-1} \partial^{I-1}(\widetilde{f}) = \partial \overline{\partial}^{J-1} \widetilde{f} \cdot \partial^{I-1} \widetilde{f}, \ \forall J, I \leqslant k+1.$$

Notice that replacing f to $\overline{\partial} f$, (2.5), (2.6) and (2.7) also hold. By our assumption at the beginning of step 2, we know (2.11) holds for $\widetilde{f} = \overline{\partial} f$, $\forall J, I \leqslant k$. So (2.10) holds when $J, I \leqslant k+1$. By symmetry of ∂ and $\overline{\partial}$, (2.11) also holds when $J, I \leqslant k+1$. And then we finish the proof of step 2.

Step 3. At last, we finish the proof of the claim. If we assume that

$$\overline{\partial}^{J} f \cdot \partial^{I} f = \overline{\partial}^{J-1} \partial^{I-1} h_{\det_{\mathscr{J}_1(E_p)}} + f \overline{\partial}^{J} \partial^{I} f, \ \forall J, I \leqslant k,$$

then by (2.10) and (2.11), we have

$$\begin{aligned}
\overline{\partial}^{J+1} f \partial^{I} f &= \overline{\partial}(\overline{\partial}^{J} f \cdot \partial^{I} f) - \overline{\partial}^{J} f \cdot \overline{\partial} \partial^{I} f \\
&= \overline{\partial}(\overline{\partial}^{J-1} \partial^{I-1} h_{\det_{\mathscr{J}_1(E_p)}} + f \cdot \overline{\partial}^{J} \partial^{I} f) - \overline{\partial}^{J} f \cdot \overline{\partial} \partial^{I} f \\
&= \overline{\partial}^{J} \partial^{I-1} h_{\det_{\mathscr{J}_1(E_p)}} + f \cdot \overline{\partial}^{J+1} \partial^{I} f + \overline{\partial} f \cdot \overline{\partial}^{J} \partial^{I} f - \overline{\partial}^{J} f \cdot \overline{\partial} \partial^{I} f \\
&= \overline{\partial}^{J} \partial^{I-1} h_{\det_{\mathscr{J}_1(E_p)}} + f \cdot \overline{\partial}^{J+1} \partial^{I} f, \ \forall J, I \leqslant k,
\end{aligned}$$

and

$$\overline{\partial}^{J} f \partial^{I+1} f = \overline{\partial}^{J-1} \partial^{I} h_{\det_{\mathscr{J}_1(E_p)}} + f \cdot \overline{\partial}^{J} \partial^{I+1} f, \ \forall J, I \leqslant k;$$

$$\overline{\partial}^{J+1} f \cdot \partial^{I+1} f = \overline{\partial}^{J} \partial^{I} h_{\det_{\mathscr{J}_1(E_p)}} + f \overline{\partial}^{J+1} \partial^{I+1} f, \ \forall J, I \leqslant k.$$

By mathematical induction, we finish the proof of the claim.

By the claim, for any $J, I \in \mathbb{Z}^{+}$, we can express $\overline{\partial}^{J} f \partial^{I} f$ in terms of f, $\overline{\partial}^{j} \partial^{i} h_{\det_{\mathscr{J}_1(E_p)}}$ and $\overline{\partial}^{j} \partial^{i} f$. Similarly, we can express $\overline{\partial}^{J} g \partial^{I} g$ in terms of g, $\overline{\partial}^{j} \partial^{i} h_{\det_{\mathscr{J}_1(E_q)}}$ and $\overline{\partial}^{j} \partial^{i} g$.

Secondly, we will give the proof of sufficient part. Notice that p and q have order of contact α at λ_0, so it follows that

$$\overline{\partial}^{J} \partial^{I} f(|\lambda_0|) = \langle \partial^{I} e_p(\lambda_0), \partial^{J} e_p(\lambda_0) \rangle = \langle \partial^{I} e_q(\lambda_0), \partial^{J} e_q(\lambda_0) \rangle = \overline{\partial}^{J} \partial^{I} g(|\lambda_0|), \ \forall I, J \leqslant \alpha.$$

If

$$f(|\lambda_0|) = g(|\lambda_0|), \overline{\partial}^{J} \partial^{I} h_{\det_{\mathscr{J}_1(E_p)}}(\lambda_0) = \overline{\partial}^{J} \partial^{I} h_{\det_{\mathscr{J}_1(E_q)}}(\lambda_0), \ \forall J, I \leqslant \alpha - 1,$$

then

$$f^{JI}(\lambda_0) = g^{JI}(\lambda_0), \ \forall J, I \leqslant \alpha.$$

Since $\overline{\partial}^{J} P(\lambda_0) \partial^{I} P(\lambda_0) = f^{JI}(\lambda_0) P(\lambda_0), \overline{\partial}^{J} Q(\lambda_0) \partial^{I} Q(\lambda_0) = g^{JI}(\lambda_0) Q(\lambda_0)$, $P(\lambda_0) \overset{u}{\sim} Q(\lambda_0)$, then we have

$$\overline{\partial}^{J} P(\lambda_0) \partial^{I} P(\lambda_0) \overset{u}{\sim} \overline{\partial}^{J} Q(\lambda_0) \partial^{I} Q(\lambda_0), \ \forall J, I \leqslant \alpha.$$

So we finish the proof of sufficient part. And the necessary part is obvious. □

By using Lemma 1.6 and Theorem 2.7, we have the following corollary:

COROLLARY 2.8. *Let P, $Q \in \mathscr{A}_\alpha(\Omega, \mathscr{U})$ be extended holomorphic curves in the sense of Theorem 2.7, then $P \overset{u}{\sim} Q$ if and only if*

$$\overline{\partial}^J \partial^I h_{det_{\mathscr{J}_1(E_p)}}(\lambda) = \overline{\partial}^J \partial^I h_{det_{\mathscr{J}_1(E_q)}}(\lambda),\ \forall J, I \leqslant \alpha - 1,\ \forall \lambda \in \Omega.$$

REFERENCES

[1] C. APOSTOL AND M. MARTIN, *A C^*-algebra approach to the Cowen-Douglas theory*, Topics in modern operator theory (Timisoara/Herculane, 1980), pp. 45–51, Operator Theory: Adv. Appl., 2, Birkhäuser, Basel-Boston, Mass., 1981.

[2] M. J. COWEN AND R. G. DOUGLAS, *Complex geometry and operator theory*, Acta Math. 141 (1978), no. 3–4, 187–261.

[3] K. JI, *Similarity Classification and Properties of Some Extended Holomorphic Curves*, Integr. Equ. Oper. Theory, **69** (2011), no. 1, 133–148.

[4] H. KWON AND S. TREIL, *Similarity of operators and geometry of eigenvector bundles*, Publ. Mat. (2009) Vol. 53, no. 2, 417–438.

[5] L. LI, *On the classification of simple C^*-algebras: Inductive limits of matrix algebras over trees*, Mem. Amer. Math. Soc., no. 605, vol. 127, (1997).

[6] M. MARTIN AND N. SALINAS, *Flag Manifolds and the Cowen-Douglas theory*, J. Operator theory. **38** (1997), no. 2, 329–365.

[7] M. MARTIN AND N. SALINAS, *The canonical complex structure of flag manifolds in a C^*-algebra*, Nonselfadjoint operator algebras, operator theory, and related topics, 173–187, Oper. Theory Adv. Appl., 104, Birkhäuser, Basel, 1998.

[8] M. MARTIN, *Hermitian geometry and involutive algebras*, Math. Z. 188 (1985), no. 3, 359–382.

[9] M. MARTIN AND N. SALINAS, *Differential geometry of generalized Grassmann manifolds in C^*-algebras*, Operator theory and boundary eigenvalue problems (Vienna, 1993), 206–243, Oper. Theory Adv. Appl., 80, Birkhäuser, Basel, 1995.

[10] G. MISRA, *Curvature and backward shift operators*, Proc. Amer. Math. Soc. 91 (1984), no. 2, 105–107.

[11] G. MISRA, *Curvature inequality and extremal properties of bundle shifts*, J. Operator Theory 11 (1984), no. 1, 305–317.

[12] M. RORDAM, F. LARSEN AND N. J. LAUSTSEN, *An introduction to K-Theory for C^*-algebras*, London Mathematical Society Student Texts 49, Cambridge University Press, 2000.

[13] N. SALINAS, *The Grassmann manifold of a C^*-algebra, and Hermitian holomorphic bundles*, Special classes of linear operators and other topics (Bucharest, 1986), 267–289, Oper. Theory Adv. Appl., 28, Birkhäuser, Basel, 1988.

[14] M. UCHIYAMA, *Curvatures and similarity of operators with holomorphic eigenvectors*, Tran. Amer. Math. Soc. 319 (1990), no. 1, 405–415.

Yingli Hou
Department of Mathematics
Hebei Normal University
Shijiazhuang, China, 050024
and
Department of Mathematics
Hebei University of technology
Tianjin, China, 300401
e-mail: houyingli@gmail.com

Kui Ji
Department of Mathematics
Hebei Normal University
Shijiazhuang, China, 050024
e-mail: jikuikui@gmail.com

Operators and Matrices
www.ele-math.com
oam@ele-math.com