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SINGULAR VALUE INEQUALITIES FOR MATRICES WITH NUMERICAL RANGES IN A SECTOR

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Abstract. Let $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{22} is $q \times q$, be an $n \times n$ complex matrix such that the numerical range of A is contained in $S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leqslant (\Re z)\tan\alpha\}$ for some $\alpha \in [0, \pi/2)$. We obtain the following singular value inequality:

$$\sigma_j(A/A_{11}) \leqslant \sec^2(\alpha)\sigma_j(A_{22}), \qquad j = 1, \ldots, q,$$

where $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$ and $\sigma_j(\cdot)$ means the j-th largest singular value. This strengthens some recent results on determinantal inequalities. We also prove

$$\sigma_j(A) \leqslant \sec^2(\alpha)\lambda_j(\Re A), \qquad j = 1, \ldots, n,$$

where $\lambda_j(\cdot)$ denotes the j-th largest eigenvalue, complementing a result of Fan and Hoffman.

1. Introduction

We start by fixing some notation. The set of $n \times n$ complex matrices is denoted by $\mathbb{M}_n$. The identity matrix of $\mathbb{M}_n$ is I_n. Let $M \in \mathbb{M}_n$. If the eigenvalues of M are all real, then we denote its j-th largest eigenvalue by $\lambda_j(M)$. The j-th largest singular value of M is denoted by $\sigma_j(M)$. Note that $\sigma_j(M) = \sqrt{\lambda_j(M^*M)}$, where M^* means the conjugate transpose of M. For two Hermitian matrices $M, N \in \mathbb{M}_n$, we write $N \leqslant M$ to mean $M - N$ is positive semidefinite, so $M \geqslant 0$ means M is positive semidefinite. We also denote $\Re M = \frac{1}{2}(M + M^*)$ and $\Im M = \frac{1}{2i}(M - M^*)$.

The numerical range of an $n \times n$ matrix M is defined by

$$W(M) = \{x^*Mx : x \in \mathbb{C}^n, x^*x = 1\}.$$

Also, we define a sector on the complex plane

$$S_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leqslant (\Re z)\tan\alpha\}, \qquad \alpha \in [0, \pi/2).$$

Consider $A \in \mathbb{M}_n$ partitioned as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \text{where } A_{22} \in \mathbb{M}_q, q \leqslant \lfloor n/2 \rfloor. \tag{1.1}$$

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Here the floor function notation $\lfloor n/2 \rfloor$ means the integer part of $n/2$. When $W(A)$ is contained in the first quadrant of the complex plane (in this case, A is called accretive-dissipative), it was conjectured in [8] that

$$|\det A| \leqslant 2^q |\det A_{11}| \cdot |\det A_{22}|. \tag{1.2}$$

More generally, suppose $W(A) \subset S_\alpha$, it was conjectured in [3] that

$$|\det A| \leqslant (\sec \alpha)^{2q} |\det A_{11}| \cdot |\det A_{22}|. \tag{1.3}$$

The coefficient $(\sec \alpha)^{2q}$ in (1.3) is known to be optimal in the sense that for some A with $W(A) \subset S_\alpha$, $\dfrac{|\det A|}{|\det A_{11}| \cdot |\det A_{22}|} = (\sec \alpha)^{2q}$. Very recently, the conjectured inequalities (1.2) and (1.3) were proved by Li and Sze in [6] via obtaining the optimal containment region for a certain generalized eigenvalue problem.

Suppose $W(A) \subset S_\alpha$, it is clear that $W(A_{11}) \subset S_\alpha$, thus A_{11} is invertible. The Schur complement of A_{11} in A is defined by $A/A_{11} := A_{22} - A_{21}A_{11}^{-1}A_{12}$.

Note that (1.3) is equivalent to

$$|\det A/A_{11}| \leqslant (\sec \alpha)^{2q} |\det A_{22}|.$$

i.e.,

$$\prod_{j=1}^{q} \sigma_j(A/A_{11}) \leqslant (\sec \alpha)^{2q} \prod_{j=1}^{q} \sigma_j(A_{22}). \tag{1.4}$$

In this paper, we show a strengthening of (1.4). One of our main results is the following.

THEOREM 1.1. *Let $A \in \mathbb{M}_n$ be partitioned as in (1.1) and $W(A) \subset S_\alpha$. Then*

$$\sigma_j(A/A_{11}) \leqslant \sec^2(\alpha)\sigma_j(A_{22}), \qquad j = 1, \ldots, q. \tag{1.5}$$

The proof of this theorem is given in Section 2.

2. Proof of Theorem 1.1

We start with some lemmas. The first one is due to Fan and Hoffman and can be found in [1, p. 73].

LEMMA 2.1. *For every $A \in \mathbb{M}_n$,*

$$\lambda_j(\Re A) \leqslant \sigma_j(A), \qquad j = 1, \ldots, n. \tag{2.1}$$

The second lemma makes use of an explicit expression for the inverse of a 2×2 partitioned matrix; see [5, p. 18].

LEMMA 2.2. *Let $A \in \mathbb{M}_n$ be invertible and partitioned as in (1.1). Also, we partition A^{-1} conformally as A and assume A_{11} is invertible. Then the $(2,2)$ block of A^{-1}, denoted by $(A^{-1})_{22}$, is equal to $(A/A_{11})^{-1}$.*

The third lemma is a technical one.

LEMMA 2.3. *Let* $X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} \in \mathbb{M}_n$, *where* X_2, Y_2 *are* $q \times n$, *such that* $YX^* = I_n$. *Then*

$$\lambda_j(X_2X_2^*)\lambda_{q+1-j}(Y_2Y_2^*) \geqslant 1, \qquad j = 1, \ldots, q. \tag{2.2}$$

Proof. As $YX^* = I_n$, we have $Y_2X_2^* = I_q$.

Note that $\lambda_{q+1-j}(Y_2Y_2^*) = \dfrac{1}{\lambda_j\left((Y_2Y_2^*)^{-1}\right)}$, so (2.2) is the same as

$$\lambda_j(X_2X_2^*) \geqslant \lambda_j\left((Y_2Y_2^*)^{-1}\right), \qquad j = 1, \ldots, q. \tag{2.3}$$

We shall prove something stronger

$$X_2X_2^* \geqslant (Y_2Y_2^*)^{-1}. \tag{2.4}$$

As

$$0 \leqslant \begin{bmatrix} Y_2 \\ X_2 \end{bmatrix} \begin{bmatrix} Y_2 \\ X_2 \end{bmatrix}^* = \begin{bmatrix} Y_2Y_2^* & Y_2X_2^* \\ X_2Y_2^* & X_2X_2^* \end{bmatrix} = \begin{bmatrix} Y_2Y_2^* & I_q \\ I_q & X_2X_2^* \end{bmatrix},$$

using a well known characterization of positivity in terms of the Schur complement (see [5, p. 472]), (2.4) follows. □

Now we are in a position to prove our main result.

Proof of Theorem 1.1. By [3, Lemma 1.1], we can write A in the following form

$$A = XZX^* \tag{2.5}$$

for some invertible $X \in \mathbb{M}_n$ and $Z = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with $|\theta_j| \leqslant \alpha$ for all j. Let $Y = (X^*)^{-1}$ and partition X, Y as in Lemma 2.3. Then from (2.5), we have $A_{22} = X_2ZX_2^*$.

By Lemma 2.1,

$$\sigma_j(X_2ZX_2^*) \geqslant \lambda_j(X_2(\Re Z)X_2^*) \geqslant \lambda_j(X_2X_2^*)\cos(\alpha), \qquad j = 1, \ldots, q.$$

Similarly, we have

$$\sigma_j(Y_2Z^*Y_2^*) \geqslant \lambda_j(Y_2Y_2^*)\cos(\alpha), \qquad j = 1, \ldots, q.$$

Combining these two inequalities and by Lemma 2.3, we get

$$\sigma_j(X_2ZX_2^*)\sigma_{q+1-j}(Y_2Z^*Y_2^*) \geqslant \cos(\alpha)^2,$$

i.e.,

$$\sigma_j(X_2ZX_2^*)\sec(\alpha)^2 \geqslant \frac{1}{\sigma_{q+1-j}(Y_2Z^*Y_2^*)} = \sigma_j\left((Y_2Z^*Y_2^*)^{-1}\right).$$

But by Lemma 2.2, $(Y_2Z^*Y_2^*)^{-1} = A/A_{11}$, and the desired result follows.

3. A reversed inequality

In this section, we prove the following reversed inequality to (2.1).

THEOREM 3.1. *Let $A \in \mathbb{M}_n$ be such that $W(A) \subset S_\alpha$. Then*

$$\sigma_j(A) \leqslant \sec^2(\alpha)\lambda_j(\Re A), \qquad j = 1, \dots, n. \tag{3.1}$$

Proof. Since $W(A^{-1}) \subset S_\alpha$, it follows that $\Re(A^{-1})$ is positive definite. Hence, we may apply Lemma 2.1 to A^{-1}, yielding

$$\begin{aligned} \frac{1}{\lambda_{n+1-j}((\Re(A^{-1}))^{-1})} &= \lambda_j(\Re(A^{-1})) \\ &\leqslant \sigma_j(A^{-1}) = \frac{1}{\sigma_{n+1-j}(A)}, \qquad \text{for } j = 1, \dots, n, \end{aligned}$$

or equivalently

$$\sigma_j(A) \leqslant \lambda_j((\Re(A^{-1}))^{-1}), \qquad \text{for } j = 1, \dots, n,$$

after replacing j with $n+1-j$. It remains to show that

$$\lambda_j((\Re(A^{-1}))^{-1}) \leqslant \lambda_j(\Re A)\sec(\alpha)^2.$$

To decode this, let $A = P + iQ$ with P positive definite and Q Hermitian. Then by [8, Lemma 4],

$$\Re(A^{-1}) = (P + QP^{-1}Q)^{-1}$$

and therefore $\left(\Re(A^{-1})\right)^{-1} = P + QP^{-1}Q$. Again applying (2.5), we can write $P = X\operatorname{diag}(\cos(\theta_1), \dots, \cos(\theta_n))X^*$ and $Q = X\operatorname{diag}(\sin(\theta_1), \dots, \sin(\theta_n))X^*$ for some invertible X. But then

$$\begin{aligned} \left(\Re(A^{-1})\right)^{-1} &= X\operatorname{diag}(\sec(\theta_1), \dots, \sec(\theta_n))X^* \\ &\leqslant \sec(\alpha)^2 X\operatorname{diag}(\cos(\theta_1), \dots, \cos(\theta_n))X^* \\ &= \sec(\alpha)^2 \Re A. \end{aligned}$$

and the result follows. □

We have the following corollary.

COROLLARY 3.2. *Let $X, Y \in \mathbb{M}_n$ be positive semidefinite. Then*

$$\sigma_j(X + iY) \leqslant \sqrt{2}\lambda_j(X + Y), \qquad j = 1, \dots, n.$$

Proof. By a continuity argument, we can assume without loss of generality that X and Y are positive definite. Then $(1-i)(X+iY) = (X+Y)+i(Y-X)$ has its numerical range in $S_{\frac{\pi}{4}}$. It follows that

$$\sqrt{2}\sigma_j(X+iY) = \sigma_j((X+Y)+i(Y-X)) \leqslant 2\lambda_j(X+Y),$$

as required. □

We borrow the following simple example from [2] to show that in general $\sigma_j(X+iY) \leqslant \lambda_j(X+Y)$ fails for $X,Y \geqslant 0$.

EXAMPLE 3.3. Take

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

A calculation shows $\sigma_2(X+iY) \approx 0.4569 > \lambda_2(X+Y) \approx 0.3820$.

The following follows immediately from Corollary 3.2.

COROLLARY 3.4. *Let $X,Y,Z \in \mathbb{M}_n$ such that $0 \leqslant X \leqslant Z$, $0 \leqslant Y \leqslant Z$. Then*

$$\sigma_j(X+iY) \leqslant 2\sqrt{2}\lambda_j(Z), \qquad j = 1,\ldots,n. \tag{3.2}$$

We present an example showing that the coefficient $2\sqrt{2}$ on the right hand side of (3.2) is optimal.

EXAMPLE 3.5. Take

$$X = \begin{bmatrix} 1-\varepsilon & \sqrt{(1-\varepsilon)t} \\ \sqrt{(1-\varepsilon)t} & t \end{bmatrix}, \quad Y = \begin{bmatrix} 1-\varepsilon & -\sqrt{(1-\varepsilon)t} \\ -\sqrt{(1-\varepsilon)t} & t \end{bmatrix}$$

and

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & t/\varepsilon \end{bmatrix},$$

where $0 < \varepsilon < 1, t > 0$. It is easy to verify that $0 \leqslant X \leqslant Z$, $0 \leqslant Y \leqslant Z$. We find

$$\sigma_2(X+iY) = \sqrt{(t+1)^2 - \sqrt{1+6t+t^2}(t-1)}, \qquad \text{as } \varepsilon \to 0.$$

Now by letting $t \to \infty$, we have $\sigma_2(X+iY) = 2\sqrt{2} = 2\sqrt{2}\lambda_2(Z)$.

4. Comments

1. Recall that a norm $\|\cdot\|$ on the algebra of $\mathbb{M}_n$ is unitarily invariant if $\|X\| = \|UXV\|$ for all unitaries U and V and all $X \in \mathbb{M}_n$. The inequality (1.5) is strong enough to imply the following norm inequality.

THEOREM 4.1. *Let $A \in \mathbb{M}_n$ be as in (1.1) and $W(A) \subset S_\alpha$. Then*

$$\|A/A_{11}\| \leqslant \sec^2(\alpha)\|A_{22}\|,$$

for any unitarily invariant norm $\|\cdot\|$. In particular, when A is accretive-dissipative, we have

$$\|A/A_{11}\| \leqslant 2\|A_{22}\|.$$

Various norm inequalities for accretive-dissipative matrices (operators) have been recently considered in [9].

2. Reversed inequalities to that of (1.2) have been given in [7] and further improved in [4]. Similarly, one may consider reversed analogue of (1.5). Without doubt, it requires further information like the condition number of A. We decide to leave it as a research problem for interested readers.

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