A SIMPLE SUFFICIENT CONDITION FOR COMPLETE POSITIVITY

WASIN SO AND CHANGQING XU

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Abstract. We use row sums and rank to give a sufficient condition on the diagonal entries of a doubly nonnegative matrix for it to be completely positive and its cp-rank equal to its rank.

1. Introduction

Let $\mathbf{R}$ be the set of real numbers, and $\mathbf{R}_+$ be the set of nonnegative real numbers. An $n \times n$ real matrix A is called *completely positive* (CP) if there is some entrywise nonnegative $m \times n$ matrix B such that $A = B^T B$. The minimum m is called the *cp-rank* of A and is denoted by cp-rank(A), the corresponding B is called a *minimal cp-factor* of A. An $n \times n$ entrywise nonnegative matrix is called *doubly nonnegative* (DN) if it is also positive semi-definite. As usual, we denote DN_n the set of DN matrices of order $n \geqslant 1$, and CP_n the set of CP matrices of order $n \geqslant 1$.

A CP matrix is obviously a DN matrix, but the converse is generally not true for matrices of order greater than four [3]. There are three basic problems about CP matrices:

1. Determine which DN matrices are CP.

2. Determine cp-rank(A) if A is CP.

3. Find a minimal cp-factor if cp-rank(A) is known.

There are many partial solutions to these problems in literature, see [1] for a comprehensive survey up to 2003. We mention the one by Kaykobad [4] using diagonally dominance: if a DN mairx $A = [a_{ij}]$ satisfies the condition $|a_{ii}| \geqslant \sum_{j \neq i} |a_{ij}|$ for all i then A is CP with cp-rank $\leqslant$ number of nonzero entries above the diagonal + number of strictly diagonally dominant rows. Kaykobad's result can be restated as follows: if $A = [a_{ij}] \in DN_n$ is such that, for all i,

$$a_{ii} \geqslant \frac{1}{2} R_i$$

where $R_i = \sum_{j=1}^n a_{ij}$ is the i-th row sum, then $A \in CP_n$. Therefore large diagonal entries (relative to row sums) guarantee complete positivity. In Section 2, we prove our main result which states that small diagonal entries (relative to row sums) also guarantee complete positivity. In Section 3, we compare our result with a recent result of Reams [7].

2. Main result

In this section, we give a sufficient condition on the diagonal entries of a doubly nonnegative matrix A of rank r for A to be completely positive with cp-rank equals r. The proof is based on the following consequence of Cauchy-Schwarz inequality:

$$(z_1 + \cdots + z_k)^2 \leqslant k(z_1^2 + \cdots + z_k^2)$$

where $z_1, \ldots, z_k$ are real numbers. Let e be the vector in $\mathbf{R}^r$ with all entries equal to 1, $\langle \cdot, \cdot \rangle$ be the standard inner product, and $\| \cdot \|$ be the Euclidean norm of a vector. The next lemma shows that any vector with a small angle with e also has nonnegative entries.

LEMMA 2.1. *Let $z \in \mathbf{R}^r$ be such that $\langle z, e \rangle \geqslant \sqrt{\frac{r-1}{r}} \|z\| \|e\|$. Then $z \in \mathbf{R}_+^r$.*

Proof. Without loss of generality, we assume $z = [z_1, z_2, \ldots, z_r]^T$ where $z_1 \geqslant z_2 \geqslant \cdots \geqslant z_r$. Then the hypothesis is

$$z_1 + z_2 + \cdots + z_r \geqslant \sqrt{(r-1)(z_1^2 + \cdots + z_r^2)} \geqslant 0,$$

and so

$$(z_1 + z_2 + \cdots + z_r)^2 \geqslant (r-1)(z_1^2 + \cdots + z_r^2).$$

Now suppose the contrary that there exists $1 \leqslant k \leqslant r-1$ such that $z_1 \geqslant \cdots \geqslant z_k \geqslant 0 > z_{k+1} \geqslant \cdots \geqslant z_r$. Hence, $z_1 + z_2 + \cdots + z_k \geqslant 0$ and $-(z_{k+1} + \cdots + z_r) > 0$. It follows from the Cauchy-Schwarz inequality above that

$$\begin{aligned}
(z_1 + \cdots + z_k + z_{k+1} + \cdots + z_r)^2 &\leqslant (z_1 + \cdots + z_k)^2 + (z_{k+1} + \cdots + z_r)^2 \\
&\leqslant k(z_1^2 + \cdots + z_k^2) + (r-k)(z_{k+1}^2 + \cdots + z_r^2).
\end{aligned}$$

Combining with the hypothesis, we have

$$(r-1)(z_1^2 + \cdots + z_r^2) \leqslant k(z_1^2 + \cdots + z_k^2) + (r-k)(z_{k+1}^2 + \cdots + z_r^2),$$

and so

$$(r-1-k)(z_1^2 + \cdots + z_k^2) + (r-1-r+k)(z_{k+1}^2 + \cdots + z_r^2) \leqslant 0.$$

Because of $z_{k+1} < 0$ and $z_1 + z_2 + \cdots + z_r \geqslant 0$, it follows that $r - 1 - k = 0$ and $r - 1 - r + k = 0$, i.e., $k = 1$ and $r = 2$. Consequently, we have $z_1 \geqslant 0 > z_2$, and so $z_1 z_2 < 0$. On the other hand, the hypothesis gives $z_1 + z_2 \geqslant \sqrt{z_1^2 + z_2^2}$, and so $2z_1 z_2 \geqslant 0$, a contradiction! $\square$

EXAMPLE 2.2. This example shows that the parameter $\sqrt{\frac{r-1}{r}}$ in Lemma 2.1 is optimal. For $c < \sqrt{\frac{r-1}{r}}$, choose small $\varepsilon > 0$ such that $-\varepsilon + \sqrt{r-1}\sqrt{1-\varepsilon^2} > c\sqrt{r}$. Take

$$z = [-\varepsilon\sqrt{r-1}, \sqrt{1-\varepsilon^2}, \cdots, \sqrt{1-\varepsilon^2}] \notin \mathbf{R}_+^r$$

and so $\|z\| = \sqrt{r-1}$, $\|e\| = \sqrt{r}$,

$$\begin{aligned}
\langle z, e \rangle &= -\varepsilon\sqrt{r-1} + (r-1)\sqrt{1-\varepsilon^2} \\
&= \sqrt{r-1}(-\varepsilon + \sqrt{r-1}\sqrt{1-\varepsilon^2}) > c\|z\|\|e\|.
\end{aligned}$$

LEMMA 2.3. *Given vectors $\beta_i \in R^r$. If there exists a nonzero vector $x \in R^r$ such that*

$$\langle \beta_i, x \rangle \geqslant \sqrt{\frac{r-1}{r}}\|\beta_i\|\|x\|$$

for all i, then there exists an orthogonal matrix Q such that $Q\beta_i \in R^r_+$ for all i.

Proof. Case 1: x is a positive multiple of e, i.e., $x = \frac{\|x\|}{\sqrt{r}}e$. Then take Q to be the identity matrix I, and we have $Qx = x = \frac{\|x\|}{\sqrt{r}}e \in R^r_+$.

Case 2: x is not a positive multiple of e. Then $v = x - \frac{\|x\|}{\sqrt{r}}e \neq 0$. Take $Q = I - \frac{2}{v^T v}vv^T$. Hence, Q is orthogonal since

$$\begin{aligned}
Q^T Q = Q^2 &= \left(I - \frac{2}{v^T v}vv^T\right)^2 \\
&= I - \frac{4}{v^T v}vv^T + \frac{4}{(v^T v)^2}vv^T vv^T \\
&= I - \frac{4}{v^T v}vv^T + \frac{4}{v^T v}vv^T = I.
\end{aligned}$$

Moreover $v^T x = x^T x - \frac{\|x\|}{\sqrt{r}}e^T x$ and

$$\begin{aligned}
v^T v &= \left(x^T - \frac{\|x\|}{\sqrt{r}}e^T\right)\left(x - \frac{\|x\|}{\sqrt{r}}e\right) \\
&= x^T x - \frac{2\|x\|}{\sqrt{r}}e^T x + \frac{\|x\|^2}{r}e^T e \\
&= 2\left(x^T x - \frac{\|x\|}{\sqrt{r}}e^T x\right) \\
&= 2v^T x.
\end{aligned}$$

And so

$$\begin{aligned}
Qx &= x - \frac{2}{v^T v}vv^T x \\
&= v + \frac{\|x\|}{\sqrt{r}}e - \frac{2}{v^T v}vv^T x \\
&= \left(1 - \frac{2}{v^T v}v^T x\right)v + \frac{\|x\|}{\sqrt{r}}e \\
&= \frac{\|x\|}{\sqrt{r}}e.
\end{aligned}$$

Finally, in both cases, we have

$$\frac{\langle Q\beta_i, e\rangle}{\|Q\beta_i\|\|e\|} = \frac{\langle Q\beta_i, \frac{\|x\|}{\sqrt{r}}e\rangle}{\|Q\beta_i\|\|\frac{\|x\|}{\sqrt{r}}e\|} = \frac{\langle Q\beta_i, Qx\rangle}{\|Q\beta_i\|\|Qx\|} = \frac{\langle \beta_i, x\rangle}{\|\beta_i\|\|x\|} \geqslant \sqrt{\frac{r-1}{r}},$$

and so $Q\beta_i \in \mathbf{R}_+^r$ by Lemma 2.1. $\square$

The following result of Gray and Wilson [2] can be proved by Lemma 2.3.

COROLLARY 2.4. *Let $A \in DN_n$ with* $\mathrm{rank}(A) = 2$. *Then $A \in CP_n$ with* cp-rank$(A) =$ 2.

Proof. Since A is a positive semidefinite matrix of rank 2, we have $A = B^T B$ for some $2 \times n$ matrix $B = [v_1, v_2, \ldots, v_n]$ where $v_i \in \mathbf{R}^2$. Moreover A is nonnegative, and so $v_i^T v_j = \langle v_i, v_j\rangle \geqslant 0$. Take $x \in \mathbf{R}^2$ to be the angle bisector of the largest angle among all pairs of vectors. Then $\langle v_i, x\rangle \geqslant \sqrt{\frac{1}{2}}\|v_i\|\|x\|$ for all i. By Lemma 2.3, there exists orthogonal Q such that $Qv_i \geqslant 0$. Consequently, $A = B^T B = (QB)^T(QB)$ is CP with cp-rank$(A) = 2$. $\square$

Now we give a simple sufficient condition for complete positivity.

THEOREM 2.5. *Let $A = [a_{ij}] \in DN_n$ with* $\mathrm{rank}(A) = r$, *and denote R_i the i-th row sum of A. If*

$$rR_i^2 \geqslant (r-1)a_{ii}(R_1 + \cdots + R_n)$$

for all i then $A \in CP_n$ with cp-rank$(A) = r$.

Proof. Since A is a positive semidefinite matrix of rank r, $A = B^T B$ for some $r \times n$ matrix $B = [\beta_1, \beta_2, \ldots, \beta_n]$ where $\beta_i \in \mathbf{R}^r$. Note that $\beta_i = Be_i$ where e_i is the i-th column of the $n \times n$ identity matrix. Take $x = \beta_1 + \beta_2 + \cdots + \beta_n = Be_1 + Be_2 + \cdots + Be_n = B(e_1 + \cdots + e_n) = Be$. Now

$$\langle \beta_i, x\rangle = \langle Be_i, Be\rangle = e_i^T B^T Be = e_i^T Ae = R_i,$$

$$\|\beta_i\|^2 = \langle \beta_i, \beta_i\rangle = \langle Be_i, Be_i\rangle = e_i^T B^T Be_i = e_i^T Ae_i = a_{ii},$$

and

$$\|x\|^2 = \langle x, x\rangle = \langle Be, Be\rangle = e^T B^T Be = e^T Ae = R_1 + \cdots + R_n.$$

Hence, by hypothesis,

$$\frac{\langle \beta_i, x\rangle}{\|\beta_i\|\|x\|} = \frac{R_i}{\sqrt{a_{ii}}\sqrt{R_1 + \cdots + R_n}} \geqslant \sqrt{\frac{r-1}{r}}.$$

By Lemma 2.3, there exists an $r \times r$ orthogonal matrix Q such that $Q\beta_i \geqslant 0$, and so QB is an $r \times n$ nonnegative matrix. Consequently, $A = B^T B = B^T Q^T QB = (QB)^T(QB)$ is CP with cp-rank$(A) \leqslant r$. On the other hand, $r = \mathrm{rank}(A) \leqslant$ cp-rank(A), so we have cp-rank$(A) = r$. $\square$

Note that the proofs of Lemma 2.3 and Theorem 2.5 indeed provide a method to find a minimal cp-factor. The next two examples show how to use the sufficient condition in Theorem 2.5 to determine a given DN matrix of rank 3 to be CP with cp-rank being 3.

EXAMPLE 2.6. Consider matrix

$$A = \begin{bmatrix} 93 & 27 & 55 & 45 \\ 27 & 45 & 33 & 51 \\ 55 & 33 & 41 & 43 \\ 45 & 51 & 43 & 62 \end{bmatrix}.$$

It is easy to check that $A \in DN_4$ with $\mathrm{rank}(A) = 3$. Using the result of Maxfield and Minc [6] that any 4×4 DN matrix is CP (whose cp-rank is less than or equal to 4), we know that A is CP, but we don't know its exact cp-rank. However, by Theorem 2.5, we know that $A \in CP_4$ with cp-rank$(A) = 3$.

EXAMPLE 2.7. Consider matrix

$$A = \begin{bmatrix} 41 & 43 & 80 & 56 & 50 \\ 43 & 62 & 89 & 78 & 51 \\ 80 & 89 & 162 & 120 & 93 \\ 56 & 78 & 120 & 104 & 62 \\ 50 & 51 & 93 & 62 & 65 \end{bmatrix}.$$

Then $A \in DN_5$ with $\mathrm{rank}(A) = 3$. But the result of Maxifeld and Minc [6] cannot be applied. Nonetheless, it can be easily checked that the sufficient condition in Theorem 2.5 is satisfied and so $A \in CP_5$ with cp-rank$(A) = 3$.

REMARK 2.8. Unfortunately, the sufficient condition in Theoem 2.5 is far from necessary. For example, the diagonal matrix $\begin{bmatrix} 100 & 0 \\ 0 & 1 \end{bmatrix} \in DN_2 = CP_2$ but it fails the condition in Theorem 2.5.

Without using the rank of the matrix, we have the following weaker result.

COROLLARY 2.9. *If $A \in DN_n$ is such that $a_{ii} \leqslant \frac{nR_i^2}{(n-1)(R_1+R_2+\cdots+R_n)}$ for all i then $A \in CP_n$.*

Note that the matrix A in Example 2.7 fails the sufficient condition in Corollary 2.9, but satisfies the sufficient condition in Theorem 2.5. For matrix with constant row sums, the sufficient condition looks even simpler.

COROLLARY 2.10. *Let $A \in DN_n$ with $\mathrm{rank}(A) = r$. If A has constant row sum R such that $a_{ii} \leqslant \frac{rR}{(r-1)n}$ for all i, then $A \in CP_n$ with cp-rank$(A) = r$.*

3. Discussion

Recently, Reams [7] gave a sufficient condition for complete positivity. We restate his result as follows.

THEOREM 3.1. *If $A = [a_{ij}] \in DN_n$ has the perron value $\rho > 0$ and perron vector $v = [v_i] > 0$ with $v_p = \min_i v_i$ such that*

$$a_{ii} \leqslant \frac{\rho v_i^2}{\sum_{j \neq p} v_j^2}$$

for all i, then $A \in CP_n$. Indeed if $A = C^2$ for some positive semi-definite C then $C \in DN_n$.

Theorem 2.5 and Theorem 3.1 are similar in the sense that both sufficient conditions involve smallness of the diagonal entries. Moreover, the following result of Marcus and Minc [5] can be deduced from either Theorem 2.5 (use Corollary 2.9) or Theorem 3.1 [7, Corollary 2].

COROLLARY 3.2. *If $A = [a_{ij}] \in DN_n$ has constant row sum 1 with $a_{ii} \leqslant \frac{1}{n-1}$ then $A \in CP_n$.*

On the other hand, Reams' result asserts that $A = C^2$ for some DN matrix C but it says nothing about cp-rank of A. And, our result asserts that cp-rank$(A) = $ rank(A) but say nothing about nonnegativity of A's square root. Next example shows that Theorem 2.5 applies, but Theorem 3.1 does not apply.

EXAMPLE 3.3. Consider the matrix

$$A = \begin{bmatrix} 93 & 27 & 55 & 45 \\ 27 & 45 & 33 & 51 \\ 55 & 33 & 41 & 43 \\ 45 & 51 & 43 & 62 \end{bmatrix}.$$

The perron value is $\rho = 191.0752$, and the perron vector is

$$v^T = \begin{bmatrix} 0.6307 & 0.3962 & 0.4571 & 0.5193 \end{bmatrix}.$$

Then

$$a_{11} = 93 \nleqslant 82.5969 = \frac{\rho v_1^2}{\sum_{j \neq 2} v_j^2}$$

i.e., Theorem 3.1 fails. However, from Example 2.6, A is CP by Theorem 2.5.

Finally we remark that we could find NO example such that Theorem 3.1 can be applied but Theorem 2.5 cannot.

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Wasin So
Department of Mathematics
San Jose State University
San Jose, CA 95120
USA
e-mail: `wasin.so@sjsu.edu`

Changqing Xu
Department of Applied Mathematics
Suzhou University of Science and Technology
Suzhou, 215009
China
e-mail: `cqxurichard@mail.usts.edu.cn.`

Operators and Matrices
www.ele-math.com
oam@ele-math.com