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ON THE KERNEL OF A SINGULAR INTEGRAL OPERATOR WITH NON–CARLEMAN SHIFT AND CONJUGATION

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Abstract. On the Hilbert space $\widetilde{L}_2(\mathbb{T})$ the singular integral operator with non-Carleman shift and conjugation $K = P_+ + (aI + AC)P_-$ is considered, where $P_\pm$ are the Cauchy projectors, $A = \sum_{j=0}^{m} a_j U^j$, a, a_j, $j = \overline{1,m}$, are continuous functions on the unit circle $\mathbb{T}$, U is the shift operator and C is the operator of complex conjugation. Some estimates for the dimension of the kernel of the operator K are obtained.

1. Introduction

Let $\mathbb{T}$ denote the unit circle in the complex plane, $\mathbb{T}_+$ and $\mathbb{T}_-$ denote the interior and the exterior (∞ included) of the unit disk, respectively. On the Hilbert Space $L_2(\mathbb{T})$ we consider the singular integral operator with Cauchy kernel, defined almost everywhere on $\mathbb{T}$ by

$$(S\varphi)(t) = (\pi i)^{-1} \int_{\mathbb{T}} \varphi(\tau)(\tau - t)^{-1} d\tau,$$

where the integral is understood in the sense of its principal value. The operator S is a bounded linear involutive operator ($S^2 = I$, where I is the identity operator on $L_2(\mathbb{T})$). Then it is possible to define in $L_2(\mathbb{T})$ a pair of complementary projection operators in $L_2(\mathbb{T})$,

$$P_\pm = \frac{1}{2}(I \pm S),$$

and to decompose $L_2(\mathbb{T}) = L_2^+(\mathbb{T}) \oplus \overset{\circ}{L}{}_2^-(\mathbb{T})$, with $L_2^+(\mathbb{T}) = \operatorname{im} P_+$ and $\overset{\circ}{L}{}_2^-(\mathbb{T}) = \operatorname{im} P_-$. We also set $L_2^-(\mathbb{T}) = \overset{\circ}{L}{}_2^-(\mathbb{T}) \oplus \mathbb{C}$.

As usual, $L_\infty(\mathbb{T})$ denotes the space of all essentially bounded functions on $\mathbb{T}$ and $H_\infty(\mathbb{T})$ the class of all bounded and analytic functions in $\mathbb{T}_+$.

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Let $H_{r,\theta}(\mathbb{T})$ denote the set of all functions of $H_\infty(\mathbb{T})$ that can be represented by the product of a rational outer function r (i.e., r has all the zeros and poles in $\mathbb{T}_-$) and an inner function θ (i.e., θ is a bounded and analytic function on $\mathbb{T}_+$, such that its modulus is equal to one a.e. on $\mathbb{T}$). Let $\mathscr{C}(\mathbb{T})$ denote the algebra of all continuous functions on $\mathbb{T}$, $\mathscr{R}(\mathbb{T})$ denote the algebra of rational functions without poles on $\mathbb{T}$, and $\mathscr{R}^\pm(\mathbb{T})$ denote the subsets of $\mathscr{R}(\mathbb{T})$ whose elements have no poles in $\mathbb{T}_\pm$, respectively.

Now let us introduce the concept of matrix function generalized factorization (see, for instance, [3] and [23]): we say that a matrix function $c \in L_\infty^{n\times n}(\mathbb{T})$ admits a right (left) generalized factorization in $L_2(\mathbb{T})$ if it can be represented as

$$c = c_-\Lambda c_+ \quad (c_+\Lambda c_-), \tag{1}$$

where

$$c_-^{\pm 1} \in \left[L_2^-(\mathbb{T})\right]^{n\times n}, \quad c_+^{\pm 1} \in \left[L_2^+(\mathbb{T})\right]^{n\times n}, \quad \Lambda(t) = \operatorname{diag}\{t^{\varkappa_j}\},$$

$\varkappa_j \in \mathbb{Z}$, $j = \overline{1,n}$, with $\varkappa_1 \geqslant \varkappa_2 \geqslant \ldots \geqslant \varkappa_n$, and $c_-P_+c_+I$ $(c_+P_+c_-I)$ represents a bounded linear operator in $L_2^n(\mathbb{T})$; the number $\varkappa = \sum\limits_{j=1}^{n} \varkappa_j$ is called the factorization index of the determinant of the matrix function c. The integers $\varkappa_j$ are uniquely defined by the matrix function c and are called its right (left) partial indices. If $\varkappa_j = 0$, $j = \overline{1,n}$, then c is said to admit a right (left) canonical generalized factorization in $L_2(\mathbb{T})$.

Any non-singular continuous matrix function $c \in \mathscr{C}^{n\times n}(\mathbb{T})$ admits a generalized factorization (1) in $L_2(\mathbb{T})$ (see, for instance, the above cited [3] and [23]).

Any non-singular rational matrix function $c \in \mathscr{R}^{n\times n}(\mathbb{T})$ admits a factorization of the form (1) (see, for instance, [12]), where

$$c_-^{\pm 1} \in \left[\mathscr{R}^-(\mathbb{T})\right]^{n\times n}, \quad c_+^{\pm 1} \in \left[\mathscr{R}^+(\mathbb{T})\right]^{n\times n}. \tag{2}$$

For the particular scalar case we note that $\varkappa = \operatorname{ind} c$ if $c \in \mathscr{C}(\mathbb{T})$; as usual, $\operatorname{ind}\varphi$ denotes the Cauchy index of a continuous function $\varphi \in \mathscr{C}(\mathbb{T})$, i.e.,

$$\operatorname{ind}\varphi = \frac{1}{2\pi}\{\arg\varphi(t)\}_{t\in\mathbb{T}}.$$

If $c^\pm \in \mathscr{R}(\mathbb{T})$, then $\varkappa = z_+ - p_+$, where z_+ is the number of zeros of c in $\mathbb{T}_+$ and p_+ is the number of poles of c in $\mathbb{T}_+$ (with regard to their multiplicities) (see, for instance, [6])

Now let ω be a homeomorphism of $\mathbb{T}$ onto itself, which is differentiable on $\mathbb{T}$ and whose derivative does not vanish there. The function $\omega : \mathbb{T} \to \mathbb{T}$ is called a shift function or simply a shift on $\mathbb{T}$. By

$$\omega_k(t) \equiv \omega[\omega_{k-1}(t)], \quad \omega_1(t) \equiv \omega(t), \quad \omega_0(t) \equiv t, \quad t \in \mathbb{T},$$

we denote the k-th iteration of the shift, $k \geqslant 2$, $k \in \mathbb{N}$.

A shift ω is called a (generalized) Carleman shift of order $n \in \mathbb{N}\setminus\{1\}$ if $\omega_n(t) \equiv t$, but $\omega_k(t) \not\equiv t$ for $k = \overline{1,n-1}$. Otherwise, if ω is not a Carleman shift, it is called a non-Carleman shift.

In what follows we will consider a linear fractional non-Carleman shift preserving the orientation on $\mathbb{T}$

$$\alpha(t) = \frac{\mu t + \nu}{\overline{\nu} t + \overline{\mu}}, \quad t \in \mathbb{T}, \tag{3}$$

where $\mu, \nu \in \mathbb{C}$: $|\mu|^2 - |\nu|^2 = 1$. This shift has two fixed points, τ_1 and τ_2, given by the formula

$$\tau_{1,2} = \frac{\mu - \overline{\mu} \pm \sqrt{(\mu + \overline{\mu})^2 - 4}}{2\overline{\nu}}. \tag{4}$$

Obviously $\tau_1 \neq \tau_2$ if $|\mathrm{Re}\,\mu| \neq 1$.

The rational shift function α admits the factorization (1)

$$\alpha(t) = \alpha_+(t) t \alpha_-(t),$$

where

$$\alpha_+(t) = \frac{1}{\overline{\nu} t + \overline{\mu}}, \quad \alpha_-(t) = \frac{\mu t + \nu}{t} = (\overline{\alpha_+(t)})^{-1}.$$

We see that the functions $\alpha_\pm, \alpha_\pm^{-1}$ are analytic in $\mathbb{T}_\pm$ and continuous in the closure of $\mathbb{T}_\pm$, respectively.

Let $a, a_0, a_1, \ldots, a_m \in \mathscr{C}(\mathbb{T})$ be given continuous functions on $\mathbb{T}$. As usual, let $\widetilde{L}_2(\mathbb{T})$ denote the real space of all Lebesgue measurable square summable complex valued functions on $\mathbb{T}$. On $\widetilde{L}_2(\mathbb{T})$, associated with the shift α, we consider the shift operator U defined by

$$(U\varphi)(t) = \alpha_+(t)\varphi[\alpha(t)].$$

The shift operator U satisfies the properties:

i) U is isometric, i.e., $\|U\varphi\| = \|\varphi\|$;

ii) $US = SU$.

We also consider the following two operators on $\widetilde{L}_2(\mathbb{T})$: the bounded linear involutive operator of complex conjugation C,

$$(C\varphi)(t) = t^{-1}\overline{\varphi(t)}, \tag{5}$$

and the functional operator

$$A = \sum_{j=0}^{m} a_j U^j.$$

The operators $P_\pm$, U and C, verify the properties

$$CU = UC, \quad UP_\pm = P_\pm U, \quad CP_\pm = P_\mp C. \tag{6}$$

In this work we will study the singular integral operator (SIO) with non-Carleman shift and conjugation defined in the unit circle

$$K = P_+ + (aI + AC)P_-. \tag{7}$$

The history of SIOs with shift, as well as related singular integral equations with shift and related boundary value problems with shift, is rich. These problems were studied during the second half of the last century till the present time. Ilya Vekua's book [28] (first edition in 1959) played a key role in this process; in this and in other similar books (see e.g. [29]), it has been shown how some mathematical physics problems lead to the solvability of boundary value problems with shift. The Fredholm theory of SIOs with Carleman shift was constructed in the sixties and the seventies of the XX century (see [21]). For the case of non-Carleman shift, the theory was completed in the eighties (see [17]). However, more interesting questions about the solvability of boundary value problems with shifts, have been considered only with very restrictive conditions on the respective coefficients (see [21]). Recent progress in the study of the spectral properties of SIOs with linear fractional Carleman shift and conjugation (see [11], [14], [15], [16], and [27]) makes it possible to study the solvability of the related boundary value problems (see [22]). For non-Carleman shift, the question about the solvability of this type of problems remains open (see [2], [18], and [24]).

In [19] we studied a generalized Riemann boundary value problem with a non-Carleman shift and conjugation on the real line, through the study of the kernel of the operator (7) (considering the shift operator $(U_r\varphi)(t) = \varphi(t+\mu)$, μ is a fixed real number, and the operator of complex conjugation $(C_r\varphi)(t) = \overline{\varphi(t)}$).

In the present paper we consider the operator (7) on the unit circle. The estimate for the dimension of its kernel,

$$\dim\ker K \leqslant l(f) + \max(\varkappa - k, 0) + \max(\varkappa + k, 0) + 1,$$

is obtained (see formula (29) bellow). It is interesting to note that, besides the terms present when we had considered K on the real line, there is a new term in the right member of (29). As we see bellow, this term, 1, appears as a consequence of the weight t^{-1} in the definition of the operator of complex conjugation. The influence of the coefficients $a_1, a_2, \ldots, a_m$ is restricted to the term $l(f)$ only; the terms $\varkappa$ and k depend only on the coefficients a and a_0.

In Section 2 we present some estimates for the dimension of the kernel of the operator K. The most general case is considered in Subsection 2.1. A special case, when K has some bounded and analytic coefficients in $\mathbb{T}_+$, is described in Subsection 2.2.

In recent years, several software applications with extensive capabilities of symbolic computation were made available to the general public. These computer algebra systems (CAS) allow to delegate to a computer all, or a significant part, of the symbolic and numeric calculations present in many mathematical algorithms. In our work we use the CAS *Mathematica* to implement, on a computer, some of our analytical algorithms within operator theory (see [5], [6], [9], and [25]).

In Section 3 we show how symbolic computation can be used to explore the dimension of the kernel of the operator K. The rational case is considered in Subsection 3.1. In Subsubsection 3.1.1 we present the analytical algorithm [ARFact-Scalar] that gives explicit factorizations for any factorable rational function defined on the unit circle. The [ARFact-Matrix] algorithm that computes explicit factorizations for non-singular rational matrix functions defined on the unit circle is presented in Subsubsection 3.1.2. The inner-outer factorization concept is considered in Subsection 3.2 to obtain particular results on the estimate of the dimension of the kernel of operator K. The Subsubsection 3.2.1 describe the generalized factorization algorithm [AFact], for special classes of factorable essentially bounded hermitian matrix functions. The [AKer-$H\varphi^* H\varphi$] algorithm that computes the kernel of SIOs related with Hankel operators is presented in Subsubsection 3.2.2. Subsubsection 3.2.3 contains some particular cases. These four analytical algorithms allows us to design a new algorithm to estimate the dimension of the kernel of operator K. Several nontrivial examples are presented to illustrate the importance of symbolic computation on the study of this kind of problems.

2. On the dimension of the kernel of the operator K

In this section we present some estimates for the dimension of the kernel of the operator (7). The most general case is considered in Subsection 2.1. A special case, when K has some bounded and analytic coefficients in $\mathbb{T}_+$, is described in Subsection 2.2.

2.1. The case $\mathscr{C}(\mathbb{T})$

PROPOSITION 2.1. *Let $K_1 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ be the SIO with shift*

$$K_1 = M_1 P_+ + M_2 P_-,$$

where M_1, M_2, are the functional operators

$$M_1 = \begin{pmatrix} I & A \\ 0 & t^{-1}\overline{a}I \end{pmatrix}, \quad M_2 = \begin{pmatrix} aI & 0 \\ \widetilde{A} & I \end{pmatrix}, \quad \widetilde{A} = \sum_{j=0}^{m} t^{-1}\overline{a_j}U^j;$$

then

$$\dim\ker K = \frac{1}{2}\dim\ker K_1. \tag{8}$$

Proof. Making use of the properties (6) and $C^2 = I$, we obtain the following relation between the operators K and K_1, similar to the Gohberg-Krupnik matrix equality (see [13]),

$$N \operatorname{diag}\{K, \widetilde{K}\} N^{-1} = K_1,$$

where

$$\widetilde{K} = P_+ + (aI - AC)P_-,$$

and N is the following invertible operator in $\widetilde{L}_2^2(\mathbb{T})$

$$N = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ C & -C \end{pmatrix}.$$

We then have

$$\dim\ker K + \dim\ker\widetilde{K} = \dim\ker K_1.$$

Since $(iI)^{-1}K(iI) = \widetilde{K}$, then $\dim\ker K = \dim\ker\widetilde{K}$.

Thus

$$2\dim\ker K = \dim\ker K_1,$$

i.e., (8). $\square$

In what follows, all the continuous functions a are invertible on $\mathbb{T}$. Then M_1, M_2 are invertible operators, so the operator K_1 is Fredholm in $\widetilde{L}_2^2(\mathbb{T})$ (see [17]).

Let us consider the operator

$$K_2 = M_2^{-1}K_1. \tag{9}$$

Simple computations show that

$$K_2 = \sum_{j=0}^{2m} \widetilde{b}_j U^j P_+ + P_-, \tag{10}$$

where

$$\widetilde{b}_j = \operatorname{diag}\{1, t^{-1}\} b_j, \; j = \overline{0, 2m},$$

with

$$b_0 = \begin{pmatrix} a^{-1} & a^{-1}a_0 \\ -a^{-1}\overline{a_0} & \overline{a} - a^{-1}|a_0|^2 \end{pmatrix}, \tag{11}$$

$$b_1 = \begin{pmatrix} 0 & a^{-1}a_1 \\ -a^{-1}(\alpha)\overline{a_1} & -a^{-1}\overline{a_0}a_1 - a^{-1}(\alpha)\overline{a_1}a_0(\alpha) \end{pmatrix},$$

$$b_2 = \begin{pmatrix} 0 & a^{-1}a_2 \\ -a^{-1}(\alpha_2)\overline{a_2} & -a^{-1}\overline{a_0}a_2 - a^{-1}(\alpha)\overline{a_1}a_1(\alpha) - a^{-1}(\alpha_2)\overline{a_2}a_0(\alpha_2) \end{pmatrix},$$

$$\ldots,$$

$$b_m = \begin{pmatrix} 0 & a^{-1}a_m \\ -a^{-1}(\alpha_m)\overline{a_m} & -a^{-1}\overline{a_0}a_m - \cdots - a^{-1}(\alpha_m)\overline{a_m}a_0(\alpha_m) \end{pmatrix},$$

$$b_{m+1} = \begin{pmatrix} 0 & 0 \\ 0 & -a^{-1}(\alpha)\overline{a_1}a_m(\alpha) - \cdots - a^{-1}(\alpha_m)\overline{a_m}a_1(\alpha_m) \end{pmatrix},$$

$$\ldots,$$

$$b_{2m} = \begin{pmatrix} 0 & 0 \\ 0 & -a^{-1}(\alpha_m)\overline{a_m}a_m(\alpha_m) \end{pmatrix}.$$

Taking into account Proposition 2.1 and (9) we have

PROPOSITION 2.2. *Let $K_2 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ be the SIO with shift defined by* (10)*, then*

$$\dim\ker K = \frac{1}{2}\dim\ker K_2.$$

We proceed with the following result.

PROPOSITION 2.3. *Let* $K_3 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ *be the SIO with shift*

$$K_3 = \sum_{j=0}^{2m} b_j U^j P_+ + P_-,$$

then

$$\dim\ker K \leqslant \frac{1}{2}(\dim\ker K_3 + 2).$$

Proof. Let

$$d = \operatorname{diag}\{1,t\}, \quad d^{-1} = \operatorname{diag}\{1,t^{-1}\}.$$

The operator K_2 defined by (10) can be written by the product

$$K_2 = d^{-1}\left(\sum_{j=0}^{2m} b_j U^j P_+ + dP_-\right)$$

we obtain that

$$\dim\ker K_2 = \dim\ker\left(\sum_{j=0}^{2m} b_j U^j P_+ + dP_-\right).$$

Taking into account Proposition 2.3, in [18], it follows the inequality[1]

$$\dim\ker\left(\sum_{j=0}^{2m} b_j U^j P_+ + dP_-\right) \leqslant \dim\ker\left(\sum_{j=0}^{2m} b_j U^j P_+ + P_-\right) + 2,$$

i.e.,

$$\dim\ker K_2 \leqslant \dim\ker K_3 + 2.$$

With Proposition 2.2 we are done. $\square$

Let e_n denote the $(n \times n)$ identity matrix and, for simplicity, $e \equiv e_2$.

PROPOSITION 2.4. *Let* $K_4 : \widetilde{L}_2^{4m}(\mathbb{T}) \to \widetilde{L}_2^{4m}(\mathbb{T})$ *be the SIO with shift*

$$K_4 = (c_0 I + c_1 U)P_+ + P_-, \tag{12}$$

where c_0 *and* c_1 *are the* $(4m \times 4m)$ *matrix functions*

$$c_0 = \operatorname{diag}\{b_0, e_{4m-2}\}$$

and

$$c_1 = \begin{pmatrix} b_1 & b_2 & \cdots & b_{2m-1} & b_{2m} \\ -e & 0 & \cdots & 0 & 0 \\ 0 & -e & \ddots & \vdots & \vdots \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & -e & 0 \end{pmatrix}. \tag{13}$$

[1] Here we have "2" in the left member of the inequality because the operators act in the real space $\widetilde{L}_2^2(\mathbb{T})$.

Then

$$\dim\ker K \leqslant \frac{1}{2}(\dim\ker K_4 + 2).$$

Proof. We take into account the result formulated in [19], Proposition 2.3, on the real line, considering the shift $\beta_r(t) = t + \mu$, $t \in \overset{\circ}{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, μ is a fixed real number. Since $U_\eta P_+ = P_+ U_\eta$, in a similar way, on the unit circle, we show that

$$\dim\ker K_3 = \dim\ker K_4.$$

With Proposition 2.3 the result follows. □

Now we analyze the matrix function b_0 defined by (11) in more detail,

$$b_0 = \begin{pmatrix} a^{-1} & a^{-1}a_0 \\ -a^{-1}\overline{a_0} & -a^{-1}|a_0|^2 + \overline{a} \end{pmatrix}.$$

Note that $\det b_0(t) \neq 0$ for all $t \in \mathbb{T}$. So the non-singular continuous matrix function b_0 admits a right generalized factorization (1) in $L_2(\mathbb{T})$

$$b_0 = b_- \Lambda b_+. \tag{14}$$

It is assumed that

$$b_\pm^{\pm 1} \in C^{2\times 2}(\mathbb{T}).$$

For the continuous function a_0 let us denote its projections by

$$(a_0)_\pm := P_\pm(a_0).$$

Considering the decomposition $a_0 = (a_0)_+ + (a_0)_-$ we start to consider the particular case when the function $(a_0)_-$ is the null function. This is an interesting case since the matrix function b_0 admits the right generalized factorization (14), where

$$b_- = \begin{pmatrix} a_-^{-1} & 0 \\ -a_-^{-1}\overline{(a_0)_+} & \overline{a_+} \end{pmatrix}, \quad b_+ = \begin{pmatrix} a_+^{-1} & a_+^{-1}(a_0)_+ \\ 0 & \overline{a_-} \end{pmatrix},$$

and

$$\Lambda(t) = \operatorname{diag}\{t^{-\varkappa}, t^{-\varkappa}\}.$$

Obviously, we get the following result

PROPOSITION 2.5. *Let a be an invertible continuous function on $\mathbb{T}$, and let*

$$a(t) = a_-(t)t^{\varkappa}a_+(t), \quad \varkappa = \operatorname{ind} a,$$

be a generalized factorization (1) of a in $L_2(\mathbb{T})$. Then the right partial indices of the matrix function b_0 are $\varkappa_1 = -\varkappa$ and $\varkappa_2 = -\varkappa$.

For the general case, we must consider the function

$$u := (a_0)_-(\overline{a_-}a_+)^{-1},$$

its projections

$$u_\pm := P_\pm u,$$

and the Hankel operator (acting in the Hardy class $H_2(\mathbb{T})$) with symbol $\varphi \in L_\infty(\mathbb{T})$

$$H_\varphi = P_-\overline{\varphi}P_+.$$

On the right partial indices of the matrix function b_0 we get the following result

PROPOSITION 2.6. *Let a be an invertible continuous function on $\mathbb{T}$, and let*

$$a(t) = a_-(t)t^{\varkappa}a_+(t), \quad \varkappa = \operatorname{ind} a,$$

be a generalized factorization (1) of a in $L_2(\mathbb{T})$. Then the right partial indices of the matrix function b_0 are

$$\varkappa_1 = -\varkappa + k, \quad \varkappa_2 = -\varkappa - k,$$

where

$$k = \dim\ker(H^*_{\overline{u_-}}H_{\overline{u_-}} - I). \tag{15}$$

Proof. The matrix function b_0 can be written as

$$b_0(t) = t^{-\varkappa}c_-(t)g(t)c_+(t),$$

where

$$c_- = \begin{pmatrix} 1 & 0 \\ -\overline{(a_0)_+} & -1 \end{pmatrix}\begin{pmatrix} a_-^{-1} & 0 \\ 0 & \overline{a_+} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \overline{u_+} & 1 \end{pmatrix},$$

$$c_+ = \begin{pmatrix} 1 & u_+ \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_+^{-1} & 0 \\ 0 & \overline{a_-} \end{pmatrix}\begin{pmatrix} 1 & (a_0)_+ \\ 0 & 1 \end{pmatrix},$$

and

$$g = \begin{pmatrix} 1 & u_- \\ \overline{u_-} & |u_-|^2 - 1 \end{pmatrix}. \tag{16}$$

The hermitian matrix function g admits the following right generalized factorization in $L_2(\mathbb{T})$

$$g(t) = g_-(t)\operatorname{diag}\{t^k, t^{-k}\}g_+(t).$$

In [4] it is proved that the right partial index k is equal to the dimension of the kernel of the selfadjoint operator $H^*_{\overline{u_-}}H_{\overline{u_-}} - I$, i.e., (15). Since $c_-^{\pm 1} \in \left[L_2^-(\mathbb{T})\right]^{2\times 2}$ and $c_+^{\pm 1} \in \left[L_2^+(\mathbb{T})\right]^{2\times 2}$, a right generalized factorization of the matrix function b_0 is

$$b_0(t) = c_-(t)g_-(t)\operatorname{diag}\{t^{-\varkappa+k}, t^{-\varkappa-k}\}g_+(t)c_+(t),$$

and so $\varkappa_1 = -\varkappa + k$, $\varkappa_2 = -\varkappa - k$ are its right partial indices. $\square$

REMARK. In [23] it is proved that the integer k is equal to the dimension of the kernel of the operator $I - P_-u_-P_+\overline{u_-}P_-$. We consider the operator $H^*_{\overline{u_-}}H_{\overline{u_-}}$ instead of the operator $P_-u_-P_+\overline{u_-}P_-$ since in [7] it is described an analytical algorithm to compute explicitly a left generalized factorization of factorable essentially bounded matrix functions of the form

$$\mathscr{A}_\gamma(u) = \begin{pmatrix} 1 & u \\ \overline{u} & |u|^2 + \gamma \end{pmatrix}, \tag{17}$$

where $u \in H_{r,\theta}(\mathbb{T})$ and $\gamma \in \mathbb{C} \setminus \{0\}$. Obviously the algorithm also computes the left partial indices of $\mathscr{A}_\gamma(u)$ (that is, the right partial indices of $\mathscr{A}^T_\gamma(u)$). In addition, in [9] it is described an analytical algorithm to compute explicitly the kernel of operators of the form $H^*_\varphi H_\varphi + \gamma I$, where $\varphi \in H_{r,\theta}(\mathbb{T})$.

PROPOSITION 2.7. *Let a be an invertible continuous function on $\mathbb{T}$ and c_1 be the matrix function defined by* (13)*. Let b_0 be the matrix function defined by* (11) *and* (14) *a right generalized factorization of b_0 in $L_2(\mathbb{T})$. Then*

$$\dim\ker K \leqslant \frac{1}{2}(\dim\ker K_5 - 2\varkappa_1^- - 2\varkappa_2^- + 2), \tag{18}$$

where $K_5 : \widetilde{L}_2^{4m}(\mathbb{T}) \to \widetilde{L}_2^{4m}(\mathbb{T})$ is the SIO with shift

$$K_5 = (I + fU)P_+ + P_-, \tag{19}$$

f is the $(4m \times 4m)$ matrix function

$$f = \mathrm{diag}\{\Lambda_-^{-1}b_-^{-1}, e_{4m-2}\}c_1\,\mathrm{diag}\{b_+^{-1}(\alpha)\Lambda_+^{-1}(\alpha), e_{4m-2}\}, \tag{20}$$

with

$$\Lambda_\pm = \mathrm{diag}\{t^{\varkappa_1^\pm}, t^{\varkappa_2^\pm}\},\ \varkappa_j^\pm = \frac{1}{2}(\varkappa_j \pm |\varkappa_j|), \quad j = 1,2.$$

Proof. The operator K_4 defined by (12) admits the factorization

$$K_4 = \mathrm{diag}\{b_-, e_{4m-2}\}\widetilde{K}_4[\mathrm{diag}\{b_+, e_{4m-2}\}P_+ + \mathrm{diag}\{b_-^{-1}, e_{4m-2}\}P_-], \tag{21}$$

where

$$\widetilde{K}_4 = \left[\mathrm{diag}\{\Lambda, e_{4m-2}\}I + \widetilde{f}U\right]P_+ + P_-,$$

with

$$\widetilde{f} = \mathrm{diag}\{b_-^{-1}, e_{4m-2}\}c_1\,\mathrm{diag}\{b_+^{-1}(\alpha), e_{4m-2}\}.$$

The first and the third operators in (21) are invertible, therefore

$$\dim\ker K_4 = \dim\ker \widetilde{K}_4. \tag{22}$$

Now we consider the left invertible operators

$$K_- = P_+ + \mathrm{diag}\{\Lambda_-, e_{4m-2}\}P_-, \quad K_+ = \mathrm{diag}\{\Lambda_+, e_{4m-2}\}P_+ + P_-,$$

and the operator

$$\widetilde{K}_5 = \left[\operatorname{diag}\{\Lambda_+, e_{4m-2}\}I + \operatorname{diag}\{\Lambda_-^{-1}, e_{4m-2}\}\widetilde{f}U\right]P_+ + P_-.$$

The following equalities hold

$$\widetilde{K}_4 K_- = \operatorname{diag}\{\Lambda_-, e_{4m-2}\}\widetilde{K}_5, \tag{23}$$

$$\widetilde{K}_5 = K_5 K_+, \tag{24}$$

where K_5 is the operator defined by (19).

It follows from (23) that

$$\dim\ker \widetilde{K}_4 \leqslant \dim\ker \widetilde{K}_5 + \dim\operatorname{coker} K_-, \tag{25}$$

and from (24)

$$\dim\ker \widetilde{K}_5 \leqslant \dim\ker K_5. \tag{26}$$

It is known that (see [26])[2]

$$\dim\operatorname{coker} K_- = -2\varkappa_1^- - 2\varkappa_2^-. \tag{27}$$

Putting together (22), (25), (26), and (27) we obtain

$$\dim\ker K_4 \leqslant \dim\ker K_5 - 2\varkappa_1^- - 2\varkappa_2^-.$$

With Proposition 2.4 we are done. $\square$

Thus, it remains to estimate $\dim\ker K_5$. As usual, let $\sigma(\xi)$ and $\|\xi\|_2$ denote the spectrum and the spectral norm of a matrix $\xi \in \mathbb{C}^{n\times n}$, respectively. Now we will make use of some results from [18]; recall that $\tau_{1,2}$ are the fixed points (4) of the shift α defined by (3).

LEMMA 2.8. [18] *For any continuous matrix function $h \in \mathscr{C}^{n\times n}(\mathbb{T})$ such that*

$$\sigma[h(\tau_j)] \subset \mathbb{T}_+, \quad j = 1,2,$$

there exists a polynomial matrix s satisfying the conditions

$$\max_{t\in\mathbb{T}} \left\|s(t)h(t)s^{-1}(\alpha(t))\right\|_2 < 1$$

and

$$P_+ s^{\pm 1} P_+ = s^{\pm 1} P_+.$$

[2] As before (see the footnote in the proof of Proposition 2.3), we have "2" in the left member of the equality because the operator acts in the real space $\widetilde{L}_2^{4m}(\mathbb{T})$.

Let R_h denote the set of all such polynomial matrices s,

$$l_1(s) = \sum_{i=1}^{n} \max_{j=\overline{1,n}} l_{i,j},$$

where $l_{i,j}$ is the degree of the element $s_{i,j}(t)$ of the polynomial matrix s and

$$l(h) = \min_{s\in R_h}\{l_1(s)\}. \tag{28}$$

LEMMA 2.9. [18] *Let* $T = (I - cU)P_+ + P_- : L_2^n(\mathbb{T}) \to L_2^n(\mathbb{T})$, *where the matrix function* c *satisfies the conditions of the Lemma 2.8, and let* $l(c)$ *be the number defined by* (28) *for the matrix function* c. *Then the following estimate holds*

$$\dim\ker T \leqslant l(c).$$

PROPOSITION 2.10. *Let the conditions of Proposition 2.7 be satisfied and* K_5 *the SIO defined by* (19). *Then*

$$\dim\ker K_5 \leqslant 2l(f),$$

where $l(f)$ *is the number defined by* (28) *for the matrix function* f.

Proof. Taking into account Lemmas 2.8 and 2.9, it suffices to show that

$$\sigma[f(\tau_j)] \subset \mathbb{T}_+, \quad j = 1, 2.$$

Without loss of generality we can suppose that the fixed points of the shift lie in the real line, i.e., $\tau_1 = 1$ and $\tau_2 = -1$ (see p. 8 in [2]).

From the factorization $b_0 = b_-\Lambda b_+$ of the matrix function b_0, we have at the fixed point $\tau_1 = 1$ (analogously we proceed at the fixed point $\tau_2 = -1$)

$$b_0(1) = b_-(1)b_+(1),$$

so

$$b_+^{-1}(1) = b_0^{-1}(1)b_-(1).$$

Now recalling (20), we can write

$$f(1) = \mathrm{diag}\{b_-^{-1}(1), e_{4m-2}\}c_1(1)\,\mathrm{diag}\{b_+^{-1}(1), e_{4m-2}\}$$

and so

$$f(1) = \mathrm{diag}\{b_-^{-1}(1), e_{4m-2}\}c_1(1)\,\mathrm{diag}\{b_0^{-1}(1), e_{4m-2}\}\,\mathrm{diag}\{b_-(1), e_{4m-2}\},$$

which means that the matrices $f(1)$ and $c_1(1)\,\mathrm{diag}\{b_0^{-1}(1), e_{4m-2}\}$ are similar.

From here, doing exactly as in the proof of Proposition 2.6 in [19], we show that all the eigenvalues of the matrix

$$c_1(1)\,\mathrm{diag}\{b_0^{-1}(1), e_{4m-2}\} = \begin{pmatrix} b_1(1)b_0^{-1}(1) & b_2(1) & b_3(1) & \cdots & b_{2m-1}(1) & b_{2m}(1) \\ -b_0^{-1}(1) & 0 & 0 & \cdots & 0 & 0 \\ 0 & -e & 0 & \cdots & 0 & 0 \\ 0 & 0 & -e & \ddots & \vdots & \vdots \\ \vdots & & & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & -e & 0 \end{pmatrix},$$

are equal to 0.

Thus

$$\sigma[f(1)] = \{0\}. \quad \square$$

Finally, Propositions 2.7 and 2.10 allow us to establish our main result on the estimate of the dimension of the kernel of K.

THEOREM 2.11. *Let K be the SIO with non-Carleman shift and conjugation defined by (7). Let a be an invertible continuous function on $\mathbb{T}$, $\varkappa = \mathrm{ind}\, a$, and k be the number defined by (15); let f be the matrix function defined by (20), and $l(f)$ be the number defined by (28) for the matrix function f. Then the following estimate holds*

$$\dim\ker K \leqslant l(f) + \max(\varkappa - k, 0) + \max(\varkappa + k, 0) + 1. \tag{29}$$

REMARK. By Proposition 2.6, the partial indices of the matrix function b_0 are $\varkappa_1 = -\varkappa + k$ and $\varkappa_2 = -\varkappa - k$. Therefore the estimate (29) can be written as

$$\dim\ker K \leqslant l(f) + \max(-\varkappa_1, 0) + \max(-\varkappa_2, 0) + 1. \tag{30}$$

The next result shows that, if the matrix function b_0 has negative partial indices then the estimates (29) does not depends (directly) on the value of k.

COROLLARY 2.12. *If the partial indice $\varkappa_1$ of the matrix function b_0 is negative, then*

$$\dim\ker K \leqslant l(f) + 2\varkappa + 1. \tag{31}$$

Proof. Since $\varkappa_1 < 0$, then $\varkappa > k$. So,

$$\max(\varkappa - k, 0) = \varkappa - k \quad \text{and} \quad \max(\varkappa + k, 0) = \varkappa + k. \quad \square$$

The estimate (29) depends on the integer constants $l(f)$, $\varkappa$, and k. In Section 3 we will see that some analytical algorithms implemented with *Mathematica* can be used to determine the constants $\varkappa$ and k. In addition, it is important to note the possibility of the constants to be equal zero, only one, or more than one simultaneously. The following results and examples in Section 3 illustrate these possibilities.

From (15) we can derive the following result:

COROLLARY 2.13. *If* $1 \notin \sigma(H^*_{\overline{u_-}} H_{\overline{u_-}})$, *then*

$$\dim\ker K \leqslant l(f) + 2\max(\varkappa, 0) + 1. \tag{32}$$

Proof. From (15) we get that $k = 0$. So,

$$\max(\varkappa - k, 0) = \max(\varkappa + k, 0) = \max(\varkappa, 0). \quad \square$$

Due to the possibility of the constant $\varkappa$ to be equal zero, independently of the value of the constant k we get the following result:

COROLLARY 2.14. *If the Cauchy index of the function a is equal to zero, then*

$$\dim\ker K \leqslant l(f) + k + 1. \tag{33}$$

Using the factorization concept we get

COROLLARY 2.15. *If a and g admits right canonical generalized factorizations, then*

$$\dim\ker K \leqslant l(f) + 1. \tag{34}$$

Obviously, the constant $l(f)$ can be equal to zero. Using Lemma 2.8 and formula (28), we obtain the following

COROLLARY 2.16. *If* $\max_{t\in\mathbb{T}} \left\| s(t) f(t) s^{-1}(\alpha(t)) \right\|_2 < 1$ *for a constant matrix s, then*

$$\dim\ker K \leqslant \max(\varkappa - k, 0) + \max(\varkappa + k, 0) + 1. \tag{35}$$

In the following subsection we consider the operator K with coefficients a_j, $j = \overline{0,m}$ in a more restricted class of continuous functions on $\mathbb{T}$ and some results for which the constant $l(f)$ is equal to zero are obtained.

2.2. The case $H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T})$

We are considering the SIO with non-Carleman shift and conjugation defined by (7), with $a^{\pm 1} \in \mathscr{C}(\mathbb{T})$.

It is easy to see that for the case when the function a_0 is a continuous, bounded and analytic function in $\mathbb{T}_+$, the selfadjoint operator $H^*_{\overline{u_-}} H_{\overline{u_-}} - I$ has a trivial kernel. Thus, according to the Proposition 2.5 and the Theorem 2.11, we get the following result

COROLLARY 2.17. *If* $a_0 \in H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T})$, *then*

$$\dim\ker K \leqslant l(f) + 2\max(\varkappa, 0) + 1. \tag{36}$$

Let us consider now the operator (7),

$$K = P_+ + (aI + AC)P_-, \tag{37}$$

with $a_j \in H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T}),\ j = \overline{1,m}$.

Let $N_1 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ be the invertible operator

$$N_1 = \begin{pmatrix} I & A - a_0 I \\ 0 & -I \end{pmatrix} P_+ + \begin{pmatrix} -I & 0 \\ \widetilde{A} - t^{-1}\overline{a_0} I & I \end{pmatrix} P_-.$$

Recall that $A = \sum\limits_{j=0}^{m} a_j U^j$ and $\widetilde{A} = \sum\limits_{j=0}^{m} t^{-1}\overline{a_j} U^j$.

We define the operator $T_1 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ by

$$T_1 = K_1 N_1,$$

where K_1 is the operator defined in Proposition 2.1. It is easily seen that

$$T_1 = \begin{pmatrix} 1 & -a_0 \\ 0 & -t^{-1}\overline{a} \end{pmatrix} P_+ + \begin{pmatrix} -a & 0 \\ -t^{-1}\overline{a_0} & 1 \end{pmatrix} P_-.$$

Therefore, according to Proposition 2.1, we have

$$\dim\ker K = \frac{1}{2} \dim\ker T_1. \tag{38}$$

Then we have

$$\begin{aligned} T_1 &= \begin{pmatrix} -a & 0 \\ -t^{-1}\overline{a_0} & 1 \end{pmatrix} \left[\begin{pmatrix} -a & 0 \\ -t^{-1}\overline{a_0} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & -a_0 \\ 0 & -t^{-1}\overline{a} \end{pmatrix} P_+ + P_- \right] \\ &= \begin{pmatrix} -a & 0 \\ -t^{-1}\overline{a_0} & 1 \end{pmatrix} \left[\begin{pmatrix} -a^{-1} & a^{-1}a_0 \\ -t^{-1}a^{-1}\overline{a_0} & t^{-1}(a^{-1}|a_0|^2 - \overline{a}) \end{pmatrix} P_+ + P_- \right]; \end{aligned}$$

therefore

$$\dim\ker T_1 = \dim\ker T_2, \tag{39}$$

where

$$T_2 = \begin{pmatrix} -a^{-1} & a^{-1}a_0 \\ -t^{-1}a^{-1}\overline{a_0} & t^{-1}(a^{-1}|a_0|^2 - \overline{a}) \end{pmatrix} P_+ + P_-.$$

The following results take place

PROPOSITION 2.18. *Let $T_3 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ be the SIO with shift*

$$T_3 = b_{00} P_+ + P_-, \tag{40}$$

where

$$b_{00} = \begin{pmatrix} -a^{-1} & a^{-1}a_0 \\ -a^{-1}\overline{a_0} & a^{-1}|a_0|^2 - \overline{a} \end{pmatrix}.$$

Then

$$\dim\ker K \leqslant \frac{1}{2}(\dim\ker T_3 + 2).$$

Proof. Proceeding analogously to the proof of Proposition 2.3, let

$$d = \operatorname{diag}\{1,t\} \quad and \quad d^{-1} = \operatorname{diag}\{1,t^{-1}\}.$$

Since

$$T_2 = d^{-1}\left[b_{00}P_+ + dP_-\right],$$

then

$$\dim\ker T_2 = \dim\ker\left[b_{00}P_+ + dP_-\right] \leqslant \dim\ker T_3 + 2.$$

Taking into account (38) and (39), the result follows. $\square$

PROPOSITION 2.19. *Let $T_3 : \widetilde{L}_2^2(\mathbb{T}) \to \widetilde{L}_2^2(\mathbb{T})$ be the SIO with shift defined by* (40)*; then*

$$\dim\ker T_3 = 2\left[\max(-\varkappa_1,0)+\max(-\varkappa_2,0)\right].$$

Proof. Since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} b_{00} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = b_0,$$

the partial indices of the matrix functions b_{00} and b_0 are the same. $\square$

Thus, with Propositions 2.18 and 2.19, we get the following estimate

PROPOSITION 2.20. *Let K be the SIO with non-Carleman shift and conjugation defined by* (37) *and $\varkappa = \operatorname{ind} a$. Then the following estimate holds*

$$\dim\ker K \leqslant \max(-\varkappa_1,0)+\max(-\varkappa_2,0)+1 \tag{41}$$

COROLLARY 2.21. *Let K be the SIO with non-Carleman shift and conjugation defined by* (37)*, $\varkappa = \operatorname{ind} a$, and $a_0 \in H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T})$. Then the following estimate holds*

$$\dim\ker K \leqslant 2\max(\varkappa,0)+1. \tag{42}$$

REMARK. By Proposition 2.5, the partial indices of the matrix function b_0 are $\varkappa_1 = \varkappa_2 = -\varkappa$. Therefore the estimate (42) can be written as

$$\dim\ker K \leqslant 2\max(-\varkappa_1,0)+1.$$

Due to the possibility of the constant $\varkappa$ to be equal zero, independently of the function a_0 we get:

COROLLARY 2.22. *If the Cauchy index of the function a is equal to zero and $a_j \in H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T})$, $j = \overline{0,m}$, then*

$$\dim\ker K \leqslant 1. \tag{43}$$

3. Symbolic computation on the dimension of the kernel of the operator K

In this section we show how symbolic computation can be used to explore the dimension of the kernel of the operator (7). The rational case is considered in Subsection 3.1. The inner-outer factorization concept is considered in Subsection 3.2 to obtain particular results on the estimates of the dimension of the kernel of operator K. The analytical algorithms described in Subsection 3.1 and Subsection 3.2 allows us to design a new algorithm to estimate the dimension of the kernel of operator K.

3.1. The case $\mathscr{R}(\mathbb{T})$

In this subsection we consider the rational case. This is an important case that shows the importance of the symbolic computation on the estimation of the dimension of the kernel of operator K since if $a^{\pm 1} \in \mathscr{R}(\mathbb{T})$, the constant $\varkappa$ can always be explicitly determined using the [ARFact-Scalar] algorithm (see [9]). Concerning the matrix functions related with the estimate (29), another analytical algorithm can be considered. If $g \in \mathscr{R}^{2\times 2}(\mathbb{T})$, then the [ARFact-Matrix] algorithm (see [6]) can be used to compute constant k. In addition, if $b_0 \in \mathscr{R}^{2\times 2}(\mathbb{T})$, then its right partial indices $\varkappa_1$ and $\varkappa_2$ can also be determined by [ARFact-Matrix] algorithm.

3.1.1. The [ARFact-Scalar] algorithm

The [ARFact-Scalar] algorithm was implemented with *Mathematica* and computes explicit factorizations for any factorable rational function defined on the unit circle. The [ARFact-Scalar] source code was used in the design of spectral and kernel algorithms. [6] and [9] contain several nontrivial examples computed with this algorithm.

The [ARFact-Scalar] algorithm has a rather simple structure and, in fact, it consists essentially in the computation of factorization (1) with factors (2), using formulæ

$$c_+(t) = \lambda\, \frac{\prod_{i=1}^{z_-}(t - z_i^-)}{\prod_{j=1}^{p_-}(t - p_j^-)}, \quad c_-(t) = \frac{\prod_{i=1}^{z_+}(1 - t^{-1} z_i^+)}{\prod_{j=1}^{p_+}(1 - t^{-1} p_j^+)}, \quad \kappa = z_+ - p_+,\ \lambda \in \mathbb{C},$$

where $z_i^\pm$ $(i = 1, \cdots, z_\pm)$ denote all the zeros of the rational function c in $\mathbb{T}_\pm$ (with regard to their multiplicities) and $p_j^\pm$ $(j = 1, \cdots, p_\pm)$ denote all the poles of c in $\mathbb{T}_\pm$ (with regard to their multiplicities).

The symbolic computation capabilities of *Mathematica*, and the *pretty-print* functionality[3], allow the [ARFact-Scalar] code to be very simple and syntactically similar to its analytical counterpart.

We note that, since the zeros and poles of $c(t)$ are a crucial information for this calculation technique, the success of the [ARFact-Scalar] algorithm depends on the possibility of finding those zeros and poles by solving polynomial equations. This can be a

[3] The pretty-print functionality allows to write on the computer screen scientific formulas in the traditional format, as if one was using pencil and paper.

serious limitation when working with polynomials of the fifth degree or higher. However, even in this case, thanks to the symbolic and numeric capabilities of *Mathematica*, it is still possible to obtain an explicit, and for all purposes exact, rational factorization.

Mathematica uses `Root` objects to represent solutions of algebraic equations in one variable, when it is impossible to find explicit formulas for these solutions. The `Root` object is not a mere denoting symbol but rather an expression that can be symbolically manipulated and numerically evaluated. In particular, it is still possible to know if any given `Root` lies in the unit circle, in the interior or in the exterior of the unit circle, which is all the information the [ARFact-Scalar] algorithm needs to construct the factors $c_{\pm}(t)$ and to compute the factorization index. Note that the factorization index is always obtained explicitly. So, in the rational case, the constant $\varkappa$ that appears in the estimate of the Theorem 2.11 can be always determined by the [ARFact-Scalar] algorithm.

The implementation of the [ARFact-Scalar] algorithm potentiates the future design of algorithms dedicated to specific domains of application and, in particular, algorithms to compute the kernel of singular integral integrals.

EXAMPLE. Let K be the SIO with non-Carleman shift and conjugation defined by (7). Let

$$a(t) = (t+\frac{5}{2})(t^5+2t^4+3)^{-1}.$$

In this case, using the [ARFact-Scalar] algorithm, we get that $\varkappa = \operatorname{ind} a = 0$ and the estimate

$$\dim\ker K \leqslant l(f)+k+1$$

holds.

Taking into account Corollary 2.17 and Corollary 2.22 two particular cases must be considered:

- If $a_0 \in H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T})$, then

$$\dim\ker K \leqslant l(f)+1.$$

- If $a_j \in H_\infty(\mathbb{T}) \cap \mathscr{C}(\mathbb{T}),\ j = \overline{0,m}$, then

$$\dim\ker K \leqslant 1.$$

REMARK. This example shows that, in spite of the impossibility of computing, in an explicit way, the roots of the high degree polynomial t^5+2t^4+3, the factorization index of the rational function $a(t)$ can always be obtained explicitly by the [ARFact-Scalar] algorithm. Here *Mathematica* uses the objects $\mathbf{Root}[\#1^5+2\#1^4+3\&, i]$ to represent the solutions of t^5+2t^4+3. As we noted before it is possible to know if any root lies in $\mathbb{T}$, in $\mathbb{T}_+$, or in $\mathbb{T}_-$:

In[]:= $\mathbf{Table}[\mathbf{ABS}[\mathbf{Root}[\#1^5+2\#1^4+3\&, i]] < 1, \{i,5\}]$

Out[]:= {False,False,False,False,False}

3.1.2. The [ARFact-Matrix] algorithm

The [ARFact-Matrix] algorithm was implemented with *Mathematica* and computes explicit factorizations for non-singular rational matrix functions defined on the unit circle (see [6]). Parts of the code of the [ARFact-Matrix] algorithm were used in the design of spectral and kernel algorithms (see [9]).

Similar to the scalar case, the success of the [ARFact-Matrix] algorithm depends on the possibility of finding solutions of polynomial equations. However, due to the complexity of the matrix case, it is not as feasible as before to use the `Root` objects to obtain an explicit matrix factorization when working with polynomials of a high degree. In fact, one crucial step of this algorithm is finding the zeros of the determinant of a matrix function. This means that the dimension of the matrix function is also a limiting factor, even when its entries are rational functions with low degree polynomials.

Some examples computed with this algorithm are presented in [6].

Since the estimate of the kernel of the SIO (7) can be obtained by computing (directly) the right partial indices $\varkappa_1$ and $\varkappa_2$ of the matrix function b_0 (if b_0 belongs to a class for which a factorization method is known), the [ARFact-Matrix] algorithm can be very helpful in the rational case.

EXAMPLE. Let K be the SIO with non-Carleman shift and conjugation defined by (7). Let

$$a(t) \equiv 1 \text{ and } a_0(t) = \frac{1-2t}{t}.$$

In this case, using the [ARFact-Matrix] algorithm, we get the factorization (14), where

$$b_-(t) = \begin{pmatrix} 1 & -\frac{1}{2} \\ 0 & -1 \end{pmatrix}, \quad \Lambda(t) = \operatorname{diag}\{t, t^{-1}\},$$

and

$$b_+(t) = \begin{pmatrix} \frac{1}{2} & -1 \\ (t-2)t & -2(t-1)^2 \end{pmatrix},$$

that is, the right partial indices of the matrix function b_0 are $\varkappa_1 = 1$ and $\varkappa_2 = -1$. Through inequality (29) the following estimate

$$\dim\ker K \leqslant l(f) + 2$$

holds.

3.2. The case $H_{r,\theta}(\mathbb{T})$

In this subsection the inner-outer factorization concept is considered to obtain particular results on the estimates of the dimension of the kernel of operator K.

This is another important case to illustrate the importance of the use of symbolic computation on the study of this kind of problems since some of our analytical algorithms, concerning these kind of factorization, and implemented with *Mathematica* allows us to design a new algorithm to estimate the dimension of the kernel of operator

K. In fact, if $|a|$ is a constant function and $\overline{a_0} \in H_{r,\theta}(\mathbb{T})$, then the [AFact] algorithm (see [7]) can be used to determine the right partial indices $\varkappa_1$ and $\varkappa_2$ of the matrix function b_0 (given by formula (11)). On the other hand, if the function $\overline{u_-}$ that appears in the matrix function g, defined by (16), belongs to $H_{r,\theta}(\mathbb{T})$, then the constant k can be determined by the [AKer-$H\varphi * H\varphi$] algorithm (see [9]). So, the algorithms [AFact] and [AKer-$H\varphi * H\varphi$] can be used to obtain concrete estimates for the dimension of the kernel of operator (7).

3.2.1. The [AFact] algorithm

In general, it is possible to show, that the study of the factorability of essentially bounded Hermitian second-order matrix functions with negative determinant and definite diagonal elements, can be reduced to the study of matrix functions of the form (17), for $u \in L_\infty(\mathbb{T})$ (see, for instance, [7] and [23]). In addition, a canonical generalized factorization of matrix functions of the type (17) has applications in several scientific research areas (see, for instance, [1], [8], [10], [20], and [22]).

In [4], [5], and [7] we present necessary and sufficient conditions for the existence of a canonical generalized factorization $A_\gamma(u) = A_\gamma^+ A_\gamma^-$. In addition, explicit formulas for the factors A_γ^+ and A_γ^- are presented. In [7] it is presented an analytical algorithm to study the factorability of matrix functions of the class (17), with $u \in L_\infty(\mathbb{T})$. For the case when $\overline{u_-} \in H_{r,\theta}$ the [AFact] algorithm computes a left generalized factorization of matrix functions of type (17). For instance, if $-\gamma \notin \sigma\left(H^*_{\overline{u_-}} H_{\overline{u_-}}\right)$, the [AFact] algorithm computes a canonical generalized factorization of $A_\gamma(\overline{u_-})$ by solving singular integral equations of type $\left(H^*_{\overline{u_-}} H_{\overline{u_-}} + \gamma I\right) \omega_+(t) = g_+(t)$ for $g_+(t) \equiv 1$ and $g_+(t) = b$.

The programming features and the built-in functions of the CAS *Mathematica* were used to compute the extensive symbolic calculations demanded by the [AFact] algorithm. As a final result, we obtained a *Mathematica* notebook that automate the factorization process as a whole. Due to its innovative character, the implementation of [AFact] potentiates the future design of algorithms dedicated to specific domains of application.

EXAMPLE. Let K be the SIO with non-Carleman shift and conjugation considered in the example of the subsubsection 3.1.2. Since b_0 is a matrix function of the class (17) the [AFact] algorithm can be used to determine explicit factorizations (14), where

$$b_-(t) = \begin{pmatrix} 1 + \frac{\alpha}{t^2} & -\frac{1}{2} \\ \frac{2\alpha}{t^2} & -1 \end{pmatrix}, \quad \Lambda(t) = \mathrm{diag}\{t, t^{-1}\},$$

and

$$b_+(t) = \begin{pmatrix} \frac{1}{2} & -1 \\ (t-2)t + \alpha & -2[(t-1)^2 + \alpha] \end{pmatrix}.$$

and to obtain, obviously, the same right partial indices of the matrix function b_0 and the same estimate

$$\dim \ker K \leqslant l(f) + 2$$

holds.

3.2.2. The [AKer-$H\varphi^*H\varphi$] algorithm

In [9] it is showed how *Mathematica* can be used to explore the spectra and the kernel of some classes of SIOs, defined on the unit circle, related with Hankel operators. The [AKer-$H\varphi^*H\varphi$] algorithm computes the kernel of the operators of the type $H^*_{\overline{u_-}}H_{\overline{u_-}} + \gamma I$, where γ is a non-null complex constant and $\overline{u_-} \in H_{r,\theta}$, and can be applied to particular functions θ or it can compute the closed form of the kernel as a general expression in θ.

EXAMPLE. Let K be the SIO with non-Carleman shift and conjugation defined by (7). Let us consider continuous functions a and a_0 such that

$$u(t) = 3\frac{\overline{\theta(t)}t}{1-2t} + \frac{3}{2}\overline{\theta(0)}.$$

In this case, $P_-u = u$, we get

$$\overline{u_-}(t) = 3\frac{\theta(t)}{t-2} + \frac{3}{2}\theta(0).$$

Since the function $\overline{u_-} - \frac{3}{2}\theta(0) \in H_{r,\theta}$ the [AKer-$H\varphi^*H\varphi$] algorithm can be used[4], considering the inner-outer factorization

$$\overline{u_-} - \tfrac{3}{2}\theta(0) = r\theta,$$

for $r(t) = 3(t-2)^{-1}$ and θ an arbitrary inner function, to determine the constant k on the estimate (29).

Taking into account that the [AKer-$H\varphi^*H\varphi$] algorithm can be applied to a general function θ, two particular cases[5] must be considered:

- If $\theta'(-1) = 0$, then $k = 1$ and
$$\dim\ker K \leqslant l(f) + \max(\varkappa - 1, 0) + \max(\varkappa + 1, 0) + 1.$$
- If $\theta'(-1) \neq 0$ and $\theta(-1) - 3\theta'(-1) \neq 0$, then $k = 0$ and
$$\dim\ker K \leqslant l(f) + 2\max(\varkappa, 0) + 1.$$

3.2.3. Particular cases

This subsubsection consider some particular cases where the inner-outer factorization concept is considered to easily obtain particular, and interesting, results on the estimates of the dimension of the kernel of operator (7).

Let us consider the SIO with non-Carleman shift and conjugation defined by (7).

Case 1. Let θ_1 be an arbitrary inner function and r_1 a rational function without poles on the unit circle (that is, $r_1 \in \mathscr{R}(\mathbb{T})$). For a non-negative integer constant q such that $r_1 t^{-q} \in \mathscr{R}_-(\mathbb{T})$ it can be considered an inner-outer factorization of the rational function $\overline{r_1 t^{-q}}$ to prove the following result:

[4]Since $H\varphi^*H\varphi = H(\varphi+\lambda)^*H(\varphi+\lambda)$, where λ is a complex constant.

[5]The conditions are provided explicitly in the output of the [AKer-$H\varphi^*H\varphi$] algorithm. They arise from the construction of an homogeneous linear system not uniquely solvable.

PROPOSITION 3.1. *Let* $a_0 \in \mathscr{C}(\mathbb{T})$ *and* $a_- t^{\varkappa} a_+$ *be a generalized factorization of* $a \in \mathscr{C}(\mathbb{T})$*. If* $p \in \mathbb{N}_0,\ p \geqslant q:\ P_- a_0 = a_+ \overline{a_-}\,\overline{\theta_1} t^{-p} r_1$*, then* $\overline{u_-} \in H_{r,\theta}(\mathbb{T})$*.*

REMARK. In the conditions of Proposition 3.1, $k = \dim\ker(H^*_{\overline{u_-}} H_{\overline{u_-}} - I)$ can be determined by the [AKer-$H\varphi^* H\varphi$] algorithm, for $\gamma = -1$. In addition, when θ_1 is a rational function, the [ARFact-Matrix] algorithm can be used to determine k as the non-negative right partial indice of the matrix function g.

EXAMPLE. Let K be the SIO with non-Carleman shift and conjugation defined by (7). Let $a_- t^{\varkappa} a_+$ be a generalized factorization of $a \in \mathscr{C}(\mathbb{T})$. Let a_0 be a continuous function satisfying the condition $(a_0)_- = a_+ \overline{a_-}\,\overline{\theta_1} t^{-p}(1-2t)$, for some $p \in \mathbb{N}$.

Using the [AKer-$H\varphi^* H\varphi$] algorithm, for any inner function $\theta = \theta_1 t^{p-1}$, we obtain:

- If $\theta'(1) = 0$, then $\ker(H^*_{\overline{u_-}} H_{\overline{u_-}} - I) = \operatorname{span}\left\{ \dfrac{\theta(1) + (t-2)t\theta(t)}{(t-1)^2\theta(1)} \right\}$
- If $\theta'(1) \neq 0$, then the operator $H^*_{\overline{u_-}} H_{\overline{u_-}} - I$ has a trivial kernel

So, for any SIO with non-Carleman shift and conjugation K with coefficient a_0 satisfying the conditions of Proposition 3.1 we get, for every inner function θ_1, the estimate

- If $\theta'(1) = 0$, then

$$\dim\ker K \leqslant l(f) + \max(\varkappa - 1, 0) + \max(\varkappa + 1, 0) + 1.$$

- If $\theta'(1) \neq 0$, then

$$\dim\ker K \leqslant l(f) + 2\max(\varkappa, 0) + 1.$$

Case 2. Let θ_1 be an arbitrary inner function and r_1 a rational function without poles on the unit circle and in the exterior of the unit circle. Considering an inner-outer factorization of the rational function $\overline{r_1}$ the next result can be proved.

PROPOSITION 3.2. *Let* $a_0 \in \mathscr{C}(\mathbb{T})$ *and* $a_- t^{\varkappa} a_+$ *be a generalized factorization of* $a \in \mathscr{C}(\mathbb{T})$*. If* $p \in \mathbb{N}_0$*:* $P_- a_0 = a_+ \overline{a_-}\,\overline{\theta_1} t^{-p} r_1$*, then* $\overline{u_-} \in H_{r,\theta}(\mathbb{T})$*.*

REMARK. In the conditions of Proposition 3.2, $k = \dim\ker(H^*_{\overline{u_-}} H_{\overline{u_-}} - I)$ can be determined by the [AKer-$H\varphi^* H\varphi$] algorithm, for $\gamma = -1$. In addition, when θ_1 is a rational function, the [ARFact-Matrix] algorithm can be used to determine k as the non-negative right partial index of the matrix function g.

EXAMPLE. Let K be the SIO with non-Carleman shift and conjugation defined by (7). Let $a_- t^{\varkappa} a_+$ be a generalized factorization of $a \in \mathscr{C}(\mathbb{T})$. Let a_0 be a continuous function such that $(a_0)_- = a_+ \overline{a_-}\,\overline{\theta_1} t^{-p}(1-2t)^{-1}$, for some $p \in \mathbb{N}_0$.

Using the [AKer-$H\varphi^* H\varphi$] algorithm, for any inner function $\theta = \theta_1 t^{p+1}$, we obtain that $H^*_{\overline{u_-}} H_{\overline{u_-}} - I$ has a trivial kernel.

So, for any SIO with non-Carleman shift and conjugation K with coefficient a_0 satisfying the conditions of Proposition 3.2 we get, for every inner function θ_1, the estimate

$$\dim\ker K \leqslant l(f) + 2\max(\varkappa, 0) + 1\,.$$

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