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ON THE INVERSE–CLOSEDNESS OF MATRIX SUBALGEBRAS

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Abstract. In this short article we construct a counterexample, which gives a negative answer to an old question, formulated in several publications, on inverse-closedness of matrix subalgebras. We also present and prove several related results.

1. Introduction

Given an unital algebra $\mathscr{A}$ we denote by $M_n(\mathscr{A})$ the algebra of all $n \times n$ matrices with entries from $\mathscr{A}$; by $G(\mathscr{A})$ all elements $a \in \mathscr{A}$ invertible in $\mathscr{A}$; by $F(\mathscr{A})$ all elements $a \in \mathscr{A}$ at least one-side invertible in $\mathscr{A}$. Recall the following

DEFINITION 1.1. A subalgebra $\mathscr{A} (\subset \mathscr{B})$ is *inverse-closed* in $\mathscr{B}$ if

$$x \in \mathscr{A} \cap G(\mathscr{B}) \Longrightarrow x^{-1} \in \mathscr{A}. \tag{1.1}$$

In this paper $\mathscr{A}$ denote an unital Banach subalgebra of an unital Banach algebra $\mathscr{B}$ over the field $\mathbb{C}$ and $\operatorname{spec}(x, \mathscr{A})$ denote the spectrum of an element x in algebra $\mathscr{A}$. We assume that algebras $\mathscr{B}$ and $\mathscr{A}$ have same norm and same unit element e,

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2. The open question and some illustrative examples

The question we are concerned about is the following one.

QUESTION 2.1. *Let $\mathscr{A}$ be an inverse-closed subalgebra of $\mathscr{B}$, and let $n \in \mathbb{N}$. Is $M_n(\mathscr{A})$ inverse-closed in $M_n(\mathscr{B})$?*

This question was formulated in several publications. See, for example, [5 (1988), Remark 1], [1 (2003)], [8 (2011), before Lemma 1.2.34].

The next several statements, obtained in [5], illustrate some examples, for which the answer to Question 2.1 is positive.

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THEOREM 2.2. *Let $\mathscr{A}$ be a commutative algebra inverse-closed in $\mathscr{B}$. Then $M_n(\mathscr{A})$ is inverse-closed in $M_n(\mathscr{B})$ for all $n \in N$.*

THEOREM 2.3. *Let $F(\mathscr{A})$ be dense in $\mathscr{A}$. Then $M_n(\mathscr{A})$ is inverse-closed in $M_n(\mathscr{B})$ for any Banach algebra $\mathscr{B}$ containing $\mathscr{A}$ and for any $n \in \mathbb{N}$.*

Some comments to Theorems 2.2–2.3 are given in Section 4.

COROLLARY 2.4. *Let $G(\mathscr{A})$ be dense in $\mathscr{A}$. Then $M_n(\mathscr{A})$ is inverse-closed in $M_n(\mathscr{B})$ for all $n \in \mathbb{N}$.*

COROLLARY 2.5. *Let $\forall x \in \mathscr{A}$ the spectrum* $\operatorname{spec}(x, \mathscr{A})$ *does not contain interior points. Then $M_n(\mathscr{A})$ is inverse-closed in $M_n(\mathscr{B})$ for all $n \in \mathbb{N}$.*

Indeed, in this case the set $G(\mathscr{A})$ is dense in $\mathscr{A}$.

3. Construction of a counterexample. Answer to Question 2.1.

We start with some preparatory statements.

LEMMA 3.1. *Let $\mathscr{B}$ be an arbitrary unital Banach algebra and $\mathscr{T}_0$ the Banach algebra of all upper (or lower) triangular $n \times n$ matrices $(a_{ik})_{i,k=1}^n$, such that $a_{ik} \in \mathscr{B}$ and $a_{m,m} = c_m e$, where $c_m \in \mathbb{C}$. Then $\mathscr{T}_0$ is inverse-closed in $M_n(\mathscr{B})$.*

Proof. Let $T \in \mathscr{T}_0 \cap G(M_n(\mathscr{B}))$. It can be easily checked (by induction), that $c_i \neq 0$, $i = 1, \dots, n$. Denote by D the diagonal matrix $D := \operatorname{diag}(c_1, c_2, \dots, c_n) \in G(\mathscr{T}_0)$, by S the operator $S := D^{-1}T \in \mathscr{T}_0$ and by E the unit matrix in $M_n(\mathbb{C})$. Then $(S - E)^n = 0$, and hence $S^{-1} = P(S)$, where $P(S) \in \mathscr{T}_0$ is a polynomial. Thus $T^{-1} = S^{-1}D^{-1} \in \mathscr{T}_0$. $\square$

Recall the following two properties of invertible lower triangular matrices (See, for example, [3, Theorem 1.5, page 5]).

PROPOSITION 3.2. *Let $(a_{ik})_{i,k=1}^n \in \mathscr{T} \cap G(M_n(\mathscr{B}))$. Then*
(i) The element a_{11} is right invertible and a_{nn} is left invertible in $\mathscr{B}$;
(ii) If $a_{11}, \dots, a_{pp}$ and $a_{qq}, \dots, a_{nn}$ $(1 < p < q < n)$ are invertible, then $a_{p+1,p+1}$ is right invertible and $a_{q-1,q-1}$ is left invertible in $\mathscr{B}$.

THEOREM 3.3. *Let $\mathscr{B}$ be an arbitrary unital Banach algebra and $\mathscr{T}$ the Banach algebra of all upper (or lower) triangular $n \times n$ matrices with entries from $\mathscr{B}$. The algebra $\mathscr{T}$ is inverse-closed in $M_n(\mathscr{B})$ if and only if $F(\mathscr{B}) = G(\mathscr{B})$.*

Proof. For definiteness we assume that $\mathscr{T}$ is the algebra of all *lower* triangular matrices.

1. Suppose that $F(\mathscr{B}) = G(\mathscr{B})$ and let $T := (a_{ik})_{i,k=1}^{n} \in \mathscr{T} \cap G(M_n(\mathscr{B}))$. It follows from the statement (i) in Proposition 3.2, that a_{11} and a_{nn} are one side invertible. Since $F(\mathscr{B}) = G(\mathscr{B})$, it follows that a_{11} and a_{nn} are invertible. Using (step by step) the property (ii), we obtain, that $a_{kk} \in G(\mathscr{B})$ for all k, and hence the diagonal matrix $D := \mathrm{diag}(a_{11},...,a_{nn}) \in G(\mathscr{T})$. Let $R := D^{-1}T$, then (by Lemma 3.1) $R \in G(\mathscr{T}_0) \subset G(\mathscr{T})$, and hence $T = DR \in G(\mathscr{T})$. This proves that $\mathscr{T}$ is inverse-closed in $M_n(\mathscr{B})$.

2. Suppose that $F(\mathscr{B}) \neq G(\mathscr{B})$. Let $a,b \in \mathscr{B}$, $ab = e$ and $ba \neq e$. Then

$$E = \begin{bmatrix} a & 0 & 0 \\ 0 & E_0 & 0 \\ e-ba & 0 & b \end{bmatrix} \begin{bmatrix} b & 0 & e-ba \\ 0 & E_0 & 0 \\ 0 & 0 & a \end{bmatrix} := TT^{-1}, \tag{3.1}$$

where E and E_0 are unit matrices respectively in $M_n(\mathscr{B})$ and $M_{n-2}(\mathscr{B})$.
Here $T \in \mathscr{T} \cap G(M_n(\mathscr{B}))$, but $T^{-1} \notin \mathscr{T}$, because $\mathscr{T}$ is the algebra of lower triangular matrices. This proves that algebra $\mathscr{T}$ is not inverse-closed. $\square$

Theorem 3.3 has a following useful application:

THEOREM 3.4. *Suppose that a unital Banach algebra* Ω *possesses the following properties:*

$$F(\Omega) = G(\Omega), \quad \textit{but} \quad F(M_2(\Omega)) \neq G(M_2(\Omega)). \tag{3.2}$$

Let $\mathscr{B}$ *denote the Banach algebra* $M_2(\Omega)$, *and* $\mathscr{A}$ $(\subset \mathscr{B})$ *the subalgebra of all lower triangular* 2×2 *matrices with entries from* Ω. *Then*

(i) *Algebra* $\mathscr{A}$ *is inverse-closed in* $\mathscr{B}$, **but**

(ii) *Algebra* $M_2(\mathscr{A})$ *is* **not** *inverse-closed in* $M_2(\mathscr{B})$.

Proof. Statement (i) follows directly from Theorem 3.3. The statement (ii) follows from Theorem 3.3, too! Indeed, it is not difficult to check, that algebra $M_2(\mathscr{A})$ is isomorphic to the algebra of 2×2 lower triangular block matrices with entries from $M_2(\Omega)$. Since $F(M_2(\Omega)) \neq G(M_2(\Omega))$, it follows from Theorem 3.3, that $M_2(\mathscr{A})$ is not inverse-closed in $M_2(\mathscr{B})$. $\square$

Let us show that there exist Banach algebras which satisfy both relations in (3.2). Consider the following example.

EXAMPLE 3.5. Let Ω denote the C^*-algebra generated by all two-dimensional singular integral operators with continuous symbols and all compact operators[1], acting on Hilbert space $H := L_2(\mathbb{R}^2)$, where $\mathbb{R}^2$ is the two dimensional Euclidean space. See, for example, [7, Ch. XI, Section 2] .

PROPOSITION 3.6. *Algebra* Ω *from Example 3.5 satisfies both relations in* (3.2)

[1]Note that *compact* operators in some publications (for example, in the book [7]) are caled *completely continuous* operators.

Proof.

1. It is well known (see, for example, [7, Ch.XI, Section 10]), that the operator $U_1U_2 - U_2U_1$ is a compact operator for each $U_1, U_2 \in \Omega$. Assume that U is left invertible in Ω, then there exists an operator $S \in \Omega$ such that $SU = I$. Then $US = I + T$, where T is a compact operator. Thus, (see [2, Ch. 4, Theorem 7.1]) U is a Fredholm operator. It is also well known (see, for example, [7, Ch. XII, Theorem 3.1]), that $\operatorname{ind} U = 0$ for each Fredholm operator $U \in \Omega$. The left invertible Fredholm operator U with $\operatorname{ind} U = 0$ is invertible in H and we showed above that $U^{-1} = S\ (\in \Omega)$. We proved that

$$F(\mathscr{D}) = G(\mathscr{D}) \tag{3.3}$$

2. Let $\mathscr{R}$ denote the C^*-algebra $M_2(\Omega)$. Algebra $\mathscr{R}$ contains Fredholm operators with non-zero index. This was first obtained in the (classical) papers [9], [10] (see also [6] and [7, page 378]). It follows from here (see, for example, [2, Theorem IV.6.2]) that algebra $\mathscr{R}$ contains a Fredholm operator W left (and only left!) invertible in $M_2(L(H))$. In particular, $\dim ker W = 0$. Let us show, that the operator W is left invertible in $\mathscr{R}$. Operator $C := W^*W$ is Fredholm operator with $\operatorname{ind} C = 0$, and $\dim ker C = 0$. The last equality follows from the relation $Cx = 0 \Longrightarrow 0 = (W^*Wx, x) = \|Wx\|^2 \Longrightarrow x = 0$. Thus C is invertible in $M_2(L(H))$. It is well known, that any C^*- Banach subalgebra of the Banach algebra $\mathscr{M}_2(L(H))$ is inverse-closed in $\mathscr{M}_2(L(H))$ (see, for example, [8, Theorem 1.2.38]). In particular, $\mathscr{R}$ is inverse closed in $M_2(L(H))$. Therefore $C \in G(\mathscr{R})$ and $W \in F(\mathscr{R}) = F(M_2(\Omega))$. But $W \notin G(\mathscr{R}) = G(M_2(\Omega))$, and we proved the relation

$$F(M_2(\Omega)) \neq G(M_2(\Omega)). \quad \square \tag{3.4}$$

Now we a ready to prove the following (main)

THEOREM 3.7. *For any number $n > 1$ $(n \in \mathbb{N})$ there exists a unital Banach algebra $\mathscr{B}$ and its Banach subalgebra $\mathscr{A}$ such that $\mathscr{A}$ is inverse-closed in $\mathscr{B}$, but $M_n(\mathscr{A})$ is not inverse-closed in $M_n(\mathscr{B})$.*

Proof. For $n = 2$ this theorem follows from Theorem 3.4 and Proposition 3.6. To pass from the case $n = 2$ to $n > 2$ it is enough to use the following evident statement. Let $M_n(\mathscr{A})$ be inverse-closed in $M_n(\mathscr{B})$, then $M_k(\mathscr{A})$ is inverse-closed in $M_k(\mathscr{B})$ for all $1 \leqslant k \leqslant n$. $\square$

We conclude this section with the following

REMARK 3.8. It would be interesting to construct a more elementary example of a Banach algebra Ω, which satisfies both conditions in (3.2). The counterexample will benefit from this.

4. Some comments

PROPOSITION 4.1. *Theorem 2.2 holds for unital algebras without any topology.*

This statement can be easily deduced from the following theorem.

THEOREM 4.2. *Let $\mathscr{K}$ be an associative and, generally speaking, non-commutative ring with identity e. Assume that $a_{mk} \in \mathscr{K}$ $(m,k \leqslant n)$ for some $n \in \mathbb{N}$, and $a_{mk}a_{pq} = a_{pq}a_{mk}$ $\forall m,k,p,q = 1,\ldots,n$. Then the matrix $T := [a_{mk}]_{m,k=1}^{n}$ is invertible in $M_n(\mathscr{K})$ if and only if the element $\Delta := \det T$ is invertible in $\mathscr{K}$.*

This result was obtained in [4]. It proved to be useful for many classes of equations and has been used in many publications. For the proof of the Theorem 4.2 in this original formulation see [3, Theorem 1.1]. The deduction of Proposition 4.1 from Theorem 4.2 uses same arguments as the proof of Theorem 1 in [5, page 94] (see also [8, Proposition 1.2.35]).

In this paper we discus the inverse-closedness of *Banach subalgebras* $\mathscr{A}$ of Banach algebras $\mathscr{B}$. But sometimes it is useful to consider first the inverse-closedness of a dense subalgebra, or even of a dense subset X, with the standard definition of the inverse-closedness. This gives the following benefit:

PROPOSITION 4.3. *Let X be a dense subset of the Banach algebra $\mathscr{A}$. If X is inverse-closed in $\mathscr{B}$, then $\mathscr{A}$ is inverse-closed in $\mathscr{B}$.*

Proof. Let $a \in \mathscr{A} \cap G(\mathscr{B})$. There exists a sequence $\{x_k\} \in X \cap G(\mathscr{B})$ such that $\lim x_k = a$. Since X is inverse-closed in $\mathscr{B}$ it follows that $x_k^{-1} \in X$, and hence, $a^{-1} = \lim x_k^{-1} \in \operatorname{clos} X = \mathscr{A}$. $\square$

REMARK 4.4. Theorem 2.3 is proved in [5, Theorem 2, Section 1]. Here we propose a shorter proof of this theorem.

Proof of Theorem 2.3. It is given that the set $X = F(\mathscr{A})$ is dense in $\mathscr{A}$ and we have to prove that $M_n(\mathscr{A})$ is inverse-closed in $M_n(\mathscr{B})$ for any $n \in \mathbb{N}$ and any Banach algebra $\mathscr{B}$, such that $\mathscr{A} \subset \mathscr{B}$. By Proposition 4.3 it is enough to show that the set $M_n(X)$ is inverse-closed in $M_n(\mathscr{B})$.

So we prove Theorem 2.3 by induction. Let $n = 1$ and $x \in X \cap G(\mathscr{B})$. Assume (for definiteness) that x is left invertible. Then there exists $w \in \mathscr{B}$ and $z \in X$ such tat $wx = xw = e$ and $zx = e$. It follows from here that $w = z \in X$. This proves the theorem for $n = 1$. Let $A := [a_{ik}]_{i,k=1}^{n} \in M_n(X) \cap G(M_n(\mathscr{B}))$ and let (for definiteness) the entry $a := a_{11}$ is left invertible, i.e. there exists $y \in X$, such that $ya = e$. We represent the matrix A as a 2×2 block matrix (with different sizes of the blocks), and use the following its factorization:

$$A = \begin{bmatrix} a & b \\ c & D \end{bmatrix} = \begin{bmatrix} e & 0 \\ cy & E \end{bmatrix} \begin{bmatrix} e & 0 \\ 0 & D - cyb \end{bmatrix} \begin{bmatrix} a & b \\ 0 & E \end{bmatrix}, \tag{4.1}$$

where E is the unit matrix in $M_{n-1}(\mathscr{B})$. It follows from representation (4.1) that the matrix $D - cyb \in M_{n-1}(X) \cap G(M_{n-1}(\mathscr{B}))$, and by induction $A \in G(M_n(X))$. $\square$

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