INJECTIVITY IN THE QUANTUM SPACE FRAMEWORK

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Abstract. In this paper we investigate injectivity of quantum (or local operator) spaces in terms of their bounded parts. A multinormed W^*-algebra with its injective domain turns out be injective if and only if its bounded part is injective in the normed sense. We prove that each locally finite domain is injective and propose an example of a non-injective domain based on affine schemes. Hamana-Ruan type formula has been obtained for quantum spaces but in a slightly different shape.

1. Introduction

The injectivity is one of the fundamental properties of operator spaces and C^*-algebras. The Arveson-Hahn-Banach-Wittstock Theorem asserts that the C^*-algebra $\mathscr{B}(H)$ of all bounded linear operators on a Hilbert space H is an injective operator space. The family of (operator) norms from $\mathscr{B}(H^n)$, $n \in \mathbb{N}$ define the canonical norm $\|\cdot\|$ on the space $M(\mathscr{B}(H))$ of all finite size matrices over $\mathscr{B}(H)$ called the matrix norm on $\mathscr{B}(H)$. The space $\mathscr{B}(H)$ equipped with this matrix norm is an operator space (or normed quantum space). Thus $\mathscr{B}(H)$ is an injective object in the category of operator spaces and matrix (or complete) contractions. The structure of injective operator spaces and the unique existence of injective envelopes of operator spaces have been investigated in [18] and [23] by M. Hamana and Z.-J. Ruan independently. An injective operator space V is matrix isometric to $pA(1-p)$ for an injective C^*-algebra A and a projection $p \in A$. Moreover, each operator space V has the injective envelope $I(V)$ defined up to a matrix isometry. The injective space $I(V)$ is uniquely defined by the following property. The identity mapping over $I(V)$ is the only matrix contraction extending the identity mapping over V. The known (see [3, 4.4.3]) Hamana-Ruan formula figures out that $I(V)$ is the upper right corner of the injective C^*-algebra $I(\mathscr{P}_V)$, where $\mathscr{P}_V = \begin{bmatrix} \mathbb{C} & V \\ V^* & \mathbb{C} \end{bmatrix}$ is the Paulsen system of V which is a self-adjoint subspace in the space $M_2(V)$ of all 2×2-matrices over V. Thus

$$I(V) = pI(\mathscr{P}_V)(1-p)$$

with $p = 1_H \oplus 0$ and $1 - p = 0 \oplus 1_H$.

The quantum spaces come up as inverse limits of operator spaces naturally [14], [15], [4], [5]. Thus a quantum space V is a linear space equipped with a (separating) family $\{\|\cdot\|_e : e \in Y\}$ of matrix seminorms. We say that Y is a defining family of matrix seminorms of V. A linear mapping $\varphi : V \to W$ between quantum spaces with their defining families Y and X fixed up respectively, is said to be *a quantum contraction* if for each $f \in X$ there corresponds a finite subset $\kappa \subseteq Y$ such that $\left\|\varphi^{(\infty)}(v)\right\|_f \leqslant \|v\|_\kappa$, $v \in M(V)$, where $\varphi^{(\infty)} : M(V) \to M(W)$ is the canonical extension of φ to the matrix spaces over the indicated spaces, and $\|v\|_\kappa = \sup_{e \in \kappa} \|v\|_e$. If $X = Y$ then φ is called *a quantum Y-isometry* if $\left\|\varphi^{(\infty)}(v)\right\|_e = \|v\|_e$ for all $v \in M(V)$ and $e \in Y$. What are the injective objects in the category of quantum spaces and quantum contractions, that is the main problem we have interest in the present paper.

Another motivation to the problem has been observed in the theory of $*$-algebras of unbounded operators in a Hilbert space (see also [6], [8]). The representation theorem obtained in [10] asserts that each quantum space V with its defining family Y of matrix seminorms can be identified with a concrete quantum space in a certain $*$-algebra. Namely, Y is identified with *a (quantum) domain*, which is a commutative set of projections in $\mathscr{B}(H)$ such that $\vee Y = 1_H$, where $\vee Y$ the least upper bound in $\mathscr{B}(H)$ of the projection set. Thereby the algebraic sum $\mathscr{Y} = \sum_{e \in Y} \mathrm{im}(e)$ is a dense subspace in H. If $L(\mathscr{Y})$ is the algebra of all linear transformations on $\mathscr{Y}$, then the algebra of *all noncommutative continuous functions on Y* is defined [5], [7], as the operator $*$-algebra

$$C_Y^*(\mathscr{Y}) = \{T \in L(\mathscr{Y}) : eT \subseteq Te, Te \in \mathscr{B}(H), e \in Y\}.$$

Each unbounded operator $T \in C_Y^*(\mathscr{Y})$ admits an unbounded dual $T^\star$ such that $\mathscr{Y} \subseteq \mathrm{dom}(T^\star)$, $T^\star(\mathscr{Y}) \subseteq \mathscr{Y}$ and $T^* = T^\star|Y \in C_Y^*(\mathscr{Y})$ [5]. Moreover, the $*$-algebra $C_Y^*(\mathscr{Y})$ equipped with the family $\|T\|_e = \|Te\|$, $T \in C_Y^*(\mathscr{Y})$, $e \in Y$ of C^*-seminorms turns out to be a unital multinormed C^*-algebra. Thus V is embedded into $C_Y^*(\mathscr{Y})$ up to a quantum Y-isometry. Note that if $Y = \{1_H\}$ then $C_Y^*(\mathscr{Y})$ is reduced to $\mathscr{B}(H)$. In the general case $C_Y^*(\mathscr{Y})$ is the multinormed completion of the commutant $Y' \subseteq \mathscr{B}(H)$ with respect to the family $\|T\|_e = \|Te\|$, $T \in Y'$, $e \in Y$ of C^*-seminorms. Actually it is a multinormed W^*-algebra (an inverse limit of W^*-algebras with W^*-continuous connecting homomorphisms) played the role of $\mathscr{B}(H)$ in the locally convex setting. The possible injectivity of $C_Y^*(\mathscr{Y})$ presents an interest in the theory of quantum spaces. A domain Y with injective $C_Y^*(\mathscr{Y})$ is called *an injective domain*. A countable domain Y is always injective [5].

In the present paper we investigate the injectivity in the quantum space framework establishing a link between injectivity in the normed sense and the injectivity in quantum sense. It turns out that many multinormed C^*-algebras appear as multinormed completions of injective C^*-algebras. For example, the commutant Y' in $\mathscr{B}(H)$ of a domain $Y \subseteq \mathscr{B}(H)$ is an injective von Neumann algebra, and $C_Y^*(\mathscr{Y})$ is just a multinormed completion of Y'. It is reasonable to ask whether a multinormed completion of an injective operator space is an injective quantum space. The reverse implication always true. Namely, if $\mathscr{A}$ is an injective multinormed C^*-algebra then its bounded part $\mathfrak{b}(\mathscr{A})$ is an injective C^*-algebra (see below Proposition 3.1), where

$\mathfrak{b}(\mathscr{A}) = \{a \in \mathscr{A} : \|a\| = \sup_e \|a\|_e < \infty\}$ is the set of all bounded elements in $\mathscr{A}$. Note that $\mathfrak{b}(\mathscr{A})$ is dense in $\mathscr{A}$. For example, $\mathfrak{b}(C_Y^*(\mathscr{Y})) = Y'$.

Our first central result asserts that the injectivity problem for a multinormed W^*-algebras can be reduced to the injectivity of the relevant domain. Namely, let $\mathscr{A}$ be a multinormed W^*-algebra. The multinormed W^*-algebras are precisely central completions of W^*-algebras due to [10, Proposition 2.1]. Then $\mathscr{A}$ is identified with the completion $\widehat{A}$ of a von Neumann algebra $A \subseteq \mathscr{B}(H)$ with its subset $Y \subseteq A$ of central projections such that $\vee Y = 1_H$ called *the domain of* $\mathscr{A}$. The completion is defined by means of the C^*-seminorms $\|b\|_e = \|be\|$, $b \in A$, $e \in Y$. Note that $\mathfrak{b}(\mathscr{A}) = A \subseteq Y' \subseteq \mathscr{B}(H)$. We prove (see Theorem 4.1) that if the domain Y of $\mathscr{A}$ is injective, then the (normed) injectivity of A implies injectivity of its completion $\mathscr{A}$ in the quantum sense. Moreover, Hamana-Ruan type formula remains true in the following case. Let Y be an injective domain and $V \subseteq C_Y^*(\mathscr{Y})$ a quantum space such that $YV \subseteq V$. Then V is an injective quantum space if and only if

$$V = p\mathscr{A}(1 - p)$$

up to a matrix Y-isometry for a certain injective multinormed C^*-algebra $\mathscr{A} \subseteq M_2(C_Y^*(\mathscr{Y}))$ enveloping Y, and a (bounded) projection p in $\mathscr{A}$ (see Corollary 4.1).

Thus injective domains play a fundamental role in the injectivity problem of quantum spaces. Which domains are injective, and is there a non-injective domain at all? That is the last problem we deal with in the paper. A domain Y is said to be a *locally finite domain* if for each $e \in Y$ we have $ef = 0$ for all $f \in Y$ except finitely many of them. Obviously, each orthogonal (in particular, countable) domain is a locally finite one. We prove (see Theorem 5.1) that a locally finite domain is injective. If the domain is not locally finite then it may not be injective. The relevant example has been proposed based on affine schemes. Namely, we generate domains using the spectra of commutative rings. The idea proposed below plays a critical role in noncommutative algebraic geometry, namely, in the theory of noncommutative schemes for Lie-complete rings developed in [11]. Thus the domain Y generated by an uncountable maximal spectrum of a commutative ring A is not injective (see Theorem 5.2), that is, the multinormed (W^*-algebra) completion $C_Y^*(\mathscr{Y})$ of Y' is not an injective quantum space though its bounded part Y' is an injective von Neumann algebra.

2. Preliminaries

In this section we provide the paper with some preliminaries. The set of all finite subsets of a set X is denoted by $X^\dagger$. The identity operator on a linear space V is denoted by 1_V. The unit ball of a normed space V is denoted by ball V. If $\mathscr{A}$ is a multinormed C^*-algebra with its defining family $\{\|\cdot\|_e : e \in X\}$ of C^*-seminorms then $\mathfrak{b}(\mathscr{A})$ denotes the set of all bounded elements in $\mathscr{A}$, that is, $a \in \mathfrak{b}(\mathscr{A})$ iff $\|a\| = \sup_{e \in X} \|a\|_e < \infty$. Actually (see [24], [19]), $\mathfrak{b}(\mathscr{A})$ equipped with the norm $\|\cdot\|$ is a C^*-algebra called *the bounded part of* $\mathscr{A}$.

2.1. The matrix spaces

The linear space of all $m \times n$-matrices $x = [x_{ij}]$ over a linear space V is denoted by $M_{m,n}(V)$, and we set $M_m(V) = M_{m,m}(V)$ and $M_{m,n} = M_{m,n}(\mathbb{C})$. Further, $M(V)$ (respectively, M) denotes the linear space of all infinite (respectively, scalar) matrices $[x_{ij}]$, $x_{ij} \in V$, where all but finitely many entry x_{ij} are zeros. Each $M_{m,n}(V)$ is a subspace in $M(V)$ comprising those matrices $x = [x_{ij}]$ with $x_{ij} = 0$ whenever $i > m$ or $j > n$. Note that M possesses the operator norm being identified with finite-rank operators on a separable Hilbert space. If $v \in M_{s,t}(V)$ and $w \in M_{m,n}(V)$ then we have their direct sum $v \oplus w = \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \in M_{s+m,t+n}(V)$. If $a \in M_{m,s}$, $v \in M_{s,t}(V)$ and $b \in M_{t,n}$, then we have their matrix product $avb = \left[\sum_{k,l} a_{ik} v_{kl} b_{lj} \right]_{i,j} \in M_{m,n}(V)$. A linear mapping $\varphi : V \to W$ has the canonical linear extensions $\varphi^{(n)} : M_n(V) \to M_n(W)$ (respectively, $\varphi^{(\infty)} : M(V) \to M(W)$) over all matrix spaces defined as $\varphi^{(n)}([x_{ij}]) = [\varphi(x_{ij})]$ (respectively, $\varphi^{(\infty)}|M_n(V) = \varphi^{(n)}$). One can easily verify that $\varphi^{(\infty)}$ preserves just introduced quantum (or matrix) operations.

2.2. Quantum spaces

Let V be a linear space. By a *quantum set* $\mathfrak{B}$ *on* V we mean a collection $\mathfrak{B} = (\mathfrak{b}_n)$ of subsets $\mathfrak{b}_n \subseteq M_n(V)$, $n \geqslant 1$. A quantum set $\mathfrak{B}$ in $M(V)$ is said to be *absolutely matrix convex* [15] if $\mathfrak{B} \oplus \mathfrak{B} \subseteq \mathfrak{B}$ and $a\mathfrak{B}b \subseteq \mathfrak{B}$, $a, b \in$ ball M. One can easily derive that an absolutely matrix convex set $\mathfrak{B}$ turns out to be an absolutely convex subset in $M(V)$ in classical sense as well [14], [9]. The Minkowski functional of an absorbent (in $M(V)$) absolutely matrix convex set is called *a matrix seminorm* on V. A polynormed (or locally convex) topology defined by a separating family of matrix seminorms is called *a quantum topology*, and the linear space V equipped with a quantum topology is called *a quantum space*. A quantum space whose quantum topology is determined by a matrix norm is a called *an abstract operator* (or *quantum normed space*). The subspaces of the C^*-algebra $\mathscr{B}(H)$ of all bounded linear operators on a Hilbert space H are (concrete) operator spaces with their matrix norms inherited from the original matrix norm on $\mathscr{B}(H)$ which is due to the identifications $M_n(\mathscr{B}(H)) = \mathscr{B}(H^n)$ for all $n \geqslant 1$. The morphisms between quantum spaces are *quantum continuous linear mappings*. A linear mapping $\varphi : V \to V'$ between quantum spaces is quantum continuous iff $\varphi^{(\infty)} : M(V) \to M(V')$ is a continuous linear mapping of the relevant polynormed spaces. The matrix seminorms being fixed up allow to define a quantum contraction between quantum spaces (see Section 1). Obviously, a superposition of quantum contractions turns out to be a quantum contraction. For the operator spaces V and V' we have the *matrix* (or *completely*) *bounded linear mappings*. Thus $\|\varphi\|_{mb} = \sup_n \left\| \varphi^{(n)} \right\| < \infty$. A linear mapping $\varphi : V \to V'$ between operator spaces is called *a matrix isometry* if $\left\| \varphi^{(\infty)}(v) \right\|_{M(V')} = \|v\|_{M(V)}$ for all $v \in M(V)$. Finally, if $(V_\kappa)_{\kappa \in \Lambda}$ is a family of quantum spaces then $V = \text{op}\prod_{\kappa \in \Xi} V_\kappa$ denotes their direct product equipped with the initial quantum topology such that all canonical projections $V \to V_\kappa$ are quantum continuous. If $\{V_\alpha, \varphi_{\alpha\beta}\}$ is a projective system of quantum spaces and quantum continuous lin-

ear mappings, then $V = \underleftarrow{\text{oplim}}\left\{V_\alpha, \varphi_{\alpha\beta}\right\}$ denotes *the quantum inverse limit* which is a quantum subspace in $\text{op}\prod_\alpha V_\alpha$. Note that each quantum space is an inverse limit of operator spaces [14], [9].

2.3. The point weak operator topology

Let $A \subseteq \mathscr{B}(H)$ be a von Neumann algebra on a Hilbert space H, $Y \subseteq A$ a subset of its central projections, and let R be the unital subring in A generated by Y. Thus R is a commutative ring and A is an algebraic module over the ring R. The Banach algebra of all bounded linear operators on A (with the operator norm $\|\cdot\|$) is denoted by $\mathscr{B}(A)$. An element $T \in \mathscr{B}(A)$ is said to be *a R-homomorphism* if $T(ea) = eT(a)$ for all $e \in R$ and $a \in A$. The set of all bounded R-homomorphism $A \to A$ is a closed subalgebra in $\mathscr{B}(A)$ denoted by $\mathscr{B}_R(A)$. The set of of all matrix (or completely) bounded mapping on A is denoted by $\mathscr{M}\mathscr{B}(A)$ whereas $\mathscr{M}\mathscr{B}_R(A)$ denotes its subset of all matrix bounded R-homomorphisms. Note that $\mathscr{M}\mathscr{B}(A)$ equipped with the matrix norm $\|\cdot\|_{mb}$ is a Banach algebra as well, and $\mathscr{M}\mathscr{B}_R(A)$ is its closed subalgebra. If Y consists of only the unit element of A then $\mathscr{B}_R(A) = \mathscr{B}(A)$ and $\mathscr{M}\mathscr{B}_R(A) = \mathscr{M}\mathscr{B}(A)$. The set ball $\mathscr{M}\mathscr{B}_R(A)$ consists of all matrix contractive R-homomorphisms $\varphi : A \to A$, $\|\varphi\|_{mb} \leqslant 1$. The family of seminorms $w_{b,x,y}(T) = |\langle T(b)x, y \rangle|$, $b \in \text{ball}A$, $x, y \in H$, defines a Hausdorff polynormed topology in $\mathscr{B}_R(A)$ called the point-weak operator topology (briefly p-WOT). In particular, ball $\mathscr{B}_R(A)$ is a topological space with its subspace ball $\mathscr{M}\mathscr{B}_R(A)$.

LEMMA 2.1. *The space* ball$\mathscr{B}_R(A)$ *is a (p-WOT) compact space and* ball$\mathscr{M}\mathscr{B}_R(A)$ *is its closed subspace.*

Proof. Take an ultrafilter $\mathfrak{F}$ in ball$\mathscr{B}_R(A)$. For each $b \in \text{ball}A$ we have a well defined mapping $\widehat{b} : \text{ball}\mathscr{B}_R(A) \to \text{ball}A$, $\widehat{b}(T) = T(b)$. In particular, the range $\mathfrak{F}(b)$ of $\mathfrak{F}$ by means of the mapping $\widehat{b}$ is an ultrafilter base in ball$A$. But ball$A$ being WOT-closed subset in ball$\mathscr{B}(H)$ is an WOT-compact set. Hence there exists $\varphi(b) = \text{WOT-}\lim\mathfrak{F}(b) \in \text{ball}A$. In particular, we have a well defined mapping $\varphi \in \text{ball}\mathscr{B}(A)$ and $\varphi = \text{p-WOT-}\lim\mathfrak{F}$. Note that $\varphi(ea) = \text{WOT-}\lim\mathfrak{F}(ea) = \text{WOT-}\lim e\mathfrak{F}(a) = e\,\text{WOT-}\lim\mathfrak{F}(a) = e\varphi(a)$ for all $e \in R$ and $a \in A$. Hence ball$\mathscr{B}_R(A)$ is a (p-WOT)-compact space.

Finally, let us prove that ball$\mathscr{M}\mathscr{B}_R(A)$ is a p-WOT-closed subspace in ball$\mathscr{B}_R(A)$. Take a net $(\varphi_\lambda) \subseteq \text{ball}\mathscr{M}\mathscr{B}_R(A)$ with $\varphi = \text{p-WOT-}\lim\varphi_\lambda$. If $b = [b_{ij}] \in M_n(A)$, $x = (x_i)$, $y = (y_i) \in H^n$ then

$$w_{b,x,y}\left(\varphi^{(n)} - \varphi_\lambda^{(n)}\right) = \left|\left\langle\left(\varphi^{(n)} - \varphi_\lambda^{(n)}\right)(b)x, y\right\rangle\right| = \left|\sum_{i,k}^{n}\langle(\varphi - \varphi_\lambda)(b_{ik})x_k, y_i\rangle\right|$$

$$\leqslant \sum_{i,k}^{n} w_{b_{ik},x_k,y_i}(\varphi - \varphi_\lambda),$$

therefore $\varphi^{(n)}(b) = \text{WOT-lim}_\lambda \varphi_\lambda^{(n)}(b)$ in $\mathscr{B}(H^n)$. Using (WOT) semicontinuity of the norm in $\mathscr{B}(H^n)$, we obtain that $\left\|\varphi^{(n)}(b)\right\| \leqslant \limsup_\lambda \left\|\varphi_\lambda^{(n)}(b)\right\| \leqslant \|b\|$ for each $b \in M_n(A)$. Hence $\varphi \in \text{ball}\,\mathscr{MB}_R(A)$, and $\text{ball}\,\mathscr{MB}_R(A)$ is a (p-WOT) compact subspace in $\text{ball}\,\mathscr{B}_R(A)$. $\quad\square$

2.4. Quantum domains

Let H be a Hilbert space. By a *quantum domain on H* we mean a subset $X \subseteq \mathscr{B}(H)$ of projections such that $\vee X = 1_H$, where $\vee X$ is the least upper bound (that is, $\sup X$) in $\mathscr{B}(H)$ of the projection set X. If X is a commutative family of projections then we briefly say that X is *a domain in H*. Consider the algebraic sum $\mathscr{X} = \sum_{e \in X} \text{im}(e)$ of a quantum domain X, which is a dense subspace in H. If $L(\mathscr{X})$ is the algebra of all linear transformations on the space $\mathscr{X}$ then *the algebra of all non-commutative continuous functions on X* (or on $\mathscr{X}$) is defined [5] as the $*$-algebra (see Section 1)

$$C_X^*(\mathscr{X}) = \{T \in L(\mathscr{X}) : eT \subseteq Te, Te \in \mathscr{B}(H), e \in X\}.$$

equipped with the family $\|T\|_e = \|Te\|$, $T \in C_X^*(\mathscr{X})$, $e \in X$ of C^*-seminorms turns out to be a unital multinormed C^*-algebra. For a finite subset $\alpha \subseteq X$ of a domain X we have a continuous C^*-seminorm $\|T\|_\alpha = \|T \cdot \vee \alpha\|$, $T \in C_X^*(\mathscr{X})$ on the algebra $C_X^*(\mathscr{X})$ too. Actually, $\|T\|_\alpha = \sup_{e \in \alpha} \|T\|_e$, $T \in C_X^*(\mathscr{X})$ (see [10]) if X is a domain. Thus $\left\{\|\cdot\|_\alpha : \alpha \in X^\dagger\right\}$ is an upward filtered family of C^*-seminorms on $C_X^*(\mathscr{X})$ which defines the original topology.

A linear subspace $V \subseteq C_X^*(\mathscr{X})$ is called *a quantum* (or *local operator*) *space* whereas a unital selfadjoint subspace $V \subseteq C_X^*(\mathscr{X})$ is called *a quantum system* [5]. In particular, for each $e \in X$ the space $Ve = \{Te : T \in V\}$ is an operator space (resp., operator system) in $\mathscr{B}(H)$ or in $\mathscr{B}(\text{im}(e))$. If $\mathscr{A}$ is a unital multinormed C^*-algebra with its family $\{q_e : e \in X\}$ of C^*-seminorms then X is identified with a certain quantum domain and there exists $*$-homomorphism $\varphi : \mathscr{A} \to C_X^*(\mathscr{X})$ such that $\|\varphi(a)\|_e = q_e(a)$, $a \in \mathscr{A}$ for all $e \in X$ (see [21], [5]). Moreover as shown in [10] the quantum domain X can be assumed to be commutative, that is, X is a domain.

Now let X be a domain on H. The commutant X' of X in $\mathscr{B}(H)$ is a unital von Neumann algebra on H, and the family $\|u\|_e = \|ue\|$, $u \in X'$, $e \in X$ of C^*-seminorms defines a polynormed topology on X'. The completion of X' with respect to this topology is reduced to the algebra $C_X^*(\mathscr{X})$ (see [10]). Moreover, $\mathfrak{b}(C_X^*(\mathscr{X})) = X'$ [10]. If X is an orthogonal family of projections (in this case $\sum X = 1_H$ (WOT)) then we say that X is *a graded domain*. For a graded domain X, we have $C_X^*(\mathscr{X}) = \prod_{e \in X} \mathscr{B}(\text{im}(e))$ (see [9]) is the direct product of C^*-algebras equipped with the direct product topology. Note that $C_X^*(\mathscr{X}) = \mathscr{B}(H)$ whenever $X = \{1_H\}$.

REMARK 2.1. If $X = \{e_n : n \in \mathbb{N}\}$ is a countable quantum domain in $\mathscr{B}(H)$ then it can be reduced to the graded one without changing the relevant $*$-algebra $C_X^*(\mathscr{X})$. Indeed, put $f_n = (1 - e_{n-1})e_n$, $n \in \mathbb{N}$. Then $e_n = \sum_{k=1}^n f_k$ and $Y = \{f_n : n \in \mathbb{N}\}$ is an orthogonal family of projections such that $\sum_n f_n = 1_H$. Moreover, $X' = Y'$ in $\mathscr{B}(H)$,

$\mathscr{X} = \mathscr{Y}$ in H, and $\|T\|_{e_n} = \sup_{1 \leqslant k \leqslant n} \|T\|_{f_k}$ for all $T \in C_X^*(\mathscr{X})$. Thereby $C_X^*(\mathscr{X}) = C_Y^*(\mathscr{Y})$.

2.5. Local positivity

Let X be a quantum domain in $\mathscr{B}(H)$. Take $a \in M_n(C_X^*(\mathscr{X}))$. We say that a is *locally matrix positive* if $ae^{\oplus n} \geqslant 0$ in $\mathscr{B}(H^n)$, or $a|\operatorname{im}(e)^n \geqslant 0$ in $\mathscr{B}(\operatorname{im}(e)^n)$ for a certain $e \in X$, where $e^{\oplus n} = e \oplus \cdots \oplus e$. In this case we write $a \geqslant_e 0$. Similarly, for $\alpha \in X^\dagger$, we write $a \geqslant_\alpha 0$ if $a \cdot (\vee \alpha)^{\oplus n} \geqslant 0$. In particular, the notation $a =_e 0$ indicates to the equality $ae^{\oplus n} = 0$. Now let Y be another quantum domain with its algebraic sum $\mathscr{Y} = \sum_{f \in Y} \operatorname{im}(f)$. A linear mapping $\varphi : W \to V$ between quantum systems $W \subseteq C_Y^*(\mathscr{Y})$ and $V \subseteq C_X^*(\mathscr{X})$ is said to be *a quantum positive* if for each $e \in X$ there corresponds $\kappa \in Y^\dagger$ such that $a \in M(W)$, $a \geqslant_\kappa 0$ (resp., $a =_\kappa 0$) implies that $\varphi^{(\infty)}(a) \geqslant_e 0$ (resp., $\varphi^{(\infty)}(a) =_e 0$) [5], where $\varphi^{(\infty)} : M(W) \to M(V)$, $\varphi^{(\infty)}[w_{ij}] = [\varphi(w_{ij})]$ is the canonical extension of φ. Thus we have a well defined matrix (or completely) positive mapping $\varphi_{e\kappa} : W \cdot \vee \kappa \to Ve$, $\varphi_{e\kappa}(a \cdot \vee \kappa) = \varphi(a)e$ of the operator systems. A unital quantum positive mapping is called *a quantum morphism*. Similarly, a linear mapping $\varphi : W \to V$ between quantum spaces $W \subseteq C_Y^*(\mathscr{Y})$ and $V \subseteq C_X^*(\mathscr{X})$ is a quantum contraction if for each $e \in X$ there corresponds $\kappa \in Y^\dagger$ such that $\left\| \varphi^{(\infty)}(w) \right\|_e \leqslant \|w\|_\kappa$, $w \in M(W)$, that is, we have a well defined matrix (or complete) contraction $\varphi_{e\kappa} : W \cdot \vee \kappa \to Ve$, $\varphi_{e\kappa}(a \cdot \vee \kappa) = \varphi(a)e$ of the operator spaces. As it is well (especially in the normed case) known (see [5, Corollary 4.1]) a unital linear mapping $\varphi : W \to V$ of quantum systems is quantum positive iff it is a quantum contraction.

REMARK 2.2. Actually, a quantum positive mapping $\varphi : W \to V$ is a quantum continuous mappings with $\left\| \varphi^{(\infty)}(w) \right\|_e \leqslant \|\varphi_{e\kappa}(\vee \kappa)\| \|w\|_\kappa = \|\varphi(1_W)e\| \|w\|_\kappa$, $w \in M(W)$ (see [5, Lemma 4.4]). In particular, if a quantum positive mapping $\varphi : W \to V$ is a local contraction (or unital) then it is a quantum contraction. Indeed, by its very definition for each $e \in X$ there corresponds another $\iota \in Y^\dagger$ such that $\|\varphi(w)\|_e \leqslant \|w\|_\iota$, $w \in M(W)$. It follows that $\sup_{e \in X} \|\varphi(1_W)e\| = \sup_{e \in X} \|\varphi(1_W)\|_e \leqslant \sup_{\iota \in X^\dagger} \|1_W\|_\iota = 1$. Therefore $\left\| \varphi^{(\infty)}(w) \right\|_e \leqslant \|w\|_\kappa$, $w \in M(W)$.

Now let $\lambda \in M_n$ and $p \in \mathscr{B}(H)$. Based on the canonical identification $\mathbb{C}^n \otimes H = H^n$, we have the following matrix $\lambda p^{\oplus n} = \lambda \otimes p = [\lambda_{ij}p]_{i,j} \in M_n(\mathscr{B}(H))$ in its various shapes. The following lemma is trivial but for the sake of a reader we provide its proof.

LEMMA 2.2. *Let $\lambda \in M_n$ and $p \in \mathscr{B}(H)$ be a nonzero projection. Then $\lambda p^{\oplus n} \geqslant 0$ (resp., hermitian) in $M_n(\mathscr{B}(H))$ if and only if $\lambda \geqslant 0$ (resp., hermitian) in M_n.*

Proof. Note that M_n is a nuclear C^*-algebra and $M_n(\mathscr{B}(H)) = M_n \otimes \mathscr{B}(H)$ is the tensor product of C^*-algebras equipped with its unique C^*-norm. It follows that $\lambda p^{\oplus n} = \lambda \otimes p \geqslant 0$ if $\lambda \geqslant 0$ in M_n. Conversely, assume that $\lambda \otimes p \geqslant 0$. Then

$(\lambda^* - \lambda) \otimes p = 0$, which in turn implies that $(\lambda^* - \lambda) \otimes 1 = 0$ in $\mathscr{B}(\mathrm{im}(p)^n)$. Therefore $\lambda^* = \lambda$. Finally, a negative eigenvalue $\lambda(z) = rz$, $r < 0$, $z \in \mathbb{C}^n \setminus \{0\}$, would lead to a similar one $\lambda \otimes p(z \otimes x) = -rz \otimes x$, $x \in \mathrm{im}(p) \setminus \{0\}$. Actually, the equality of spectra $\sigma(\lambda \otimes p) = \sigma(\lambda)\sigma(p) \geqslant 0$ implies that $\sigma(\lambda) \geqslant 0$. Hence $\lambda \geqslant 0$ in M_n. $\quad\square$

3. Injective module envelopes

In this section we introduce injectivity in the quantum space setting. It turns out that this locally convex injectivity deals with the normed injectivity equipped with some algebraic module structures. Therefore in this section we mainly develop a normed background of locally convex injectivity.

3.1. The rigid inclusions for quantum spaces

Let X be a domain and $V \subseteq C_X^*(\mathscr{X})$ a quantum system. We say that V is *an injective quantum system* if for a quantum system inclusion $W_0 \subseteq W$ and a quantum morphism $\varphi : W_0 \to V$ there exists a quantum morphism $\Phi : W \to V$ extending φ. A subspace $V \subseteq C_X^*(\mathscr{X})$ is said to be *an injective quantum space* (or *strong injective local operator space*) if for a quantum space inclusion $W_0 \subseteq W$, and a quantum contraction $\varphi : W_0 \to V$ there exists a quantum contraction $\Phi : W \to V$ (with respect to the same family of matrix seminorms) extending φ (see [5]). Finally, a multinormed C^*-algebra $\mathscr{A}$ is said to be *injective* if for each quantum system W_0 in a multinormed C^*-algebra $\mathscr{B}$ and a quantum positive mapping $\varphi : W_0 \to \mathscr{A}$ there exists a quantum positive mapping $\Phi : \mathscr{B} \to \mathscr{A}$ extending φ.

REMARK 3.1. Each injective multinormed C^*-algebra is an injective quantum system. Moreover, if a quantum system is an injective quantum space then it turns out to be an injective quantum system. For a unital linear mapping between quantum systems is a quantum morphism iff it is a quantum contraction [5, Corollary 4.1].

The C^*-algebra $\mathscr{B}(H)$ is an injective quantum space. Indeed, if $W_0 \subseteq W \subseteq C_Y^*(\mathscr{Y})$ is a quantum space inclusions, and $\varphi : W_0 \to \mathscr{B}(H)$ is a quantum contraction, then $\left\|\varphi^{(\infty)}(w)\right\| \leqslant \|w\|_\kappa$, $w \in M(W_0)$ for some $\kappa \in Y^\dagger$. Using Arveson-Hahn-Banach-Wittstock theorem, we derive that φ has an extension $\Phi : W \to \mathscr{B}(H)$ such that $\left\|\Phi^{(\infty)}(w)\right\| \leqslant \|w\|_\kappa$, $w \in M(W)$, that is, Φ is a quantum contractive extension of φ. In this case, the same matrix seminorm $\|\cdot\|_\kappa$ has been preserved for the extension Φ of φ. In particular, each injective operator space is an injective quantum space.

Let $V \subseteq W \subseteq C_X^*(\mathscr{X})$ be the inclusions of quantum spaces. As in the normed case, we say that the inclusion $V \subseteq W$ is *rigid* if we have only the identity quantum contractive (with respect to the fixed family of matrix seminorms) mapping over W extending the identity mapping over V. If $V \subseteq W$ is rigid then W is the only possible injective quantum space within V and W. Indeed, if W_0 is an injective quantum space and $V \subseteq W_0 \subseteq W$ then the identity mapping $W_0 \to W_0$ has a quantum contractive ex-

tension $\varphi : W \to W_0 \subseteq W$. But $\varphi|V = \mathrm{id}$ and $V \subseteq W$ is rigid, therefore $\varphi = \mathrm{id}$, that is, $W \subseteq W_0$. Therefore $W_0 = W$.

If $V \subseteq W_1$ and $V \subseteq W_2$ are rigid inclusions, and both W_1 and W_2 are injective quantum spaces then $W_1 = W_2$ up to a quantum isomorphism. Indeed, the inclusion $V \subseteq W_2$ is extended to a quantum contractive mapping $\varphi_1 : W_1 \to W_2$ such that $\varphi_1|V = \mathrm{id}$. Similarly, we have a quantum contraction $\varphi_2 : W_2 \to W_1$ such that $\varphi_2|V = \mathrm{id}$. Then $\varphi_2 \varphi_1$ and $\varphi_1 \varphi_2$ are both quantum contractions over W_1 and W_2, respectively, acting as the identity mappings over V. Whence $\varphi_2 \varphi_1 = \mathrm{id}$ and $\varphi_1 \varphi_2 = \mathrm{id}$, that is, $W_1 = W_2$ up to a quantum isomorphism. Certainly all just considered inclusions can be replaced by quantum isometric embeddings (fixing up the family of seminorms). This allows to define *the injective envelope of V* as an injective quantum space $I(V)$ with the rigid inclusion $V \subseteq I(V)$ up to a quantum isomorphism.

REMARK 3.2. Assume V is a quantum space with the rigid inclusion $V \subseteq I(V)$ into its injective envelope. If $V \subseteq W$ and W is an injective quantum space then $I(V) \subseteq W$ up to a quantum isomorphism. Indeed, the inclusion $V \subseteq W$ is extended to a quantum contraction $\varphi : I(V) \to W$, for W is injective. Similarly, $I(V)$ being injective, the inclusion $V \subseteq I(V)$ is extended to a quantum contraction $\psi : W \to I(V)$. But $\psi \varphi : I(V) \to I(V)$ is a quantum contractive extension of the identity mapping over V. By the rigidity property of the injective envelope, we have $\psi \varphi = \mathrm{id}$, which in turn implies that φ is a quantum isomorphic embedding. Thus $I(V) \subseteq W$.

A dense inclusion into is always rigid. So is the inclusion $Y' = \mathfrak{b}(C_Y^*(\mathscr{Y})) \subseteq C_Y^*(\mathscr{Y})$. One has to concern to the latter inclusion with precautions, for in many cases Y' is an injective operator space (therefore injective quantum space) but it does not imply the injectivity of $C_Y^*(\mathscr{Y})$ though $Y' \subseteq C_Y^*(\mathscr{Y})$ is rigid. The latter inclusion is considered in the category of polynormed spaces (not normed ones), the identity mapping $Y' \to Y'$ is not quantum contraction if the initial Y' equipped with the polynormed topology from $C_Y^*(\mathscr{Y})$ but the terminal Y' with the normed topology from $\mathscr{B}(H)$.

PROPOSITION 3.1. *If $\mathscr{A}$ is an injective multinormed C^*-algebra then $\mathfrak{b}(\mathscr{A})$ is an injective C^*-algebra.*

Proof. Assume $\{q_e : e \in X\}$ is a defining family of C^*-seminorms on $\mathscr{A}$. As we have mentioned above, the set X is identified with a quantum domain on a Hilbert space H such that $\mathscr{A}$ turns out to be a closed $*$-subalgebra in $C_X^*(\mathscr{X})$. Thus $q_e = \|\cdot\|_e$ on $\mathscr{A}$ for all $e \in X$. Let W_0 be an operator system in a unital C^*-algebra B and let $\varphi : W_0 \to \mathfrak{b}(\mathscr{A})$ be a matrix (or completely) positive mapping. For each positive $a \in M(W_0)$ its range $\varphi^{(\infty)}(a)$ is positive in $M(\mathfrak{b}(\mathscr{A}))$, in particular, $\varphi^{(\infty)}(a) \geqslant_e 0$ in $M(\mathscr{A})$ for all $e \in X$. Thus $\varphi : W_0 \to \mathfrak{b}(\mathscr{A}) \subseteq \mathscr{A}$ is a quantum positive mapping, which is extended to a quantum positive mapping $\Phi : B \to \mathscr{A}$, for $\mathscr{A}$ is an injective multinormed C^*-algebra. By its very definition, each mapping $\Phi_e : B \to \mathscr{A}e$, $\Phi_e(b) = \Phi(b)e$ is matrix positive. Then $\|\Phi_e\|_{mb} = \|\Phi_e\| = \|\Phi_e(1_B)\|_e = q_e(\Phi(1_B))$ (see [12, Lemma 5.1.1] and Remark 2.2), that is, $q_e\left(\Phi^{(\infty)}(b)\right) \leqslant q_e(\Phi(1_B))\|b\|$ for all $b \in M(B)$, $e \in X$.

But $\Phi(1_B) = \varphi(1_B) \in \mathfrak{b}(\mathscr{A})$. It follows that

$$\left\| \Phi^{(\infty)}(b) \right\| = \sup_{e \in X} q_e \left(\Phi^{(\infty)}(b) \right) \leqslant \sup_{e \in X} q_e(\varphi(1_B)) \|b\| = \|\varphi(1_B)\| \|b\|$$

for all $b \in M(B)$. Thus $\Phi(B) \subseteq \mathfrak{b}(\mathscr{A})$ and Φ is a matrix bounded. It remains to prove that $\Phi : B \to \mathfrak{b}(\mathscr{A})$ is matrix positive. Take a positive matrix $b \in M_n(B)$. Then $\varphi^{(n)}(b) \in M_n(\mathfrak{b}(\mathscr{A})) = \mathfrak{b}(M_n(\mathscr{A}))$. So, we can assume that $n = 1$. Since $\Phi : B \to \mathscr{A}$ is quantum positive, it follows that $\Phi(b) \geqslant_e 0$ for all $e \in X$. Then $\Phi(b)$ is hermitian (see [5, Lemma 4.3]). But $\mathscr{A}$ is an Arens-Michael algebra being the inverse limit of C^*-algebras $A_e = \mathscr{A}/\ker q_e$, $e \in X$. Therefore $\sigma_{\mathscr{A}}(\Phi(b)) = \cup_{e \in X} \sigma_{A_e}(\Phi_e(b)) \geqslant 0$ (see [20, 5.2.12]), where σ indicates to spectra in the relevant algebras. But $\sigma_{\mathfrak{b}(\mathscr{A})}(\Phi(b))$ is the closure of $\sigma_{\mathscr{A}}(\Phi(b))$ [22, Proposition 1.11]. Hence $\sigma_{\mathfrak{b}(\mathscr{A})}(\Phi(b)) \geqslant 0$, that is, $\Phi(b)$ is positive in $\mathfrak{b}(\mathscr{A})$. $\quad\square$

3.2. The graded domains

The following assertion states injectivity of $C_X^*(\mathscr{X})$ for a graded domain X.

LEMMA 3.1. *Let A_e, $e \in X$ be a family of injective C^*-algebras (resp., injective operator spaces). Then $\mathscr{A} = \prod_{e \in X} A_e$ is an injective multinormed C^*-algebra (resp., injective quantum space). In particular, if X is a graded domain in a Hilbert space H then $C_X^*(\mathscr{X})$ is an injective multinormed C^*-algebra and injective quantum space simultaneously.*

Proof. First, assume that all A_e, $e \in X$ are injective C^*-algebras. Let W_0 be a quantum system in a unital multinormed C^*-algebra $\mathscr{B}$ with its defining family $\{q_f : f \in Y\}$ of C^*-seminorms. One can assume [21], [10] that $\mathscr{B} \subseteq C_Y^*(\mathscr{Y})$ is a closed $*$-subalgebra and $q_f(b) = \|b\|_f$, $b \in \mathscr{B}$ (Y is identified with is a quantum domain). If $\varphi : W_0 \to \mathscr{A}$ is a quantum positive mapping, then for each $e \in X$ there corresponds $\kappa \in Y^{\dagger}$ such that the mapping $\varphi_{e\kappa} : W_0 \cdot \vee \kappa \to A_e$, $\varphi_{e\kappa}(w \cdot \vee \kappa) = \pi_e \varphi(w)$ is a matrix positive mapping of the operator systems, where $\pi_e : \mathscr{A} \to A_e$ is the canonical projection. Note that $\mathscr{B} \cdot \vee \kappa$ is a C^*-algebra (see [24]) and $W_0 \cdot \vee \kappa$ is an operator system in $\mathscr{B} \cdot \vee \kappa$. Taking into account the injectivity of A_e, we obtain a matrix positive mapping $\Phi_{e\kappa} : \mathscr{B} \cdot \vee \kappa \to A_e$ extending $\varphi_{e\kappa}$. In particular, the mapping $\Phi_e : \mathscr{B} \to A_e$, $\Phi_e(b) = \Phi_{e\kappa}(b \cdot \vee \kappa)$ is quantum positive. Consider the mapping $\Phi : \mathscr{B} \to \prod_{e \in X} A_e = \mathscr{A}$, $\Phi(b) = (\Phi_e(b))_e$. Since $\Phi^{(\infty)}(b) = \left(\Phi_e^{(\infty)}(b) \right)_e$ (up to the canonical identification), $b \in M(\mathscr{B})$, we derive that Φ is a quantum positive mapping. If $w \in W_0$ then

$$\Phi(w) = (\Phi_e(w))_e = (\Phi_{e\kappa}(w \cdot \vee \kappa))_e = (\varphi_{e\kappa}(w \cdot \vee \kappa))_\kappa = (\pi_e \varphi(w))_e = \varphi(w)$$

(κ depends upon e) that is, $\Phi(w) = \varphi(w)$. Thus $\mathscr{A}$ is an injective multinormed C^*-algebra. In particular, if X is a graded domain in a Hilbert space H then $C_X^*(\mathscr{X}) =$

$\prod_{e \in X} \mathscr{B}(\mathrm{im}(e))$. But each $\mathscr{B}(\mathrm{im}(e))$ is an injective C^*-algebra thanks to Arveson-Hahn-Banach-Wittstock theorem. Therefore $C_X^*(\mathscr{X})$ is an injective multinormed C^*-algebra.

Finally, assume that all A_e, $e \in X$ are injective operator spaces, $W_0 \subseteq W \subseteq C_Y^*(\mathscr{Y})$ a quantum space inclusions. If $\varphi : W_0 \to \mathscr{A}$ is a quantum contraction then for each $e \in X$ there corresponds $\kappa \in Y^\dagger$ such that the mapping $\varphi_{e\kappa} : W_0 \cdot \vee \kappa \to A_e$, $\varphi_{e\kappa}(w \cdot \vee \kappa) = \pi_e \varphi(w)$ is a matrix contraction (see Remark 2.2). Based on the injectivity of A_e, we obtain a matrix contractive $\Phi_{e\kappa} : W \cdot \vee \kappa \to A_e$ extension of $\varphi_{e\kappa}$. In particular, the mapping $\Phi_e : W \to A_e$, $\Phi_e(b) = \Phi_{e\kappa}(b \cdot \vee \kappa)$ is a quantum contraction. As above the mapping $\Phi : W \to \prod_{e \in X} A_e = \mathscr{A}$, $\Phi(w) = (\Phi_e(w))_e$ is a quantum contraction extending φ. In particular, $C_X^*(\mathscr{X}) = \mathrm{op} \prod_{e \in X} \mathscr{B}(\mathrm{im}(e))$ is an injective quantum space. □

REMARK 3.3. Note that $\mathfrak{b}(\mathscr{A}) = \bigoplus_{e \in X}^{\infty} A_e$ whenever $\mathscr{A} = \prod_{e \in X} A_e$, and $X' = \mathfrak{b}(C_X^*(\mathscr{X}))$. Based on Proposition 3.1, we obtain that $\bigoplus_{e \in X}^{\infty} A_e$ is an injective C^*-algebra (see also [2, IV. 2.1.2 (ii)]).

PROPOSITION 3.2. *Let V be a quantum space (resp., system). Then V is injective if and only if it can be identified with a quantum space (resp., system) in some $C_X^*(\mathscr{X})$ with a graded domain X such that $V = P(C_X^*(\mathscr{X}))$ for a certain quantum contractive projection (resp., quantum morphism-projection) $P : C_X^*(\mathscr{X}) \to C_X^*(\mathscr{X})$.*

Proof. Using [10], we can assume that $V \subseteq C_Y^*(\mathscr{Y})$ for a certain domain Y. Put $H = \oplus_{e \in Y} \mathrm{im}(e)$ and let X be the family of canonical projections in $\mathscr{B}(H)$ generated by the latter decomposition. Thus X is a graded domain in H. Consider the *-homomorphism $i : C_Y^*(\mathscr{Y}) \to C_X^*(\mathscr{X})$, $i(T) = (T|\mathrm{im}(e))_{e \in Y}$. Then $\|T|\mathrm{im}(e)\| = \|Te\| = \|T\|_e$, $T \in C_Y^*(\mathscr{Y})$, $e \in Y$, that is, i is a quantum Y-isometry. Thus V is identified with a quantum space or system in $C_X^*(\mathscr{X})$ for a certain graded domain X. Since V is an injective quantum space (resp., system), it follows that the identity mapping $V \to V$ is extended up to a quantum contraction (resp., quantum morphism-projection) $P : C_X^*(\mathscr{X}) \to C_X^*(\mathscr{X})$ onto the subspace V.

Conversely, if $V = P(C_X^*(\mathscr{X}))$ for a certain quantum contraction (resp., quantum morphism-projection) $P : C_X^*(\mathscr{X}) \to C_X^*(\mathscr{X})$, then V is injective. Indeed, fix a quantum space (resp., system) inclusion $W_0 \subseteq W$ and a quantum contraction (resp., quantum morphism) $\varphi : W_0 \to V \subseteq C_X^*(\mathscr{X})$. By Lemma 3.1, $C_X^*(\mathscr{X})$ is an injective quantum space. It is an injective quantum system either (see Remark 3.1). Therefore φ admits a quantum contractive (resp., quantum morphism) extension $\psi : W \to C_X^*(\mathscr{X})$. Put $\Phi = P\psi : W \to V$ which is a quantum contraction (resp., quantum morphism), that is, V is injective. □

As we have seen in [10] the multinormed C^*-algebra $C_X^*(\mathscr{X})$ for a (commutative) domain X inherits many properties of $\mathscr{B}(H)$ in the locally convex theory. But the property to be injective quantum space (or system) we have seen just in the graded ("extreme") case. Later on we generalize the result to locally finite domains. Finally,

note the rigidity introduced above is too general to be pushed forward. It is not even clear whether it does exist. Some additional module structures that we have seen in multinormed W^*-algebras allow to introduce the rigidity which is closed to the rigidity of operator spaces.

3.3. Injective R-envelope of an operator space

Now let $A \subseteq \mathcal{B}(H)$ be an injective (unital) von Neumann algebra on a Hilbert space H, and let $Y \subseteq A$ be a subset of its central projections. For the union $Y \cup \{1_H\}$ we use the notation Y_+, which is a commutative set of projections in $\mathcal{B}(H)$ as well. In particular, it generates the commutative subring $R(Y_+)$ in A, which consists of all $\mathbb{Z}$-finite sums of monomials (or words) in elements of Y_+. For brevity we use the notation R instead of $R(Y_+)$. Since $RA = AR \subseteq A$, the algebra A is equipped with canonical R-module structure. Thus A is an algebraic R-module.

Let $V \subseteq A$ be a subspace such that $YV \subseteq V$. Thus $eV \subseteq V$ for all $e \in Y$. It follows that $(1-e)V \subseteq V$, $e \in Y$, and $e_1 \cdots e_n V \subseteq V$ for all $e_i \in Y$, that is, V is an algebraic R-module. Thus V is (an algebraic) R-module iff $YV \subseteq V$. If W is an injective subspace in A such that $YW \subseteq W$ then W is called *an injective R-module*. So, is the enveloping algebra A. We say that the inclusion $V \subseteq W$ of an operator space V into a R-submodule W in A is *R-rigid* if we have only the identity matrix contractive R-homomorphism over W extending the identity mapping over V.

LEMMA 3.2. *Let A be an injective von Neumann algebra with its subset $Y \subseteq A$ of central projections, R the unital subring in A generated by Y, and let $W \subseteq A$ be an operator space. Then W is an injective R-module iff $W = \mathrm{im}(P)$ is the range of a certain projection $P \in \mathrm{ball}\, \mathcal{MB}_R(A)$.*

Proof. First assume that $W = \mathrm{im}(P)$ is the range of a certain projection $P \in \mathrm{ball}\, \mathcal{MB}_R(A)$. Since A is injective, it is the range of a certain morphism-projection $\mathcal{B}(H) \to \mathcal{B}(H)$ [17]. Therefore A is an injective operator space. In particular, W is an injective operator space [12, 4.1.6]. Moreover, W is a R-submodule in A being the range of a R-homomorphism. Thus W is an injective R-module.

Conversely, assume that W is an injective R-module. The identity mapping over W is extended to a matrix contractive projection onto W. Thus $W = \mathrm{im}(P)$ for a certain projection $P \in \mathrm{ball}\, \mathcal{MB}(A)$. Pick $x \in A$ and a projection e from the ring R. By assumption, $RW \subseteq W$, therefore $ex \in A$ and $(1-e)P(ex) \in W$. But $P|W = \mathrm{id}$. Therefore $P((1-e)P(ex)) = (1-e)P(ex)$ and we can use the argument from [13, Proof of Theorem 2.5], which is due to Effros, Ozawa and Ruan. Namely,

$$
\begin{aligned}
(1+\lambda)^2 \|(1-e)P(ex)\|^2 &= \|(1-e)P(ex + \lambda(1-e)P(ex))\|^2 \\
&\leqslant \|ex + \lambda(1-e)P(ex)\|^2 \\
&\leqslant \|ex\|^2 + \lambda^2 \|(1-e)P(ex)\|^2,
\end{aligned}
$$

which implies that $(1+2\lambda)\|(1-e)P(ex)\|^2 \leqslant \|ex\|^2$ for all real λ. Then $(1-e)P(ex) = 0$ or $P(ex) = eP(ex)$ for all x. Interchanging the role of $1-e$ and e, we obtain that

$eP((1-e)x) = 0$ or $eP(x) = eP(ex)$. Hence $eP(x) = P(ex)$ for all $x \in A$. Since Y is commutative, the elements of the ring R is a $\mathbb{Z}$-sum of projections from R, therefore $eP(x) = P(ex)$ for all $e \in R$ and $x \in A$. Thus $P : A \to A$ is a R-module homomorphism, that is, $P \in \mathscr{MB}_R(A)$. $\square$

Now let $V \subseteq A$ be an operator subspace. We define $\mathscr{E}_V = \{\varphi \in \text{ball } \mathscr{MB}_R(A) : \varphi|_V = \text{id}\}$ to be a nonempty subset in $\text{ball } \mathscr{MB}_R(A)$, for $1_A \in \mathscr{E}_V$. For the elements φ, $\psi \in \mathscr{E}_V$, we set $\varphi \preceq \psi$ iff $\|\varphi(a)\| \leqslant \|\psi(b)\|$, $a \in A$. That is a reflexive and transitive relation on $\mathscr{E}_V$, and we put $\varphi \approx \psi$ iff $\varphi \preceq \psi$ and $\psi \preceq \varphi$. An element $\varphi \in \mathscr{E}_V$ is said to be *minimal* if $\psi \preceq \varphi$ implies $\psi \approx \varphi$. The set of all its minimal elements is denoted by $\min \mathscr{E}_V$.

The forthcoming lemma is a modified version of the known Hamana-Ruan extension lemma [12, 6.1.5], [18], [23].

LEMMA 3.3. *The set* $\min \mathscr{E}_V$ *is not empty and it consists of projections in* $\text{ball } \mathscr{MB}_R(A)$. *For each* $\varphi \in \min \mathscr{E}_V$ *the inclusion* $V \subseteq \text{im}(\varphi)$ *is* R-*rigid, and* $\text{im}(\varphi) \subseteq W$ *up to a matrix isometry for each injective* R-*submodule* $W \subseteq A$ *with the inclusion* $V \subseteq W$. *In particular,* $\text{im}(\varphi) = W$ *up to a matrix isometry for each injective* R-*submodule* $W \subseteq A$ *with the* R-*rigid inclusion* $V \subseteq W$.

Proof. We use the arguments very similar to [12, 6.1.5, 6.2.1]. First note that $\mathscr{E}_V$ is a (p-WOT) closed subset in $\text{ball } \mathscr{MB}_R(A)$. Indeed, if $(\varphi_\lambda)_\lambda \subseteq \mathscr{E}_V$ is a net with $\varphi = (\text{p-WOT})\lim_\lambda \varphi_\lambda \in \text{ball } \mathscr{MB}_R(A)$ (see Lemma 2.1) then $\varphi(v) = (\text{WOT})\lim_\lambda \varphi_\lambda(v) = (\text{WOT})\lim_\lambda v = v$ for all $v \in V$. Thus $\mathscr{E}_V$ is a (p-WOT) compact space thanks to Lemma 2.1. Moreover, each section $F_\varphi = \{\psi \in \mathscr{E}_V : \psi \preceq \varphi\}$ is closed out of semi-continuity of the (operator) norm with respect to WOT on A. Whence $\min \mathscr{E}_V \neq \varnothing$ [12, 6.1.4]. Take $\varphi \in \min \mathscr{E}_V$. Then $\psi_n = n^{-1}\sum_{k=1}^n \varphi^k \in \text{ball } \mathscr{MB}_R(A)$ (confirm that $\mathscr{MB}_R(A)$ is a subalgebra in $\mathscr{MB}(A)$) and $\psi_n|_V = \text{id}$, that is, $\psi_n \in \mathscr{E}_V$ and $\psi_n \preceq \varphi$. Therefore $\psi_n \approx \varphi$ or $\|\varphi(a)\| = \|\psi_n(a)\|$ for all $a \in A$. It follows that $\|\varphi(a) - \varphi^2(a)\| = \|\varphi(a - \varphi(a))\| = \|\psi_n(a - \varphi(a))\| = n^{-1}\|\sum_{k=1}^n \varphi^k(a - \varphi(a))\| = n^{-1}\|\varphi(a) - \varphi^{n+1}(a)\| \leqslant 2n^{-1}\|a\|$, that is, φ is a projection in $\text{ball } \mathscr{MB}_R(A)$. In particular, $\text{im}(\varphi)$ is an injective R-module (see Lemma 3.2). Actually, the inclusion $V \subseteq \text{im}(\varphi)$ is R-rigid. Indeed, assume $\tau \in \text{ball } \mathscr{MB}_R(\text{im}(\varphi))$ with $\tau|_V = \text{id}$. Then $\psi = \tau\varphi \in \text{ball } \mathscr{MB}_R(A)$, and $\psi(v) = \tau(\varphi(v)) = \tau(v) = v$, $v \in V$, which means that $\psi \in \mathscr{E}_V$. But $\|\psi(a)\| = \|\tau\varphi(a)\| \leqslant \|\varphi(a)\|$, $a \in A$, that is, $\psi \preceq \varphi$. Therefore $\psi \in \min \mathscr{E}_V$ as well. Thus ψ is a projection, $\|\psi(a)\| = \|\varphi(a)\|$, $a \in A$, and $\text{im}(\psi) \subseteq \text{im}(\varphi)$. Then $\varphi\psi = \psi$ and $\|\varphi(a) - \psi(a)\| = \|\varphi(a - \psi(a))\| = \|\psi(a - \psi(a))\| = 0$ for all $a \in A$. Hence $\varphi = \psi$ or $\tau = \text{id}$. Thus $V \subseteq \text{im}(\varphi)$ is R-rigid.

Finally, assume that W is an injective R-submodule in A with the inclusion $V \subseteq W$. By Lemma 3.2, $W = \text{im}(P)$ for a certain projection $P \in \text{ball } \mathscr{MB}_R(A)$. Then $P|_V = \text{id}$, therefore $P \in \mathscr{E}_V$. Consider the mapping $\varphi P \varphi \in \mathscr{MB}(A)$. Since both projections are R-homomorphisms, we derive that $\varphi P \varphi \in \text{ball } \mathscr{MB}_R(A)$. In particular, $\varphi P \varphi \in \text{ball } \mathscr{MB}_R(\text{im}(\varphi))$ and $\varphi P \varphi(v) = \varphi(P(v)) = \varphi(v)$ for all $v \in V$. Since $V \subseteq \text{im}(\varphi)$ is R-rigid, it follows that $\varphi P \varphi = \text{id}$ over $\text{im}(\varphi)$, or $\varphi(P(x)) = x$ for all $x \in \text{im}(\varphi)$. Taking into account that both φ and P are matrix contractions, we conclude that $P : \text{im}(\varphi) \to \text{im}(P)$ is a matrix isometric embedding. $\square$

Based on Lemma 3.3, we define *an injective R-envelope $I_R(V)$ of a subspace $V \subseteq A$* to be the range $\text{im}(\varphi)$ of a minimal projection from the set $\mathscr{E}_V$. Thus $I_R(V)$ is uniquely defined up to a matrix isometry. The injective R-envelope is minimal in the sense that if $V \subseteq W \subseteq I_R(V)$ for some injective R-submodule $W \subseteq A$ then $W = I_R(V)$ (up to a matrix isometry) thanks to Lemma 3.3.

3.4. Paulsen module of an operator space

Let Y be a domain in $\mathscr{B}(H)$ and $A = Y'$ its commutant in $\mathscr{B}(H)$. Each monomial in elements of Y_+ turns out to be a projection. Thereby the ring R generated by Y_+ consists of all $\mathbb{Z}$-finite sums of projections, and $A = R'$ in $\mathscr{B}(H)$. Fix $n \in \mathbb{N}$. The matrix space $M_n(A)$ has a canonical R-module structure determined by the matrix product $xe^{\oplus n}$ for all $x \in M_n(A)$ and $e \in R$. For brevity we write xe instead of $xe^{\oplus n}$. We define *the Paulsen system of the domain Y* as

$$\mathscr{P}_Y = \begin{bmatrix} \mathbb{C} & A \\ A^* & \mathbb{C} \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix} \in M_2(A) : \alpha, \beta \in \mathbb{C}, a, b \in A \right\},$$

which is an operator system in $M_2(A)$. If $V \subseteq A$ is an operator space, we define its Paulsen system as the subspace

$$\mathscr{P}_V = \begin{bmatrix} \mathbb{C} & V \\ V^* & \mathbb{C} \end{bmatrix} \subseteq \mathscr{P}_Y.$$

Note that $M_n(\mathscr{P}_V)$ is a subspace in $M_{2n}(V)$ for each n. Actually,

$$M_n(\mathscr{P}_V) = \begin{bmatrix} M_n & M_n(V) \\ M_n(V)^* & M_n \end{bmatrix} = \left\{ \begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix} \in M_{2n}(V) : \alpha, \beta \in M_n, a, b \in M_n(V) \right\}$$

up to the canonical (shuffling) identification. The subspace $M_n(\mathscr{P}_V)$ is not a R-submodule in $M_{2n}(V)$. The R-submodule in $M_{2n}(V)$ generated by $M_n(\mathscr{P}_V)$ is denoted by $M_n(\mathscr{P}_V)R$. Thus the R-module $M_n(\mathscr{P}_V)R$ consists of all finite sums $\sum_{e \in R} x_e e$ with $x_e \in M_n(\mathscr{P}_V)$. In particular,

$$\mathscr{P}_V R = \left\{ \begin{bmatrix} \sum_{e \in R} \alpha_e e & \sum_{e \in R} a_e e \\ (\sum_{e \in R} b_e e)^* & \sum_{e \in R} \beta_e e \end{bmatrix} \in M_2(V) : \alpha_e, \beta_e \in \mathbb{C}, a_e, b_e \in V \right\},$$

which is a R-submodule in $M_2(V)$ generated by $\mathscr{P}_V$ called *the Paulsen module of V*.

LEMMA 3.4. *Let Y be a domain in $\mathscr{B}(H)$. Then $M_n(\mathscr{P}_V)R = M_n(\mathscr{P}_V R)$ up to a canonical identification, and if $x \in M_n(\mathscr{P}_V R)$ then $x = \sum_{i=1}^m x_i e_i$ for some orthogonal family $(e_i)_i$ of projections from the ring R and matrices $(x_i)_i \subseteq M_n(\mathscr{P}_V)$.*

Proof. Based on the canonical identification $M_n(\mathscr{P}_V) = M_n \otimes \mathscr{P}_V$ we have

$$M_n(\mathscr{P}_V)R = (M_n \otimes \mathscr{P}_V)R = M_n \otimes \mathscr{P}_V R = M_n(\mathscr{P}_V R).$$

We need justify the second equality. Pick an elementary tensor $\lambda \otimes x \in M_n \otimes \mathscr{P}_V$ and $e \in R$. Then

$$(\lambda \otimes x)\,e = [\lambda_{ij} x]\, e^{\oplus 2n} = \left[\begin{bmatrix} \lambda_{ij}\alpha & \lambda_{ij}a \\ \lambda_{ij}b^* & \lambda_{ij}\beta \end{bmatrix}\right]_{ij} e^{\oplus 2n} = \begin{bmatrix} [\lambda_{ij}\alpha] & [\lambda_{ij}a] \\ [\lambda_{ij}b^*] & [\lambda_{ij}\beta] \end{bmatrix} e^{\oplus 2n}$$

$$= \begin{bmatrix} [\lambda_{ij}\alpha e] & [\lambda_{ij}ae] \\ [\lambda_{ij}b^* e] & [\lambda_{ij}\beta e] \end{bmatrix} = \left[\begin{bmatrix} \lambda_{ij}\alpha e & \lambda_{ij}ae \\ \lambda_{ij}b^* e & \lambda_{ij}\beta e \end{bmatrix}\right]_{ij} = [\lambda_{ij}xe] = \lambda \otimes xe,$$

that is, $(\lambda \otimes x)\,e = \lambda \otimes xe$, which results in the identification $(M_n \otimes \mathscr{P}_V)R = M_n \otimes \mathscr{P}_V R$.

Now take an element $x \in M_n(\mathscr{P}_V R)$. Since $M_n(\mathscr{P}_V R) = M_n(\mathscr{P}_V)R$, it follows that $x = \sum_{i=1}^m z_i e_i$ for some $(z_i)_i \subseteq M_n(\mathscr{P}_V)$ and $(e_i)_i \subseteq R$. But each element of the ring R is a $\mathbb{Z}$-sum of (commutative) monomials in Y_+, that is, projections from the ring R. Therefore we can assume that all $(e_i)_i$ are projections from the ring R. We proceed an orthogonalization by induction on m. If $m = 1$ the assertion is trivial. For $m = 2$ we have $x = z_1 e_1 + z_2 e_2 = z_1 f_1 + (z_1 + z_2) f_2 + z_2 f_3$, where $f_1 = e_1(1 - e_2)$, $f_2 = e_1 e_2$, $f_3 = e_2(1 - e_1)$ are projections from the ring R. Since Y is commutative, we conclude that $f_i f_j = 0$, $1 \leqslant i \neq j \leqslant 3$, that is, $(f_i)_i$ is an orthogonal family of projections from the ring R.

In the general case, based on induction hypothesis we have $x = \sum_{i=1}^m z_i e_i = \sum_{i=1}^k y_i f_i + z_m e_m$ for some orthogonal family $(f_i)_i$ of projections from R and matrices $(y_i)_i \subseteq M_n(\mathscr{P}_V)$. Then $f = \sum_{i=1}^k f_i$ is a projection from the commutative ring R. It follows that

$$x = \sum_{i=1}^k y_i f_i (1 - e_m) + y_i f_i e_m + z_m e_m (1 - f) + z_m e_m f$$

$$= \sum_{i=1}^k y_i f_i (1 - e_m) + (y_i + z_m) f_i e_m + z_m e_m (1 - f)$$

$$= \sum_{i=1}^k y_i g_i + \sum_{i=1}^k (y_i + z_m) g_{i+k} + z_m g_{2k+1}$$

with $g_i = f_i(1 - e_m)$, $g_{i+k} = f_i e_m$, $1 \leqslant i \leqslant k$, and $g_{2k+1} = e_m(1 - f)$. One can easily verify that $(g_i)_i$ is an orthogonal family of projections from the ring R. Whence $x = \sum_{i=1}^{2k+1} x_i g_i$ for some matrices $(x_i)_i \subseteq M_n(\mathscr{P}_V)$. $\quad\square$

Fix an element $x \in M_n(\mathscr{P}_V R) \setminus \{0\}$. Using Lemma 3.4, we conclude that

$$x = \sum_{i=1}^m x_i e_i = \sum_{i=1}^m \begin{bmatrix} \alpha_i e_\iota & a_i e_\iota \\ b_i^* e_\iota & \beta_i e_\iota \end{bmatrix} = \begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix},$$

where $\alpha = \sum_{i=1}^m \alpha_i e_\iota = \sum_{i=1}^m \alpha_i e_\iota^{\oplus n}$, $\beta = \sum_{i=1}^m \beta_i e_\iota = \sum_{i=1}^m \beta_i e_\iota^{\oplus n}$, $a = \sum_{i=1}^m a_i e_\iota$, $b = \sum_{i=1}^m b_i e_\iota$ with α_i, $\beta_i \in M_n$, a_i, $b_i \in M_n(V)$ and an orthogonal family of projections $(e_i)_i$ from the ring R. We can assume that $x_i e_i \neq 0$ for all i. In particular, if x is a hermitian element then $b = a$, which in turn implies that $a_i e_\iota^{\oplus n} = b_i e_\iota^{\oplus n}$, $1 \leqslant i \leqslant m$, due

to orthogonality of the family $(e_i)_i$. Moreover, both α and β are hermitian elements in $M_n(A)$. Using again orthogonality and Lemma 2.2, we derive that all α_i, β_i are scalar hermitian matrices in M_n.

If $x \geqslant 0$ is positive then $xe_i = x_i e_i \geqslant 0$ for all i due to the orthogonal expansion. Using again Lemma 2.2, we derive that $\alpha_i \geqslant 0$, $\beta_i \geqslant 0$, for $\alpha e_i = \lambda x e_i \lambda^* \geqslant 0$ with $\lambda = \begin{bmatrix} I_n & 0 \end{bmatrix} \in M_{n,2n}$ (similarly, $\beta e_i \geqslant 0$). Put $e = \sum_{i=1}^m e_i$, which is a projection from the ring R called *a supporting projection for* x. We set $\alpha + \varepsilon e = \sum_{i=1}^m (\alpha_i + \varepsilon I_n) e_i$, and $(\alpha + \varepsilon e)^{-1/2} = \sum_{i=1}^m (\alpha_i + \varepsilon I_n)^{-1/2} e_i$, $\varepsilon > 0$. Note that $(\alpha + \varepsilon e)^{-1/2} (\alpha + \varepsilon e)$ $(\alpha + \varepsilon e)^{-1/2} = \sum_{i=1}^m e_i^{\oplus n} = e^{\oplus n}$ thanks to the orthogonality property either. Similarly, $(\beta + \varepsilon e)^{-1/2} (\beta + \varepsilon e) (\beta + \varepsilon e)^{-1/2} = e^{\oplus n}$.

LEMMA 3.5. *Let* $x = \begin{bmatrix} \alpha & a \\ a^* & \beta \end{bmatrix} \in M_n(\mathscr{P}_V R)$ *be a hermitian element with its supporting projection* $e \in R$. *Then* $x \geqslant 0$ *if and only if* $\left\| (\alpha + \varepsilon e)^{-1/2} a (\beta + \varepsilon e)^{-1/2} \right\| \leqslant 1$ *for all* $\varepsilon > 0$.

Proof. As above put $x = \sum_{i=1}^m x_i e_i$ and $e = \sum_{i=1}^m e_i \in R$. Note that $x \geqslant 0$ iff $x + \varepsilon e^{\oplus 2n} = \sum_{i=1}^m (x_i + \varepsilon) e_i \geqslant 0$ for all $\varepsilon > 0$. Since $x + \varepsilon e^{\oplus 2n} = \begin{bmatrix} \alpha + \varepsilon e & a \\ a^* & \beta + \varepsilon e \end{bmatrix}$, it follows that $\lambda_\varepsilon (x + \varepsilon e^{\oplus 2n}) \lambda_\varepsilon^* \geqslant 0$ with $\lambda_\varepsilon = (\alpha + \varepsilon e)^{-1/2} \oplus (\beta + \varepsilon e)^{-1/2}$, whenever $x \geqslant 0$. Using again orthogonality of the family (e_i), we conclude that

$$\lambda_\varepsilon (x + \varepsilon e^{\oplus 2n}) \lambda_\varepsilon^* = \begin{bmatrix} e^{\oplus n} & a_\varepsilon \\ a_\varepsilon^* & e^{\oplus n} \end{bmatrix} = \begin{bmatrix} I & a_\varepsilon \\ a_\varepsilon^* & I \end{bmatrix} e \geqslant 0$$

for all $\varepsilon > 0$, where $a_\varepsilon = (\alpha + \varepsilon e)^{-1/2} a (\beta + \varepsilon e)^{-1/2} = \sum_{i=1}^m (\alpha_i + \varepsilon I_n)^{-1/2} a_i e_i$ $(\beta_i + \varepsilon I_n)^{-1/2}$. It follows that $\|a_\varepsilon\| = \|a_\varepsilon e\| \leqslant 1$ (see [12, Proposition 1.3.2]). $\quad\square$

3.5. Injective R-envelope of $\mathscr{P}_V$

As above let Y be a domain in $\mathscr{B}(H)$ and $A = Y'$ its commutant in $\mathscr{B}(H)$. Since Y is commutative, it follows that A is an injective von Neumann algebra [2, IV. 2.2.7]. Moreover, $M_2(A)$ is injective [2, IV. 2.1.5] as well. Using Remark 3.2, we derive that $I(\mathscr{P}_V R) \subseteq M_2(A)$ up to a matrix isometry, where $I(\mathscr{P}_V R)$ is the normed injective envelope of $\mathscr{P}_V R$.

PROPOSITION 3.3. *If* $V \subseteq A$ *is an operator space then* $I_R(\mathscr{P}_V) = I(\mathscr{P}_V R)$.

Proof. The identity mapping over $I(\mathscr{P}_V R)$ is extended to a matrix contractive projection $\Phi \in \text{ball} \mathscr{MB}(M_2(A))$ onto $I(\mathscr{P}_V R)$. Since $\mathscr{P}_V R$ is unital, it follows that Φ is a morphism projection. Moreover, $R \subseteq \mathscr{P}_V R$, therefore $\Phi(e^{\oplus 2})^* \Phi(e^{\oplus 2}) = e^{\oplus 2} e^{\oplus 2} = e^{\oplus 2} = \Phi(e^{\oplus 2*} e^{\oplus 2})$ for all $e \in Y_+$. Using Stinespring Theorem [12, Corollary 5.2.2], we obtain that $\Phi(xe^{\oplus 2}) = \Phi(x) e^{\oplus 2}$ or $\Phi(xe) = \Phi(x) e$ for all $x \in M_2(A)$ and

$e \in Y_+$. It follows that Φ is a R-module homomorphism, that is, $\Phi \in \text{ball}\, \mathscr{M}\mathscr{B}_R(M_2(A))$. Thus $I(\mathscr{P}_V R)$ is an injective R-module. But $\mathscr{P}_V \subseteq \mathscr{P}_V R$, therefore $I_R(\mathscr{P}_V) \subseteq I(\mathscr{P}_V R)$ thanks to Lemma 3.3. Further, $\mathscr{P}_V \subseteq I_R(\mathscr{P}_V)$ and $I_R(\mathscr{P}_V)$ is a R-module. In particular, $\mathscr{P}_V R \subseteq I_R(\mathscr{P}_V)$. But $I_R(\mathscr{P}_V)$ is an injective space either. Using again Lemma 3.3 (but right now in its classical version for operator spaces), we derive that $I(\mathscr{P}_V R) \subseteq I_R(\mathscr{P}_V)$. $\square$

Now let $V \subseteq A$ be an operator space which is a R-module as well, $\tau : V \to V$ a R-module homomorphism. It can canonically be extended to a linear mapping

$$\widetilde{\tau} : \mathscr{P}_V R \to \mathscr{P}_V R, \quad \widetilde{\tau}\begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \tau(a) \\ \tau(b)^* & \beta \end{bmatrix}$$

(note that $a, b \in V$ and $\alpha, \beta \in \text{span}(R)$). Obviously, $\widetilde{\tau}$ is a unital mapping such that $\widetilde{\tau}(e^{\oplus 2}) = e^{\oplus 2}$, $e \in R$. Furthermore, $\widetilde{\tau}(xe) = \widetilde{\tau}(x)e$, $x \in \mathscr{P}_V R$, $e \in R$, that is, $\widetilde{\tau}$ is a R-module homomorphism.

LEMMA 3.6. *If $\tau : V \to V$ is a matrix contractive R-module homomorphism then the mapping $\widetilde{\tau} : \mathscr{P}_V R \to \mathscr{P}_V R$ is matrix positive.*

Proof. Take a positive element $x = \begin{bmatrix} \alpha & a \\ a^* & \beta \end{bmatrix} \in M_n(\mathscr{P}_V R)$ with its supporting projection $e = \sum_{i=1}^m e_i \in R$. Note that $\widetilde{\tau}^{(n)} : M_n(\mathscr{P}_V R) \to M_n(\mathscr{P}_V R)$ is acting in the following way

$$\widetilde{\tau}^{(n)}\begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \tau^{(n)}(a) \\ \tau^{(n)}(b)^* & \beta \end{bmatrix}.$$

By Lemma 3.5, $\|a_\varepsilon\| \leqslant 1$ for all $\varepsilon > 0$, where $a_\varepsilon = (\alpha + \varepsilon e)^{-1/2} a (\beta + \varepsilon e)^{-1/2}$ and $a = \sum_{i=1}^m a_i e_i$. Note that $a_\varepsilon = \sum_{i=1}^m (\alpha_i + \varepsilon I_n)^{-1/2} a_i e_i (\beta_i + \varepsilon I_n)^{-1/2}$. Since τ is R-homomorphism, we have

$$\tau^{(n)}(a_\varepsilon) = \sum_i (\alpha_i + \varepsilon I_n)^{-1/2} \tau^{(n)}(a_i e_i)(\beta_i + \varepsilon I_n)^{-1/2}$$

$$= \sum_i (\alpha_i + \varepsilon I_n)^{-1/2} \tau^{(n)}(a_i) e_i (\beta_i + \varepsilon I_n)^{-1/2}$$

$$= (\alpha + \varepsilon e)^{-1/2} \tau^{(n)}(a)(\beta + \varepsilon e)^{-1/2} = \tau^{(n)}(a)_\varepsilon.$$

But τ is a matrix contraction, therefore $\left\| \tau^{(n)}(a)_\varepsilon \right\| = \left\| \tau^{(n)}(a_\varepsilon) \right\| \leqslant \|a_\varepsilon\| \leqslant 1$ for all $\varepsilon > 0$. Using again Lemma 3.5, we obtain that $\widetilde{\tau}^{(n)}(x) \geqslant 0$. $\square$

Fix the projection $p = 1 \oplus 0 \in M_2(A)$ and put $p' = 1 - p$. The injective R-envelope of an operator subspace in A has the following description.

THEOREM 3.1. *Let Y be a domain in $\mathscr{B}(H)$. If $V \subseteq Y'$ is an operator space then $I_R(V) = p I_R(\mathscr{P}_V) p'$ up to a matrix isometry.*

Proof. Note that $I(\mathscr{P}_V R) = \operatorname{im}(\Phi)$ is an injective C^*-algebra with respect to the multiplication $x \cdot y = \Phi(xy)$, $x, y \in I(\mathscr{P}_V R)$, where $\Phi \in \operatorname{ball} \mathscr{M}\mathscr{B}(M_2(A))$ is the projection. Since $p = 1 \oplus 0 \in \mathscr{P}_V \subseteq \mathscr{P}_V R$, we have $p \cdot p = \Phi(p^2) = \Phi(p) = p$ and $\Phi(p)^* \cdot \Phi(p) = p \cdot p = p = \Phi(p^*p)$. Using Stinespring Theorem [12, Corollary 5.2.2], we obtain that $\Phi(py) = p \cdot \Phi(y)$ for all $y \in M_2(A)$. Similarly, p' is a projection in $\mathscr{P}_V$ and $\Phi(yp') = \Phi(y) \cdot p'$ for all $y \in M_2(A)$. The mapping $i : V \to M_2(A)$, $i(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$ is a matrix isometry. Using Hamana-Ruan formula [3, 4.4.3], we derive that

$$V = i(V) \subseteq I(V) = pI(\mathscr{P}_V) p' \subseteq M$$

up to matrix isometries, where $M = pI(\mathscr{P}_V R) p'$ is the right upper-corner of $I(\mathscr{P}_V R)$. But $M = pI_R(\mathscr{P}_V) p'$ by Proposition 3.3. Therefore M is a R-module, and $I_R(V) \subseteq M$ by Lemma 3.3. Actually, the inclusion $V \subseteq M$ is R-rigid. Indeed, let $\tau : M \to M$ be a matrix contractive R-homomorphism extending the identity mapping over $i(V)$. By Lemma 3.6,

$$\widetilde{\tau} : \mathscr{P}_M R \to \mathscr{P}_M R, \quad \widetilde{\tau}\begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix} = \begin{bmatrix} \alpha & \tau(a) \\ \tau(b)^* & \beta \end{bmatrix},$$

is a unital matrix positive R-homomorphism. In particular, $\widetilde{\tau}$ is a matrix contraction. But $\mathscr{P}_V R$ is canonically identified with a R-submodule in $\mathscr{P}_M R$. Namely,

$$\mathscr{P}_V R = \begin{bmatrix} \mathbb{C} & V \\ V^* & \mathbb{C} \end{bmatrix} R \subseteq \begin{bmatrix} \mathbb{C} & M \\ M^* & \mathbb{C} \end{bmatrix} R = \mathscr{P}_M R,$$

and $\widetilde{\tau} = \operatorname{id}$ over $\mathscr{P}_V R$. Moreover,

$$\begin{aligned}
\mathscr{P}_M R &= \begin{bmatrix} \mathbb{C} & pI(\mathscr{P}_V R) p' \\ (pI(\mathscr{P}_V R) p')^* & \mathbb{C} \end{bmatrix} R \\
&\subseteq I(\mathscr{P}_V R) R = I_R(\mathscr{P}_V) R \subseteq I_R(\mathscr{P}_V) = I(\mathscr{P}_V R),
\end{aligned}$$

that is, $\mathscr{P}_M R$ is viewed as a submodule of $I(\mathscr{P}_V R)$. Based on the injectivity of $I(\mathscr{P}_V R)$, we derive that $\widetilde{\tau}$ has a matrix contractive extension $\sigma : I(\mathscr{P}_V R) \to I(\mathscr{P}_V R)$. But $\mathscr{P}_V R \subseteq \mathscr{P}_M R \subseteq I(\mathscr{P}_V R)$ and $\sigma | \mathscr{P}_V R = \widetilde{\tau} | \mathscr{P}_V R = \operatorname{id}$. Using the rigidity property, we obtain that $\sigma = \operatorname{id}$. Therefore $\tau = \operatorname{id}$ over M. Thus the inclusion $V \subseteq M$ is R-rigid. By Lemma 3.3, $M = I_R(V)$. Thus $I_R(V) = pI(\mathscr{P}_V R) p' = pI_R(\mathscr{P}_V) p'$. $\square$

4. Injective quantum modules

In this section we investigate the problem whether a multinormed completion of an injective W^*-algebra is injective in the quantum space sense. A quantum domain Y whose commutant completion $C_Y^*(\mathscr{Y})$ is an injective quantum space is called *an injective quantum domain*. Each graded domain is injective thanks to Lemma 3.1. We prove that the injectivity problem of multinormed W^*-algebras can be reduced to the injectivity problem of their domains.

4.1. Injective quantum modules

Let Y be an injective domain in $\mathscr{B}(H)$ and let R be the commutative subring in $\mathscr{B}(H)$ generated by Y_+. A subspace $V \subseteq C_Y^*(\mathscr{Y})$ with $YV \subseteq V$ (or $RV \subseteq V$) is called an algebraic R-module. Note that $YV \subseteq \mathfrak{b}(C_Y^*(\mathscr{Y})) = Y'$ whereas $(1-e)v$ may not be a bounded element in V for $v \in V$ and $e \in Y$. Actually, YV lies in the bounded part $Y' \cap V = \mathfrak{b}(V)$ of the subspace V. An injective quantum space V with the property $YV \subseteq V$ is called *an injective quantum R-module*. Note that if V is the range of a quantum contractive R-module projection $P : C_Y^*(\mathscr{Y}) \to C_Y^*(\mathscr{Y})$, then V is an injective quantum R-module (see [5, Lemma 8.1]). Indeed, since $C_Y^*(\mathscr{Y})$ is injective, we conclude that (see Proposition 3.2) $C_Y^*(\mathscr{Y}) = Q(C_X^*(\mathscr{X}))$ for a certain quantum contractive projection $Q : C_X^*(\mathscr{X}) \to C_X^*(\mathscr{X})$, where X is a graded domain. Hence V is the range of a quantum contractive projection on $C_X^*(\mathscr{X})$. By Proposition 3.2, V is an injective quantum space. Moreover, V turns out to be R-module being the range of a R-module homomorphism.

Note that each local von Neumann subalgebra in $C_Y^*(\mathscr{Y})$ [10] is a R-submodule in $C_Y^*(\mathscr{Y})$.

PROPOSITION 4.1. *Let Y be an injective domain, and let $V \subseteq C_Y^*(\mathscr{Y})$ be a subspace. Then V is an injective quantum R-module if and only if its bounded part $\mathfrak{b}(V)$ is an injective R-module and $V = \overline{\mathfrak{b}(V)}$. In this case, V is the range of a quantum contractive R-module projection $P : C_Y^*(\mathscr{Y}) \to C_Y^*(\mathscr{Y})$ such that $\left\| P^{(\infty)}(x) \right\|_e \leqslant \|x\|_e$ for all $x \in M(C_Y^*(\mathscr{Y}))$ and $e \in Y$.*

Proof. First assume that $\mathfrak{b}(V)$ is an injective R-module and $V = \overline{\mathfrak{b}(V)}$. Let us point out that we deal with the multinormed topology in $C_Y^*(\mathscr{Y})$. In particular, $YV = Y\overline{\mathfrak{b}(V)} \subseteq \overline{Y\mathfrak{b}(V)} \subseteq \overline{\mathfrak{b}(V)} \subseteq V$, that is, V is a R-module. Note that Y is a commutative set of projections, therefore Y' is an injective von Neumann algebra [2, IV. 2.2.7], and $\mathfrak{b}(V) \subseteq Y'$. By Lemma 3.2, $\mathfrak{b}(V) = \mathrm{im}(\varphi)$ is the range of a certain projection $\varphi \in \mathrm{ball}\, \mathscr{M}\mathscr{B}_R(Y')$. In particular, $\varphi(ex) = e\varphi(x)$ for all $x \in Y'$ and $e \in Y$. Since the same can be applied to all matrices too, we obtain that $e^{\oplus n}\varphi^{(n)}(x) = \varphi^{(n)}(e^{\oplus n}x)$ for all $x \in M_n(Y')$, $e \in Y$. It follows that

$$\left\| \varphi^{(n)}(x) \right\|_e = \left\| e^{\oplus n}\varphi^{(n)}(x) \right\| = \left\| \varphi^{(n)}(e^{\oplus n}x) \right\| \leqslant \left\| e^{\oplus n}x \right\| = \|x\|_e,$$

that is, $\left\| \varphi^{(\infty)}(x) \right\|_e \leqslant \|x\|_e$ for all $x \in M_n(Y')$ and $e \in Y$. Thus φ is matrix continuous relative to the quantum topology inherited from $C_Y^*(\mathscr{Y})$, therefore it has a matrix continuous extension $\Phi : C_Y^*(\mathscr{Y}) \to C_Y^*(\mathscr{Y})$. Pick $x \in C_Y^*(\mathscr{Y})$ and $e \in R$. Then $x = \lim_t x_t$ for a certain net $(x_t)_t$ in $\mathfrak{b}(C_Y^*(\mathscr{Y})) = Y'$ (out of density of Y' in $C_Y^*(\mathscr{Y})$). Using the continuity of Φ, we obtain the following $e\Phi(x) = \lim_t e\varphi(x_t) = \lim_t \varphi(ex_t) = \Phi(ex) \in V$. Thus $e^{\oplus n}\Phi^{(n)}(x) = \Phi^{(n)}(e^{\oplus n}x)$ for all $x \in M_n(C_Y^*(\mathscr{Y}))$, $e \in R$. In particular, $\left\| \Phi^{(n)}(x) \right\|_e \leqslant \|x\|_e$ for all $x \in M_n(C_Y^*(\mathscr{Y}))$ and $e \in Y$. Note that Φ^2 and Φ are compatible over Y', therefore $\Phi : C_Y^*(\mathscr{Y}) \to C_Y^*(\mathscr{Y})$ is a projection and $\Phi(C_Y^*(\mathscr{Y})) = \overline{\varphi(Y')} = \overline{\mathfrak{b}(V)} = V$. Whence V is an injective quantum R-module.

Conversely, assume that V is an injective quantum R-module. There exists a quantum contractive projection $P : C_Y^*(\mathscr{Y}) \to C_Y^*(\mathscr{Y})$ onto the subspace V. Then $P(Y') \subseteq Y'$ and $P : Y' \to Y'$ is a matrix contractive projection onto a subspace. Indeed, for each $e \in Y$ there corresponds $\kappa \in Y^\dagger$ such that $\left\| P^{(\infty)}(x) \right\|_e \leqslant \|x\|_\kappa \leqslant \|x\|$, $x \in M(Y')$. It follows that

$$\left\| P^{(\infty)}(x) \right\| = \sup_{e \in Y} \left\| P^{(\infty)}(x) \right\|_e \leqslant \sup_{\kappa \in Y^\dagger} \|x\|_\kappa \leqslant \|x\|, \quad x \in M(Y'),$$

that is, $P : Y' \to Y'$ is a matrix contractive projection. Moreover, $V = P(C_Y^*(\mathscr{Y})) = P(\overline{Y'}) \subseteq \overline{P(Y')} \subseteq \overline{\mathfrak{b}(V)}$ and $\mathfrak{b}(V) \subseteq \mathrm{im}(P|Y') \subseteq Y' \cap V = \mathfrak{b}(V)$. But V is closed being the range of a continuous projection. Hence $V = \overline{\mathfrak{b}(V)}$ and $\mathfrak{b}(V) = \mathrm{im}(P|Y')$. It follows that $\mathfrak{b}(V)$ is an injective operator space. But $Y\mathfrak{b}(V) \subseteq YV \subseteq Y' \cap V = \mathfrak{b}(V)$, that is, $\mathfrak{b}(V)$ is an injective R-module. $\square$

4.2. The domain of a multinormed W^*-algebra

A weak* continuous $*$-homomorphism between W^*-algebras is called a W^*-*homomorphism*. By a *multinormed W^*-algebra* $\mathscr{A}$ we mean an inverse limit $\mathscr{A} = \varprojlim \{A_\iota, \varphi_{\iota\kappa}\}$ of W^*-algebras A_ι such that all connecting maps $\varphi_{\iota\kappa} : A_\kappa \to A_\iota$ ($\iota \leqslant \kappa$) are W^*-homomorphisms (Fragoulopoulou [16]). Note that the multinormed W^*-algebras appear as the central completions of von Neumann algebras [10, Propositon 2.1]. Namely, let $A \subseteq \mathscr{B}(H)$ be a von Neumann algebra on a Hilbert space H, and let $Y \subseteq A$ be a subset of its central projections such that $\vee Y = 1_H$. Note that $A \subseteq Y'$. The family of C^*-seminorms $\|b\|_e = \|be\|$, $b \in Y$, $e \in Y$ defines a (Hausdorff) multinormed topology in Y' (in particular, in A) called *the central topology* (see [10]). The completion of A relative to the central topology denoted by A_Y is a multinormed W^*-algebra, and it is a pattern for the class of all multinormed W^*-algebras as shown in [10]. Thus, if $\mathscr{A}$ is a multinormed W^*-algebra then there is a W^*-algebra A and a domain Y in the center of A such that $\mathscr{A} = A_Y = \overline{A}$, which is the closure of A in $C_Y^*(\mathscr{Y})$. The domain Y is called *the domain of the multinormed W^*-algebra $\mathscr{A}$*.

THEOREM 4.1. *Let $\mathscr{A}$ be a multinormed W^*-algebra with its injective domain Y. Then $\mathscr{A}$ is injective if and only if its bounded part $\mathfrak{b}(\mathscr{A})$ is injective in the normed sense.*

Proof. If $\mathscr{A}$ is injective then so is $\mathfrak{b}(\mathscr{A})$ thanks to Proposition 3.1. Conversely, assume that $\mathscr{A}$ is a multinormed W^*-algebra with its injective domain Y, and its injective bounded part $\mathfrak{b}(\mathscr{A})$. Thus $\mathscr{A} = A_Y \subseteq C_Y^*(\mathscr{Y})$ and $A \subseteq Y'$ is a von Neumann algebra. Take $a \in \mathfrak{b}(\mathscr{A})$. Then $a = \lim_\iota a_\iota$ in $C_Y^*(\mathscr{Y})$ for a net (a_ι) in A. By its very definition of the multinormed topology in $C_Y^*(\mathscr{Y})$, we conclude that $ae_\alpha = \lim_\iota a_\iota e_\alpha$ in $\mathscr{B}(H)$ for each $\alpha \in Y^\dagger$, where $e_\alpha = \vee \alpha$. Therefore $ae_\alpha \in A$ for all α. But $a = (\mathrm{WOT})\lim_\alpha ae_\alpha$ and A is a von Neumann subalgebra. Then $a \in A$. Whence $\mathfrak{b}(\mathscr{A}) = A$ and A is an injective operator space (see to the proof of Lemma 3.2). Since

$YA \subseteq A$, it follows that A is an injective R-module. Using Proposition 4.1, we conclude that $\mathscr{A}$ being the closure of A turns out to be an injective quantum R-module. Thus $\mathscr{A}$ is an injective quantum space. In particular, $\mathscr{A}$ is an injective quantum system (see Remark 3.1). Based upon Proposition 3.2, $\mathscr{A} = P(C_X^*(\mathscr{X}))$ for a certain quantum morphism-projection $P : C_X^*(\mathscr{X}) \to C_X^*(\mathscr{X})$, where X is a graded domain. But $C_X^*(\mathscr{X})$ is an injective multinormed C^*-algebra thanks to Lemma 3.1. Whence $\mathscr{A}$ is an injective multinormed C^*-algebra. $\square$

Now we generalize the assertion of Theorem 4.1 to the injective R-envelopes of quantum spaces. Let Y be an injective domain, $V \subseteq C_Y^*(\mathscr{Y})$ a quantum space with its dense bounded part $\mathfrak{b}(V)$. We define *the injective R-envelope of the quantum space V* in the following way

$$I_R(V) = \overline{I_R(\mathfrak{b}(V))}.$$

By Proposition 4.1, V is an injective quantum R-module iff $V = I_R(V)$. By Theorem 4.1, if $V = \mathscr{A} \subseteq C_Y^*(\mathscr{Y})$ is a local von Neumann subalgebra with injective bounded part $\mathfrak{b}(\mathscr{A})$ then $I_R(\mathscr{A}) = \overline{I_R(\mathfrak{b}(\mathscr{A}))} = \overline{\mathfrak{b}(\mathscr{A})} = \mathscr{A}$.

We define *the Paulsen system $\mathscr{P}_V$ of a quantum space $V \subseteq C_Y^*(\mathscr{Y})$* as a quantum system

$$\mathscr{P}_V = \begin{bmatrix} \mathbb{C} & V \\ V^* & \mathbb{C} \end{bmatrix} \subseteq M_2(C_Y^*(\mathscr{Y})).$$

Note that $\mathfrak{b}(\mathscr{P}_V) = \mathscr{P}_{\mathfrak{b}(V)}$. Indeed, if $x = \begin{bmatrix} \alpha & a \\ b^* & \beta \end{bmatrix} \in \mathfrak{b}(\mathscr{P}_V)$ then $a = \begin{bmatrix} 1 & 0 \end{bmatrix} x \begin{bmatrix} 0 & 1 \end{bmatrix}^* \in \mathfrak{b}(C_Y^*(\mathscr{Y})) = Y'$. Similarly, $b \in Y'$.

The following assertion is a locally convex version of one proved in Theorem 3.1.

THEOREM 4.2. *Let Y be an injective domain, $V \subseteq C_Y^*(\mathscr{Y})$ a quantum space with its dense bounded part $\mathfrak{b}(V)$. Then $I_R(\mathscr{P}_V)$ is an injective multinormed C^*-algebra and*

$$I_R(V) = pI_R(\mathscr{P}_V)p'$$

up to a matrix Y-isometry, where p is a bounded projection in $I_R(\mathscr{P}_V)$ and $p' = 1 - p$.

Proof. By its very definition, $I_R(\mathscr{P}_V) = \overline{I_R(\mathfrak{b}(\mathscr{P}_V))} = \overline{I_R(\mathscr{P}_{\mathfrak{b}(V)})}$, and $I_R(\mathscr{P}_{\mathfrak{b}(V)}) = I(\mathscr{P}_{\mathfrak{b}(V)}R)$ by virtue of Proposition 3.3. Moreover, by Lemma 3.2, there is a projection $P \in \mathrm{ball}\,\mathscr{MB}_R(M_2(Y'))$ onto $I_R(\mathscr{P}_{\mathfrak{b}(V)})$. In particular, $\left\| P^{(n)}(x) \right\|_e = \left\| P^{(n)}(x)e^{\oplus 2n} \right\| \leqslant \|x\|_e$ for all $x \in M_n(M_2(Y'))$, $e \in Y$. Hence P is a quantum contractive mapping with respect to the quantum topology generated by Y. Therefore it has a unique linear extension $\Phi : M_2(C_Y^*(\mathscr{Y})) \to M_2(C_Y^*(\mathscr{Y}))$ such that $\left\| \Phi^{(\infty)}(y) \right\|_e \leqslant \|y\|_e$, $y \in M(M_2(C_Y^*(\mathscr{Y})))$, $e \in Y$. Note that $\Phi^2 = \Phi$ due to the density of $M_2(Y')$ in $M_2(C_Y^*(\mathscr{Y}))$. Moreover, Φ is a quantum positive mapping, namely, $y \geqslant_e 0$ implies that $\Phi^{(\infty)}(y) \geqslant_e 0$ for all $y \in M(M_2(C_Y^*(\mathscr{Y})))$ (see [5, Corollary 4.1]). Thus Φ is a quantum morphism and its range $\mathscr{A} = \mathrm{im}(\Phi)$ is a multinormed C^*-algebra with respect to the multiplication $a \cdot b = \Phi(ab)$, $\|a \cdot b\|_e = \|\Phi(ab)\|_e \leqslant \|ab\|_e \leqslant \|a\|_e \|b\|_e$

for all $a, b \in \mathscr{A}$, $e \in Y$ (see [5, Theorem 8.3]). Note that $\Phi(M_2(Y'))$ is dense in $\mathscr{A}$, and $\Phi(M_2(Y')) = P(M_2(Y')) = I_R(\mathscr{P}_{\mathfrak{b}(V)})$. Whence $\mathscr{A}$ is the (multinormed) closure of $I_R(\mathscr{P}_{\mathfrak{b}(V)})$ in $M_2(C_Y^*(\mathscr{Y}))$, that is, $\mathscr{A} = I_R(\mathscr{P}_V)$.

Consider the projection $p = 1 \oplus 0 \in \mathscr{P}_{\mathfrak{b}(V)}R \subseteq \mathfrak{b}(\mathscr{A})$. Note that $p \cdot p = \Phi(p^2) = \Phi(p) = P(p) = p$ and $\Phi(p)^* \cdot \Phi(p) = p \cdot p = p = \Phi(p^*p)$. Using [5, Corollary 5.5], we obtain that $\Phi(py) = p \cdot \Phi(y)$ for all $y \in M_2(C_Y^*(\mathscr{Y}))$. Similarly, $p' = 1 - p$ is a projection and $\Phi(yp') = \Phi(y) \cdot p'$ for all $y \in M_2(C_Y^*(\mathscr{Y}))$. The mapping $i : V \to M_2(C_Y^*(\mathscr{Y}))$, $i(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$, is a matrix Y-isometry. Indeed, $i^{(n)}(v) = \begin{bmatrix} 0 & v \\ 0 & 0 \end{bmatrix}$, $v \in M_n(V)$ up to the canonical shuffling. Therefore

$$\left\| i^{(n)}(v) \right\|_e = \left\| \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\|_e = \left\| \begin{bmatrix} v & 0 \\ 0 & 0 \end{bmatrix} \right\|_e = \|v\|_e$$

for all $e \in Y$. Using Theorem 3.1, we have

$$p I_R(\mathscr{P}_V) p' = p \mathscr{A} p' = p \overline{I_R(\mathscr{P}_{\mathfrak{b}(V)})} p' = \overline{p I_R(\mathscr{P}_{\mathfrak{b}(V)}) p'} = \overline{I_R(\mathfrak{b}(V))} = I_R(V)$$

up to matrix Y-isometries. Thus $I_R(V) = p I_R(\mathscr{P}_V) p'$. $\quad\square$

COROLLARY 4.1. *Let Y be an injective domain in $\mathscr{B}(H)$, R the subring in $\mathscr{B}(H)$ generated by Y_+, and $V \subseteq C_Y^*(\mathscr{Y})$ a quantum space. Then V is an injective quantum R-module if and only if $V = p \mathscr{A} p'$ up to a matrix Y-isometry for a certain injective multinormed C^*-algebra $\mathscr{A} \subseteq M_2(C_Y^*(\mathscr{Y}))$, $Y_+^{\oplus 2} \subseteq \mathscr{A}$, and a bounded projection p in $\mathscr{A}$.*

Proof. Based on Proposition 4.1 and Theorem 4.2, we conclude that if V is an injective quantum R-module then $V = I_R(V) = p \mathscr{A} p'$, where $\mathscr{A} = I_R(\mathscr{P}_V)$ is an injective multinormed C^*-algebra. Conversely, assume that $V = p \mathscr{A} p'$ for an injective multinormed C^*-algebra $\mathscr{A} \subseteq M_2(C_Y^*(\mathscr{Y}))$, $Y_+^{\oplus 2} \subseteq \mathscr{A}$, and a bounded projection p in $\mathscr{A}$. Thus $\mathscr{A} = \text{im}(\Phi)$ for a quantum positive projection $\Phi : M_2(C_Y^*(\mathscr{Y})) \to M_2(C_Y^*(\mathscr{Y}))$. Since Φ is unital, it is a quantum contraction (see Remark 2.2). Consider a new projection $P : M_2(C_Y^*(\mathscr{Y})) \to M_2(C_Y^*(\mathscr{Y}))$, $P(z) = p \Phi(z) p'$. Then $P^{(n)}(z) = p^{\oplus n} \Phi^{\oplus n}(z) p'^{\oplus n}$ for all $z \in M_n(M_2(C_Y^*(\mathscr{Y})))$. For each $e \in Y$ there corresponds $\kappa \in Y^\dagger$ such that $\left\| \Phi^{(n)}(z) \right\|_e \leqslant \|z\|_\kappa$ for all $z \in M_n(M_2(C_Y^*(\mathscr{Y})))$, which in turn implies that $\left\| P^{(n)}(z) \right\|_e = \|p^{\oplus n} \Phi^{\oplus n}(z) e p'^{\oplus n}\| \leqslant \|\Phi^{\oplus n}(z) e\| \leqslant \|z\|_\kappa$. Whence P is a quantum contractive projection. Taking into account that Y is an injective domain, we conclude that $\text{im}(P)$ is an injective quantum space. $\quad\square$

REMARK 4.1. The assertion of just can be modified slightly in the following way. Instead of the ring R one may consider another subring in $\mathscr{B}(H)$ generated by Z_+, where Z consists of some $\vee \kappa$ with $\kappa \in Y^\dagger$ covering Y. In this case, we would have an extension $\Phi : M_2(C_Y^*(\mathscr{Y})) \to M_2(C_Y^*(\mathscr{Y}))$ such that $\left\| \Phi^{(\infty)}(y) \right\|_\kappa \leqslant \|y\|_\kappa$, $y \in M(M_2(C_Y^*(\mathscr{Y})))$, $\vee \kappa \in Z$.

5. Injective domains

As we have seen above many properties on injectivity of quantum spaces are re-duced to the property of the relevant domains to be injective. In this section we investi-gate the injectivity problem for domains. So is a graded domain thanks to Lemma 3.1. We generalize this result for locally finite domains. The locally finite domains present a critical edge of injectivity, which fails to be true for non-locally finite domains.

5.1. Locally finite domains

Let Y be a domain in $\mathscr{B}(K)$. We say that Y is a *locally finite domain on* K if for each $e \in Y$ we have $ef = 0$ for all $f \in Y$ except finitely many of them. Obviously, each graded domain is a locally finite one.

THEOREM 5.1. *Let Y be a locally finite domain on a Hilbert space. The multi-normed C^*-algebra $C_Y^*(\mathscr{Y})$ is an injective quantum space (in particular, system). Thus Y is an injective domain and $I(Y') = C_Y^*(\mathscr{Y})$.*

Proof. As in the proof of Proposition 3.2, the algebra $C_Y^*(\mathscr{Y})$ is embedded into $C_X^*(\mathscr{X})$ for a certain graded domain X. Namely, we set $H = \oplus_{e \in Y} \operatorname{im}(e)$ and let $X = \overline{Y} = \{\overline{e} : e \in Y\}$ be a family of the canonical orthogonal projections in $\mathscr{B}(H)$ generated by the latter decomposition. Thus X is a graded domain in H. Consider the $*$-homomorphism $i : C_Y^*(\mathscr{Y}) \to C_X^*(\mathscr{X})$, $i(T) = (T|\operatorname{im}(f))_{f \in Y}$. Then $\|i(T)\|_{\overline{f}} = \|Tf\| = \|T\|_f$, $T \in C_Y^*(\mathscr{Y})$, $f \in Y$, that is, i is a Y-isometry. We identify $C_Y^*(\mathscr{Y})$ with its range in $C_X^*(\mathscr{X})$. Note that $T \in Y'$ iff $i(T) \in X'$. Indeed,

$$\|i(T)\| = \sup_{f \in Y} \|T|\operatorname{im}(f)\| = \sup_{f \in Y} \|Tf\| = \sup_{f \in Y} \|T\|_f = \|T\|.$$

Thus $Y' = \mathfrak{b}(C_Y^*(\mathscr{Y})) = \mathfrak{b}(C_X^*(\mathscr{X})) \cap C_Y^*(\mathscr{Y}) = X' \cap C_Y^*(\mathscr{Y})$, that is, Y' is a unital $*$-subalgebra in X'. But Y' is an injective von Neumann algebra, therefore there is a conditional expectation (projection of norm 1) $P : X' \to Y'$ from X' onto Y'. Thus [2, II, 6.10.1] $P : X' \to Y'$ is a matrix positive contraction such that $P|Y' = \operatorname{id}_{Y'}$ and

$$aP(x) = i(aP(x)) = i(a)i(P(x)) = P(i(a)x),$$
$$P(x)a = i(P(x)a) = i(P(x))i(a) = P(xi(a))$$

for all $a \in Y'$, $x \in X'$. Fix $e \in Y$. Then $eP(x) = P(i(e)x)$ for all $x \in X'$. By assump-tion, Y is a locally finite domain, in particular, $ef = 0$, $f \in Y \setminus \alpha$ for a certain $\alpha \in Y^{\mathfrak{f}}$ which depends upon e. Note that $i(e) = (e\overline{f})_f = (ef)_f \neq \overline{e}$ if α has at least two distinct elements. Then $i(e)x = (ef)_f(xf)_f = (\vee \overline{\alpha})i(e)x$ with $\vee \overline{\alpha} = \sum_{f \in \alpha} \overline{f}$, and

$$\|i(e)x\| = \sup_{f \in \alpha} \|\overline{f}i(e)x\| = \sup_{f \in \alpha} \|i(e)\overline{f}x\| \leqslant \sup_{f \in \alpha} \|\overline{f}x\| = \|x\|_{\overline{\alpha}}$$

for all $x \in X'$. It follows that

$$\|P(x)\|_{\overline{e}} = \|\overline{e}P(x)\| = \|eP(x)\| = \|i(eP(x))\| = \|P(i(e)x)\| \leqslant \|i(e)x\| \leqslant \|x\|_{\overline{\alpha}}$$

for all $x \in X'$. Thus $P : X' \to Y'$ is a continuous mapping with respect to the multi-normed topologies inherited from $C_X^*(\mathscr{X})$ and $C_Y^*(\mathscr{Y})$, respectively. In particular, it has a continuous extension $P : C_X^*(\mathscr{X}) \to C_Y^*(\mathscr{Y})$ to the relevant multinormed completions. As above, just taking into account that $P : X' \to Y'$ is a matrix contraction, we obtain that $\left\| P^{(\infty)}(x) \right\|_{\overline{e}} \leqslant \|x\|_{\overline{\alpha}}$ for all $x \in M(C_X^*(\mathscr{X}))$. Thus P is a quantum contraction. Furthermore, since $P^2 = P$ on $C_X^*(\mathscr{X})$ (out of density of X' in $C_X^*(\mathscr{X})$), it follows that the range of P is closed. Since Y' is dense in $C_Y^*(\mathscr{Y})$, we obtain that P is a projection onto $C_Y^*(\mathscr{Y})$. Using Lemma 3.1 and Proposition 3.2, we conclude that $C_Y^*(\mathscr{Y})$ is an injective quantum space. Based on Remark 3.1, we derive that $C_Y^*(\mathscr{Y})$ is an injective quantum system too.

Finally, the inclusion $Y' \subseteq C_Y^*(\mathscr{Y})$ is rigid, for Y' is dense in $C_Y^*(\mathscr{Y})$. Moreover, $C_Y^*(\mathscr{Y})$ is an injective quantum space. Therefore $I(Y') = C_Y^*(\mathscr{Y})$. $\square$

Injectivity of $C_Y^*(\mathscr{Y})$ fails to be true for a domain which is not locally finite one as shows the forthcoming construction.

5.2. The domain of an affine scheme

We generate (quantum) domains using the spectra of commutative rings. Let H be a Hilbert space with its Hilbert basis Ω. The canonical projection in $\mathscr{B}(H)$ onto a closed subspace $K \subseteq H$ is denoted by p_K. The closed subspace in H generated by a subset $e \subseteq \Omega$ is denoted by $[e]$. For brevity we use the notation p_e instead of $p_{[e]}$. Thus $p_e\left(\sum_{a \in \Omega} \lambda_a a\right) = \sum_{a \in e} \lambda_a a$, $\lambda_a \in \mathbb{C}$, $a \in \Omega$.

REMARK 5.1. If $e, f \subseteq \Omega$ then $p_e \wedge p_f = p_e p_f = p_{e \cap f}$ and $p_e \vee p_f = p_{e \cup f}$. Indeed, $p_e \wedge p_f = p_e p_f = p_{[e] \cap [f]} = p_{[e \cap f]} = p_{e \cap f}$, for Ω is a basis in H (see [2, I.5.1.3]). Furthermore, $p_e \vee p_f = p_{([e]+[f])^-}$. But $([e]+[f])^- = [e \cup f]$, therefore $p_e \vee p_f = p_{[e \cup f]} = p_{e \cup f}$. Similarly, if Y is a family of subsets in Ω then $\bigvee_{e \in Y} p_e = \bigvee_{e \in Y} p_{[e]} = p_{(\sum [e])^-} = p_{[\cup e]} = p_{\cup Y}$, that is, $\bigvee_{e \in Y} p_e = p_{\cup Y}$ (see [2, I.5.1.3]).

Now let A be a unital commutative ring, $Y = \mathrm{Max}(A)$ the set of all maximal ideals of A ordered with respect to the usual inclusion, and let $\Omega(A)$ be the set all non-unit (or non-invertible) elements of the ring A. Recall that $a \in \Omega(A)$ iff Aa is a proper ideal, that is, $Aa \subseteq e$ for some $e \in \mathrm{Max}(A)$. Thus $\Omega(A) = \cup Y$. Consider the Hilbert space $H_A = \ell_2(\Omega(A))$ with its Hilbert basis $\Omega(A)$. The origin of the Hilbert space H_A and zero of the ring A should not be mixed up. They are different elements in H_A. As above for each $e \in Y$ we have the projection $p_e \in \mathscr{B}(H_A)$. To avoid any cluttering with the notations, we identify e with the relevant projection p_e. Based on Remark 5.1, we obtain a commutative family Y of projections in $\mathscr{B}(H_A)$ such that

$$\vee Y = \bigvee_{e \in Y} e = p_{\cup Y} = p_{\Omega_A} = 1_H,$$

that is, Y is a domain on H called *the domain of the spectrum of the ring A*.

LEMMA 5.1. *Let $e, f \in Y$. Assume that there exists a finite subset $\gamma \in Y^{\dagger}$ such that $\|efT\| \leqslant \max\{\|efvT\| : v \in \gamma\}$ for all $T \in Y'$. Then $e \in \gamma$ or $f \in \gamma$.*

Proof. Put $u = e \cap f$ which is an ideal of the ring A, and let $\alpha = \cup\gamma = \cup_{\tau \in \gamma} \tau$ be the finite union of ideals in A. First let us prove that $u \subseteq \alpha$. If the latter is not the case then $u \backslash u \cap \alpha \neq \varnothing$ and $T = p_{u \backslash u \cap \alpha} \neq 0$. Using Remark 5.1, we obtain that

$$T = p_u(1_H - p_{u \cap \alpha}) = p_u\left(1_H - \bigvee_{\tau \in \gamma} p_{u \cap \tau}\right) = p_u\left(1_H - \bigvee_{\tau \in \gamma} p_u p_\tau\right)$$

and $p_u = ef$, $p_u p_\tau = efp_\tau \in Y'$ for all $\tau \in \gamma$. Furthermore Y' being a von Neumann algebra contains $\bigvee_{\tau \in \gamma} p_u p_\tau$ too. Hence $T \in Y'$. If $v \in \gamma$ then

$$efvT = p_u p_v T = p_{u \cap v} p_u \left(1_H - \bigvee_{\tau \in \gamma} p_u p_\tau\right) = p_u\left(p_{u \cap v} - p_{u \cap v} \bigvee_{\tau \in \gamma} p_{u \cap \tau}\right) = 0.$$

It follows that $\max\{\|efvT\| : v \in \gamma\} = 0$ though $efT = p_u T = T$ and $\|p_u T\| = 1$, a contradiction. Hence $u \subseteq \alpha$. But γ is a finite family of prime ideals of A and u is an ideal in $\cup\gamma$. Then $u \subseteq \varepsilon$ for a certain $\varepsilon \in \gamma$ [1, P.1.11]. Furthermore, u is the intersection of the prime ideals e and f. In particular, $e \cdot f \subseteq u \subseteq \varepsilon$, where $e \cdot f$ is the ring product of ideals from A. Therefore $e \subseteq \varepsilon$ or $f \subseteq \varepsilon$ (just use the definition of a prime ideal). But all ideals are the maximal ones either. Hence $e = \varepsilon$ or $f = \varepsilon$, that is, $e \in \gamma$ or $f \in \gamma$. $\square$

Note that Y is not a locally finite domain in H, for $ef = p_{e \cap f} \neq 0$ for all $e, f \in Y$. As above the algebraic sum of the domain Y is denoted by $\mathscr{Y}$.

THEOREM 5.2. *Let A be a unital commutative ring, whose maximal spectrum Y is uncountable. The multinormed C^*-algebra $C_Y^*(\mathscr{Y})$ is not an injective quantum space though its bounded part $\mathfrak{b}(C_Y^*(\mathscr{Y})) = Y'$ is an injective von Neumann algebra. Thus Y is not an injective domain.*

Proof. As in the proof of Theorem 5.1, we set $H = \oplus_{e \in Y} \text{im}(e)$ and let $X = \{\overline{e} : e \in Y\}$ be the family of canonical orthogonal projections on H generated by the latter decomposition. Thus X is a graded domain on H. We have the inclusion $C_Y^*(\mathscr{Y}) \subseteq C_X^*(\mathscr{X})$, $T = (T|\text{im}(f))_{f \in Y} = (T\overline{f})_{f \in Y}$ (see to the proof of Theorem 5.1). In particular, $e = (ef)_{f \in Y} = (\overline{ef})_{f \in Y}$ for each $e \in Y$. If $C_Y^*(\mathscr{Y})$ were an injective quantum space, we would have a quantum contractive projection $P : C_X^*(\mathscr{X}) \rightarrow C_Y^*(\mathscr{Y})$ onto $C_Y^*(\mathscr{Y})$. Thus for each $e \in Y$ there corresponds $\alpha \in Y^{\dagger}$ such that $\left\|P^{(\infty)}(z)\right\|_e \leqslant \|z\|_{\overline{\alpha}}$, $z \in M(C_X^*(\mathscr{X}))$, where $\|z\|_{\overline{\alpha}} = \|z \cdot (\vee\overline{\alpha})\|$ and $\vee\overline{\alpha} = \Sigma_{f \in \alpha} \overline{f}$. Since $\|z\|_e \leqslant \|z\|_{\overline{\alpha}}$ for all $z \in C_Y^*(\mathscr{Y})$, we can assume that $e \in \alpha$. In particular, $P(X') = P(\mathfrak{b}(C_X^*(\mathscr{X}))) \subseteq \mathfrak{b}(C_Y^*(\mathscr{Y})) = Y'$ and

$$\left\|P^{(\infty)}(x)\right\| = \sup_{e \in Y}\left\|P^{(\infty)}(x)\right\|_e \leqslant \sup_{\alpha} \|x\|_{\overline{\alpha}} \leqslant \|x\|$$

for all $x \in X'$, that is, $P : X' \to X'$ is a matrix contraction. But $P(x) = x$ for all $x \in Y' \subseteq C_Y^*(\mathscr{Y})$. Hence $P : X' \to X'$ is a matrix contractive projection onto Y' of norm 1. Using [2, II. 6.10.2], we conclude that P is a conditional expectation, that is, P is Y'-linear mapping. It follows that $P(ex) = eP(x)$ for all $x \in X'$ and $e \in Y$. Then $\|eP(x)\| = \|P(x)\|_e \leqslant \|x\|_{\overline{\alpha}} = \|(\vee\overline{\alpha})x\|$. Thus if $(\vee\overline{\alpha})x = 0$ then $P(ex) = 0$. Take any $x \in X'$. Then $e(1 - \vee\overline{\alpha})x = (1 - \vee\overline{\alpha})ex$ and $P(e(1 - \vee\overline{\alpha})x) = 0$. In particular, $P(ex) = P(e(\vee\overline{\alpha})x) + P(e(1 - \vee\overline{\alpha})x) = P(e(\vee\overline{\alpha})x)$.

Now pick $e, f \in Y$. There are finite subsets $\alpha, \beta \in Y^{\dagger}$ with the properties $\left\|P^{(\infty)}(z)\right\|_e \leqslant \|z\|_{\overline{\alpha}}$ and $\left\|P^{(\infty)}(z)\right\|_f \leqslant \|z\|_{\overline{\beta}}$, $z \in M(C_X^*(\mathscr{X}))$ respectively. Put $\gamma = \alpha \cap \beta$. Then

$$P(efx) = P((\vee\overline{\alpha})efx) = P(f((\vee\overline{\alpha}))ex) = P\left(\left(\vee\overline{\beta}\right)f(\vee\overline{\alpha})ex\right)$$
$$= P\left(\left(\vee\overline{\beta}\right)(\vee\overline{\alpha})efx\right) = P((\vee\overline{\gamma})efx)$$

for all $x \in X'$. In particular, $efT = P((\vee\overline{\gamma})efT)$ for all $T \in Y'$, and

$$\|efT\| = \|P((\vee\overline{\gamma})efT)\| \leqslant \|(\vee\overline{\gamma})efT\| = \max\{\|efvT\| : v \in \gamma\}.$$

By Lemma 5.1, $e \in \gamma$ or $f \in \gamma$.

Finally, fix $e \in Y$. Put $e_1 = e$, $\alpha_1 = \alpha$. Pick $e_2 \notin \alpha_1$ and again α_2 stands for the finite subset in Y that corresponds to e_2. In a similar way, we get a sequence $(e_n)_{n \in \mathbb{N}}$ such that $e_{n+1} \notin \alpha_1 \cup \ldots \cup \alpha_n$ for all n. Consider the union $\cup_{n \in \mathbb{N}}\alpha_n$ which is a countable subset in Y. By assumption, Y is not countable, therefore there is an element $f \in Y \setminus \cup_n \alpha_n$ with the relevant finite subset β in Y. For each couple e_n and f we have $f \notin \alpha_n$. Based on the fact that we have just proved above, we conclude that $e_n \in \beta$ for all n, that is, $(e_n)_{n \in \mathbb{N}} \subseteq \beta$. But β was a finite subset, a contradiction. Consequently, $C_Y^*(\mathscr{Y})$ is not an injective quantum space. $\square$

Thus locally finite domains is a reasonable class to be considered from the injectivity point of view. Thus a locally finite domain Y is injective by Theorem 5.1, whereas the domain of an uncountable affine scheme is an example of a non-injective domain thanks to Theorem 5.2.

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