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SOME INEQUALITIES FOR UNITARILY INVARIANT NORMS

JUNLIANG WU AND JIANGUO ZHAO

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Abstract. In this note, we use the convexity of the function $\varphi(v)$ to sharpen the matrix version of the Heinz means, where $\varphi(v)$ is defined as $\varphi(v) = \|A^v X B^{1-v} + A^{1-v} X B^v\|$ on $[0,1]$ for $A,B,X \in M_n$ such that A and B are positive semidefinite, and also give a refinement of the inequality [Theorem 6, SIAM J. Matrix Anal. Appl. 20 (1998), 466–470] which is due to Zhan.

1. Introduction

Throughout, let M_n, $B(H)$, C_∞ be the set of $n \times n$ complex matrices, the set of all bounded linear operators on a complex separable Hilbert space H and the class of compact operators, respectively. For a compact operator $A \in C_\infty$, let $s_1(A) \geqslant s_2(A) \geqslant \cdots \geqslant 0$ be the singular values of A, i.e., the eigenvalues of the positive operator $|A| = (A^*A)^{\frac{1}{2}}$, arranged in a decreasing order and repeated according to multiplicity. $\|\cdot\|$ denotes a unitarily invariant norm defined on a two-sided ideal $K_{\|\cdot\|}$ that is included in C_∞, which has the basic property $\|UAV\| = \|A\|$ for every $A \in K_{\|\cdot\|}$ and all unitary operators $U,V \in B(H)$. Especially well known among unitarily invariant norms are the Schatten p-norms defined as $\|A\|_p = (\sum_{i=1}^{\infty} s_i^p(A))^{\frac{1}{p}}$, for $p \geqslant 1$. The Ky-Fan norms defined as $\|A\|_{(k)} = \sum_{i=1}^{k} s_i(A)$, $k = 1,2,\cdots\infty$, represent another interesting family of unitarily invariant norms. Properties of such norms may be found in ([1], [7], [10], [14], [15]).

As is well known, the Heinz means of two nonnegative real numbers a and b are defined as

$$H_v(a,b) = \frac{a^v b^{1-v} + a^{1-v} b^v}{2},$$

for $0 \leqslant v \leqslant 1$.

It is easy to see that the following inequalities hold:

$$\sqrt{ab} \leqslant H_v(a,b) \leqslant \frac{a+b}{2}, \tag{1}$$

for $0 \leqslant v \leqslant 1$.

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The matrix version of (1) due to Bhatia and Davis [2] is the following inequalities,

$$2\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leqslant \|A^{v}XB^{1-v}+A^{1-v}XB^{v}\| \leqslant \|AX+XB\|, \tag{2}$$

where $0\leqslant v\leqslant 1$, A, B, $X\in M_n$ such that A and B are positive semidefinite. Usually, $\frac{\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\|}{2}$ are called the Heinz means of A and B.

For A, B, $X\in M_n$ such that A and B are positive semidefinite, putting

$$\varphi(v)=\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\|,$$

then the function $\varphi(v)$ is a continuous convex function on [0,1], attains its minimum at $v=\frac{1}{2}$, and attains its maximum at $v=0$ or $v=1$. Thus it is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$. Moreover, $\varphi(v)=\varphi(1-v)$, for $0\leqslant v\leqslant 1$. One may find the mentioned properties of the function $\varphi(v)$ in ([2], [3], [8], [9]).

In [6], when $\|\cdot\|$ is operator norm, Corach-Porta-Recht proved the following, so-called C-P-R inequality,

$$\|SXS^{-1}+S^{-1}XS\|\geqslant 2\|X\|, \tag{3}$$

for any invertible self-adjoint operator S and $X\in K_{\|\cdot\|}$.

They proved this inequality by using the integral representation of a self-adjoint operator with respect to a spectral measure.

An immediate consequence of the C-P-R inequality is the following,

$$\|SXT^{-1}+S^{-1}XT\|\geqslant 2\|X\|, \tag{4}$$

for invertible self-adjoint operators S, T and $X\in K_{\|\cdot\|}$. (3) and (4) also hold for other unitarily invariant norms.

In [16], by introducing two parameters r and t, Zhan proved that the following inequality

$$(2+t)\|A^{r}XB^{2-r}+A^{2-r}XB^{r}\|\leqslant 2\|A^{2}X+tAXB+XB^{2}\| \tag{5}$$

holds for any unitarily invariant norm $\|\cdot\|$, where A, B, $X\in M_n$ such that A and B are positive semidefinite matrices and $(t,r)\in(-2,2]\times[\frac{1}{2},\frac{3}{2}]$.

In this note, we use the convexity of the function $\varphi(v)=\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\|$ on [0, 1] to sharpen the inequalities (2), and then give a refinement of the inequality (5).

2. Main results

The following Lemma [13], plays an important role in our discussion.

LEMMA 1. *Let $g(x)$ be a real valued function which is convex on the interval $[a,b]$. If $p,q\geqslant 0$, and $0<y\leqslant\frac{b-a}{p+q}\min(p,q)$, then*

$$g(C)\leqslant\frac{1}{2y}\int_{C-y}^{C+y}g(t)dt\leqslant\frac{1}{2}(g(C-y)+g(C+y))\leqslant\frac{pg(a)+qg(b)}{p+q}, \tag{6}$$

where $C=\frac{pa+qb}{p+q}$.

It is worth to mention that the inequalities (6) is the Hermite-Hadamard's inequalities when $p=q=1$, and $y=\frac{b-a}{2}$.

THEOREM 1. *Let A, B and $X \in M_n$ such that A and B are positive semidefinite. For every unitarily invariant norm $\|\cdot\|$, then*
(a)

$$\begin{aligned}
\|A^{\mu}XB^{1-\mu} &+ A^{1-\mu}XB^{\mu}\| \\
&\leqslant \|A^{C_\mu^{(1)}}XB^{1-C_\mu^{(1)}}+A^{1-C_\mu^{(1)}}XB^{C_\mu^{(1)}}\| \\
&\leqslant \frac{1}{2y}\int_{C_\mu^{(1)}-y}^{C_\mu^{(1)}+y}(\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\|)dv \\
&\leqslant \frac{1}{2}(\|A^{C_\mu^{(1)}-y}XB^{1-C_\mu^{(1)}+y}+A^{1-C_\mu^{(1)}+y}XB^{C_\mu^{(1)}-y}\| \\
&\quad +\|A^{C_\mu^{(1)}+y}XB^{1-C_\mu^{(1)}-y}+A^{1-C_\mu^{(1)}-y}XB^{C_\mu^{(1)}+y}\|) \\
&\leqslant \frac{p}{p+q}\|AX+XB\|+\frac{q}{p+q}\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\| \\
&\leqslant \|AX+XB\| \qquad (7)
\end{aligned}$$

holds for $0<\mu\leqslant\frac{1}{2}$, where $p,\ q>0$, $C_\mu^{(1)}=\frac{q\mu}{p+q}$ and $0<y\leqslant\frac{\mu}{p+q}\min(p,q)$;
(b)

$$\begin{aligned}
\|A^{\mu}XB^{1-\mu} &+ A^{1-\mu}XB^{\mu}\| \\
&\leqslant \|A^{C_\mu^{(2)}}XB^{1-C_\mu^{(2)}}+A^{1-C_\mu^{(2)}}XB^{C_\mu^{(2)}}\| \\
&\leqslant \frac{1}{2y}\int_{C_\mu^{(2)}-y}^{C_\mu^{(2)}+y}(\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\|)dv \\
&\leqslant \frac{1}{2}(\|A^{C_\mu^{(2)}-y}XB^{1-C_\mu^{(2)}+y}+A^{1-C_\mu^{(2)}+y}XB^{C_\mu^{(2)}-y}\| \\
&\quad +\|A^{C_\mu^{(2)}+y}XB^{1-C_\mu^{(2)}-y}+A^{1-C_\mu^{(2)}-y}XB^{C_\mu^{(2)}+y}\|) \\
&\leqslant \frac{p}{p+q}\|A^{\mu}XB^{1-\mu}+A^{1-\mu}XB^{\mu}\|+\frac{q}{p+q}\|AX+XB\| \\
&\leqslant \|AX+XB\| \qquad (8)
\end{aligned}$$

holds for $\frac{1}{2}<\mu<1$, where $p,\ q>0$, $C_\mu^{(2)}=\frac{p\mu+q}{p+q}$ and $0<y\leqslant\frac{1-\mu}{p+q}\min(p,q)$.

Proof. We consider the case $\mu\in(0,\frac{1}{2}]$ at first.

Applying Lemma 1 to the function $\varphi(v)=\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\|$ on the interval $[0,\mu]$, we have

$$\begin{aligned}
\varphi(C_\mu^{(1)}) &\leqslant \frac{1}{2y}\int_{C_\mu^{(1)}-y}^{C_\mu^{(1)}+y}\varphi(t)dt \leqslant \frac{1}{2}(\varphi(C_\mu^{(1)}-y)+\varphi(C_\mu^{(1)}+y)) \\
&\leqslant \frac{p\varphi(0)+q\varphi(\mu)}{p+q}, \qquad (9)
\end{aligned}$$

where $C_\mu^{(1)} = \frac{q\mu}{p+q}$ and $0 < y \leqslant \frac{\mu}{p+q}\min(p,q)$.

Thus

$$\begin{aligned}
\|A^{C_\mu^{(1)}}XB^{1-C_\mu^{(1)}} &+ A^{1-C_\mu^{(1)}}XB^{C_\mu^{(1)}}\| \\
&\leqslant \frac{1}{2y}\int_{C_\mu^{(1)}-y}^{C_\mu^{(1)}+y}(\|A^vXB^{1-v}+A^{1-v}XB^v\|)dv \\
&\leqslant \frac{1}{2}(\|A^{C_\mu^{(1)}-y}XB^{1-C_\mu^{(1)}+y}+A^{1-C_\mu^{(1)}+y}XB^{C_\mu^{(1)}-y}\| \\
&\quad +\|A^{C_\mu^{(1)}+y}XB^{1-C_\mu^{(1)}-y}+A^{1-C_\mu^{(1)}-y}XB^{C_\mu^{(1)}+y}\|) \\
&\leqslant \frac{p}{p+q}\|AX+XB\|+\frac{q}{p+q}\|A^\mu XB^{1-\mu}+A^{1-\mu}XB^\mu\|. \qquad (10)
\end{aligned}$$

Noting that the function $\varphi(v) = \|A^vXB^{1-v}+A^{1-v}XB^v\|$ is decreasing on $[0,\frac{1}{2}]$ and increasing on $[\frac{1}{2},1]$, then by (10), we get the desired inequalities (7).

Likewise, if $\mu \in (\frac{1}{2},1)$, applying Lemma 1 to the function $\varphi(v) = \|A^vXB^{1-v}+A^{1-v}XB^v\|$ on the interval $[\mu,1]$, then we obtain the inequalities (8).

The proof is completed. □

REMARK 1. Putting $p = q = 1$, and $y = \frac{\mu}{2}$ when $0 < \mu \leqslant \frac{1}{2}$, $y = \frac{1-\mu}{2}$ when $\frac{1}{2} < \mu < 1$, it is easy to see that Theorem 1 is just the result of Corollary 2 in [11]. Thus Corollary 2 in [11] is a special case of Theorem 1.

REMARK 2. Theorem 1 is a refinement of the sencond inequality in (2).

Similarly, applying Lemma 1 to the function $\varphi(v) = \|A^vXB^{1-v}+A^{1-v}XB^v\|$ on $[\mu,1-\mu]$ when $\mu \in [0,\frac{1}{2})$, and on $[1-\mu,\mu]$ when $\mu \in (\frac{1}{2},1]$, respectively, we have the refinement of the first inequality in (2).

THEOREM 2. *Let A, B and $X \in M_n$ such that A and B are positive semidefinite. For every unitarily invariant norm $\|\cdot\|$, then*

$$\begin{aligned}
2\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| &\leqslant \|A^{C_\mu^{(3)}}XB^{1-C_\mu^{(3)}}+A^{1-C_\mu^{(3)}}XB^{C_\mu^{(3)}}\| \\
&\leqslant \frac{1}{2y}\int_{C_\mu^{(3)}-y}^{C_\mu^{(3)}+y}(\|A^vXB^{1-v}+A^{1-v}XB^v\|)dv \\
&\leqslant \frac{1}{2}(\|A^{C_\mu^{(3)}-y}XB^{1-C_\mu^{(3)}+y}+A^{1-C_\mu^{(3)}+y}XB^{C_\mu^{(3)}-y}\| \\
&\quad +\|A^{C_\mu^{(3)}+y}XB^{1-C_\mu^{(3)}-y}+A^{1-C_\mu^{(3)}-y}XB^{C_\mu^{(3)}+y}\|) \\
&\leqslant \|A^\mu XB^{1-\mu}+A^{1-\mu}XB^\mu\| \qquad (11)
\end{aligned}$$

holds for $\mu \in [0,1]-\{\frac{1}{2}\}$, where p, $q > 0$, $C_\mu^{(3)} = \frac{p\mu+q(1-\mu)}{p+q}$ and $0 < y \leqslant \frac{|1-2\mu|}{p+q}\min(p,q)$.

COROLLARY 1. *Let A and B be positive semidefinite matrices. For every unitarily invariant norm $\|\cdot\|$, then*

$$\begin{aligned}
2\|AB\| &\leqslant \|A^{C_\mu^{(3)}+\frac{1}{2}}B^{\frac{3}{2}-C_\mu^{(3)}}+A^{\frac{3}{2}-C_\mu^{(3)}}B^{\frac{1}{2}+C_\mu^{(3)}}\| \\
&\leqslant \frac{1}{2y}\int_{C_\mu^{(3)}-y}^{C_\mu^{(3)}+y}(\|A^{\frac{1}{2}+v}B^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}\|)dv \\
&\leqslant \frac{1}{2}(\|A^{\frac{1}{2}+C_\mu^{(3)}-y}B^{\frac{3}{2}-C_\mu^{(3)}+y}+A^{\frac{3}{2}-C_\mu^{(3)}+y}B^{\frac{1}{2}+C_\mu^{(3)}-y}\| \\
&\quad +\|A^{\frac{1}{2}+C_\mu^{(3)}+y}B^{\frac{3}{2}-C_\mu^{(3)}-y}+A^{\frac{3}{2}-C_\mu^{(3)}-y}B^{\frac{1}{2}+C_\mu^{(3)}+y}\|) \\
&\leqslant \|A^{\frac{1}{2}+\mu}B^{\frac{3}{2}-\mu}+A^{\frac{3}{2}-\mu}B^{\frac{1}{2}+\mu}\| \\
&\leqslant \frac{1}{2}\|(A+B)^2\|
\end{aligned} \tag{12}$$

holds for $\mu\in[0,1]-\{\frac{1}{2}\}$, *where* $p,\ q>0$, $C_\mu^{(3)}=\frac{p\mu+q(1-\mu)}{p+q}$ *and* $0<y\leqslant\frac{|1-2\mu|}{p+q}\min(p,q)$.

Proof. Taking $X=A^{\frac{1}{2}}B^{\frac{1}{2}}$ in (11), combining (2) with the following inequality

$$\|A^{\frac{1}{2}}(A+B)B^{\frac{1}{2}}\|\leqslant\frac{1}{2}\|(A+B)^2\|,$$

then, we can obtain the inequalities (12).

The proof is completed. $\square$

REMARK 3. Obviously, (12) is a refinement of the following inequality

$$\|AB\|\leqslant\frac{1}{4}\|(A+B)^2\|,$$

which is due to Bhatia and Kittaneh [4], where $A,\ B\in M_n$ are positive semidefinite matrices.

Next, we give the refinement of the inequality (5).

THEOREM 3. *Let A, B and $X\in M_n$ such that A and B are positive matrices and $(t,r)\in(-2,2]\times(\frac{1}{2},\frac{3}{2})$. For every unitarily invariant norm $\|\cdot\|$, the following inequalities hold,*

(a) for $r\in(\frac{1}{2},1]$

$$\begin{aligned}
2\|A^2X &+ XB^2+tAXB\| \\
&\geqslant 2(\|A^2X+XB^2+2AXB\|)-(4-2t)\|AXB\| \\
&\geqslant \frac{4p}{p+q}\|A^{\frac{3}{2}}XB^{\frac{1}{2}}+A^{\frac{1}{2}}XB^{\frac{3}{2}}\|+\frac{4q}{p+q}\|A^rXB^{2-r}+A^{2-r}XB^r\|-(4-2t)\|AXB\| \\
&\geqslant 2(\|A^{C_\mu^{(1)}-y+\frac{1}{2}}XB^{\frac{3}{2}-C_\mu^{(1)}+y}+A^{\frac{3}{2}-C_\mu^{(1)}+y}XB^{C_\mu^{(1)}-y+\frac{1}{2}}\|
\end{aligned}$$

$$
\begin{aligned}
&+\|A^{C_\mu^{(1)}+y+\frac{1}{2}}XB^{\frac{3}{2}-C_\mu^{(1)}-y}+A^{\frac{3}{2}-C_\mu^{(1)}-y}XB^{C_\mu^{(1)}+y+\frac{1}{2}}\|)-(4-2t)\|AXB\|\\
\geqslant\ & \frac{2}{y}\int_{C_\mu^{(1)}-y}^{C_\mu^{(1)}+y}(\|A^{v+\frac{1}{2}}XB^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}XB^{v+\frac{1}{2}}\|)dv-(4-2t)\|AXB\|\\
\geqslant\ & 4\|A^{C_\mu^{(1)}+\frac{1}{2}}XB^{\frac{3}{2}-C_\mu^{(1)}}+A^{\frac{3}{2}-C_\mu^{(1)}}XB^{C_\mu^{(1)}+\frac{1}{2}}\|-(4-2t)\|AXB\|\\
\geqslant\ & 4\|A^rXB^{2-r}+A^{2-r}XB^r\|-(4-2t)\|AXB\|\\
\geqslant\ & (t+2)\|A^rXB^{2-r}+A^{2-r}XB^r\|,
\end{aligned}\tag{13}
$$

where $\mu=r-\frac{1}{2},\ p,\ q>0,\ C_\mu^{(1)}=\frac{q\mu}{p+q}$ *and* $0<y\leqslant\frac{\mu}{p+q}\min(p,q)$*;*

(b) for $r\in[1,\frac{3}{2})$

$$
\begin{aligned}
2\|A^2X\ +\ & XB^2+tAXB\|\\
\geqslant\ & 2(\|A^2X+XB^2+2AXB\|)-(4-2t)\|AXB\|\\
\geqslant\ & \frac{4q}{p+q}\|A^{\frac{3}{2}}XB^{\frac{1}{2}}+A^{\frac{1}{2}}XB^{\frac{3}{2}}\|+\frac{4p}{p+q}\|A^rXB^{2-r}+A^{2-r}XB^r\|-(4-2t)\|AXB\|\\
\geqslant\ & 2(\|A^{C_\mu^{(2)}-y+\frac{1}{2}}XB^{\frac{3}{2}-C_\mu^{(2)}+y}+A^{\frac{3}{2}-C_\mu^{(2)}+y}XB^{C_\mu^{(2)}-y+\frac{1}{2}}\|\\
&+\|A^{C_\mu^{(2)}+y+\frac{1}{2}}XB^{\frac{3}{2}-C_\mu^{(2)}-y}+A^{\frac{3}{2}-C_\mu^{(2)}-y}XB^{C_\mu^{(2)}+y+\frac{1}{2}}\|)-(4-2t)\|AXB\|\\
\geqslant\ & \frac{2}{y}\int_{C_\mu^{(2)}-y}^{C_\mu^{(2)}+y}(\|A^{v+\frac{1}{2}}XB^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}XB^{v+\frac{1}{2}}\|)dv-(4-2t)\|AXB\|\\
\geqslant\ & 4\|A^{C_\mu^{(2)}+\frac{1}{2}}XB^{\frac{3}{2}-C_\mu^{(2)}}+A^{\frac{3}{2}-C_\mu^{(2)}}XB^{C_\mu^{(2)}+\frac{1}{2}}\|-(4-2t)\|AXB\|\\
\geqslant\ & 4\|A^rXB^{2-r}+A^{2-r}XB^r\|-(4-2t)\|AXB\|\\
\geqslant\ & (t+2)\|A^rXB^{2-r}+A^{2-r}XB^r\|,
\end{aligned}\tag{14}
$$

where $\mu=r-\frac{1}{2},\ p,\ q>0,\ C_\mu^{(2)}=\frac{p\mu+q}{p+q}$ *and* $0<y\leqslant\frac{1-\mu}{p+q}\min(p,q)$.

Proof. Putting $\mu=r-\frac{1}{2}$, then $\mu\in(0,1)$. We consider the case $\mu\in(0,\frac{1}{2}]$ at first. Using the refinements of the Heinz means (7), we have

$$
\begin{aligned}
\|A^{\frac{1}{2}}XB^{-\frac{1}{2}}\ +\ & A^{-\frac{1}{2}}XB^{\frac{1}{2}}\|\\
\geqslant\ & \frac{p}{p+q}\|A^{\frac{1}{2}}XB^{-\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{1}{2}}\|+\frac{q}{p+q}\|A^{\mu-\frac{1}{2}}XB^{\frac{1}{2}-\mu}+A^{\frac{1}{2}-\mu}XB^{\mu-\frac{1}{2}}\|\\
\geqslant\ & \frac{1}{2}(\|A^{C_\mu^{(1)}-y-\frac{1}{2}}XB^{\frac{1}{2}-C_\mu^{(1)}+y}+A^{\frac{1}{2}-C_\mu^{(1)}+y}XB^{C_\mu^{(1)}-y-\frac{1}{2}}\|\\
&+\|A^{C_\mu^{(1)}+y-\frac{1}{2}}XB^{\frac{1}{2}-C_\mu^{(1)}-y}+A^{\frac{1}{2}-C_\mu^{(1)}-y}XB^{C_\mu^{(1)}+y-\frac{1}{2}}\|)\\
\geqslant\ & \frac{1}{2y}\int_{C_\mu^{(1)}-y}^{C_\mu^{(1)}+y}(\|A^{v-\frac{1}{2}}XB^{\frac{1}{2}-v}+A^{\frac{1}{2}-v}XB^{v-\frac{1}{2}}\|)dv\\
\geqslant\ & \|A^{C_\mu^{(1)}-\frac{1}{2}}XB^{\frac{1}{2}-C_\mu^{(1)}}+A^{\frac{1}{2}-C_\mu^{(1)}}XB^{C_\mu^{(1)}-\frac{1}{2}}\|\\
\geqslant\ & \|A^{\mu-\frac{1}{2}}XB^{\frac{1}{2}-\mu}+A^{\frac{1}{2}-\mu}XB^{\mu-\frac{1}{2}}\|.
\end{aligned}\tag{15}
$$

Since the following equality holds

$$AXB^{-1}+A^{-1}XB+2X=A^{\frac{1}{2}}(A^{\frac{1}{2}}XB^{-\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{1}{2}})B^{-\frac{1}{2}}+A^{-\frac{1}{2}}(A^{\frac{1}{2}}XB^{-\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{1}{2}})B^{\frac{1}{2}},$$

utilizing the generalized version of C-P-R inequality (4) for unitarily invariant norms, we obtain

$$\|AXB^{-1}+A^{-1}XB+2X\| \geqslant 2\|A^{\frac{1}{2}}XB^{-\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{1}{2}}\|. \tag{16}$$

On the other hand, due to

$$AXB^{-1}+A^{-1}XB+2X=AXB^{-1}+A^{-1}XB+tX+(2-t)X,$$

we have

$$\|AXB^{-1}+A^{-1}XB+2X\| \leqslant \|AXB^{-1}+A^{-1}XB+tX\|+(2-t)\|X\|. \tag{17}$$

Again, from the generalized version of C-P-R inequality (4) for unitarily invariant norms, it is easy to see that if $s\in R$,

$$\|A^sXB^{-s}+A^{-s}XB^s\| \geqslant 2\|X\|.$$

Noting that $t-2\leqslant 0$, thus

$$(t-2)\|A^sXB^{-s}+A^{-s}XB^s\| \leqslant 2(t-2)\|X\|,$$

which is equivalent to

$$4\|A^sXB^{-s}+A^{-s}XB^s\|-4\|X\|+2t\|X\| \geqslant (t+2)\|A^sXB^{-s}+A^{-s}XB^s\|. \tag{18}$$

Combining (15), (16), (17) with (18), we can deduce

$$\begin{aligned}
2\|AXB^{-1} &+ A^{-1}XB+tX\| \\
&\geqslant 2\|AXB^{-1}+A^{-1}XB+2X\|-(4-2t)\|X\| \\
&\geqslant 4\|A^{\frac{1}{2}}XB^{-\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{1}{2}}\|-(4-2t)\|X\| \\
&\geqslant \frac{4p}{p+q}\|A^{\frac{1}{2}}XB^{-\frac{1}{2}}+A^{-\frac{1}{2}}XB^{\frac{1}{2}}\| \\
&\quad +\frac{4q}{p+q}\|A^{\mu-\frac{1}{2}}XB^{\frac{1}{2}-\mu}+A^{\frac{1}{2}-\mu}XB^{\mu-\frac{1}{2}}\|-(4-2t)\|X\| \\
&\geqslant 2(\|A^{C_\mu^{(1)}-y-\frac{1}{2}}XB^{\frac{1}{2}-C_\mu^{(1)}+y}+A^{\frac{1}{2}-C_\mu^{(1)}+y}XB^{C_\mu^{(1)}-y-\frac{1}{2}}\| \\
&\quad +\|A^{C_\mu^{(1)}+y-\frac{1}{2}}XB^{\frac{1}{2}-C_\mu^{(1)}-y}+A^{\frac{1}{2}-C_\mu^{(1)}-y}XB^{C_\mu^{(1)}+y-\frac{1}{2}}\|)-(4-2t)\|X\| \\
&\geqslant \frac{2}{y}\int_{C_\mu^{(1)}-y}^{C_\mu^{(1)}+y}(\|A^{v-\frac{1}{2}}XB^{\frac{1}{2}-v}+A^{\frac{1}{2}-v}XB^{v-\frac{1}{2}}\|)dv-(4-2t)\|X\| \\
&\geqslant 4\|A^{C_\mu^{(1)}-\frac{1}{2}}XB^{\frac{1}{2}-C_\mu^{(1)}}+A^{\frac{1}{2}-C_\mu^{(1)}}XB^{C_\mu^{(1)}-\frac{1}{2}}\|-(4-2t)\|X\|
\end{aligned}$$

$$\geqslant 4\|A^{\mu-\frac{1}{2}}XB^{\frac{1}{2}-\mu}+A^{\frac{1}{2}-\mu}XB^{\mu-\frac{1}{2}}\|-(4-2t)\|X\|$$
$$\geqslant (t+2)\|A^{\mu-\frac{1}{2}}XB^{\frac{1}{2}-\mu}+A^{\frac{1}{2}-\mu}XB^{\mu-\frac{1}{2}}\|. \tag{19}$$

Replacing X by AXB and μ by $r-\frac{1}{2}$ in (19), respectively, we have (13). Finally, (14) is obtained analogously.

The proof is completed. □

REMARK 4. By continuity, the condition positive in Theorem 3 can be replaced by positive semidefinite.

Taking $r=1$ in (13), we can get the following corollary.

COROLLARY 2. *Let A, B and $X\in M_n$ such that A and B are positive semidefinite. For every unitarily invariant norm $\|\cdot\|$, the following inequalities hold,*

$$\begin{aligned}
2\|A^2X &+ XB^2+tAXB\| \\
&\geqslant 2(\|A^2X+XB^2+2AXB\|)-(4-2t)\|AXB\| \\
&\geqslant \frac{4p}{p+q}\|A^{\frac{3}{2}}XB^{\frac{1}{2}}+A^{\frac{1}{2}}XB^{\frac{3}{2}}\|+\frac{8q}{p+q}\|AXB\|-(4-2t)\|AXB\| \\
&\geqslant 2(\|A^{C^{(4)}-y+\frac{1}{2}}XB^{\frac{3}{2}-C^{(4)}+y}+A^{\frac{3}{2}-C^{(4)}+y}XB^{C^{(4)}-y+\frac{1}{2}}\| \\
&\quad +\|A^{C^{(4)}+y+\frac{1}{2}}XB^{\frac{3}{2}-C^{(4)}-y}+A^{\frac{3}{2}-C^{(4)}-y}XB^{C^{(4)}+y+\frac{1}{2}}\|)-(4-2t)\|AXB\| \\
&\geqslant \frac{2}{y}\int_{C^{(4)}-y}^{C^{(4)}+y}(\|A^{v+\frac{1}{2}}XB^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}XB^{v+\frac{1}{2}}\|)dv-(4-2t)\|AXB\| \\
&\geqslant 4\|A^{C^{(4)}+\frac{1}{2}}XB^{\frac{3}{2}-C^{(4)}}+A^{\frac{3}{2}-C^{(4)}}XB^{C^{(4)}+\frac{1}{2}}\|-(4-2t)\|AXB\| \\
&\geqslant 2(t+2)\|AXB\|,
\end{aligned} \tag{20}$$

where $-2<t\leqslant 2$, $p,\ q>0$, $C^{(4)}=\frac{q}{2(p+q)}$ *and* $0<y\leqslant\frac{1}{2(p+q)}\min(p,q)$.

REMARK 5. Taking $t=0$ in (20), it is just a refinement of the famous Arithmetic-Geometric mean inequality

$$2\|AXB\|\leqslant\|A^2X+XB^2\|,$$

for A, B, $X\in M_n$ such that A and B are positive semidefinite.

It is worth to point out that techniques from [5] are used to sharpen the inequality (5).

REMARK 6. Putting $p=q=1$, and $y=\frac{\mu}{2}$ when $0<\mu\leqslant\frac{1}{2}$, $y=\frac{1-\mu}{2}$ when $\frac{1}{2}<\mu<1$, respectively, then Theorem 3 is just the result of Theorem 2.1 in [5]. Thus Theorem 2.1 in [5] is a special case of Theorem 3.

In [8], Hiai and Kosaki (Corollary 2.3) proved

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\|\leqslant\|\int_0^1 A^tXB^{1-t}dt\|\leqslant\left\|\frac{AX+XB}{2}\right\| \tag{21}$$

for A, B, $X \in M_n$ such that A, B are positive semidefinite matrices and every unitarily invariant norm $\|\cdot\|$.

In [2], the following inequality (p. 164, Exercise 5.4.8)

$$\frac{1}{2}\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\| \leqslant \|\int_0^1 A^{t}XB^{1-t}dt\|, \tag{22}$$

holds for $\frac{1}{4} \leqslant v \leqslant \frac{3}{4}$ and every unitarily invariant norm $\|\cdot\|$, where A, B, $X \in M_n$ such that A, B are positive semidefinite matrices.

It follows from (2), (21), and (22)

$$\|A^{\frac{1}{2}}XB^{\frac{1}{2}}\| \leqslant \frac{1}{2}\|A^{v}XB^{1-v}+A^{1-v}XB^{v}\| \leqslant \|\int_0^1 A^{t}XB^{1-t}dt\| \leqslant \left\|\frac{AX+XB}{2}\right\|, \tag{23}$$

where $\frac{1}{4} \leqslant v \leqslant \frac{3}{4}$.

THEOREM 4. *Let A and B be positive semidefinite matrices. Then for every unitarily invariant norm $\|\cdot\|$, the following inequalities*

$$\begin{aligned}\|AB\| &\leqslant \frac{1}{2}\|A^{\frac{1}{2}+v}B^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}\| \leqslant \|\int_0^1 A^{\frac{1}{2}+t}B^{\frac{3}{2}-t}dt\| \leqslant \left\|\left(\frac{A+B}{2}\right)^2\right\| \\ &\leqslant \frac{1}{4}\|A^2+B^2+2AB\|,\end{aligned} \tag{24}$$

holds for $\frac{1}{4} \leqslant v \leqslant \frac{3}{4}$.

Proof. Putting $X = A^{\frac{1}{2}}B^{\frac{1}{2}}$ in (23), we have

$$\|AB\| \leqslant \frac{1}{2}\|A^{\frac{1}{2}+v}B^{\frac{3}{2}-v}+A^{\frac{3}{2}-v}B^{\frac{1}{2}+v}\| \leqslant \|\int_0^1 A^{\frac{1}{2}+t}B^{\frac{3}{2}-t}dt\|. \tag{25}$$

In [17], Zou and He got (Theorem 2.1, its equivalent form)

$$\|\int_0^1 A^{\frac{1}{2}+t}B^{\frac{3}{2}-t}dt\| \leqslant \left\|\left(\frac{A+B}{2}\right)^2\right\| \tag{26}$$

for every unitarily invariant norm $\|\cdot\|$ and positive semidefinite matrices A and B.

On the other hand, in [12], for every unitarily invariant norm $\|\cdot\|$, Matharu and Aujla obtained

$$\|(A+B)(A+B)^*\| \leqslant \|AA^*+BB^*+2AB^*\|, \tag{27}$$

where A, $B \in M_n$.

Thus, the desired inequalities (24) follows from (25), (26) and (27).

The proof is completed. □

REMARK 7. (24) is a refinement of the following inequality

$$4\|AB\| \leqslant \|A^2+B^2+2AB\|,$$

which is due to (5) when $X = I$ (the identity matrix), $r = 1$, and $t = 2$.

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Junliang Wu
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331, P.R. China
e-mail: jlwu678@163.com

Jianguo Zhao
College of Mathematics and Statistics
Chongqing University
Chongqing, 401331, P.R. China
and
College of Science, Shihezi University
Shihezi, Xinjiang, 832003, P.R. China
e-mail: zhaojianguo@shzu.edu.cn