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CONDITIONS C_p, C'_p, AND C''_p FOR p–OPERATOR SPACES

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Abstract. Conditions C, C', and C'' were introduced for operator spaces in an attempt to study local reflexivity and exactness of operator spaces [4, Chapter 14]. For example, it is known that an operator space W is locally reflexive if and only if W satisfies condition C'' [4, Theorem 14.3.1] and an operator space V is exact if and only if V satisfies condition C' [4, Theorem 14.4.1]. It is also known that an operator space V satisfies condition C if and only if it satisfies conditions C' and C'' [4, Lemma 14.2.1], [7, Theorem 5]. In this paper, we define p-operator space analogues of these definitions, which will be called conditions C_p, C'_p, and C''_p, and show that a p-operator space on L_p space satisfies condition C_p if and only if it satisfies both conditions C'_p and C''_p. The p-operator space injective tensor product of p-operator spaces plays a key role.

1. Introduction to p-operator spaces

A *concrete operator space* V is defined to be a closed subspace of $\mathscr{B}(H)$, where $\mathscr{B}(H)$ denotes the space of all bounded linear operators on a Hilbert space H. For each $n \in \mathbb{N}$, the matrix algebra $\mathbb{M}_n(\mathscr{B}(H))$ with entries in $\mathscr{B}(H)$ can be identified with $\mathscr{B}(\underbrace{H \oplus \cdots \oplus H}_{n})$ via *matrix multiplication*

$$\begin{bmatrix} & & \\ & T_{ij} & \\ & & \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n T_{1j}h_j \\ \vdots \\ \sum_{j=1}^n T_{nj}h_j \end{bmatrix}, \quad [T_{ij}] \in \mathbb{M}_n(\mathscr{B}(H)), \quad h_j \in H,$$

and this gives rise to a norm $\|\cdot\|_n$ on $\mathbb{M}_n(V)$, which we denote by $M_n(V)$. It is then easy to verify that the following two properties (called Ruan's axioms) hold:

$\mathscr{D}_\infty$ for $u \in M_n(V)$ and $v \in M_m(V)$, we have $\left\| \begin{bmatrix} u & 0 \\ 0 & v \end{bmatrix} \right\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$.

$\mathscr{M}$ for $u \in M_m(V)$, $\alpha \in \mathbb{M}_{n,m}(\mathbb{C})$, and $\beta \in \mathbb{M}_{m,n}(\mathbb{C})$, we have $\|\alpha u \beta\|_n \leqslant \|\alpha\| \|u\|_m \|\beta\|$, where $\|\alpha\|$ is the norm of α as a member of $\mathscr{B}(\ell_2^m, \ell_2^n)$, and similarly for β.

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An *abstract operator space* is a Banach space X together with a family of norms $\|\cdot\|_n$ defined on $\mathbb{M}_n(X)$ satisfying the conditions $\mathscr{D}_\infty$ and $\mathscr{M}$ above. In [13], Ruan showed that these two concepts coincide and after Ruan's characterization, operator space theory has really been taken off and quickly developed into an active research area in modern analysis. Many important applications have been found in some related areas. For example, let G be a locally compact group. It is well known that G is amenable if and only if the convolution algebra $L_1(G)$ is amenable as a Banach algebra [9]. We consider another Banach algebra called the *Fourier algebra* $A(G)$ which consists of all coefficient functions of the left regular representation λ of G, i.e.,

$$A(G) = \{\omega(\cdot) = \langle\lambda(\cdot)\xi, \eta\rangle : \xi, \eta \in L_2(G)\}.$$

By [5], $A(G)$ is a commutative Banach algebra with respect to pointwise multiplication and can be regarded as the predual of $VN(G)$, the group von Neumann algebra of G. If G is abelian, then its dual group $\hat{G}$ is also abelian and we have the isometric isomorphism $A(G) \cong L_1(\hat{G})$, and this suggests a relationship between the amenability of G and the amenability (as a Banach algebra) of $A(G)$. Indeed, if $A(G)$ is amenable, then G is amenable. In the opposite direction, Johnson showed that the Banach algebra $A(G)$ fails to be amenable even in the case of very simple compact groups, such as $SU(2,\mathbb{C})$ [10].

In [14], Ruan studied the *operator amenability* of $A(G)$ which can be regarded as the amenability of $A(G)$ in the category of operator spaces, and proved that a locally compact group G is amenable if and only if $A(G)$ is operator amenable. This suggests that $A(G)$ is better viewed as an operator space, and motivated by this observation, there has been some research [3, 1] to study *Figà-Talamanca-Herz Algebra* $A_p(G)$, which can be regarded as an L_p space generalization of the Fourier algebra $A(G)$ (The reader is referred to [6, 8] for more details on $A_p(G)$), in the framework of L_p space generalization of operator spaces. This leads to the definition of p-operator spaces we will give below. Throughout this paper, we let $1 < p < \infty$.

DEFINITION 1.1. Let SQ_p denote the collection of subspaces of quotients of L_p spaces. A Banach space X is called a *concrete p-operator space* if X is a closed subspace of $\mathscr{B}(E)$ for some $E \in SQ_p$, where $\mathscr{B}(E)$ denotes the space of all bounded linear operators on E.

Let $\mathbb{M}_n(X)$ denote the linear space of all $n \times n$ matrices with entries in X. For a concrete p-operator space $X \subseteq \mathscr{B}(E)$ and for each $n \in \mathbb{N}$, define a norm $\|\cdot\|_n$ on $\mathbb{M}_n(X)$ by identifying $\mathbb{M}_n(X)$ as a subspace of $\mathscr{B}(\ell_p^n(E))$, and let $M_n(X)$ denote the corresponding normed space. The norms $\|\cdot\|_n$ then satisfy

$\mathscr{D}_\infty$ for $u \in M_n(X)$ and $v \in M_m(X)$, we have $\|u \oplus v\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$.

$\mathscr{M}_p$ for $u \in M_m(X)$, $\alpha \in \mathbb{M}_{n,m}(\mathbb{C})$, and $\beta \in \mathbb{M}_{m,n}(\mathbb{C})$, we have $\|\alpha u \beta\|_n \leqslant \|\alpha\|\|u\|_m\|\beta\|$, where $\|\alpha\|$ is the norm of α as a member of $\mathscr{B}(\ell_p^m, \ell_p^n)$, and similarly for β.

REMARK 1.2. When $p = 2$, these are Ruan's axioms and 2-operator spaces are simply operator spaces because the SQ_2 spaces are exactly Hilbert spaces.

As in operator spaces, we can also define abstract p-operator spaces.

DEFINITION 1.3. An *abstract p-operator space* is a Banach space X together with a sequence of norms $\|\cdot\|_n$ defined on $\mathbb{M}_n(X)$ satisfying the conditions $\mathscr{D}_\infty$ and $\mathscr{M}_p$ above.

Thanks to the following theorem by Le Merdy, we do not distinguish between concrete p-operator spaces and abstract p-operator spaces, so from now on we will merely speak of p-operator spaces.

THEOREM 1.4. [11, Theorem 4.1] *An abstract p-operator space X can be isometrically embedded in $\mathscr{B}(E)$ for some $E \in SQ_p$ in such a way that the canonical norms on $\mathbb{M}_n(X)$ arising from this embedding agree with the given norms.*

Note that a linear map $u : X \to Y$ between p-operator spaces X and Y induces a map $u_n : M_n(X) \to M_n(Y)$ by applying u entrywise. We say that u is *p-completely bounded* if $\|u\|_{pcb} := \sup_n \|u_n\| < \infty$. Similarly, we define *$p$-completely contractive*, *p-completely isometric*, and *p-completely quotient* maps. We write $\mathscr{CB}_p(X,Y)$ for the space of all p-completely bounded maps from X into Y, and to turn the mapping space $\mathscr{CB}_p(X,Y)$ into a p-operator space, we define a norm on $\mathbb{M}_n(\mathscr{CB}_p(X,Y))$ by identifying this space with $\mathscr{CB}_p(X,M_n(Y))$. Using Le Merdy's theorem, one can show that $\mathscr{CB}_p(X,Y)$ itself is a p-operator space. In particular, the *p-operator dual space* of X is defined to be $\mathscr{CB}_p(X,\mathbb{C})$. The next lemma by Daws shows that we may identify the Banach dual space X' of X with the p-operator dual space $\mathscr{CB}_p(X,\mathbb{C})$ of X.

LEMMA 1.5. [3, Lemma 4.2] *Let X be a p-operator space, and let $\varphi \in X'$, the Banach dual of X. Then φ is p-completely bounded as a map to $\mathbb{C}$. Moreover, $\|\varphi\|_{pcb} = \|\varphi\|$.*

If $E = L_p(\mu)$ for some measure μ and $X \subseteq \mathscr{B}(E) = \mathscr{B}(L_p(\mu))$, then we say that X is a *p-operator space on L_p space*. These p-operator spaces are often easier to work with. For example, let $\kappa_X : X \to X''$ denote the canonical inclusion from a p-operator space X into its second dual. Contrary to operator spaces, κ_X is *not* always p-completely isometric. Thanks to the following theorem by Daws, however, we can easily characterize those p-operator spaces with the property that the canonical inclusion is p-completely isometric.

PROPOSITION 1.6. [3, Proposition 4.4] *Let X be a p-operator space. Then κ_X is a p-complete contraction. Moreover, κ_X is a p-complete isometry if and only if $X \subseteq \mathscr{B}(L_p(\mu))$ p-completely isometrically for some measure μ.*

Conditions C, C', and C'' for operator spaces were introduced and studied in [4, Chapter 14] and [7] and they play an important role in understanding local reflexivity and exactness of operator spaces. For example, it is known that an operator space is locally reflexive if and only if it satisfies condition C'' [4, Theorem 14.3.1]. It is also known that an operator space is exact if and only if it satisfies condition C' [4, Theorem 14.4.1]. In this paper, we define p-operator space analogues of these conditions, which will be called conditions C_p, C'_p, and C''_p, and show that a p-operator space on L_p space satisfies condition C_p if and only if it satisfies both conditions C'_p and C''_p.

2. Tensor product of p-operator spaces

In this section, we recall basic properties of tensor products on p-operator spaces studied in [3, 1]. We mainly focus on p-projective tensor product and p-injective tensor product.

DEFINITION 2.1. Let X,Y be p-operator spaces. Let $X\otimes Y$ denote the algebraic tensor product of X and Y. For $u\in\mathbb{M}_n(X\otimes Y)$, let

$$\|u\|_{\wedge_p}=\inf\{\|\alpha\|\|v\|\|w\|\|\beta\| : u=\alpha(v\otimes w)\beta\},$$

where the infimum is taken over $r,s\in\mathbb{N}$, $\alpha\in M_{n,r\times s}$, $v\in M_r(X)$, $w\in M_s(Y)$, and $\beta\in M_{r\times s,n}$.

Daws defined and studied the p-projective tensor product [3]. Note that $\|\cdot\|_{\wedge_p}$ gives the algebraic tensor product $X\otimes Y$ a p-operator space structure [3, Proposition 4.8]. Furthermore, $\|\cdot\|_{\wedge_p}$ is the largest subcross p-operator space norm on $X\otimes Y$ in the sense that $\|x\otimes y\|\leqslant\|x\|_r\|y\|_s$ for all $x\in M_r(X)$ and all $y\in M_s(Y)$ [3, Proposition 4.8]. The *p-operator space projective tensor product* is defined to be the completion of $X\otimes Y$ with respect to this norm and is denoted by $X\overset{\wedge_p}{\otimes}Y$.

REMARK 2.2.

a. One can show that p-operator space projective tensor product is commutative, i.e., $X\overset{\wedge_p}{\otimes}Y=Y\overset{\wedge_p}{\otimes}X$ p-completely isometrically.

b. By universality of the Banach space projective tensor product $\overset{\pi}{\otimes}$ [11, A.3.3], we have

$$\|u\|_{\wedge_p}\leqslant\|u\|_\pi$$

for all $u\in X\otimes Y$.

Let V,W, and Z be p-operator spaces, and let $\psi : V\times W\to Z$ be a bilinear map. Define bilinear maps $\psi_{r,s;t,u}$ by

$$\psi_{r,s;t,u} : M_{r,s}(V)\times M_{t,u}(W)\to M_{r\times t,s\times u}(Z),\qquad (v,w)\mapsto(\psi(v_{i,j},w_{k,l})),$$

and let $\psi_{r;s}=\psi_{r,r;s,s}$. Finally define

$$\|\psi\|_{jpcb}=\sup\{\|\psi_{r;s}\| : r,s\in\mathbb{N}\}.$$

We say that ψ is *jointly p-completely bounded* (respectively, *jointly p-completely contractive*) if $\|\psi\|_{jpcb} < \infty$ (respectively, $\|\psi\|_{jpcb} \leqslant 1$). The space of all jointly p-completely bounded maps from $V \times W$ to Z will be denoted by $\mathscr{CB}_p(V \times W, Z)$ and this space can be turned into a p-operator space in the same way as for $\mathscr{CB}_p(V,W)$. Here we collect some results on the p-projective tensor product for convenience.

PROPOSITION 2.3. [3, Proposition 4.9] *Let X, Y, and Z be p-operator spaces. Then we have natural p-completely isometric identifications*

$$\mathscr{CB}_p(X \overset{\wedge_p}{\otimes} Y, Z) = \mathscr{CB}_p(X \times Y, Z) = \mathscr{CB}_p(X, \mathscr{CB}_p(Y,Z)).$$

In particular,

$$(X \overset{\wedge_p}{\otimes} Y)' = \mathscr{CB}_p(X, Y').$$

As in operator spaces, the p-operator space projective tensor product is projective in the following sense:

PROPOSITION 2.4. [3, Proposition 4.10] *Let X, X_1, Y, and Y_1 be p-operator spaces. If $u : X \to X_1$ and $v : Y \to Y_1$ are p-complete quotient maps, then $u \otimes v$ extends to a p-complete quotient map $u \otimes v : X \overset{\wedge_p}{\otimes} Y \to X_1 \overset{\wedge_p}{\otimes} Y_1$.*

We now briefly introduce the p-operator space injective tensor product.

DEFINITION 2.5. Let X, Y be p-operator spaces. Regarding the algebraic tensor product $X \otimes Y$ as a subspace of $\mathscr{CB}_p(X', Y)$, we define the *p-operator space injective tensor product* $X \overset{\vee_p}{\otimes} Y$ to be the completion of $X \otimes Y$ in $\mathscr{CB}_p(X', Y)$.

To be precise, for $u = [u_{ij}] \in \mathbb{M}_n(X \otimes Y)$ with $u_{ij} = \sum_{k=1}^{N_{ij}} x_k^{ij} \otimes y_k^{ij}$, the p-operator space injective tensor product norm $\|u\|_{\vee_p}$ is defined by

$$\begin{aligned}\|u\|_{\vee_p} &= \|u\|_{M_n(\mathscr{CB}_p(X',Y))} = \|u\|_{\mathscr{CB}_p(X', M_n(Y))} \\ &= \sup\left\{ \left\| \left[\sum_{k=1}^{N_{ij}} f_{st}(x_k^{ij}) y_k^{ij} \right] \right\|_{M_{mn}(Y)} : m \in \mathbb{N}, f = [f_{st}] \in M_m(X')_1 \right\},\end{aligned} \tag{2.1}$$

where $M_m(X')_1$ denotes the closed unit ball of $M_m(X') = \mathscr{CB}_p(X, M_m)$.

PROPOSITION 2.6. *Suppose that X, X_1, Y, and Y_1 are p-operator spaces. Given p-complete contractions $\varphi : X \to X_1$ and $\psi : Y \to Y_1$, the mapping*

$$\varphi \otimes \psi : X \otimes Y \to X_1 \otimes Y_1$$

extends to a p-complete contraction

$$\varphi \otimes \psi : X \overset{\vee_p}{\otimes} Y \to X_1 \overset{\vee_p}{\otimes} Y_1.$$

Proof. Since $\varphi \otimes \psi = (id_{X_1} \otimes \psi) \circ (\varphi \otimes id_Y)$, it suffices to show that $\varphi \otimes id_Y$ and $id_{X_1} \otimes \psi$ extend to p-complete contractions. Let $u = [u_{ij}] \in M_n(X \otimes Y)$. Let us write $u_{ij} = \sum_k^{N_{ij}} x_k^{ij} \otimes y_k^{ij}$ for each u_{ij}. Since

$$(\varphi \otimes id_Y)_n(u) = \left[\sum_k^{N_{ij}} \varphi(x_k^{ij}) \otimes y_k^{ij}\right] \in M_n(X_1 \otimes Y),$$

from (2.1) it follows that

$$\|(\varphi \otimes id_Y)_n(u)\|_{\vee_p} = \sup\left\{\left\|\left[\sum_{k=1}^{N_{ij}} g_{st}(\varphi(x_k^{ij}))y_k^{ij}\right]\right\|_{M_{mn}(Y)} : m \in \mathbb{N}, g = [g_{st}] \in M_m(X_1')_1\right\}.$$

Define $h_{st} = g_{st} \circ \varphi$ for $1 \leqslant s,t \leqslant m$, then $h = [h_{st}] = g \circ \varphi \in M_m(X')_1$ and we have

$$\|(\varphi \otimes id_Y)_n(u)\|_{\vee_p} \leqslant \|u\|_{\vee_p}.$$

To show that $id_{X_1} \otimes \psi$ is also p-completely contractive, let $v = [v_{ij}] \in M_n(X_1 \otimes Y)$. Writing $v_{ij} = \sum_k^{N_{ij}} w_k^{ij} \otimes y_k^{ij}$, we have

$$\|v\|_{\vee_p} = \sup\left\{\left\|\left[\sum_{k=1}^{N_{ij}} f_{st}(w_k^{ij})y_k^{ij}\right]\right\|_{M_{mn}(Y)} : m \in \mathbb{N}, f = [f_{st}] \in M_m(X_1')_1\right\}.$$

On the other hand,

$$\begin{aligned}
&\|(id_{X_1} \otimes \psi)_n(v)\|_{\vee_p} \\
&= \sup\left\{\left\|\left[\sum_{k=1}^{N_{ij}} f_{st}(w_k^{ij})\psi(y_k^{ij})\right]\right\|_{M_{mn}(Y_1)} : m \in \mathbb{N}, f = [f_{st}] \in M_m(X_1')_1\right\} \\
&= \sup\left\{\left\|\psi_{mn}\left(\left[\sum_{k=1}^{N_{ij}} f_{st}(w_k^{ij})y_k^{ij}\right]\right)\right\|_{M_{mn}(Y_1)} : m \in \mathbb{N}, f = [f_{st}] \in M_m(X_1')_1\right\} \\
&\leqslant \|\psi\|_{pcb}\|v\|_{\vee_p}. \quad \square
\end{aligned} \tag{2.2}$$

REMARK 2.7.

a. By definition of the Banach space injective tensor product $\overset{\varepsilon}{\otimes}$, we have

$$\|u\|_\varepsilon = \|u\|_{\mathscr{B}(X',Y)} \leqslant \|u\|_{\mathscr{CB}_p(X',Y)} = \|u\|_{\vee_p}$$

for every $u \in X \otimes Y$.

b. Let $u \in \mathbb{M}_n(X \otimes Y)$. If $Y \subseteq \mathscr{B}(L_p(\nu))$ for some measure ν, then by Definition 2.5 and [3, Theorem 4.3, Proposition 4.4]

$$\begin{aligned}
\|u\|_{\vee_p} &= \sup\{\|\psi(\varphi_{st}(u_{ij}))\|_{M_{rmn}} : m,k \in \mathbb{N}, \varphi = [\varphi_{st}] \in M_m(X')_1, \psi \in M_k(Y')_1\} \\
&= \sup\{\|(\varphi \otimes \psi)_n(u)\| : m,k \in \mathbb{N}, \varphi \in M_m(X')_1, \psi \in M_k(Y')_1\}.
\end{aligned}$$

c. Let $F : X \otimes Y \to Y \otimes X$ denote the "flip", that is, $F(\sum x_i \otimes y_i) = \sum y_i \otimes x_i$. If $Y \subseteq \mathscr{B}(L_p(\nu))$ for some measure ν, then by (b) above, for every $u \in \mathbb{M}_n(X \otimes Y)$, we get

$$\|u\|_{\vee_p} = \sup\{\|(\varphi \otimes \psi)_n(u)\| : m,k \in \mathbb{N}, \varphi \in M_m(X')_1, \psi \in M_k(Y')_1\}.$$

On the other hand, if $X \subseteq \mathscr{B}(L_p(\mu))$ for some measure μ as well, then

$$\|F_n(u)\|_{\vee_p} = \sup\{\|(\psi \otimes \varphi)_n(F_n(u))\| : m,k \in \mathbb{N}, \varphi \in M_m(X')_1, \psi \in M_k(Y')_1\}$$

and it follows that $X \overset{\vee_p}{\otimes} Y = Y \overset{\vee_p}{\otimes} X$ p-completely isometrically.

d. $M_r \overset{\vee_p}{\otimes} M_s$ is p-completely isometrically isomorphic to M_{rs}. This follows immediately from [1, Theorem 3.2].

At this moment, we do not know whether the p-operator space injective tensor product is injective, that is, if $u : X \to \tilde{X}$ and $v : Y \to \tilde{Y}$ are p-completely isometric injections, then we do not know whether $u \otimes v$ always extends to a p-completely isometric injection $u \otimes v : X \overset{\vee_p}{\otimes} Y \to \tilde{X} \overset{\vee_p}{\otimes} \tilde{Y}$. But if we assume that all the p-operator spaces under consideration are on L_p space, then we can show that $u \otimes v : X \overset{\vee_p}{\otimes} Y \to \tilde{X} \overset{\vee_p}{\otimes} \tilde{Y}$ is a p-complete isometry as in the following proposition. This fact supports that the terminology p-injective tensor product is still reasonable.

PROPOSITION 2.8. *Let μ_1, μ_2 be measures. For $i = 1,2$, suppose $X_i \subseteq Y_i \subseteq \mathscr{B}(L_p(\mu_i))$. Then*

$$X_1 \overset{\vee_p}{\otimes} X_2 \subseteq Y_1 \overset{\vee_p}{\otimes} Y_2$$

p-completely isometrically.

Proof. For $i = 1,2$, let $\varphi_i : X_i \hookrightarrow Y_i$ denote the (p-completely isometric) inclusion. Since $\varphi_1 \otimes \varphi_2 = (\varphi_1 \otimes id_{Y_2}) \circ (id_{X_1} \otimes \varphi_2)$, by Remark 2.7 (c) above, it suffices to show that

$$id_{X_1} \otimes \varphi_2 : X_1 \overset{\vee_p}{\otimes} X_2 \to X_1 \overset{\vee_p}{\otimes} Y_2$$

is p-completely isometric. Note that the following diagram commutes:

$$\begin{tikzcd}
X_1 \overset{\vee_p}{\otimes} X_2 \arrow[r, "id_{X_1} \otimes \varphi_2"] \arrow[d, hook] & X_1 \overset{\vee_p}{\otimes} Y_2 \arrow[d, hook] \\
\mathscr{CB}_p(X_1', X_2) \arrow[r, hook] & \mathscr{CB}_p(X_1', Y_2)
\end{tikzcd}$$

Since $X_1 \overset{\vee_p}{\otimes} X_2 \subseteq \mathscr{CB}_p(X_1', X_2)$, $X_1 \overset{\vee_p}{\otimes} Y_2 \subseteq \mathscr{CB}_p(X_1', Y_2)$, and $\mathscr{CB}_p(X_1', X_2) \subseteq \mathscr{CB}_p(X_1', Y_2)$ p-completely isometrically, we conclude that $id_{X_1} \otimes \varphi_2$ is p-completely isometric. $\square$

3. Conditions C_p', C_p'', and C_p for p-operator spaces

In this section, we define conditions C_p', C_p'', and C_p for p-operator spaces and prove the main result. Throughout the section, μ and ν will denote measures.

LEMMA 3.1. *Let V and W be p-operator spaces. Then the bilinear mapping*

$$\tilde{\Psi} : V' \times W' \to (V \overset{\vee_p}{\otimes} W)', \qquad (f,g) \mapsto f \otimes g$$

is jointly p-completely contractive and hence the canonical mapping $\Psi : V' \overset{\wedge_p}{\otimes} W' \to (V \overset{\vee_p}{\otimes} W)'$ is p-completely contractive.

Proof. We identify $[f_{ij}] \in M_r(V')$ with an operator $F \in \mathscr{CB}_p(V, M_r)$, and likewise $[g_{kl}] \in M_s(W')$ with $G \in \mathscr{CB}_p(W, M_s)$. We have the identification $M_{rs}((V \overset{\vee_p}{\otimes} W)') = \mathscr{CB}_p(V \overset{\vee_p}{\otimes} W, M_{rs})$. Let H be the map $[f_{ij} \otimes g_{kl}] : V \overset{\vee_p}{\otimes} W \to M_{rs}$. Then by Proposition 2.6 and Remark 2.7 (d) we have the commutative diagram

$$\begin{tikzcd} V \overset{\vee_p}{\otimes} W \arrow[r, "H"] \arrow[d, "F\otimes G"'] & M_{rs} \\ M_r \overset{\vee_p}{\otimes} M_s \arrow[ur, "\cong"'] & \end{tikzcd}$$

with $\|F \otimes G\|_{pcb} \leqslant \|F\|_{pcb} \|G\|_{pcb}$, and it follows that $\|[f_{ij} \otimes g_{kl}]\| = \|H\|_{pcb} \leqslant \|F \otimes G\|_{pcb} \leqslant \|F\|_{pcb} \|G\|_{pcb}$ as required. $\square$

LEMMA 3.2. *Let V and W be p-operator spaces. Then $\|\cdot\|_{\vee_p}$ is a subcross matrix norm. In particular, for every $u \in M_n(V \otimes W)$, we have $\|u\|_{\vee_p} \leqslant \|u\|_{\wedge_p}$.*

Proof. Just to fix notation, we identify $M_r(V) \otimes M_q(W)$ with $M_{rq}(V \otimes W)$ by $(v_{ij}) \otimes (w_{kl}) \mapsto (v_{ij} \otimes w_{kl})_{(i,k),(j,l)}$ where we have the ordering $(1,1) \leqslant (1,2) \leqslant \cdots \leqslant (1,q) \leqslant (2,1) \leqslant \cdots \leqslant (r,q)$. Hence $I_r \otimes w \in M_r \otimes M_q(W) = M_{rq}(W)$ is identified with a block matrix in $M_r(M_q(W))$ which has r copies of w down the diagonal and 0 elsewhere. Applying axiom $\mathscr{D}_\infty$ repeatedly hence shows that $\|I_r \otimes w\|_{rq} = \|w\|_q$. Then, for $\alpha \in M_r$, the matrix $\alpha \otimes w \in M_r \otimes M_q(W) = M_{rq}(W)$ is the product $(\alpha \otimes I_q)(I_r \otimes w)$ which has norm at most $\|\alpha\|_r \|w\|_q$ by axiom $\mathscr{M}_p$. Now let $v \in M_r(V)$ and $w \in M_q(W)$, and consider $v \otimes w \in M_{rq}(V \overset{\vee_p}{\otimes} W)$. This tensor induces the operator $T \in \mathscr{CB}_p(V', M_{rq}(W))$ given by $T(f) = (f(v_{ij}) w_{kl})_{(i,k),(j,l)} = (f(v_{ij})) \otimes w$. For $f = (f_{ab}) \in M_n(V')$, we see that $T_n(f) = \langle\langle f, v \rangle\rangle \otimes w \in M_{nrq}(W)$, which by the previous paragraph has norm at most $\|\langle\langle f, v\rangle\rangle\|_{nr} \|w\|_q \leqslant \|f\|_n \|v\|_r \|w\|_q$. Hence $\|T\|_{pcb} \leqslant \|v\|_r \|w\|_q$ as required. $\square$

Let V and W be p-operator spaces and fix $\varphi \in (V \overset{\vee_p}{\otimes} W)'$. For $v_0 \in V$, we define a bounded linear functional ${}_{v_0}\varphi$ on W by

$$ {}_{v_0}\varphi(w) = \varphi(v_0 \otimes w), \qquad w \in W. $$

In general, when $v_0 = [v_{ij}] \in M_r(V)$ and $\varphi = [\varphi_{kl}] \in M_n((V \overset{\vee_p}{\otimes} W)')$, we define ${}_{v_0}\varphi = [{}_{v_{ij}}\varphi_{kl}] \in M_{rn}(W')$. Similarly, for $w_0 \in W$, we define $\varphi_{w_0} \in V'$ by

$$ \varphi_{w_0}(v) = \varphi(v \otimes w_0), \qquad v \in V. $$

As in ${}_{v_0}\varphi$ above, we can extend the definition of φ_{w_0} for $w_0 \in M_r(W)$ and $\varphi \in M_n((V \overset{\vee_p}{\otimes} W)')$. Define a linear map $\Phi^R_{V,W} : V \otimes W'' \to (V \overset{\vee_p}{\otimes} W)''$ by

$$ \Phi^R_{V,W}(v \otimes w'')(\varphi) = \langle {}_v\varphi, w'' \rangle_{W',W''}, \qquad v \in V, \quad w'' \in W'', \quad \varphi \in (V \overset{\vee_p}{\otimes} W)'. $$

Similarly, define a linear map $\Phi^L_{V,W} : V'' \otimes W \to (V \overset{\vee_p}{\otimes} W)''$ by

$$ \Phi^L_{V,W}(v'' \otimes w)(\varphi) = \langle \varphi_w, v'' \rangle_{V',V''}, \qquad v'' \in V'', \quad w \in W, \quad \varphi \in (V \overset{\vee_p}{\otimes} W)'. $$

LEMMA 3.3. *The map $\Phi^R_{V,W}$ (respectively, $\Phi^L_{V,W}$) defined above extends to a p-completely contractive map $\Phi^R_{V,W} : V \overset{\wedge_p}{\otimes} W'' \to (V \overset{\vee_p}{\otimes} W)''$ (respectively, $\Phi^L_{V,W} : V'' \overset{\wedge_p}{\otimes} W \to (V \overset{\vee_p}{\otimes} W)''$).*

Proof. Consider the bilinear map $\Phi : V \times W'' \to (V \overset{\vee_p}{\otimes} W)''$ given by

$$ (v, w'') \mapsto (\varphi \mapsto \langle {}_v\varphi, w'' \rangle_{W',W''}), $$

then we get

$$ \Phi_{r;s} : M_r(V) \times M_s(W'') \to M_{rs}((V \overset{\vee_p}{\otimes} W)''), \qquad ([v_{ij}], [w_{kl}{}'']) \mapsto [\Phi(v_{ij}, w_{kl}{}'')] $$

and

$$ \|[\Phi(v_{ij}, w_{kl}{}'')]\| = \sup_n \left\{ \|\langle\langle \Phi_{r;s}(v, w''), \varphi \rangle\rangle\| : \varphi \in M_n((V \overset{\vee_p}{\otimes} W)'), \|\varphi\| \leqslant 1 \right\}. $$

Since $\langle\langle \Phi_{r;s}(v, w''), \varphi \rangle\rangle = \langle\langle {}_v\varphi, w'' \rangle\rangle$, we have

$$ \|\langle\langle \Phi_{r;s}(v, w''), \varphi \rangle\rangle\| = \|\langle\langle {}_v\varphi, w'' \rangle\rangle\| \leqslant \|{}_v\varphi\|_{M_{rn}(W')} \cdot \|w''\|_{M_s(W'')} $$

and the result follows because $\overset{\vee_p}{\otimes}$ is a subcross matrix norm and hence

$$ \begin{aligned} \|{}_v\varphi\|_{M_{rn}(W')} &= \sup_m \{ \|\langle\langle {}_v\varphi, w \rangle\rangle\|_{M_{rnm}} : w \in M_m(W), \|w\| \leqslant 1 \} \\ &= \sup_m \{ \|\langle\langle \varphi, v \otimes w \rangle\rangle\|_{M_{rnm}} : w \in M_m(W), \|w\| \leqslant 1 \} \\ &\leqslant \|\varphi\| \cdot \|v\| \\ &\leqslant \|v\|. \quad \square \end{aligned} $$

REMARK 3.4. Let α be a general subcross matrix norm.

a. We have a natural p-complete contraction $V \overset{\wedge p}{\otimes} W \to V \otimes_\alpha W$ and the adjoint gives a contraction $(V \otimes_\alpha W)' \to \mathscr{CB}_p(V, W') \subseteq \mathscr{B}(V, W')$ given by

$$\varphi \mapsto L_\varphi, \quad \langle L_\varphi(v), w\rangle = \varphi(v \otimes w), \quad \varphi \in (V \otimes_\alpha W)' \quad v \in V, \quad w \in W.$$

b. Using the natural p-complete contraction $V \overset{\wedge p}{\otimes} W \to V \otimes_\alpha W$, each member in $(V \otimes_\alpha W)'$ can be regarded as a member in $(V \overset{\wedge p}{\otimes} W)'$.

c. We can define $\Phi^R_{V,W} : V \otimes W'' \to (V \otimes_\alpha W)''$ and $\Phi^L_{V,W} : V'' \otimes W \to (V \otimes_\alpha W)''$ for a general subcross norm α and Lemma 3.3 remains valid if $\overset{\vee p}{\otimes}$ is replaced by $\otimes_\alpha$.

Let $\Psi : V' \overset{\wedge p}{\otimes} W' \to (V \overset{\vee p}{\otimes} W)'$ denote the canonical map, and consider the following commutative diagram

$$\begin{tikzcd}
V \otimes W'' \arrow[r, "\Phi^R_{V,W}"] \arrow[d, equal] & (V \overset{\vee p}{\otimes} W)'' \arrow[r, "\Psi'"] & (V' \overset{\wedge p}{\otimes} W')' \arrow[d, equal] \\
\mathscr{CB}^\sigma_{p,F}(V', W'') \arrow[rr, hook, "\iota"] & & \mathscr{CB}_p(V', W'')
\end{tikzcd},$$

where $\mathscr{CB}^\sigma_{p,F}(V', W'')$ denotes the space of all weak*-continuous p-completely bounded finite rank maps from V' to W'' and ι denotes the inclusion map. This commutative diagram shows that $\Phi^R_{V,W}$ is one-to-one, so one can equip $V \otimes W''$ with the p-operator space norm inherited from $(V \overset{\vee p}{\otimes} W)''$, which will be denoted by, following the notation in [4], $V \overset{\vee p}{\otimes} : W''$. We say that V satisfies *condition* C'_p (or V has *property* C'_p) if this induced norm coincides with the p-operator space injective tensor product norm for every $W \subseteq \mathscr{B}(L_p(\nu))$.

Similarly, the following diagram

$$\begin{tikzcd}
V'' \otimes W \arrow[r, "\Phi^L_{V,W}"] \arrow[d, equal] & (V \overset{\vee p}{\otimes} W)'' \arrow[r, "\Psi'"] & (V' \overset{\wedge p}{\otimes} W')' \arrow[d, equal] \\
\mathscr{CB}^\sigma_{p,F}(W', V'') \arrow[rr, hook, "\iota"] & & \mathscr{CB}_p(W', V'')
\end{tikzcd}$$

is also commutative, $\Phi^L_{V,W}$ is one-to-one, and one can hence equip $V'' \otimes W$ with the p-operator space norm inherited from $(V \overset{\vee p}{\otimes} W)''$, which will be denoted by $V'' : \overset{\vee p}{\otimes} W$. We say that V satisfies *condition* C''_p (or V has *property* C''_p) if this induced norm coincides with the injective tensor product norm for every $W \subseteq \mathscr{B}(L_p(\nu))$.

EXAMPLE 3.5. We say that a p-operator space X is *reflexive* if the canonical isometric inclusion $\kappa_X : X \to X''$ is a p-completely isometric isomorphism from X onto X''. It is easy to verify that a p-operator space X is reflexive if and only if X is reflexive as a Banach space and there is a measure μ such that $X \subseteq \mathscr{B}(L_p(\mu))$. In particular, for any measure μ, $L_p^c(\mu)$ and $L_{p'}^r(\mu)$ are reflexive, where $L_p^c(\mu)$ (respectively, $L_{p'}^r(\mu)$) denotes the p-operator space structure given on $L_p(\mu)$ (respectively, $L_{p'}(\mu)$) by the identification $L_p(\mu) = \mathscr{B}(\mathbb{C}, L_p(\mu))$ (respectively, $L_{p'}(\mu) = \mathscr{B}(L_p(\mu), \mathbb{C})$). It is clear that every reflexive p-operator space satisfies condition C''_p.

In order to define *condition C_p* for p-operator spaces, we need the natural map from $V'' \otimes W''$ to $(V \overset{\vee_p}{\otimes} W)''$. To do this, let α be a general subcross matrix norm on $V \otimes W$ and consider the diagram

$$\begin{tikzcd}
 & (V \overset{\wedge_p}{\otimes} W'')'' \arrow[dr, "(\Phi^R_{V,W})''"] & & \\
V'' \otimes W'' \arrow[ur, "\Phi^L_{V,W''}"] \arrow[dr, "\Phi^R_{V'',W}"'] & & (V \otimes_\alpha W)'''' \arrow[r, "P"] & (V \otimes_\alpha W)'' \\
 & (V'' \overset{\wedge_p}{\otimes} W)'' \arrow[ur, "(\Phi^L_{V,W})''"'] & &
\end{tikzcd} , \tag{3.1}$$

where P is the restriction mapping and $(\Phi^R_{V,W})''$ and $(\Phi^L_{V,W})''$ are from Remark 3.4 (c).

Consider the following p-complete contraction:

$$(V \overset{\wedge_p}{\otimes} W)' \cong \mathscr{CB}_p(V, W') \xrightarrow{\text{adj}} \mathscr{CB}_p(W'', V') \cong (V \overset{\wedge_p}{\otimes} W'')'.$$

For $\varphi \in (V \overset{\wedge_p}{\otimes} W)'$, let $\varphi^\wedge \in (V \overset{\wedge_p}{\otimes} W'')'$ denote the image of φ under this map. Then we have

$$\varphi^\wedge(v \otimes w'') = \langle {}_v\varphi, w'' \rangle_{W', W''} = \Phi^R_{V,W}(v \otimes w'')(\varphi), \qquad v \in V, \quad w'' \in W''.$$

Moreover, $\varphi^\wedge$ is weak*-continuous in the second variable. Similarly, we also consider the p-complete contraction

$$(V \overset{\wedge_p}{\otimes} W)' \cong \mathscr{CB}_p(W, V') \xrightarrow{\text{adj}} \mathscr{CB}_p(V'', W') \cong (V'' \overset{\wedge_p}{\otimes} W)'$$

and define ${}^\wedge\varphi$, and then we get that

$${}^\wedge\varphi(v'' \otimes w) = \langle \varphi_w, v'' \rangle_{V', V''} = \Phi^L_{V,W}(v'' \otimes w)(\varphi), \qquad v'' \in V'', \quad w \in W,$$

and that ${}^\wedge\varphi$ is weak*-continuous in the first variable.

REMARK 3.6. Let α be a general subcross matrix norm. By Remark 3.4 (b), we can still define $\varphi^\wedge \in (V \overset{\wedge_p}{\otimes} W'')'$ for any $\varphi \in (V \otimes_\alpha W)'$. Similarly, we can define ${}^\wedge\varphi \in (V'' \overset{\wedge_p}{\otimes} W)'$ for any $\varphi \in (V \otimes_\alpha W)'$.

The next result follows by Remarks 3.4 and 3.6, and the same argument as in the proof of [7, Theorem 1].

THEOREM 3.7. *Let V and W be p-operator spaces. Let α be a subcross matrix norm on $V \otimes W$ and denote by $V \otimes_\alpha W$ the resulting normed space. Then the following are equivalent.*

a. There exists a separately weak-continuous extension*

$$\Phi : V'' \otimes W'' \to (V \otimes_\alpha W)''$$

of the natural inclusion $\iota : V \otimes W \to (V \otimes_\alpha W)''$.

b. The following diagram commutes

$$\begin{tikzcd}
& (V \overset{\wedge p}{\otimes} W'')'' \arrow[dr, "(\Phi^R_{V,W})''"] & & \\
V'' \otimes W'' \arrow[ur, "\Phi^L_{V,W''}"] \arrow[dr, "\Phi^R_{V'',W}"'] & & (V \otimes_\alpha W)'''' \arrow[r, "P"] & (V \otimes_\alpha W)'' \\
& (V'' \overset{\wedge p}{\otimes} W)'' \arrow[ur, "(\Phi^L_{V,W})''"'] & &
\end{tikzcd} .$$

c. For every $\varphi \in (V \otimes_\alpha W)'$, two functionals $({}^\wedge\varphi)^\wedge$ and ${}^\wedge(\varphi^\wedge)$ coincide on $V'' \otimes W''$.

d. For every $\varphi \in (V \otimes_\alpha W)'$, $L_\varphi : V \to W'$ is weakly compact, where $\langle L_\varphi(v), w\rangle = \varphi(v \otimes w)$, $v \in V$, $w \in W$.

THEOREM 3.8. *Let $V \subseteq \mathscr{B}(L_p(\mu))$ and $W \subseteq \mathscr{B}(L_p(\nu))$. For every $\varphi \in (V \overset{\vee p}{\otimes} W)'$, L_φ is weakly compact, where L_φ is as in Theorem 3.7 (d).*

Proof. Without loss of generality, we may assume $\|\varphi\| (= \|\varphi\|_{pcb}) \leqslant 1$. Let Φ^V (respectively Φ^W) denote the embedding $\Phi^V : V \hookrightarrow \mathscr{B}(L_p(\mu))$ (respectively, $\Phi^W : W \hookrightarrow \mathscr{B}(L_p(\nu))$). By Proposition 2.8 and [1, Theorem 3.2], we have p-completely isometric embeddings

$$V \overset{\vee p}{\otimes} W \hookrightarrow \mathscr{B}(L_p(\mu)) \overset{\vee p}{\otimes} \mathscr{B}(L_p(\nu)) \hookrightarrow \mathscr{B}(L_p(\mu \times \nu)).$$

Consider the diagram below:

$$\begin{tikzcd} V \overset{\vee_p}{\otimes} W \arrow[rr, "\varphi"] \arrow[d, hook, "\Phi^V \otimes \Phi^W"'] & & \mathbb{C} \\ \mathscr{B}(L_p(\mu)) \overset{\vee_p}{\otimes} \mathscr{B}(L_p(\nu)) \arrow[d, hook] & & \\ \mathscr{B}(L_p(\mu \times \nu)) \arrow[uurr, dashed, "\tilde{\varphi}"'] & & \end{tikzcd}$$

By Hahn-Banach Theorem, φ extends to $\tilde{\varphi} : \mathscr{B}(L_p(\mu \times \nu)) \to \mathbb{C}$. Applying the same technique as in the proof of [1, Theorem 3.6], we can find a measure space (Ω, Σ, θ) together with two vectors $\xi \in L_p(\theta)$, $\eta \in L_{p'}(\theta)$, and a unital p-completely contractive homomorphism $\pi : \mathscr{B}(L_p(\mu \times \nu)) \to \mathscr{B}(L_p(\theta))$ such that $\tilde{\varphi}(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle$.

Define $T : \mathscr{B}(L_p(\mu)) \to \mathscr{B}(L_p(\nu))'$ by

$$\langle T(x), y \rangle = \tilde{\varphi}(x \otimes y), \qquad x \in \mathscr{B}(L_p(\mu)), \quad y \in \mathscr{B}(L_p(\nu)).$$

Then it is easy to check that the following diagram is commutative:

$$\begin{tikzcd} V \arrow[r, "L_\varphi"] \arrow[d, hook, "\Phi^V"'] & W' \\ \mathscr{B}(L_p(\mu)) \arrow[r, "T"] & \mathscr{B}(L_p(\nu))' \arrow[u, "(\Phi^W)'"'] \end{tikzcd}$$

Define $R : \mathscr{B}(L_p(\mu)) \to L_p(\theta)$ and $S : \mathscr{B}(L_p(\nu)) \to L_{p'}(\theta)$ by

$$R(x) = \pi(x \otimes 1)\xi, \quad x \in \mathscr{B}(L_p(\mu)), \qquad \text{and} \qquad S(y) = (\pi(1 \otimes y))'\eta, \quad y \in \mathscr{B}(L_p(\nu)),$$

then the diagram

$$\begin{tikzcd} \mathscr{B}(L_p(\mu)) \arrow[rr, "T"] \arrow[dr, "R"'] & & \mathscr{B}(L_p(\nu))' \\ & L_p(\theta) \arrow[ur, "S'"'] & \end{tikzcd}$$

is commutative, because

$$\begin{aligned} \langle S'R(x), y \rangle &= \langle R(x), S(y) \rangle = \langle \pi(x \otimes 1)\xi, (\pi(1 \otimes y))'\eta \rangle = \langle \pi(x \otimes y)\xi, \eta \rangle \\ &= \tilde{\varphi}(x \otimes y) = \langle T(x), y \rangle. \end{aligned}$$

Combining these two commutative diagrams, we finally have $L_\varphi = (\Phi^W)'S'R\Phi^V$, that is, L_φ is factorized through a reflexive Banach space $L_p(\theta)$, so L_φ is a weakly compact operator [12, Propositions 3.5.4 and 3.5.11]. $\square$

COROLLARY 3.9. *Let V,W be p-operator spaces on L_p space. Then there exists a (necessarily unique) separately weak*-continuous extension*

$$\Phi : V'' \otimes W'' \to (V \overset{\vee_p}{\otimes} W)''$$

of the natural inclusion $\iota : V \otimes W \to (V \overset{\vee_p}{\otimes} W)''$.

Proof. Combine Theorem 3.7 and Theorem 3.8. Uniqueness follows from separate weak*-continuity. $\square$

Now we are ready to define condition C_p for p-operator spaces. Let Φ be as in Corollary 3.9. The following commutative diagram

$$\begin{tikzcd}
V'' \otimes W'' \arrow[r, "\Phi"] \arrow[d, equal] & (V \overset{\vee_p}{\otimes} W)'' \arrow[r, "\Psi'"] & (V' \overset{\wedge_p}{\otimes} W')' \arrow[d, equal] \\
\mathscr{CB}^{\sigma}_{p,F}(V', W'') \arrow[rr, hook, "\iota"] & & \mathscr{CB}_p(V', W'')
\end{tikzcd}$$

shows that Φ is injective. Thus we can equip $V'' \otimes W''$ with the p-operator space structure induced by Φ, which will be denoted by $V'' : \overset{\vee_p}{\otimes} : W''$. We say that $V \subseteq \mathscr{B}(L_p(\mu))$ satisfies *condition* C_p (or has *property* C_p) if the map Φ is isometric with respect to the injective tensor product norm for every $W \subseteq \mathscr{B}(L_p(\nu))$.

PROPOSITION 3.10. *Suppose that $V \subseteq \mathscr{B}(L_p(\mu))$. Then V satisfies condition C_p if and only if V satisfies both conditions C'_p and C''_p.*

Proof. Suppose that V satisfies condition C_p and $W \subseteq \mathscr{B}(L_p(\nu))$. By Proposition 2.8 and [3, Theorem 4.3], we have a p-completely isometric embedding $V \overset{\vee_p}{\otimes} W'' \subseteq V'' \overset{\vee_p}{\otimes} W''$ and the bottom row in the following commutative diagram

$$\begin{tikzcd}
V \overset{\vee_p}{\otimes} : W'' \arrow[r] \arrow[d, hook] & V \overset{\vee_p}{\otimes} W'' \arrow[d, hook] \\
V'' : \overset{\vee_p}{\otimes} : W'' \arrow[r] & V'' \overset{\vee_p}{\otimes} W''
\end{tikzcd}$$

is isometric. Therefore the top row is also isometric and hence V satisfies condition C'_p. That V satisfies condition C''_p can be proved using a similar argument.

On the other hand, if V satisfies condition C''_p, we get

$$V'' \overset{\vee_p}{\otimes} W'' = V'' : \overset{\vee_p}{\otimes} : W'' \hookrightarrow (V \overset{\vee_p}{\otimes} W'')''.$$

If V also satisfies condition C_p', then

$$V \overset{\vee_p}{\otimes} W'' = V \overset{\vee_p}{\otimes} : W'' \hookrightarrow (V \overset{\vee_p}{\otimes} W)'',$$

and hence we have isometric inclusion

$$V'' \overset{\vee_p}{\otimes} W'' \hookrightarrow (V \overset{\vee_p}{\otimes} W)''''.$$

Since $V'' \overset{\vee_p}{\otimes} W'' \subset (V \overset{\vee_p}{\otimes} W)''$ and $(V \overset{\vee_p}{\otimes} W)'' \hookrightarrow (V \overset{\vee_p}{\otimes} W)''''$ isometrically, the inclusion $V'' \overset{\vee_p}{\otimes} W'' \subseteq (V \overset{\vee_p}{\otimes} W)''$ must be isometric. $\square$

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