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DERIVABLE MAPS AND GENERALIZED DERIVATIONS

ZHIDONG PAN

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Abstract. Let $\mathscr{A}$ be a unital algebra, $\mathscr{M}$ be an $\mathscr{A}$-bimodule, $L(\mathscr{A},\mathscr{M})$ be the set of all linear maps from $\mathscr{A}$ to $\mathscr{M}$, and $\mathscr{R}_{\mathscr{A}}$ be a relation on $\mathscr{A}$. A map $\delta \in L(\mathscr{A},\mathscr{M})$ is called *derivable on* $\mathscr{R}_{\mathscr{A}}$ if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $(A,B) \in \mathscr{R}_{\mathscr{A}}$. One purpose of this paper is to propose the study of derivable maps on a new, but natural, relation $\mathscr{R}_{\mathscr{A}}$. Moreover, we give a characterization of generalized derivations on $\mathscr{M}_n(\mathbb{C})$, the $n \times n$ matrix algebra over the complex numbers; specifically, a linear map δ on $\mathscr{M}_n(\mathbb{C})$ is a generalized derivation iff there exists an $M \in \mathscr{M}_n(\mathbb{C})$ such that $\delta(AB) = \delta(A)B + A\delta(B)$, for all $A,B \in \mathscr{M}_n(\mathbb{C})$ satisfying $AMB = 0$; in this case $\delta(I) = cM$, for some $c \in \mathbb{C}$.

1. Introduction

Let $\mathscr{A}$ be a unital algebra, $\mathscr{M}$ be an $\mathscr{A}$-bimodule, and $L(\mathscr{A},\mathscr{M})$ be the set of all linear maps from $\mathscr{A}$ to $\mathscr{M}$. A map $\delta \in L(\mathscr{A},\mathscr{M})$ is called a *derivation* if for all $A,B \in \mathscr{A}$, $\delta(AB) = \delta(A)B + A\delta(B)$. Let $\mathscr{R}_{\mathscr{A}}$ be a relation on $\mathscr{A}$, i.e. $\mathscr{R}_{\mathscr{A}}$ is a nonempty subset of $\mathscr{A} \times \mathscr{A}$. We say $\delta \in L(\mathscr{A},\mathscr{M})$ is *derivable on* $\mathscr{R}_{\mathscr{A}}$ if $\delta(AB) = \delta(A)B + A\delta(B)$ for all $(A,B) \in \mathscr{R}_{\mathscr{A}}$; for convenience, such a δ will be called a *partial derivation* . There have been many papers studying when a partial derivation is a derivation. Jordan derivations have been extensively studied (see, e.g. [2], [4], [6], [10], and [12]), these are partial derivations that are derivable on $\mathscr{R}_{\mathscr{A}} = \{(A,B) \in \mathscr{A} \times \mathscr{A} : A = B\}$. Recently, many have considered partial derivations that are derivable on $\mathscr{R}_{\mathscr{A}} = \{(A,B) \in \mathscr{A} \times \mathscr{A} : AB = C\}$, for some fixed $C \in \mathscr{A}$ (see, e.g. [1], [3], [5], [7–11], and 13–15]).

In general, partial derivations are not necessarily derivations. Examples of such partial derivations include generalized derivations. Recall that a map $\delta \in L(\mathscr{A},\mathscr{M})$ is called a *generalized derivation* if for all $A,B \in \mathscr{A}$, $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$, where I is the unit of $\mathscr{A}$. For any $M \in \mathscr{M}$, we define a right multiplier M_r from $\mathscr{A}$ to $\mathscr{M}$ by $M_r(A) = AM$, $\forall\, A \in \mathscr{A}$ and a left multiplier M_l from $\mathscr{A}$ to $\mathscr{M}$ by $M_l(A) = MA$, $\forall\, A \in \mathscr{A}$. If $\delta \in L(\mathscr{A},\mathscr{M})$ and $M = \delta(I)$, then one can easily check that δ is a generalized derivation iff $\delta - M_r$ is a derivation iff $\delta - M_l$ is a derivation. That is, generalized derivations can be viewed as a sum of a derivation and a right (or left) multiplier. If $\delta \in L(\mathscr{A},\mathscr{M})$ is a generalized derivation, let $M = \delta(I)$ and $\mathscr{R}_{\mathscr{A}}(M,0) = \{(A,B) \in \mathscr{A} \times \mathscr{A} : AMB = 0\}$. Clearly, δ is derivable on $\mathscr{R}_{\mathscr{A}}(M,0)$.

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Naturally, this raises the following question: For any $M \in \mathscr{M}$, if $\delta \in L(\mathscr{A},\mathscr{M})$ is derivable on $\mathscr{R}_{\mathscr{A}}(M,0)$, is δ a generalized derivation? In this paper, we show this is the case when $\mathscr{A} = \mathscr{M} = \mathscr{M}_n(\mathbb{C})$, the $n\times n$ matrix algebra over the complex numbers. In this case, for simplicity, we will use $\mathscr{M}_n$ for $\mathscr{M}_n(\mathbb{C})$ and for any $M \in \mathscr{M}_n$ let $\mathscr{R}(M,0) = \{(A,B) \in \mathscr{M}_n \times \mathscr{M}_n : AMB = 0\}$.

2. Characterization of generalized derivations on $\mathscr{M}_n$

The following is our main result.

THEOREM 2.1. *If $\delta \in L(\mathscr{M}_n,\mathscr{M}_n)$, then δ is a generalized derivation iff there exists an $M \in \mathscr{M}_n$ such that δ is derivable on $\mathscr{R}(M,0)$; in this case $\delta(I) = cM$, for some $c \in \mathbb{C}$.*

We begin with two simple reduction lemmas.

LEMMA 2.2. *Suppose $\mathscr{A}$ is a unital algebra and $\mathscr{M}$ is an $\mathscr{A}$-bimodule. Let $\Delta \in L(\mathscr{A},\mathscr{M})$, $M \in \mathscr{M}$, $T \in \mathscr{A}$ be invertible in $\mathscr{A}$, and $\delta(A) = T^{-1}\Delta(TAT^{-1})T$, $\forall\, A \in \mathscr{A}$. Then $\delta(I) = T^{-1}\Delta(I)T$, and*

(i) Δ is derivable on $\mathscr{R}_{\mathscr{A}}(M,0)$ iff δ is derivable on $\mathscr{R}_{\mathscr{A}}(T^{-1}MT,0)$.

(ii) Δ is a generalized derivation iff δ is a generalized derivation.

Proof. For any $A,B \in \mathscr{A}$, let $A_1 = T^{-1}AT$ and $B_1 = T^{-1}BT$. A routine calculation shows $(A,B) \in \mathscr{R}_{\mathscr{A}}(M,0)$ iff $(A_1,B_1) \in \mathscr{R}_{\mathscr{A}}(T^{-1}MT,0)$ and

$$\delta(A_1B_1) - \delta(A_1)B_1 - A_1\delta(B_1) = T^{-1}[\Delta(AB) - \Delta(A)B - A\Delta(B)]T.$$

Thus (i) follows.

Similarly, (ii) follows from

$$\delta(A_1B_1) - \delta(A_1)B_1 - A_1\delta(B_1) + A_1\delta(I)B_1 = T^{-1}[\Delta(AB) - \Delta(A)B - A\Delta(B) + A\Delta(I)B]T.$$

□

LEMMA 2.3. *If $\Delta \in L(\mathscr{M}_n,\mathscr{M}_n)$, $n \geqslant 2$ and E_{ij} are the matrix units of $\mathscr{M}_n$, then there exists a $\delta \in L(\mathscr{M}_n,\mathscr{M}_n)$ such that $\delta - \Delta$ is an inner derivation and $E_{ii}\delta(E_{jj})E_{jj} = 0$, for all $i \neq j$.*

Proof. Take $K = \sum_{i=1}^n \Delta(E_{ii})E_{ii}$ and define $\delta_K \in L(\mathscr{M}_n,\mathscr{M}_n)$ by $\delta_K(A) = KA - AK$, $\forall\, A \in \mathscr{M}_n$. Let $\delta = \Delta - \delta_K$, then $\forall\, j$,

$$\delta(E_{jj}) = \Delta(E_{jj}) - (KE_{jj} - E_{jj}K) = \Delta(E_{jj}) - \Delta(E_{jj})E_{jj} + E_{jj}\sum_{i=1}^n \Delta(E_{ii})E_{ii}.$$

It follows that for any $i \neq j$,

$$E_{ii}\delta(E_{jj})E_{jj} = 0. \tag{2.0}$$

□

LEMMA 2.4. *If $\delta \in L(\mathscr{M}_n, \mathscr{M}_n)$, $n \geqslant 2$ satisfies Equation (2.0), $J \in \mathscr{M}_n$ is a Jordan matrix, and δ is derivable on $\mathscr{R}(J,0)$, then $\delta(E_{kl}) = E_{kk}\delta(E_{kl})(E_{ll}+E_{l+1l+1})$, $\forall\, l < n$ and $\delta(E_{kn}) = E_{kk}\delta(E_{kn})E_{nn}$.*

Proof. For any $k < j$ or $k \geqslant j+2$, then $E_{ij}JE_{kl} = 0$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$0 = \delta(E_{ij}E_{kl}) = \delta(E_{ij})E_{kl} + E_{ij}\delta(E_{kl}). \tag{2.1}$$

In particular, $\delta(E_{ij})E_{kk} + E_{ij}\delta(E_{kk}) = 0$, thus $\delta(E_{ij})E_{kk} + E_{ij}\delta(E_{kk})E_{kk} = 0$. Combining with (2.0), we see $\delta(E_{ij})E_{kk} = 0$, which implies

$$\delta(E_{ij})E_{kl} = 0. \tag{2.2}$$

By (2.1) and (2.2),

$$E_{ij}\delta(E_{kl}) = 0. \tag{2.3}$$

We will first prove

$$\delta(E_{kl}) = E_{kk}\delta(E_{kl}). \tag{$*$}$$

The conclusion of the lemma follows directly from (2.2) and $(*)$. We will prove $(*)$ by induction on k.

If $k = 1$ then by (2.3),

$$\delta(E_{1l}) = I\delta(E_{1l}) = (\sum_{i=1}^{n} E_{ii})\delta(E_{1l}) = E_{11}\delta(E_{1l}).$$

Suppose $k \geqslant 2$ and

$$\delta(E_{k-1l}) = E_{k-1k-1}\delta(E_{k-1l}). \tag{2.4}$$

By (2.3),

$$\delta(E_{kl}) = (E_{k-1k-1} + E_{kk})\delta(E_{kl}). \tag{2.5}$$

Let $J = (a_{ij})$, there are two possible cases.

Case 1. $a_{k-1k} = 0$

In this case, $E_{k-1k-1}JE_{kl} = 0$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$0 = \delta(E_{k-1k-1}E_{kl}) = \delta(E_{k-1k-1})E_{kl} + E_{k-1k-1}\delta(E_{kl}). \tag{2.6}$$

In particular,

$$0 = \delta(E_{k-1k-1})E_{kk} + E_{k-1k-1}\delta(E_{kk}). \tag{2.7}$$

Multiplying E_{kk} from the right of (2.7) and applying (2.0) gives $0 = \delta(E_{k-1k-1})E_{kk}$, which implies $\delta(E_{k-1k-1})E_{kl} = \delta(E_{k-1k-1})E_{kk}E_{kl} = 0$. Plugging this in (2.6), we get $E_{k-1k-1}\delta(E_{kl}) = 0$. Putting this in (2.5) gives $(*)$.

Case 2. $a_{k-1k} = 1$

If $a_{kk} = 0$ then $E_{kk}JE_{kl} = 0$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$\delta(E_{kk}E_{kl}) = \delta(E_{kk})E_{kl} + E_{kk}\delta(E_{kl}).$$

Multiplying E_{k-1k-1} from the left and applying (2.0) gives

$$E_{k-1k-1}\delta(E_{kk}E_{kl}) = E_{k-1k-1}\delta(E_{kk})E_{kl} = E_{k-1k-1}\delta(E_{kk})E_{kk}E_{kl} = 0,$$

and putting this in (2.5) gives $(*)$.

If $a_{kk} \neq 0$, then $a_{k-1k-1} = a_{kk} \neq 0$. Let $a = a_{k-1k-1}$ then $E_{k-1k-1}J(aE_{kl} - E_{k-1l}) = 0$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$\delta[E_{k-1k-1}(aE_{kl} - E_{k-1l})] = \delta(E_{k-1k-1})(aE_{kl} - E_{k-1l}) + E_{k-1k-1}\delta(aE_{kl} - E_{k-1l}).$$

Combining with (2.4), we get

$$0 = \delta(E_{k-1k-1})(aE_{kl} - E_{k-1l}) + aE_{k-1k-1}\delta(E_{kl}). \tag{2.8}$$

In particular,

$$0 = \delta(E_{k-1k-1})(aE_{kk} - E_{k-1k}) + aE_{k-1k-1}\delta(E_{kk}).$$

Multiplying E_{kk} from the right and applying (2.0) gives $0 = \delta(E_{k-1k-1})(aE_{kk} - E_{k-1k})$. Thus $\delta(E_{k-1k-1})(aE_{kl} - E_{k-1l}) = \delta(E_{k-1k-1})(aE_{kk} - E_{k-1k})E_{kl} = 0$. This, together with (2.8), gives $E_{k-1k-1}\delta(E_{kl}) = 0$. Now, $(*)$ follows from (2.5). $\square$

LEMMA 2.5. *If $\delta \in L(\mathscr{M}_n, \mathscr{M}_n)$, $n \geqslant 2$ satisfies (2.0), $J \in \mathscr{M}_n$ is a Jordan matrix, and δ is derivable on $\mathscr{R}(J,0)$, then $\delta(E_{ij})E_{j+1j+1} = E_{ij}\delta(E_{jj})E_{j+1j+1}$, $\forall\, j < n$.*

Proof. Let $J = (a_{ij})$ and fix a $j < n$.

If $a_{jj+1} = 0$ then $E_{ij}JE_{j+1j+1} = 0$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$0 = \delta(E_{ij}E_{j+1j+1}) = \delta(E_{ij})E_{j+1j+1} + E_{ij}\delta(E_{j+1j+1}).$$

Applying Lemma 2.4, we get $\delta(E_{ij})E_{j+1j+1} = 0$, in particular, $\delta(E_{jj})E_{j+1j+1} = 0 = \delta(E_{ij})E_{j+1j+1}$.

If $a_{jj+1} = 1$ then $E_{ij}J(a_{jj}E_{j+1j} - E_{jj}) = 0$. Since δ is derivable on $\mathscr{R}(J,0)$, we have $\delta[E_{ij}(a_{jj}E_{j+1j} - E_{jj})] = \delta(E_{ij})(a_{jj}E_{j+1j} - E_{jj}) + E_{ij}\delta(a_{jj}E_{j+1j} - E_{jj})$. Applying Lemma 2.4, we get $-\delta(E_{ij}) = \delta(E_{ij})(a_{jj}E_{j+1j} - E_{jj}) - E_{ij}\delta(E_{jj})$. Multiplying E_{j+1j+1} from the right yields the conclusion. $\square$

LEMMA 2.6. *If $\delta \in L(\mathscr{M}_n, \mathscr{M}_n)$, $n \geqslant 2$ satisfies (2.0), $J \in \mathscr{M}_n$ is a Jordan matrix, and δ is derivable on $\mathscr{R}(J,0)$, then for each i, $\delta(E_{ii}) = c_iE_{ii}J$, for some $c_i \in \mathbb{C}$.*

Proof. Any matrix $T = (t_{ij})$ can be viewed as a linear operator on $\mathbb{C}^n$ with standard column vectors $\{e_1, \cdots, e_n\}$ as basis, that is, for any column vector $x \in \mathbb{C}^n$, we can define $Tx = (t_{ij})x$. The range and kernel of T will be denoted by $\operatorname{ran}(T)$ and $\ker(T)$, respectively. Fix any i, then $\operatorname{ran}(E_{ii}J) \subseteq \mathbb{C}e_i$. By Lemma 2.4, $\delta(E_{ii}) = E_{ii}\delta(E_{ii})$, so $\operatorname{ran}(\delta(E_{ii})) \subseteq \mathbb{C}e_i$. Thus, $E_{ii}J$ and $\delta(E_{ii})$ are operators of rank at most one, with range contained in the same one-dimensional vector space.

If $E_{ii}J = 0$, then $E_{ii}JE_{k1} = 0$, for all $k = 1, \cdots, n$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$\delta(E_{ii}E_{k1}) = \delta(E_{ii})E_{k1} + E_{ii}\delta(E_{k1}) = \delta(E_{ii})E_{k1} + E_{ii}E_{kk}\delta(E_{k1}).$$

Thus $0=\delta(E_{ii})E_{k1}$ and $\delta(E_{ii})=0$. In this case, $\delta(E_{ii})=cE_{ii}J$, for any $c\in\mathbb{C}$.

To complete the proof, we only need to show $\ker(E_{ii}J)\subseteq\ker(\delta(E_{ii}))$, when $E_{ii}J\neq 0$.

Suppose $J=(a_{ij})$ and $E_{ii}J\neq 0$.

If $i=n$, then $E_{nn}J=a_{nn}E_{nn}\neq 0$ implies $\ker(E_{nn}J)=\text{span}\{e_1,e_2,\cdots,e_{n-1}\}$. By Lemma 2.4, $\delta(E_{nn})=\delta(E_{nn})E_{nn}$. Thus $\ker(E_{nn}J)\subseteq\ker(\delta(E_{nn}))$

If $i<n$, then $E_{ii}J=a_{ii}E_{ii}+a_{ii+1}E_{ii+1}$. It follows that $e_k\in\ker(E_{ii}J)$, $\forall\, k<i$ or $k\geqslant i+2$ and $a_{ii+1}e_i-a_{ii}e_{i+1}\in\ker(E_{ii}J)$.

Since $E_{ii}J\neq 0,\ker(E_{ii}J)=\text{ span }\{e_1,\cdots,e_{i-1},a_{ii+1}e_i-a_{ii}e_{i+1},e_{i+2},\cdots,e_n\}$. Note that $E_{ii}J(a_{ii+1}E_{i1}-a_{ii}E_{i+11})=0$, and δ is derivable on $\mathscr{R}(J,0)$,

$$\delta[E_{ii}(a_{ii+1}E_{i1}-a_{ii}E_{i+11})]=\delta(E_{ii})(a_{ii+1}E_{i1}-a_{ii}E_{i+11})+E_{ii}\delta(a_{ii+1}E_{i1}-a_{ii}E_{i+11}).$$

Combining the above equation with Lemma 2.4, we get $0=\delta(E_{ii})(a_{ii+1}E_{i1}-a_{ii}E_{i+11})$. Thus $\delta(E_{ii})(a_{ii+1}e_i-a_{ii}e_{i+1})=\delta(E_{ii})(a_{ii+1}E_{i1}-a_{ii}E_{i+11})e_1=0$, i.e. $a_{ii+1}e_i-a_{ii}e_{i+1}\in\ker(\delta(E_{ii}))$. By Lemma 2.4, $\delta(E_{ii})=\delta(E_{ii})(E_{ii}+E_{i+1i+1})$, thus $e_k\in\ker(\delta(E_{ii}))$, $\forall\, k<i$ or $k\geqslant i+2$, and $\ker(E_{ii}J)\subseteq\ker(\delta(E_{ii}))$. $\square$

LEMMA 2.7. *If $\delta\in L(\mathscr{M}_n,\mathscr{M}_n)$, $n\geqslant 2$ satisfies (2.0), $J\in\mathscr{M}_n$ is a Jordan matrix, and δ is derivable on $\mathscr{R}(J,0)$, then there exists a $c\in\mathbb{C}$ such that $\delta(E_{ii})=cE_{ii}J$, for all $i=1,\cdots,n$; as a consequence $\delta(I)=cJ$.*

Proof. For any $i\neq k$, by Lemma 2.6, there exist $c_i,c_k\in\mathbb{C}$ such that $\delta(E_{ii})=c_iE_{ii}J$ and $\delta(E_{kk})=c_kE_{kk}J$. If $E_{ii}J=0$ we can choose c_i to be any number, in particular, take $c_i=c_k$. Similarly, if $E_{kk}J=0$, we can take $c_k=c_i$. Let $J=(a_{ij})$. Fix any i and k, without loss of generality, we assume $i<k$, $E_{ii}J\neq 0$ and $E_{kk}J\neq 0$. Thus a_{ii} and a_{ii+1} are not both zero, and a_{kk} and a_{kk+1} are not both zero. For any j with $E_{jj}J\neq 0$, define $j^*=j$ if $a_{jj}\neq 0$; otherwise $j^*=j+1$. Thus $a_{jj^*}\neq 0$, in particular, $a_{ii^*}\neq 0$. and $a_{kk^*}\neq 0$; moreover, if $k=n$ then $E_{nn}J=a_{nn}E_{nn}\neq 0$ implies $n^*=n$.

Claim: $a_{ki^*}=0$; indeed, since $i<k$, it follows $i^*\leqslant k$. If $i^*<k$ then $a_{ki^*}=0$ since J is a Jordan matrix. By the definition of i^*, $i^*=k$ precisely when $a_{ii}=0$ and $i^*=i+1=k$. In this case, since $E_{ii}J\neq 0$ and J is a Jordan matrix, $a_{ii+1}=1$ and $a_{ki^*}=a_{i+1i+1}=a_{ii}=0$.

By the claim,

$$E_{ik}JE_{i^*k^*}=a_{ki^*}E_{ik^*}=0. \tag{2.9}$$

We will proceed by considering two separate cases: $a_{ik^*}=0$ and $a_{ik^*}\neq 0$

Case 1. $a_{ik^*}=0$.

In this case,

$$E_{ii}JE_{k^*k^*}=a_{ik^*}E_{ik^*}=0. \tag{2.10}$$

It follows from Eqs. (2.9) and (2.10),

$$(a_{kk^*}E_{ii}+a_{ii^*}E_{ik})J(E_{i^*k^*}-E_{k^*k^*})=a_{kk^*}E_{ii}JE_{i^*k^*}-a_{ii^*}E_{ik}JE_{k^*k^*}=0. \tag{2.11}$$

Since δ is derivable on $\mathscr{R}(J,0)$, by (2.9), (2.10), and (2.11) we have

$$\delta(E_{ik}E_{i^*k^*})=\delta(E_{ik})E_{i^*k^*}+E_{ik}\delta(E_{i^*k^*}),$$

$$\delta(E_{ii}E_{k^*k^*}) = \delta(E_{ii})E_{k^*k^*} + E_{ii}\delta(E_{k^*k^*}),$$

and

$$\begin{aligned}\delta[(a_{kk^*}E_{ii}+a_{ii^*}E_{ik})(E_{i^*k^*}-E_{k^*k^*})] = {}& \delta(a_{kk^*}E_{ii}+a_{ii^*}E_{ik})(E_{i^*k^*}-E_{k^*k^*})\\ &+(a_{kk^*}E_{ii}+a_{ii^*}E_{ik})\delta(E_{i^*k^*}-E_{k^*k^*}).\end{aligned}$$

The last three equations give us

$$\begin{aligned}a_{kk^*}\delta(E_{ii}E_{i^*k^*}) - a_{ii^*}\delta(E_{ik}E_{k^*k^*}) = {}& a_{kk^*}\delta(E_{ii})E_{i^*k^*} - a_{ii^*}\delta(E_{ik})E_{k^*k^*}\\ &+a_{kk^*}E_{ii}\delta(E_{i^*k^*}) - a_{ii^*}E_{ik}\delta(E_{k^*k^*}).\end{aligned}$$

By Lemma 2.4, $\delta(E_{ii}E_{i^*k^*}) = E_{ii}\delta(E_{i^*k^*})$. Thus

$$-a_{ii^*}\delta(E_{ik}E_{k^*k^*}) = a_{kk^*}\delta(E_{ii})E_{i^*k^*} - a_{ii^*}\delta(E_{ik})E_{k^*k^*} - a_{ii^*}E_{ik}\delta(E_{k^*k^*}). \tag{2.12}$$

If $k^* = k$, by Eq. (2.12),

$$-a_{ii^*}\delta(E_{ik}) = a_{kk}\delta(E_{ii})E_{i^*k} - a_{ii^*}\delta(E_{ik})E_{kk} - a_{ii^*}E_{ik}\delta(E_{kk}).$$

Multiplying E_{kk} from the right of this equation gives

$$0 = a_{kk}\delta(E_{ii})E_{i^*k} - a_{ii^*}E_{ik}\delta(E_{kk})E_{kk}.$$

By Lemma 2.6,

$$\begin{aligned}0 &= a_{kk}\delta(E_{ii})E_{i^*k} - a_{ii^*}E_{ik}\delta(E_{kk})E_{kk} = a_{kk}c_iE_{ii}JE_{i^*k} - a_{ii^*}E_{ik}c_kE_{kk}JE_{kk}\\ &= a_{kk}c_ia_{ii^*}E_{ik} - a_{ii^*}c_ka_{kk}E_{ik}.\end{aligned}$$

Since $a_{ii^*} \neq 0$ and $a_{kk} = a_{kk^*} \neq 0$, we get $c_i = c_k$.

If $k^* = k+1$, by Lemma 2.4 and Eq. (2.12),

$$0 = a_{kk+1}\delta(E_{ii})E_{i^*k+1} - a_{ii^*}\delta(E_{ik})E_{k+1k+1}.$$

Combining this with Lemmas 2.5 and 2.6, we have

$$\begin{aligned}0 &= a_{kk+1}\delta(E_{ii})E_{i^*k+1} - a_{ii^*}\delta(E_{ik})E_{k+1k+1} = a_{kk+1}\delta(E_{ii})E_{i^*k+1} - a_{ii^*}E_{ik}\delta(E_{kk})E_{k+1k+1}\\ &= a_{kk+1}c_iE_{ii}JE_{i^*k+1} - a_{ii^*}E_{ik}c_kE_{kk}JE_{k+1k+1} = a_{kk+1}c_ia_{ii^*}E_{ik+1} - a_{ii^*}c_ka_{kk+1}E_{ik+1}.\end{aligned}$$

Since $a_{ii^*} \neq 0$ and $a_{kk+1} = a_{kk^*} \neq 0$, we have $c_i = c_k$.

Case 2. $a_{ik^*} \neq 0$.

This case can only happen when $k^* = k = i+1$, thus $a_{kk} \neq 0$ and $a_{ii+1} = a_{ik} = 1$. It follows that $(a_{kk}E_{ii} - E_{ik})JE_{kk} = 0$. Since δ is derivable on $\mathscr{R}(J,0)$,

$$\delta[(a_{kk}E_{ii} - E_{ik})E_{kk}] = \delta(a_{kk}E_{ii} - E_{ik})E_{kk} + (a_{kk}E_{ii} - E_{ik})\delta(E_{kk}).$$

By Lemma 2.4,

$$\delta(-E_{ik}) = \delta(a_{kk}E_{ii})E_{kk} - \delta(E_{ik})E_{kk} - E_{ik}\delta(E_{kk}).$$

Multiplying E_{kk} from the right, we get $0 = a_{kk}\delta(E_{ii})E_{kk} - E_{ik}\delta(E_{kk})E_{kk}$. By Lemma 2.6,

$$\begin{aligned} 0 &= a_{kk}\delta(E_{ii})E_{kk} - E_{ik}\delta(E_{kk})E_{kk} = a_{kk}c_iE_{ii}JE_{kk} - E_{ik}c_kE_{kk}JE_{kk} \\ &= a_{kk}c_ia_{ik}E_{ik} - c_ka_{kk}E_{ik} = a_{kk}c_iE_{ik} - c_ka_{kk}E_{ik}. \end{aligned}$$

Therefore, $c_i = c_k$. $\square$

Proof of Theorem 2.1. The statement is clearly true when $n = 1$, so we assume $n \geqslant 2$. With one direction being clear, we only need to prove that if $\delta \in L(\mathscr{M}_n, \mathscr{M}_n)$ is derivable on $\mathscr{R}(M,0)$, for some $M \in \mathscr{M}_n$, then δ is a generalized derivation with $\delta(I) = cM$, for some $c \in \mathbb{C}$. By Lemmas 2.2 and 2.3, we can assume δ satisfies Eq. (2.0) and δ is derivable on $\mathscr{R}(J,0)$, where J is a Jordan matrix of M. Let $S = \delta(I)$ and define $S_r \in L(\mathscr{M}_n, \mathscr{M}_n)$ by $S_r(A) = AS$, $\forall\, A \in \mathscr{M}_n$. Let $\tau = \delta - S_r$. Then τ is derivable on $\mathscr{R}(J,0)$ and, by Lemma 2.7, $\tau(E_{jj}) = 0$, $\forall\, j = 1,2,...,n$; in particular, τ satisfies Eq. (2.0). For any $j < n$, by Lemma 2.5, $\tau(E_{ij})E_{j+1j+1} = E_{ij}\tau(E_{jj})E_{j+1j+1} = 0$. Thus, by Lemma 2.4, $\tau(E_{ij}) = \tau(E_{ij})(E_{jj} + E_{j+1j+1}) = \tau(E_{ij})E_{jj}$, $\forall\, j < n$ and $\tau(E_{in}) = \tau(E_{in})E_{nn}$. It follows that for any i,j,l,

$$\tau(E_{ij}E_{ll}) = \tau(E_{ij})E_{ll}. \tag{2.13}$$

We will show τ is a derivation by showing for any i,j,k,l,

$$\tau(E_{ij}E_{kl}) = \tau(E_{ij})E_{kl} + E_{ij}\tau(E_{kl}). \tag{2.14}$$

Eq. (2.13) implies Eq. (2.14) holds for $k = l$.

If $j \neq k$ then by Lemma 2.4 $E_{ij}\tau(E_{kl}) = 0$. By Eq. (2.13), $\tau(E_{ij})E_{kl} = \tau(E_{ij})E_{jj}E_{kl} = 0$. Thus Eq. (2.14) holds for $j \neq k$. In particular, if $k \neq l$, then

$$\tau(E_{il}E_{kl}) = \tau(E_{il})E_{kl} + E_{il}\tau(E_{kl}). \tag{2.15}$$

It remains to show Eq. (2.14) holds for $j = k$ and $k \neq l$. Let $J = (a_{ij})$.

If $a_{kl} \neq 0$, then $E_{ik}J(a_{kl}E_{kl} - a_{kk}E_{ll}) = 0$. Since τ is derivable on $\mathscr{R}(J,0)$,

$$\tau[E_{ik}(a_{kl}E_{kl} - a_{kk}E_{ll})] = \tau(E_{ik})(a_{kl}E_{kl} - a_{kk}E_{ll}) + E_{ik}\tau(a_{kl}E_{kl} - a_{kk}E_{ll}).$$

Applying Eq. (2.13) to this equation, we have $\tau(E_{ik}E_{kl}) = \tau(E_{ik})E_{kl} + E_{ik}\tau(E_{kl})$.

Similarly, if $a_{lk} \neq 0$, then $(a_{lk}E_{ik} - a_{kk}E_{il})JE_{kl} = 0$. Since τ is derivable on $\mathscr{R}(J,0)$,

$$\delta[(a_{lk}E_{ik} - a_{kk}E_{il})E_{kl}] = \delta(a_{lk}E_{ik} - a_{kk}E_{il})E_{kl} + (a_{lk}E_{ik} - a_{kk}E_{il})\delta(E_{kl}).$$

Combining this with Eq. (2.15), we get $\tau(E_{ik}E_{kl}) = \tau(E_{ik})E_{kl} + E_{ik}\tau(E_{kl})$.

Suppose $a_{kl} = a_{lk} = 0$. If $a_{ll} \neq 0$, then note $(a_{ll}E_{ik} - a_{kk}E_{il})J(E_{kl} + E_{ll}) = 0$. Since τ is derivable on $\mathscr{R}(J,0)$,

$$\tau[(a_{ll}E_{ik} - a_{kk}E_{il})(E_{kl} + E_{ll})] = \tau(a_{ll}E_{ik} - a_{kk}E_{il})(E_{kl} + E_{ll}) + (a_{ll}E_{ik} - a_{kk}E_{il})\tau(E_{kl} + E_{ll}).$$

Combining this with Eqs. (2.13) and (2.15) gives $\tau(E_{ik}E_{kl}) = \tau(E_{ik})E_{kl} + E_{ik}\tau(E_{kl})$.

Finally, if $a_{ll}=0$ then for any positive integers $s,t \leqslant n$, $E_{sl}JE_{lt}=0$. Since τ is derivable on $\mathscr{R}(J,0)$,

$$\tau(E_{sl}E_{lt})=\tau(E_{sl})E_{lt}+E_{sl}\tau(E_{lt}). \tag{2.16}$$

By Eqs. (2.13) and (2.16),

$$\begin{aligned}\tau(E_{ik}E_{kl}) &= \tau(E_{il})E_{ll}=\tau(E_{il})E_{lk}E_{kl}=[\tau(E_{il}E_{lk})-E_{il}\tau(E_{lk})]E_{kl}\\ &= \tau(E_{ik})E_{kl}-E_{il}\tau(E_{lk})E_{kl}=\tau(E_{ik})E_{kl}-E_{ik}E_{kl}\tau(E_{lk})E_{kl}\\ &= \tau(E_{ik})E_{kl}-E_{ik}[\tau(E_{kl}E_{lk})-\tau(E_{kl})E_{lk}]E_{kl}=\tau(E_{ik})E_{kl}-E_{ik}[0-\tau(E_{kl})E_{ll}]\\ &= \tau(E_{ik})E_{kl}+E_{ik}\tau(E_{kl}).\end{aligned}$$

The equation $\delta(I)=cJ$ for some $c\in\mathbb{C}$ is proved in Lemma 2.7. $\square$

COROLLARY 2.8. *A linear map $\delta\in L(\mathscr{M}_n,\mathscr{M}_n)$ is a generalized derivation iff $\delta(AB)=\delta(A)B+A\delta(B)$, for all $A,B\in\mathscr{M}_n$ satisfying $A\delta(I)B=0$.*

2.1. Remarks

For an algebra $\mathscr{A}$ and an $\mathscr{A}$-bimodule $\mathscr{M}$, we call a relation $\mathscr{R}_{\mathscr{A}}\subseteq\mathscr{A}\times\mathscr{A}$ a *derivational set* of $L(\mathscr{A},\mathscr{M})$ if whenever $\delta\in L(\mathscr{A},\mathscr{M})$ is derivable on $\mathscr{R}_{\mathscr{A}}$ it implies δ is a derivation. For $\mathscr{A}=\mathscr{M}=\mathscr{M}_n$ and any $0\neq M\in\mathscr{M}_n$, Theorem 2.1 implies $\mathscr{R}(M,0)$ is a maximal non-derivational set of $L(\mathscr{M}_n,\mathscr{M}_n)$ as illustrated in the following corollary.

COROLLARY 2.9. *Given any $0\neq M\in\mathscr{M}_n$, every relation $\mathscr{R}$ on $\mathscr{M}_n$ such that $\mathscr{R}(M,0)\subsetneq\mathscr{R}$ is a derivational set of $L(\mathscr{M}_n,\mathscr{M}_n)$.*

Proof. If $\delta\in L(\mathscr{M}_n,\mathscr{M}_n)$ is derivable on $\mathscr{R}$ then it is derivable on $\mathscr{R}(M,0)$. By Theorem 2.1, δ is a generalized derivation such that $\delta(I)=cM$, for some $c\in\mathbb{C}$. Thus $\delta(AB)=\delta(A)B+A\delta(B)-cAMB$ for all $(A,B)\in\mathscr{M}_n\times\mathscr{M}_n$. In particular, $\delta(A_1B_1)=\delta(A_1)B_1+A_1\delta(B_1)-cA_1MB_1$ for any $(A_1,B_1)\in\mathscr{R}$ and $(A_1,B_1)\notin\mathscr{R}(M,0)$. On the other hand, since δ is derivable on $\mathscr{R}$, $\delta(A_1B_1)=\delta(A_1)B_1+A_1\delta(B_1)$. Thus $cA_1MB_1=0$. Since $(A_1,B_1)\notin\mathscr{R}(M,0)$, $c=0$. $\square$

For a Banach algebra $\mathscr{A}$ and $M\in\mathscr{A}$, let $\mathscr{R}_{\mathscr{A}}(M,0)=\{(A,B)\in\mathscr{A}\times\mathscr{A}:AMB=0\}$.

2.2. Question

For what Banach algebra $\mathscr{A}$ does it hold that $\delta\in L(\mathscr{A},\mathscr{A})$ is a generalized derivation iff δ is derivable on $\mathscr{R}_{\mathscr{A}}(M,0)$ for some $M\in\mathscr{A}$?

In particular, we do not know if the above holds when $\mathscr{A}$ is a C^*-algebra, a von Neumann algebra, a CSL-algebra, a nest algebra, even $B(H)$, the algebra of all bounded linear operators on a Hilbert space H.

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Zhidong Pan
Department of Mathematics
Saginaw Valley State University
University Center, MI 48710, USA
e-mail: pan@svsu.edu