HOCHSCHILD COHOMOLOGY OF TYPE II$_1$ VON NEUMANN ALGEBRAS WITH PROPERTY Γ

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Abstract. In this paper, Property Γ for a type II$_1$ von Neumann algebra is introduced as a generalization of Murray and von Neumann's Property Γ for a type II$_1$ factor. The main result of this paper is that if a type II$_1$ von Neumann algebra $\mathcal{M}$ with separable predual has Property Γ, then the continuous Hochschild cohomology group $H^k(\mathcal{M}, \mathcal{M})$ vanishes for every $k \geqslant 2$. This gives a generalization of an earlier result in [4].

1. Introduction

The continuous Hochschild cohomology of von Neumann algebras was initialized by Johnson, Kadison and Ringrose in [14], [15], [11], where it was conjectured that the k-th continuous Hochschild cohomology group $H^k(\mathcal{M}, \mathcal{M})$ is trivial for any von Neumann algebra $\mathcal{M}$, $k \geqslant 1$. In the case $k = 1$, this conjecture, which is equivalent to the problem of whether a derivation of a von Neumann algebra into itself is inner, had been solved by Kadison and Sakai independently in [13], [21]. In the following we focus on the case when $k \geqslant 2$. In [11], it was shown that $H^k(\mathcal{M}, \mathcal{M}) = 0$ for $k \geqslant 2$ if $\mathcal{M}$ is an injective von Neumann algebra. It follows that if $\mathcal{M}$ is a type I von Neumann algebra, then $H^k(\mathcal{M}, \mathcal{M}) = 0$ for $k \geqslant 2$.

Significant progress was made after the introduction of completely bounded Hochschild cohomology groups for von Neumann algebras ([2], [3], [4], [5], [6], [7], [18], [19], [23], [24]). It was shown in [6], [8] (see also [22]) that the completely bounded Hochschild cohomology group $H^k_{cb}(\mathcal{M}, \mathcal{M}) = 0$ for $k \geqslant 2$. As a consequence of results in [2], if $\mathcal{M}$ is a type II$_\infty$ or type III von Neumann algebra, then $H^k(\mathcal{M}, \mathcal{M}) = 0$ for $k \geqslant 2$. In the case that $\mathcal{M}$ is a type II$_1$ von Neumann algebra, many results as listed below have also been obtained. (We refer to a wonderful book [22] by Sinclair and Smith for a survey of Hochschild cohomology theory for von Neumann algebras and proofs of most of the following results.)

   (i) $H^k(\mathcal{M}, \mathcal{M}) = 0$ for $k \geqslant 2$ if the type II$_1$ central summand in the type decomposition $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_{c_1} \oplus \mathcal{M}_{c_\infty} \oplus \mathcal{M}_\infty$ of the von Neumann algebra $\mathcal{M}$ satisfies $\mathcal{M}_{c_1} \otimes \mathcal{R} \cong \mathcal{M}_{c_1}$, where $\mathcal{R}$ is the hyperfinite type II$_1$ factor ([2]).

(ii) $H^k(\mathscr{M},\mathscr{M}) = 0$ for $k \geqslant 2$ if $\mathscr{M}$ is a type II_1 von Neumann algebra with a Cartan subalgebra and separable predual ([3], [18], [23], [24]); it was shown later in [1] that $H^k(\mathscr{M},\mathscr{M}) = 0$ if $\mathscr{M}$ is a type II_1 factor with a Cartan masa.

(iii) $H^2(\mathscr{M},\mathscr{M}) = 0$ for $k \geqslant 2$ if $\mathscr{M}$ is a type II_1 factor satisfying various technical properties related to its action on $L^2(\mathscr{M},tr)$ ([18]).

(iv) $H^k(\mathscr{M},\mathscr{M}) = 0$ for $k \geqslant 2$ if $\mathscr{M}$ is a type II_1 factor with Property Γ ([4]).

(v) $H^2(\mathscr{M}_1 \otimes \mathscr{M}_2, \mathscr{M}_1 \otimes \mathscr{M}_2) = 0$ if both $\mathscr{M}_1$ and $\mathscr{M}_2$ are type II_1 von Neumann algebras ([19]).

A motivation of this paper is to generalize the listed result (iv) in [4] for type II_1 factors with Property Γ to general type II_1 von Neumann algebras with certain properties. Recall Murray and von Neumann's Property Γ for type II_1 factors as follows. *Suppose $\mathscr{A}$ is a type II_1 factor with a trace τ. Let $\|\cdot\|_2$ be the 2-norm on $\mathscr{A}$ given by $\|a\|_2 = \sqrt{\tau(a^*a)}$ for any $a \in \mathscr{A}$. Then $\mathscr{A}$ has Property Γ if, given $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathscr{A}$, there exists a unitary $u \in \mathscr{A}$ such that*

(a) $\tau(u) = 0$;

(b) $\|ua_j - a_j u\|_2 < \varepsilon, \ \forall 1 \leqslant j \leqslant k$.

An equivalent definition of Property Γ for a type II_1 factor $\mathscr{A}$ was given by Dixmier in [9]. *Suppose $\mathscr{A}$ is a type II_1 factor with a trace τ. Let $\|\cdot\|_2$ be the 2-norm on $\mathscr{A}$ given by $\|a\|_2 = \sqrt{\tau(a^*a)}$ for any $a \in \mathscr{A}$. Then $\mathscr{A}$ has Property Γ if, given $n \in \mathbb{N}$, $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathscr{A}$, there exist n orthogonal equivalent projections $\{p_1, p_2, \ldots, p_n\}$ in $\mathscr{A}$ with sum I such that*

$$\|p_i a_j - a_j p_i\|_2 < \varepsilon, \qquad \forall 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

In the paper, we extend Dixmier's equivalent definition of Murray and von Neumann's Property Γ to general von Neumann algebras as follows.

DEFINITION 3.1. Suppose $\mathscr{M}$ is a type II_1 von Neumann algebra with a predual $\mathscr{M}_\sharp$. Suppose that $\sigma(\mathscr{M},\mathscr{M}_\sharp)$ is the weak-$*$ topology on $\mathscr{M}$ induced from $\mathscr{M}_\sharp$. We say that $\mathscr{M}$ has Property $\hat{\Gamma}$ if and only if $\forall a_1, a_2, \ldots, a_k \in \mathscr{M}$ and $\forall n \in \mathbb{N}$, there exist a partially ordered set Λ and a family of projections

$$\{p_{i\lambda} : 1 \leqslant i \leqslant n; \lambda \in \Lambda\} \subseteq \mathscr{M}$$

satisfying

(i) For each $\lambda \in \Lambda$, $\{p_{1\lambda}, p_{2\lambda}, \ldots, p_{n\lambda}\}$ is a family of orthogonal equivalent projections in $\mathscr{M}$ with sum I.

(ii) For each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$,

$$\lim_\lambda (p_{i\lambda} a_j - a_j p_{i\lambda})^*(p_{i\lambda} a_j - a_j p_{i\lambda}) = 0 \qquad \text{in } \sigma(\mathscr{M},\mathscr{M}_\sharp) \text{ topology.}$$

We note that Definition 3.1 coincides with Dixmier's definition (and Murray and von Neumann's definition) when $\mathscr{M}$ is a type II$_1$ factor (see Corollary 3.5). The following theorem is our main result of this paper, which gives a generalization of an earlier result in [4].

THEOREM 6.4. *Suppose $\mathscr{M}$ is a type II$_1$ von Neumann algebra with separable predual. If $\mathscr{M}$ has Property* Γ*, then the Hochschild cohomology group*

$$H^k(\mathscr{M},\mathscr{M}) = 0, \qquad \forall\, k \geqslant 2.$$

The proof of Theorem 6.4 follows the similar line as the one in [4] besides that new tools from direct integral theory for von Neumann algebras need to be developed.

The organization of this paper is as follows. In section 3, we introduce a definition of Property Γ for type II$_1$ von Neumann algebras. In section 4, by applying the technique of direct integrals to $\mathscr{M}$, we will construct a hyperfinite subfactor $\mathscr{R}$ such that the relative commutant of $\mathscr{R}$ is the center of $\mathscr{M}$ and $\mathscr{R}$ satisfies the additional property of containing an asymptotically commuting family of projections for $\mathscr{M}$. In section 5, we will prove a Grothendieck inequality for $\mathscr{R}$-multimodular normal multilinear maps. Then, in section 6, we combine these results obtained in section 4 and section 5 to show that for a type II$_1$ von Neumann algebra $\mathscr{M}$ with separable predual, if $\mathscr{M}$ has Property Γ, then every bounded k-linear $\mathscr{R}$-multimodular separately normal map from $\mathscr{M}^k$ to $\mathscr{M}$ is completely bounded, which implies the triviality of the cohomology group $H^k(\mathscr{M},\mathscr{M})$ by Theorem 3.1.1 and Theorem 4.3.1 in [22].

2. Preliminaries

2.1. Hochschild cohomology

In this subsection, we will recall a definition of continuous Hochschild cohomology groups (see [22]).

Let $\mathscr{M}$ be a von Neumann algebra. We say that a Banach space $\mathscr{X}$ is a Banach $\mathscr{M}$-bimodule if there is a module action of $\mathscr{M}$ on both the left and right of $\mathscr{X}$ satisfying

$$\|m\xi\| \leqslant \|m\|\|\xi\|$$

and

$$\|\xi m\| \leqslant \|\xi\|\|m\|$$

for any $m \in \mathscr{M}, \xi \in \mathscr{X}$.

For each integer $k \geqslant 1$, we denote by $\mathscr{L}^k(\mathscr{M},\mathscr{X})$ the Banach space of k-linear bounded maps $\phi : \mathscr{M}^k \to \mathscr{X}$. For $k = 0$, we define $\mathscr{L}^0(\mathscr{M},\mathscr{X})$ to be $\mathscr{X}$. Then we can define coboundary operators $\partial^k : \mathscr{L}^k(\mathscr{M},\mathscr{X}) \to \mathscr{L}^{k+1}(\mathscr{M},\mathscr{X})$ as follows:

(i) when $k \geqslant 1$, for any $\phi \in \mathscr{L}^k(\mathscr{M}, \mathscr{X}), a_1, a_2, \ldots, a_k \in \mathscr{M}$,

$$\partial^k \phi(a_1, a_2, \ldots, a_{k+1}) = a_1 \phi(a_2, \ldots, a_{k+1})$$
$$+ \sum_{i=1}^{k} (-1)^k \phi(a_1, \ldots, a_{i-1}, a_i a_{i+1}, \ldots, a_{k+1})$$
$$+ (-1)^{k+1} \phi(a_1, \ldots, a_k) a_{k+1}.$$

(ii) when $k = 0$, for any $\xi \in \mathscr{X}, a \in \mathscr{M}$, $\partial^0 \xi(a) = a\xi - \xi a$.

It's easy to check that $\partial^k \partial^{k-1} = 0$ for each $k \geqslant 1$. Thus $Im\partial^{k-1}$ (the space of coboundaries) is contained in $Ker\partial^k$ (the space of cocycles). The continuous Hochschild cohomology groups $H^k(\mathscr{M}, \mathscr{X})$ are then defined to be the quotient vector spaces $Ker\partial^k / Im\partial^{k-1}, k \geqslant 1$.

2.2. Direct integral

The concepts of direct integrals of separable Hilbert spaces and von Neumann algebras acting on separable Hilbert spaces were introduced by von Neumann in [26]. General knowledge on direct integrals can be found in [26], [16]. Here, we list some lemmas which will be needed in this paper.

LEMMA 2.1. ([16]) *Suppose $\mathscr{M}$ is a von Neumann algebra acting on a separable Hilbert space H. Let $\mathscr{Z}$ be the center of $\mathscr{M}$. Then there is a direct integral decomposition of $\mathscr{M}$ relative to $\mathscr{Z}$, i.e., there exists a locally compact complete separable metric measure space (X, μ) such that*

(i) *H is (unitarily equivalent to) the direct integral of $\{H_s : s \in X\}$ over (X, μ), where each H_s is a separable Hilbert space, $s \in X$.*

(ii) *$\mathscr{M}$ is (unitarily equivalent to) the direct integral of $\{\mathscr{M}_s\}$ over (X, μ), where $\mathscr{M}_s$ is a factor in $B(H_s)$ almost everywhere. Also, if $\mathscr{M}$ is of type I_n (n could be infinite), II_1, II_∞ or III, then the components $\mathscr{M}_s$ are, almost everywhere, of type I_n, II_1, II_∞ or III, respectively.*

Moreover, the center $\mathscr{Z}$ is (unitarily equivalent to) the algebra of diagonalizable operators relative to this decomposition.

The following lemma gives a decomposition of a normal state on a direct integral of von Neumann algebras.

LEMMA 2.2. ([16]) *Suppose H is the direct integral of separable Hilbert spaces $\{H_s\}$ over (X, μ), $\mathscr{M}$ is a decomposable von Neumann algebra on H (i.e every operator in $\mathscr{M}$ is decomposable relative to the direct integral decomposition, see Definition 14.1.6 in [16]) and ρ is a normal state on $\mathscr{M}$. There is a positive normal linear functional ρ_s on $\mathscr{M}_s$ for every $s \in X$ such that $\rho(a) = \int_X \rho_s(a(s))d\mu$ for each a in $\mathscr{M}$. If $\mathscr{M}$ contains the algebra $\mathscr{C}$ of diagonalizable operators and $\rho|_{E \mathscr{M} E}$ is faithful or tracial, for some projection E in $\mathscr{M}$, then $\rho_s|_{E(s).\mathscr{M}_s E(s)}$ is, accordingly, faithful or tracial almost everywhere.*

REMARK 2.3. From the proof of Lemma 14.1.19 in [16], we obtain that if $\rho = \sum_{n=1}^{\infty} \omega_{y_n}$ on $\mathscr{M}$, where $\{y_n\}$ is a sequence of vectors in H such that $\sum_{n=1}^{\infty} \|y_n\|^2 = 1$ and ω_y is defined on $\mathscr{M}$ such that $\omega_y(a) = \langle ay, y \rangle$ for any $a \in \mathscr{M}, y \in H$, then ρ_s can be chosen to be $\sum_{n=1}^{\infty} \omega_{y_n(s)}$ for each $s \in X$.

REMARK 2.4. Let $\mathscr{M} = \int_X \oplus M_s d\mu$ and $H = \int_X \oplus H_s d\mu$ be the direct integral decompositions of $(\mathscr{M}, H)$ relative to the center $\mathscr{Z}$ of $\mathscr{M}$. By the argument in section 14.1 in [16], we can find a separable Hilbert space K and a family of unitaries $\{U_s : H_s \to K; s \in X\}$ such that $s \to U_s x(s)$ is measurable (i.e. $s \to \langle U_s x(s), y \rangle$ is measurable for any vector y in K) for every $x \in H$ and $s \to U_s a(s) U_s^*$ is measurable (i.e. $s \to \langle U_s a(s) U_s^* y, z \rangle$ is measurable for any vectors y, z in K) for every decomposable operator $a \in B(H)$.

PROPOSITION 2.5. *Let $\mathscr{M}$ be a type II$_1$ von Neumann algebra acting on a separable Hilbert space H. Let $\mathscr{M} = \int_X \oplus M_s d\mu$ and $H = \int_X \oplus H_s d\mu$ be the direct integral decompositions of $\mathscr{M}$ and H relative to the center $\mathscr{Z}$ of $\mathscr{M}$. Suppose K is a Hilbert space and $\{U_s : H_s \to K\}$ is a family of unitaries as in Remark 2.4. Denote by $\mathscr{B}$ the unit ball of $B(K)$ equipped with the $*$-strong operator topology. Suppose ρ is a faithful normal tracial state on $\mathscr{M}$. Then there is a family of positive, faithful, normal, tracial linear functionals ρ_s on $\mathscr{M}_s$ (almost everywhere) such that*

(a) $\rho(a) = \int_X \rho_s(a(s)) d\mu$ for every $a \in \mathscr{M}$;

(b) for any $a_0 \in \mathscr{M}$, there exists a Borel μ-null subset N of X such that the maps

$$(s, b) \to \rho_s((a_0(s) U_s^* b U_s - U_s^* b U_s a_0(s))^* (a_0(s) U_s^* b U_s - U_s^* b U_s a_0(s)))$$

and

$$(s, b) \to \rho_s(U_s^* b U_s)$$

are Borel measurable from $(X \setminus N) \times \mathscr{B}$ to $\mathbb{C}$.

Proof. If ρ is a faithful, normal, tracial state on $\mathscr{M}$, then there exist a sequence of vectors $\{y_n\} \subset H$ with $\sum_{n=1}^{\infty} \|y_n\|^2 = 1$ such that $\rho = \sum_{n=1}^{\infty} \omega_{y_n}$. Take $\rho_s = \sum_{n=1}^{\infty} \omega_{y_n(s)}$ for every $s \in X$. By Remark 2.3, we know, for $s \in X$ almost everywhere, ρ_s is a positive, faithful, normal, tracial linear functional on $\mathscr{M}_s$ and

$$\rho(a) = \int_X \rho_s(a(s)) d\mu \qquad \forall a \in \mathscr{M}. \tag{2.1}$$

For each $n \in \mathbb{N}$,

$$\omega_{y_n}(a) = \int_X \omega_{y_n(s)}(a(s)) d\mu \qquad \forall a \in \mathscr{M}.$$

Consider the maps $\phi_n, \psi_n : X \times \mathscr{B} \to \mathbb{C}$:

$$\phi_n(s,b) = \omega_{y_n(s)}((a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))^*(a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s)))$$

and

$$\psi_n(s,b) = \omega_{y_n(s)}(U_s^*bU_s).$$

We have

$$\begin{aligned}
\phi_n(s,b) &= \omega_{y_n(s)}((a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))^*(a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))) \\
&= \langle (a_0sU_s^*bU_s - U_s^*bU_sa_0(s))^*(a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))y_n(s), y_n(s) \rangle \\
&= \langle a_0(s)U_s^*bU_sy_n(s), a_0(s)U_s^*bU_sy_n(s) \rangle \\
&\quad - \langle a_0(s)U_s^*bU_sy_n(s), U_s^*bU_sa_0(s)y_n(s) \rangle \\
&\quad - \langle U_s^*bU_sa_0(s)y_n(s), a_0(s)U_s^*bU_sy_n(s) \rangle \\
&\quad + \langle U_s^*bU_sa_0(s)y_n(s), U_s^*bU_sa_0(s)y_n(s) \rangle \\
&= \langle bU_sy_n(s), U_sa_0^*(s)U_s^*U_sa_0(s)U_s^*bU_sy_n(s) \rangle \\
&\quad - \langle U_sa_0(s)U_s^*bU_sy_n(s), bU_sa_0(s)U_s^*U_sy_n(s) \rangle \\
&\quad - \langle bU_sa_0(s)U_s^*U_sy_n(s), U_sa_0(s)U_s^*bU_sy_n(s) \rangle \\
&\quad + \langle U_sa_0(s)U_s^*U_sy_n(s), b^*bU_sa_0(s)U_s^*U_sy_n(s) \rangle.
\end{aligned}$$

By the choice of the family $\{U_s : s \in X\}$ in Remark 2.4, the maps

$$s \to U_sa_0(s)U_s^* \tag{2.2}$$
$$s \to U_sa_0^*(s)U_s^* \tag{2.3}$$

from X to $B(K)$ and

$$s \to U_sy_n(s)$$

from X to K are measurable. Therefore by Lemma 14.3.1 in [16], there is a Borel μ-null subset $N_{n,1}$ of X such that, restricted to $X \setminus N_{n,1}$, the maps (2.2) and (2.3) are all Borel maps. It follows that the map ϕ_n is a Borel map from $(X \setminus N_{n,1}) \times \mathscr{B}$.

Since $\omega_{y_n(s)}(U_s^*bU_s) = \langle bU_sy_n(s), U_sy_n(s) \rangle$, the map $(s,b) \to \omega_{y_n(s)}(U_s^*bU_s)$ from $X \times \mathscr{B}$ to $\mathbb{C}$ is measurable by the choice of $\{U_s : s \in X\}$ in Remark 2.4. By Lemma 14.3.1 in [16], there exists a Borel μ-null subset $N_{n,2}$ of X such that the map

$$\psi_n : (s,b) \to \omega_{y_n(s)}(U_s^*bU_s)$$

is Borel measurable from $(X \setminus N_{n,2}) \times \mathscr{B}$ to $\mathbb{C}$.

By the discussions in the preceding paragraphs, the maps

$$\phi_n : (s,b) \to \omega_{y_n(s)}((a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))^*(a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s)))$$

and

$$\psi_n : (s,b) \to \omega_{y_n(s)}(U_s^*bU_s)$$

are Borel measurable from $(X \setminus N_n) \times \mathcal{B}$ to $\mathbb{C}$, where $N_n = N_{n,1} \cup N_{n,2}$ is a μ-null subset of X.

Since $\rho_s = \sum\limits_{n=1}^{\infty} \omega_{y_n(s)}$, we obtain that

$$(s,b) \rightarrow \rho_s((a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))^*(a_0(s)U_s^*bU_s - U_s^*bU_sa_0(s))) \qquad (2.4)$$

and

$$(s,b) \rightarrow \rho_s(U_s^*bU_s) \qquad (2.5)$$

are Borel measurable from $(X \setminus N) \times \mathcal{B}$ to $\mathbb{C}$, where $N = \cup_{n \in \mathbb{N}} N_n$ is a Borel μ-null subset of X. By (2.1), (2.4), and (2.5), we complete the proof of the proposition.   □

3. Property Γ for type II$_1$ von Neumann algebras

In this section, we will introduce Property Γ for general von Neumann algebras and discuss some of its properties.

Murray and von Neumann's Property Γ for a type II$_1$ factor is defined as follows. *Suppose $\mathscr{A}$ is a type II$_1$ factor with a trace τ. Let $\|\cdot\|_2$ be the 2-norm on $\mathscr{A}$ given by $\|a\|_2 = \sqrt{\tau(a^*a)}$ for any $a \in \mathscr{A}$. Then $\mathscr{A}$ has Property Γ if, given $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathscr{A}$, there exists a unitary $u \in \mathscr{A}$ such that*

(a) $\tau(u) = 0$;

(b) $\|ua_j - a_ju\|_2 < \varepsilon, \ \forall\, 1 \leqslant j \leqslant k$.

An equivalent definition of Property Γ for a type II$_1$ factor $\mathscr{A}$ was given by Dixmier in [9]. *Suppose $\mathscr{A}$ is a type II$_1$ factor with a trace τ. Let $\|\cdot\|_2$ be the 2-norm on $\mathscr{A}$ given by $\|a\|_2 = \sqrt{\tau(a^*a)}$ for any $a \in \mathscr{A}$. Then $\mathscr{A}$ has Property Γ if, given $n \in \mathbb{N}$, $\varepsilon > 0$ and $a_1, a_2, \ldots, a_k \in \mathscr{A}$, there exists n orthogonal equivalent projections $\{p_1, p_2, \ldots, p_n\}$ in $\mathscr{A}$ with sum I such that*

$$\|p_ia_j - a_jp_i\|_2 < \varepsilon, \qquad \forall\, 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k.$$

We introduce a definition of Property $\hat{\Gamma}$ for a type II$_1$ von Neumann algebra as follows.

DEFINITION 3.1. Suppose $\mathscr{M}$ is a type II$_1$ von Neumann algebra with a predual $\mathscr{M}_\sharp$. Suppose that $\sigma(\mathscr{M}, \mathscr{M}_\sharp)$ is the weak-$*$ topology on $\mathscr{M}$ induced from $\mathscr{M}_\sharp$. We say that $\mathscr{M}$ has Property $\hat{\Gamma}$ if and only if $\forall\, a_1, a_2, \ldots, a_k \in \mathscr{M}$ and $\forall\, n \in \mathbb{N}$, there exist a partially ordered set Λ and a family of projections

$$\{p_{i\lambda} : 1 \leqslant i \leqslant n; \lambda \in \Lambda\} \subseteq \mathscr{M}$$

satisfying

(i) For each $\lambda \in \Lambda$, $\{p_{1\lambda}, p_{2\lambda}, \ldots, p_{n\lambda}\}$ is a family of orthogonal equivalent projections in $\mathscr{M}$ with sum I.

(ii) For each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$,

$$\lim_{\lambda}(p_{i\lambda}a_j - a_j p_{i\lambda})^*(p_{i\lambda}a_j - a_j p_{i\lambda}) = 0 \qquad \text{in } \sigma(\mathcal{M}, \mathcal{M}_\sharp) \text{ topology.}$$

REMARK 3.2. Suppose that $\mathcal{M}$ acts on a Hilbert space H. It is well-known that $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$, the weak-$*$ topology, on the unit ball of $\mathcal{M}$ coincides with the weak operator topology on the unit ball of $\mathcal{M}$. (See Theorem 7.4.2 in [16])

Let $\mathcal{M}$ be a countably decomposable type II_1 von Neumann algebra with a faithful normal tracial state ρ. Let $\|\cdot\|_2$ be the 2-norm on $\mathcal{M}$ given by $\|a\|_2 = \sqrt{\rho(a^*a)}$, $\forall a \in \mathcal{M}$. Let $H_\rho = L^2(\mathcal{M}, \rho)$. For each element $a \in \mathcal{M}$, we denote by $\bar{a}$ the corresponding vector in H_ρ. Let π_ρ be the left regular representation of $\mathcal{M}$ on H_ρ induced by $\pi_\rho(a)(\bar{b}) = \overline{ab}, \forall a, b \in \mathcal{M}$. The vector $\bar{I}$ is cyclic for π_ρ, where I is the unit of $\mathcal{M}$. Since that ρ is faithful, we obtain that π_ρ is faithful.

The following result is well-known. For the purpose of completeness, we include a proof here.

LEMMA 3.3. *Let $\mathcal{M}$ be a countably decomposable type II_1 von Neumann algebra acting on a Hilbert space H and ρ be a faithful normal tracial state on $\mathcal{M}$. Let $\|\cdot\|_2$ be the 2-norm on $\mathcal{M}$ given by $\|a\|_2 = \sqrt{\rho(a^*a)}, \forall a \in \mathcal{M}$. Then the topology induced by $\|\cdot\|_2$ coincides with the strong operator topology on bounded subsets of $\mathcal{M}$.*

Proof. We claim that π_ρ is $WOT - WOT$ continuous on bounded subsets of $\mathcal{M}$. To show this, we first suppose $\{a_\lambda\}$ is a net in the unit ball $(\mathcal{M})_1$ of $\mathcal{M}$ such that

$$WOT - \lim_{\lambda} a_\lambda = a \in (\mathcal{M})_1.$$

Then for any $b, c \in \mathcal{M}$,

$$\lim_{\lambda}\langle \pi_\rho(a_\lambda)\pi_\rho(b)\bar{I}, \pi_\rho(c)\bar{I} \rangle = \lim_{\lambda}\rho(c^*a_\lambda b) = \rho(c^*ab) = \langle \pi_\rho(a)\pi_\rho(b)\bar{I}, \pi_\rho(c)\bar{I} \rangle.$$
(3.1)

Since the vector $\bar{I}$ is cyclic for π_ρ, we obtain from (3.1) that

$$\lim_{\lambda}\langle \pi_\rho(a_\lambda)x, y \rangle = \langle \pi_\rho(a)x, y \rangle, \qquad \forall x, y \in H_\rho.$$

Therefore $WOT - \lim_{\lambda} \pi_\rho(a_\lambda) = \pi_\rho(a)$ and π_ρ is $WOT - WOT$ continuous on bounded subsets of $\mathcal{M}$.

Since $(\mathcal{M})_1$ is WOT compact, the unit ball $(\pi_\rho(\mathcal{M}))_1 = \pi_\rho((\mathcal{M})_1)$ is WOT closed. By Kaplansky's Density Theorem, $\pi_\rho(\mathcal{M})$ is a von Neumann algebra. Hence π_ρ from $\mathcal{M}$ onto $\pi_\rho(\mathcal{M})$ is a $*$-isomorphism between von Neumann algebras. By Theorem 7.1.16 in [16], π_ρ is a $*$-homeomorphism from $(\mathcal{M})_1$ onto $(\pi_\rho(\mathcal{M}))_1$ when both are endowed with the strong operator topology.

Now we can prove the result. First suppose $\{b_\lambda\}$ is a net in $(\mathcal{M})_1$ such that $SOT - \lim_{\lambda} b_\lambda = 0$. Then $SOT - \lim_{\lambda} b_\lambda^* b_\lambda = 0$. Since ρ is SOT-continuous on $(\mathcal{M})_1$,

$\lim_\lambda \rho(b_\lambda^* b_\lambda) = 0$, which implies that $\lim_\lambda \|b_\lambda\|_2 = 0$. On the other hand, suppose $\{c_\lambda\}$ is a net in $(\mathcal{M})_1$ such that $\lim_\lambda \|c_\lambda\|_2 = 0$. Then for any $a \in \mathcal{M}$, $\lim_\lambda \|c_\lambda a\|_2 = 0$, and hence

$$\lim_\lambda \langle \pi_\rho(c_\lambda)\bar{a}, \pi_\rho(c_\lambda)\bar{a} \rangle = \lim_i \langle \pi_\rho(a^* c_\lambda^* c_\lambda a)\bar{I}, \bar{I} \rangle$$
$$= \lim_\lambda \rho(a^* c_\lambda^* c_\lambda a)$$
$$= \lim_\lambda \|c_\lambda a\|_2^2$$
$$= 0.$$

Because $\{\bar{a} : a \in \mathcal{M}\}$ is norm dense in H_ρ, it follows that $SOT - \lim_i \pi_\rho(c_\lambda) = 0$ in $B(H_\rho)$. Since π_ρ is a homeomorphism from $(\mathcal{M})_1$ onto $(\pi_\rho(\mathcal{M}))_1$ when both are endowed with the strong operator topology, $SOT - \lim_i c_\lambda = 0$ in $B(H)$. $\square$

COROLLARY 3.4. *Let $\mathcal{M}$ be a countably decomposable type II$_1$ von Neumann algebra with a faithful normal tracial state ρ. Then the following are equivalent:*

(a) *$\mathcal{M}$ has Property $\hat{\Gamma}$ (in the sense of Definition 3.1);*

(b) *Given any $\varepsilon > 0$, any positive integer n and elements $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{M}$ summing to I satisfying*

$$\|p_i a_j - a_j p_i\|_{2,\rho} < \varepsilon, \qquad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k,$$

where the 2-norm $\|\cdot\|_{2,\rho}$ on $\mathcal{M}$ is given by $\|a\|_{2,\rho} = \sqrt{\rho(a^* a)}$ for any $a \in \mathcal{M}$.

(c) *For any faithful normal tracial state $\tilde{\rho}$ on $\mathcal{M}$, any $\varepsilon > 0$, any positive integer n and elements $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{M}$ summing to I satisfying*

$$\|p_i a_j - a_j p_i\|_{2,\tilde{\rho}} < \varepsilon, \qquad 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant k,$$

where the 2-norm $\|\cdot\|_{2,\tilde{\rho}}$ on $\mathcal{M}$ is given by $\|a\|_{2,\tilde{\rho}} = \sqrt{\tilde{\rho}(a^* a)}$ for any $a \in \mathcal{M}$.

Proof. We might assume that $\mathcal{M}$ acts on a Hilbert space H.

(a)$\Rightarrow$(b) It follows directly from Definition 3.1, Remark 3.2 and Lemma 3.3.

(b)$\Rightarrow$(a) Assume that (b) holds. Let $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_k \in \mathcal{M}$. By (b), there exists a family of projections

$$\{p_{ir} : 1 \leqslant i \leqslant n; r \geqslant 1\} \subseteq \mathcal{M}$$

satisfying

(1) for each $r \geqslant 1$, $p_{1r}, p_{2r}, \ldots, p_{nr}$ is a family of orthogonal equivalent projections in $\mathcal{M}$ with sum I.

(2) for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$,

$$\lim_{r \to \infty} \rho((p_{ir}a_j - a_j p_{ir})^*(p_{ir}a_j - a_j p_{ir})) = 0. \qquad (3.2)$$

In order to show that $\mathcal{M}$ has Property $\hat{\Gamma}$, we need only to verify that the family of projections $\{p_{ir} : 1 \leqslant i \leqslant n; r \geqslant 1\}$ satisfies condition (ii) in Definition 3.1. Actually, combining equation (3.2) and Lemma 3.3, we know that, for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$, as $r \to \infty$, $p_{ir}a_j - a_j p_{ir}$ converges to 0 in strong operator topology. Therefore, for each $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant k$, as $r \to \infty$, $(p_{ir}a_j - a_j p_{ir})^*(p_{ir}a_j - a_j p_{ir})$ converges to 0 in weak operator topology and, whence in $\sigma(\mathcal{M}, \mathcal{M}_\sharp)$ by Remark 3.2. Therefore, $\mathcal{M}$ has Property $\hat{\Gamma}$.

(b)$\Leftrightarrow$(c) Suppose ρ_1 and ρ_2 are two faithful normal tracial states on $\mathcal{M}$. By Lemma 3.3, the 2-norms induced by ρ_1 and ρ_2 will give the same topology on bounded subsets of $\mathcal{M}$ (since they both coincide with the strong operator topology on bounded subsets of $\mathcal{M}$). Therefore (b) and (c) are equivalent. $\square$

COROLLARY 3.5. *Suppose that $\mathcal{M}$ is a factor of type II_1 with a tracial state τ. The following are equivalent:*

(i) *$\mathcal{M}$ has Property $\hat{\Gamma}$ in the sense of Definition 3.1.*

(ii) *$\mathcal{M}$ has Property Γ in the sense of Dixmier (equivalently, of Murray and von Neumann).*

Proof. A type II_1 factor is countably decomposable and τ is the unique faithful normal tracial state of $\mathcal{M}$. From Dixmier's Definition of Property Γ for type II_1 factors and Corollary 3.4, we know that $(i) \Leftrightarrow (ii)$. $\square$

REMARK 3.6. Because of Corollary 3.5, from now on we will use Definition 3.1 as a definition of Property Γ for type II_1 von Neumann algebras.

In the rest of the paper, we will only consider von Neumann algebras with separable predual because direct integral theory is only applied to von Neumann algebras with separable predual. Next proposition follows directly from Definition 3.1, Corollary 3.4 and the assumption that $\mathcal{M}$ is a type II_1 von Neumann algebra with separable predual.

PROPOSITION 3.7. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra with separable predual and ρ be a faithful normal tracial state on $\mathcal{M}$. Then $\mathcal{M}$ has Property Γ if and only if for any $n \in \mathbb{N}$, there exists a family of projections $\{p_{ir} : 1 \leqslant i \leqslant n, r \in \mathbb{N}\}$ such that*

(i) *for each $r \in \mathbb{N}$, $\{p_{ir} : 1 \leqslant i \leqslant n\}$ is a set of n equivalent orthogonal projections in $\mathcal{M}$ with sum I;*

(ii) *for each $1 \leqslant i \leqslant n$, $\lim_{r \to \infty} \|p_{ir}a - ap_{ir}\|_2 = 0$ for any $a \in \mathcal{M}$, where $\|\cdot\|_2$ is the 2-norm induced by ρ on $\mathcal{M}$.*

With the help of Proposition 3.7 and Corollary 3.4, we can directly get the next corollary.

COROLLARY 3.8. *Let $\mathscr{M}$ be a type II$_1$ von Neumann algebra with separable predual and ρ a faithful normal tracial state on $\mathscr{M}$. Suppose $\{a_j : j \in \mathbb{N}\}$ is a sequence of elements in $\mathscr{M}$ that generates $\mathscr{M}$ as a von Neumann algebra. Then $\mathscr{M}$ has Property Γ if and only if for any $n \in \mathbb{N}$, there exists a family of projections $\{p_{ir} : 1 \leqslant i \leqslant n, r \in \mathbb{N}\}$ such that*

(i) *for each $r \in \mathbb{N}$, $\{p_{ir} : 1 \leqslant i \leqslant n\}$ is a set of n equivalent orthogonal projections in $\mathscr{M}$ with sum I;*

(ii) *for each $1 \leqslant i \leqslant n$ and $j \in \mathbb{N}$, $\lim_{r \to \infty} \|p_{ir}a_j - a_jp_{ir}\|_2 = 0$, where $\|\cdot\|_2$ is the 2-norm induced by ρ on $\mathscr{M}$.*

EXAMPLE 3.9. Let $\mathscr{A}_1$ be a type II$_1$ factor with separable predual and $\mathscr{A}_2$ a finite von Neumann algebra with separable predual. Suppose $\mathscr{A}_1$ has Property Γ. Then the von Neumann algebra tensor product $\mathscr{A}_1 \otimes \mathscr{A}_2$ is a type II$_1$ von Neumann algebra with separable predual and Property Γ.

REMARK 3.10. Let $\mathscr{M}$ be a von Neumann algebra acting on a separable Hilbert space H and $\mathscr{Z}$ the center of $\mathscr{M}$. Suppose $\mathscr{M} = \int_X \oplus \mathscr{M}_s d\mu$ and $H = \int_X \oplus H_s d\mu$ are the direct integral decompositions of $\mathscr{M}$ and H over (X, μ) relative to $\mathscr{Z}$. Take a countable *SOT* dense self-adjoint subset $\mathscr{F}$ of $\mathscr{M}$ and let $\mathscr{S}$ be the set of all rational $*$-polynomials (i.e, coefficients from $\mathbb{Q} + i\mathbb{Q}$) with variables from $\mathscr{F}$. We observe that $\mathscr{S}$ is countable and *SOT* dense in $\mathscr{M}$. Take $\{a_j : j \in \mathbb{N}\}$ to be the unit ball of $\mathscr{S}$. By Kaplansky's Density Theorem, $\{a_j : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball $(\mathscr{M})_1$. By Definition 14.1.14 and Lemma 14.1.15 in [16], $\{a_j(s) : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball $(\mathscr{M}_s)_1$ for almost every $s \in X$. In the rest of this paper, when we mention a *SOT* dense sequence $\{a_j : j \in \mathbb{N}\}$ of $(\mathscr{M})_1$ (or $(\mathscr{M}')_1$), we always assume that this sequence has been chosen such that $\{a_j(s) : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball $(\mathscr{M}_s)_1$ (or $(\mathscr{M}'_s)_1$) for almost every $s \in X$.

LEMMA 3.11. *Suppose $H = \int_X \oplus H_s d\mu$ is a direct integral of separable Hilbert spaces and $\mathscr{A}$ is a decomposable von Neumann algebra (see definition in [16]) over H such that $\mathscr{A} \cong M_m(\mathbb{C})$, the $m \times m$ matrix algebra over $\mathbb{C}$ for some $m \in \mathbb{N}$. Then $\mathscr{A}_s \cong M_m(\mathbb{C})$ for almost every $s \in X$.*

Proof. Notice that $\mathscr{A}$ is also a finite dimensional C^*-algebra. By Theorem 14.1.13 in [16] and the fact that $\mathscr{A}$ is separable, the map from $\mathscr{A}$ to $\mathscr{A}_s$ given by $a \to a(s)$ is a unital $*$-homomorphism for almost every $s \in X$. Since $\mathscr{A} \cong M_m(\mathbb{C})$, $\mathscr{A}$ is simple. Therefore $\mathscr{A}_s \cong M_m(\mathbb{C})$ for almost every $s \in X$. $\square$

The following Proposition gives a useful characterization of a type II$_1$ von Neumann algebra with Property Γ.

PROPOSITION 3.12. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra acting on a separable Hilbert space H and $\mathcal{Z}$ the center of $\mathcal{M}$. Suppose ρ is a faithful normal tracial state on $\mathcal{M}$ and a 2-norm $\|\cdot\|_2$ on $\mathcal{M}$ is defined by $\|a\|_2 = \sqrt{\rho(a^*a)}$ for any $a \in \mathcal{M}$. Suppose $\mathcal{M} = \int_X \oplus \mathcal{M}_s d\mu$ and $H = \int_X \oplus H_s d\mu$ are the direct integral decompositions of $\mathcal{M}$ and H over (X, μ) relative to $\mathcal{Z}$. Then the following are equivalent:*

(i) *$\mathcal{M}$ has Property Γ.*

(ii) *There exists a positive integer $n_0 \geqslant 2$ such that*

> *for any $\varepsilon > 0$, and any $a_1, a_2, \ldots, a_k \in \mathcal{M}$, there exist orthogonal equivalent projections p_1, p_2, $\ldots$, p_{n_0} in $\mathcal{M}$ summing to I satisfying*

$$\|p_i a_j - a_j p_i\|_2 < \varepsilon, \qquad 1 \leqslant i \leqslant n_0, 1 \leqslant j \leqslant k.$$

(iii) *$\mathcal{M}_s$ is a type II_1 factor with Property Γ for almost every $s \in X$.*

Proof. Since $\mathcal{M}$ is a type II_1 von Neumann algebra, by Lemma 2.1, the component $\mathcal{M}_s$ is a type II_1 factor for almost every $s \in X$. We may assume $\mathcal{M}_s$ is a type II_1 factor with a trace τ_s for each $s \in X$.

By Lemma 2.2, there is a positive faithful normal tracial linear functional ρ_s on $\mathcal{M}_s$ for almost every $s \in X$ such that $\rho(a) = \int_X \rho_s(a(s)) d\mu$ for each a in $\mathcal{M}$. We may assume ρ_s is positive, faithful, normal and tracial for each $s \in X$. Hence for each $s \in X$, ρ_s is a positive scalar multiple of the unique trace τ_s on the type II_1 factor $\mathcal{M}_s$.

Let $\{a_j : j \in \mathbb{N}\}, \{a'_j : j \in \mathbb{N}\}$ be *SOT* dense subsets of the unit balls $\mathcal{M}_1, (\mathcal{M}')_1$ of $\mathcal{M}$ and $\mathcal{M}'$ respectively. By Proposition 14.1.24 in [16], we may assume that $(\mathcal{M}')_s = (\mathcal{M}_s)'$ for every $s \in X$ and we use the notation $\mathcal{M}'_s$ for both. By Remark 3.10, we may assume $\{a_j(s) : j \in \mathbb{N}\}$ and $\{a'_j(s) : j \in \mathbb{N}\}$ are *SOT* dense in $(\mathcal{M}_s)_1$ and $(\mathcal{M}'_s)_1$ for every $s \in X$.

(i)$\Rightarrow$(ii): The result is clear from Corollary 3.4, Corollary 3.5 and Remark 3.6.

(ii)$\Rightarrow$(iii): For this direction, we suppose (ii) holds. Notice that $\mathcal{M}$ acts on a separable Hilbert space, whence $\mathcal{M}$ is countably generated in strong operator topology. By (ii), there exists a sequence of systems of matrix units $\{\{e_{i,j}^{(r)}\}_{i,j=1}^{n_0} \ r \in \mathbb{N}\}$ such that

(A) for each $r \in \mathbb{N}$, we have

$$\sum_{i=1}^{n_0} e_{i,i}^{(r)} = I, \quad (e_{i,j}^{(r)})^* = e_{j,i}^{(r)} \quad \text{and} \quad e_{i,j}^{(r)} e_{k,l}^{(r)} = \delta_{jk} e_{i,l}^{(r)} \text{ for all } 1 \leqslant i, j, k, l \leqslant n_0.$$

(B) for each $1 \leqslant i \leqslant n_0$, $\lim_{r \to \infty} \|e_{i,i}^{(r)} a - a e_{i,i}^{(r)}\|_2 = 0$ for any $a \in \mathcal{M}$.

By condition (A) and Lemma 3.11, there exists a μ-null subset N_0 of X such that, for each $r \in \mathbb{N}$, $\{e_{i,j}^{(r)}(s)\}_{i,j=1}^{n_0}$ is a system of matrix units such that $\sum_{i=1}^{n_0} e_{i,i}^{(r)}(s) = I_s$ (the identity in $\mathcal{M}_s$) in $\mathcal{M}_s$ for each $s \in X \setminus N_0$. In the following, we let

$$p_{i,r} = e_{ii}^{(r)} \qquad \text{for all } 1 \leqslant i \leqslant n_0, r \in \mathbb{N}.$$

Therefore, without loss of generality, we can assume that

(I) $\{p_{1,r}(s), p_{2,r}(s), \ldots, p_{n_0,r}(s)\}$ is a set of n_0 equivalent orthogonal projections with sum I_s in $\mathcal{M}_s$ for every $r \in \mathbb{N}$ and every $s \in X$;

(II) for each $1 \leqslant i \leqslant n_0$,

$$\lim_{r \to \infty} \|p_{i,r}a - ap_{i,r}\|_2 = 0 \qquad \text{for any } a \in \mathcal{M}.$$

In the following we will use a diagonal selection process to produce a subsequence $\{r_m : m \in \mathbb{N}\}$ of $\{r : r \in \mathbb{N}\}$ and a μ-null subset X_0 of X such that

$$\lim_{m \to \infty} \|p_{i,r_m}(s)a_j(s) - a_j(s)p_{i,r_m}(s)\|_{2,s} = 0 \qquad \forall i \in \{1, 2, \ldots, n_0\} \text{ and } \forall s \in X \setminus X_0, \tag{3.3}$$

where the $\|\cdot\|_{2,s}$ is the 2-norm induced by the unique trace τ_s on each $\mathcal{M}_s$.

First, by Assumption (II), for each $i \in \{1, 2, \ldots, n_0\}$,

$$\lim_{r \to \infty} \|p_{i,r}a_1 - a_1 p_{i,r}\|_2$$

$$= \lim_{r \to \infty} \int_X \rho_s((p_{i,r}(s)a_1(s) - a_1(s)p_{i,r}(s))^*(p_{i,r}(s)a_1(s) - a_1(s)p_{i,r}(s)))d\mu = 0.$$

Therefore there exists a μ-null subset Y_1 of X and a subsequence $\{r_{1,m} : m \in \mathbb{N}\}$ of $\{r : r \in \mathbb{N}\}$ such that

$$\lim_{m \to \infty} \rho_s((p_{i,r_{1,m}}(s)a_1(s) - a_1(s)p_{i,r_{1,m}}(s))^*(p_{i,r_{1,m}}(s)a_1(s) - a_1(s)p_{1,r_{1,m}}(s))) = 0$$

for any $s \in X \setminus Y_1$ and any $i \in \{1, 2, \ldots, n_0\}$. Since ρ_s is a positive scalar multiple of the unique trace τ_s on the type II$_1$ factor $\mathcal{M}_s$, we obtain

$$\lim_{m \to \infty} \|p_{i,r_{1,m}}(s)a_1(s) - a_1(s)p_{i,r_{1,m}}(s)\|_{2,s} = 0$$

for any $i \in \{1, 2, \ldots, n\}$ and any $s \in X \setminus Y_1$, where $\|\cdot\|_{2,s}$ is the 2-norm on $\mathcal{M}_s$ induced by τ_s.

Again, there is a subsequence $\{r_{2,m} : m \in \mathbb{N}\}$ of $\{r_{1,m} : m \in \mathbb{N}\}$ and a μ-null subset Y_2 of X such that

$$\lim_{m \to \infty} \|p_{i,r_{2,m}}(s)a_2(s) - a_2(s)p_{i,r_{2,m}}(s)\|_{2,s} = 0$$

for any $i \in \{1, 2, \ldots, n_0\}$ and any $s \in X \setminus Y_2$.

Continuing in this way, we obtain a subsequence $\{r_{k,m} : m \in \mathbb{N}\}$ of $\{r_{k-1,m} : m \in \mathbb{N}\}$ and a μ-null subset Y_k for each $k \geqslant 2$, satisfying

$$\lim_{m \to \infty} \|p_{i,r_{k,m}}(s)a_k(s) - a_k(s)p_{i,r_{k,m}}(s)\|_{2,s} = 0$$

for any $i \in \{1, 2, \ldots, n_0\}$ and any $s \in X \setminus Y_k$. Now we apply the diagonal selection by letting $r_m = r_{m,m}$ for each $m \in \mathbb{N}$ to these subsequences and obtain that

$$\lim_{m \to \infty} \|p_{i,r_m}(s)a_j(s) - a_j(s)p_{i,r_m}(s)\|_{2,s} = 0 \tag{3.4}$$

for any $i \in \{1, 2, \ldots, n_0\}$, $j \in \mathbb{N}$ and $s \in X \setminus X_0$, where $X_0 = \cup_{k \in \mathbb{N}} Y_k$ is a μ-null subset of X.

Since $\{a_j : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball of $\mathcal{M}_s$ for each $s \in X$, (3.4) implies that, for any $i \in \{1, 2, \ldots, n_0\}$, $s \in X \setminus X_0$ and any $a \in \mathcal{M}_s$,

$$\lim_{m \to \infty} \| p_{i, r_m}(s)a - a p_{i, r_m}(s) \|_{2,s} = 0. \tag{3.5}$$

It follows from (3.5) and Assumption (I) that $\mathcal{M}_s$ is a type II_1 factor with Property Γ for almost every $s \in X$.

(iii) $\Rightarrow$ (i): Suppose $\mathcal{M}_s$ is a type II_1 factor with Property Γ for almost every $s \in X$. We may assume that for every $s \in X$, $\mathcal{M}_s$ is a type II_1 factor with Property Γ.

By Remark 2.4, we can obtain a separable Hilbert space K and a family of unitaries $\{U_s : H_s \to K; s \in X\}$ such that $s \to U_s x(s)$ and $s \to U_s a(s) U_s^*$ are measurable for any $x \in H$ and any decomposable operator $a \in B(H)$. Let $\mathcal{B}$ be the unit ball of self-adjoint elements in $B(K)$ equipped with the $*$-strong operator topology. Then it is metrizable by setting $d(S, T) = \sum_{m=1}^{\infty} 2^{-m}(\|(S - T)e_m\| + \|(S^* - T^*)e_m\|)$ for any $S, T \in \mathcal{B}$, where $\{e_m\}$ is an orthonormal basis of K. The metric space $(\mathcal{B}, d)$ is complete and separable. Now let $\mathcal{B}_1 = \mathcal{B}_2 = \ldots = \mathcal{B}_l = \ldots = \mathcal{B}$ and $\mathcal{C} = \prod_{l \in \mathbb{N}} \mathcal{B}_l$ provided with the product topology of the $*$-strong operator topology on each B_l. It follows that $\mathcal{C}$ is metrizable and it's also a complete separable metric space.

Replacing a_0 by a_j for $j \in \mathbb{N}$, we apply Proposition 2.5 countably many times and obtain positive, faithful, normal, tracial linear functionals ρ_s on $\mathcal{M}_s$ (almost everywhere) and a Borel μ-null subset N of X such that,

(1) $\rho(a) = \int_X \rho_s(a(s)) d\mu$ for every $a \in \mathcal{M}$;

(2) for any $j \in \mathbb{N}$, the maps

$$s \to \rho_s((a_j U_s^* b U_s - U_s^* b U_s a_j(s))^* (a_j U_s^* b U_s - U_s^* b U_s a_j(s))) \tag{3.6}$$

and

$$s \to \rho_s(U_s^* b U_s) \tag{3.7}$$

from X to $\mathbb{C}$ are Borel measurable when restricted to $X \setminus N$.

We denote by $(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots))$ an element in $X \times \mathcal{C}$. Since $b \to b^*$ and $b \to b^2$ are $*$-*SOT* continuous from $\mathcal{B}$ to $\mathcal{B}$, the maps

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots)) \to Q_{it}, \tag{3.8}$$

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots)) \to Q_{it}^2, \tag{3.9}$$

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots)) \to Q_{it}^* \tag{3.10}$$

are Borel measurable from $X \times \mathcal{C}$ to $\mathcal{B}$.

By Remark 2.4, the map $s \to U_s a_j'(s) U_s^*$ from X to $\mathcal{B}$ is measurable for every $j \in \mathbb{N}$. Therefore, by Lemma 14.3.1 in [16], there exists a Borel μ-null subset N' of

X such that the map $s \to U_s a'_j(s) U_s^*$ is Borel measurable when restricted to $X \setminus N'$ for every $j \in \mathbb{N}$. Hence the maps

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots)) \to Q_{it} U_s a'_j(s) U_s^*, \tag{3.11}$$

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots)) \to U_s a'_j(s) U_s^* Q_{it} \tag{3.12}$$

are Borel measurable when restricted to $(X \setminus N') \times \mathscr{C}$ for every $j \in \mathbb{N}$.

Since the functionals ρ_s are chosen such that the maps

$$s \to \rho_s((a_j U_s^* b U_s - U_s^* b U_s a_j(s))^* (a_j U_s^* b U_s - U_s^* b U_s a_j(s)))$$

and

$$s \to \rho_s(U_s^* b U_s)$$

are Borel measurable when restricted to $X \setminus N$, where N is a Borel μ-null subset of X, the maps

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots))$$
$$\to \rho_s((a_j(s) U_s^* Q_{it} U_s - U_s^* Q_{it} U_s a_j(s))^* (a_j(s) U_s^* Q_{it} U_s - U_s^* Q_{it} U_s a_j(s))) \tag{3.13}$$

and

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots)) \to \rho_s(U_s^* Q_{it} U_s) \tag{3.14}$$

are Borel measurable when restricted to $(X \setminus N) \times \mathscr{C}$ for each $j \in \mathbb{N}$.

Take $N_0 = N \cup N'$. Then we have the following claim.

Claim 3.12.1. N_0 *is a Borel μ-null subset of X and the maps* (3.8)–(3.14) *are Borel measurable when restricted to $X \setminus N_0$.*

Next we introduce the following subset η of $(X \setminus N_0) \times \mathscr{C}$.

Let η be a subset of $(X \setminus N_0) \times \mathscr{C}$ that consists of all these elements

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots, Q_{1t}, Q_{2t}, \ldots, Q_{nt}, \ldots)) \in (X \setminus N_0) \times \mathscr{C}$$

satisfying

(a) for any $1 \leqslant i \leqslant n, t \in \mathbb{N}$,

$$Q_{it} = Q_{it}^* = Q_{it}^2 \neq 0; \tag{3.15}$$

(b) for any $1 \leqslant i \leqslant n, \ t, j \in \mathbb{N}$,

$$Q_{it} U_s a'_j(s) U_s^* = U_s a'_j(s) U_s^* Q_{it}; \tag{3.16}$$

(c) for any $1 \leqslant i \leqslant n, \ t \in \mathbb{N}, 1 \leqslant j \leqslant t$,

$$\rho_s((a_j(s) U_s^* Q_{it} U_s - U_s^* Q_{it} U_s a_j(s))^* (a_j(s) U_s^* Q_{it} U_s - U_s^* Q_{it} U_s a_j(s))) < 1/t; \tag{3.17}$$

(d) for any $t \in \mathbb{N}$,

$$\rho_s(U_s^* Q_{1t} U_s) = \ldots = \rho_s(U_s^* Q_{nt} U_s) \ \text{and} \ Q_{1t} + Q_{2t} + \ldots + Q_{nt} = I. \quad (3.18)$$

We have the following claim.

Claim 3.12.2: *The set* η *is analytic.*

Proof of Claim 3.12.2. By Claim 3.12.1, we know the maps (3.8)–(3.14) are Borel measurable when restricted to $X \setminus N_0$. It follows that the set η is a Borel set. Thus by Theorem 14.3.5 in [16], η is analytic. The proof of Claim 3.12.2 is completed. $\square$

Claim 3.12.3: *Let* π *be the projection of* $X \times \mathscr{M}$ *onto* X. *Then* $\pi(\eta) = X \setminus N_0$.

Proof of Claim 3.12.3. Let

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots, Q_{1t}, Q_{2t}, \ldots, Q_{nt}, \ldots))$$

be an element in η. From the definitions of the set η, it's not hard to see that condition (a) is equivalent to that each Q_{it} is a nonzero projection. Since $\{a'_j(s) : j \in \mathbb{N}\}$ is *SOT* dense in $(\mathscr{M}')_1$ for each $s \in X$, condition (b) is equivalent to the condition that $U_s^* Q_{it} U_s \in \mathscr{M}_s$. Notice that $\{a_j(s) : j \in \mathbb{N}\}$ is *SOT* dense in $(\mathscr{M})_1$ for each $s \in X$, condition (c) is equivalent to

$$\lim_{t \to \infty} \rho_s((aU_s^* Q_{it} U_s - U_s^* Q_{it} U_s a)^* (aU_s^* Q_{it} U_s - U_s^* Q_{it} U_s a)) = 0$$

for any $a \in \mathscr{M}_s$. Furthermore, ρ_s is a positive scalar multiple of τ_s on $\mathscr{M}_s$ for each $s \in X$, it follows that condition (c) is equivalent to

$$\lim_{t \to \infty} \|aU_s^* Q_{it} U_s - U_s^* Q_{it} U_s a\|_{2,s} = 0$$

for any $a \in \mathscr{M}_s$. Moreover, $(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots, Q_{1t}, Q_{2t}, \ldots, Q_{nt}, \ldots))$ satisfies condition (a) and condition (d) if and only if $U_s^* Q_{1t} U_s, U_s^* Q_{2t} U_s, \ldots, U_s^* Q_{nt} U_s$ are n equivalent projections in $\mathscr{M}_s$ with sum I_s for each $n \in \mathbb{N}$ and each $t \in \mathbb{N}$.

For each $s \in X$, notice that $\mathscr{M}_s$ is a type II_1 factor with Property Γ. From the argument in the preceding paragraph, there exist projections $\{U_s^* Q_{it} U_s : 1 \leqslant i \leqslant n, t \in \mathbb{N}\}$ in $\mathscr{M}_s$ such that

$$(s, (Q_{11}, Q_{21}, \ldots, Q_{n1}, Q_{12}, Q_{22}, \ldots, Q_{n2}, \ldots, Q_{1t}, Q_{2t}, \ldots, Q_{nt}, \ldots)) \in X \times \mathscr{C}$$

satisfies conditions (a), (b), (c) and (d). Therefore the image of η under π is exactly $X \setminus N_0$. The proof of Claim 3.12.3 is completed. $\square$

Continue the proof of Proposition 3.12. By Claim 3.12.2 and Claim 3.12.3, η is analytic and the image of η under π is $X \setminus N_0$. By Theorem 14.3.6 in [16], there is a measurable mapping

$$s \to (Q_{11}^{(s)}, Q_{21}^{(s)}, \ldots, Q_{n1}^{(s)}, Q_{12}^{(s)}, Q_{22}^{(s)}, \ldots, Q_{n2}^{(s)}, \ldots)$$

from $X \setminus N_0$ to $\mathscr{C}$ such that, for $s \in X \setminus N_0$ almost everywhere,

$$(s, (Q_{11}^{(s)}, Q_{21}^{(s)}, \ldots, Q_{n1}^{(s)}, Q_{12}^{(s)}, Q_{22}^{(s)}, \ldots, Q_{n2}^{(s)}, \ldots))$$

satisfies conditions (a), (b), (c) and (d) (see (3.15), (3.16), (3.17) and (3.18)). By defining $Q_{it}^{(s)} = 0$ for any $t \in \mathbb{N}, 1 \leqslant i \leqslant n, s \in N_0$, we obtain *a measurable mapping*

$$s \to (Q_{11}^{(s)}, Q_{21}^{(s)}, \ldots, Q_{n1}^{(s)}, Q_{12}^{(s)}, Q_{22}^{(s)}, \ldots, Q_{n2}^{(s)}, \ldots) \tag{3.19}$$

from X to $\mathscr{C}$ such that such that, for $s \in X$ almost everywhere,

$$(s, (Q_{11}^{(s)}, Q_{21}^{(s)}, \ldots, Q_{n1}^{(s)}, Q_{12}^{(s)}, Q_{22}^{(s)}, \ldots, Q_{n2}^{(s)}, \ldots))$$

satisfies conditions (a), (b), (c) and (d) (see (3.15), (3.16), (3.17) and (3.18)).

By (3.19), for any $t \in \mathbb{N}, 1 \leqslant i \leqslant n$ and any vectors $y, z \in H$, we have

$$\langle U_s^* Q_{it}^{(s)} U_s y(s), z(s) \rangle = \langle Q_{it}^{(s)} U_s y(s), U_s z(s) \rangle$$

and thus the map

$$s \to \langle U_s^* Q_{it}^{(s)} U_s y(s), z(s) \rangle$$

is measurable. Since

$$|\langle U_s^* Q_{it}^{(s)} U_s y(s), z(s) \rangle| \leqslant \|y(s)\| \|z(s)\|,$$

the map $s \to \langle U_s^* Q_{it} U_s y(s), z(s) \rangle$ is integrable. By Definition 14.1.1 in [16], it follows that

$$U_s^* Q_{it}^{(s)} U_s y(s) = (p_{it} y)(s) \tag{3.20}$$

almost everywhere for some $p_{it} y$ in H. For each $t \in \mathbb{N}$, (3.20) implies that $p_{it}(s) = U_s^* Q_{it}^{(s)} U_s$ for almost every $s \in X$. Therefore $p_{it} \in \mathscr{M}$ for each $t \in \mathbb{N}$. Notice conditions (a) and (d) together imply that $U_s^* Q_{1t}^{(s)} U_s, U_s^* Q_{2t}^{(s)} U_s, \ldots, U_s^* Q_{nt}^{(s)} U_s$ are n orthogonal equivalent projections in $\mathscr{M}_s$ with sum I_s for each $t \in \mathbb{N}$. It follows that $p_{1t}, p_{2t}, \ldots, p_{nt}$ are n orthogonal equivalent projections in $\mathscr{M}$ with sum I for each $t \in \mathbb{N}$.

In order to show that $\mathscr{M}$ has Property Γ, it suffices to show that for any $i \in \{1, 2, \ldots, n\}$ and $a \in \mathscr{M}$,

$$\lim_{t \to \infty} \rho((ap_{it} - p_{it}a)^*(ap_{it} - p_{it}a)) = 0. \tag{3.21}$$

By condition (c), we obtain that for each $j \in \mathbb{N}, 1 \leqslant i \leqslant n$ and $s \in X$,

$$\lim_{t \to \infty} \rho_s((a_j(s)U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))^*(a_j(s)U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))) = 0. \tag{3.22}$$

Fix $i \in \{1, 2, \dots, n\}$ and $j \in \mathbb{N}$. For each $t \in \mathbb{N}$, define a function $f_t : X \to \mathbb{C}$ such that

$$f_t(s) = \rho_s((a_j(s)U_s^* Q_{is}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))^* (a_j(s) U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))).$$

It follows from (3.22) that

$$\lim_{t \to \infty} f_t(s) = 0 \qquad (3.23)$$

almost everywhere. By Lemma 14.1.9 in [16], for each $j \in \mathbb{N}$, $\|a_j\|$ is the essential bound of $\{\|a_j(s)\| : s \in X\}$. Therefore

$$\begin{aligned}
&\|(a_j(s)U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))^* (a_j(s) U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))\| \\
&\leqslant 4\|a_j(s)\|^2 \\
&\leqslant 4\|a_j\|^2
\end{aligned}$$

almost everywhere. Hence

$$0 \leqslant f_t(s) \leqslant 4\|a_j\|^2 \rho_s(I_s) \qquad (3.24)$$

almost everywhere. Furthermore,

$$\int_X 4\|a_j\|^2 \rho_s(I_s) d\mu = 4\|a_j\|^2 \rho(I) = 4\|a_j\|^2 \leqslant 4, \qquad (3.25)$$

by the Dominated Convergence Theorem, it follows from (3.23), (3.24) and (3.25) that

$$\lim_{t \to \infty} \int_X \rho_s((a_j(s)U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))^* (a_j(s) U_s^* Q_{it}^{(s)} U_s - U_s^* Q_{it}^{(s)} U_s a_j(s))) d\mu = 0. \qquad (3.26)$$

Since $p_{it}(s) = U_s^* Q_{it}^{(s)} U_s$ for almost every $s \in X$, (3.26) implies

$$\lim_{t \to \infty} \rho((a_j p_{it} - p_{it} a_j)^* (a(j) p_{it} - p_{it} a_j)) = 0. \qquad (3.27)$$

From the fact that $\{a_j : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball of $\mathscr{M}$, we obtain equation (3.21) from (3.27). Thus $\mathscr{M}$ is a type II_1 von Neumann algebra with Property Γ. $\square$

If $\mathscr{M}$ is a type II_1 von Neumann algebra, then by Lemma 6.5.6 in [16], for any $m \in \mathbb{N}$, there is a unital subalgebra $\mathscr{A}$ of $\mathscr{M}$ such that $\mathscr{A} \cong M_m(\mathbb{C})$.

PROPOSITION 3.13. *Suppose $\mathscr{M}$ is a type II_1 von Neumann algebra acting on a separable Hilbert space H. Suppose further $\mathscr{A}$ is a unital subalgebra of $\mathscr{M}$ such that $\mathscr{A} \cong M_m(\mathbb{C})$ for some $m \in \mathbb{N}$. Let $\mathscr{N} = \mathscr{A}' \cap \mathscr{M}$. Then $\mathscr{M}$ has Property Γ if and only if $\mathscr{N}$ has Property Γ.*

Proof. By Lemma 11.4.11 in [16], $\mathscr{M} \cong \mathscr{A} \otimes \mathscr{N} \cong M_m(\mathbb{C}) \otimes \mathscr{N}$. It is trivial to see that if $\mathscr{N}$ has Property Γ, then $\mathscr{M}$ has Property Γ. Thus we only need to show that Property Γ of $\mathscr{M}$ implies Property Γ of $\mathscr{N}$.

Suppose $\mathcal{M}$ has Property Γ. Let $\mathcal{M} = \int_X \oplus \mathcal{M}_s d\mu$ and $H = \int_X \oplus H_s d\mu$ be the direct integral decompositions relative to the center $\mathcal{Z}$ of $\mathcal{M}$. Since $\mathcal{M}$ is a type II$_1$ von Neumann algebra with Property Γ, by Proposition 3.12, $\mathcal{M}_s$ is a type II$_1$ factor with Property Γ for almost every $s \in X$. We may assume $\mathcal{M}_s$ is a type II$_1$ factor with Property Γ for every $s \in X$.

Since $\mathcal{A} \cong M_m(\mathbb{C})$, by Lemma 3.11, we may assume $\mathcal{A}_s \cong M_m(\mathbb{C})$ for every $s \in X$. Since $\mathcal{N} = \mathcal{A}' \cap \mathcal{M}$, $\mathcal{N}_s = \mathcal{A}'_s \cap \mathcal{M}_s$ for almost every $s \in X$. Then by Lemma 11.4.11 in [16],

$$\mathcal{M}_s \cong \mathcal{A}_s \otimes \mathcal{N}_s \cong M_m(\mathbb{C}) \otimes \mathcal{N}_s$$

for almost every $s \in X$. Since $\mathcal{M}_s$ is a type II$_1$ factor with Property Γ for every $s \in X$, by Lemma 5.1 in [4], $\mathcal{N}_s$ has Property Γ for almost every $s \in X$. By a similar argument as the proof of Proposition 3.12, we can conclude that $\mathcal{N}$ has Property Γ.   □

4. Hyperfinite II$_1$ subfactors in type II$_1$ von Neumann algebras

Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra with separable predual and Property Γ. We will devote this section to the construction of a hyperfinite type II$_1$ subfactor $\mathcal{R}$ of $\mathcal{M}$ such that

(I) $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$, where $\mathcal{Z}$ is the center of $\mathcal{M}$;

(II) for any given $a_1, a_2, \ldots, a_k \in \mathcal{M}, n \in \mathbb{N}$ and $\varepsilon > 0$, there exist orthogonal equivalent projections $p_1, p_2, \ldots, p_n$ in $\mathcal{R}$ with sum I such that

$$\|p_i a_j - a_j p_i\|_2 < \varepsilon, \ \ i = 1, 2, \ldots; \ \ j = 1, 2, \ldots, k,$$

where the 2-norm $\| \cdot \|_2$ is given by $\|a\|_2 = \sqrt{\rho(a^* a)}, \forall a \in \mathcal{M}$ for some faithful tracial state ρ on $\mathcal{M}$.

LEMMA 4.1. *Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra acting on a separable Hilbert space H. Let $m \in \mathbb{N}$ and $\mathcal{A}$ be a unital subalgebra of $\mathcal{M}$ such that $\mathcal{A} \cong M_m(\mathbb{C})$. Let $\mathcal{N} = \mathcal{A}' \cap \mathcal{M}$. Assume that $\mathcal{M} = \int_X \oplus \mathcal{M}_s d\mu$ and $H = \int_X \oplus H_s d\mu$ are the direct integral decompositions relative to the center $\mathcal{Z}$ of $\mathcal{M}$. Assume that ρ is a faithful normal tracial state on $\mathcal{M}$ and $\{\rho_s : s \in X\}$ is a family of positive, faithful, normal, tracial functionals as introduced in Lemma 2.2 and Proposition 2.5. If $\mathcal{M}$ has Property Γ, then*

$\forall a_1, a_2, \ldots, a_k \in \mathcal{M}$, $\forall n \in \mathbb{N}$ and $\forall \varepsilon > 0$, there exist a μ-null subset X_0 of X and a family of mutually orthogonal equivalent projections $\{p_1, p_2, \ldots, p_n\}$ in $\mathcal{N}$ with sum I such that,

$$\rho_s((p_i(s)a_j(s) - a_j(s)p_i(s))^*(p_i(s)a_j(s) - a_j(s)p_i(s))) < \varepsilon,$$

for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$, and $s \in X \setminus X_0$.

Proof. Since $\mathscr{A} \cong M_m(\mathbb{C})$ and $\mathscr{N} = \mathscr{A}' \cap \mathscr{M}$, by Lemma 11.4.11 in [16], $\mathscr{M} \cong \mathscr{A} \otimes \mathscr{N}$. Then by the discussion in section 11.2 in [16], $\mathscr{N}$ is a type II_1 von Neumann algebra. Let $\{a_{gh}\}_{g,h=1}^m$ be a system of matrix units for $\mathscr{A}$. By Lemma 3.11, we may assume that $\mathscr{A}_s \cong M_m(\mathbb{C})$ and $\mathscr{M}_s \cong \mathscr{A}_s \otimes \mathscr{N}_s$ for every $s \in X$. We also assume that, for each $s \in X$, $\{a_{gh}(s)\}_{g,h=1}^m$ is a system of matrix units for $\mathscr{A}_s$. By Proposition 3.12, we may assume that $\mathscr{M}_s$ is a type II_1 factor with Property Γ for every $s \in X$. Then by Lemma 5.1 in [4], $\mathscr{N}_s$ is a type II_1 factor with Property Γ for every $s \in X$.

Let $\{a_r' : r \in \mathbb{N}\}$ be a *SOT* dense subset in the unit ball $\mathscr{M}_1'$ of $\mathscr{M}'$. By Proposition 14.1.24 in [16], we may assume that $(\mathscr{M}')_s = (\mathscr{M}_s)'$ for every $s \in X$ and we use the notation $\mathscr{M}_s'$ for both. Therefore by Remark 3.10, we may assume $\{a_r'(s) : r \in \mathbb{N}\}$ is *SOT* dense in $(\mathscr{M}_s')_1$ for every $s \in X$.

Take a separable Hilbert space K and a family of unitaries $\{U_s : H_s \to K; s \in X\}$ as in Remark 2.4 such that $s \to U_s x(s)$ and $s \to U_s a(s) U_s^*$ are measurable for any $x \in H$ and any decomposable operator $a \in B(H)$. Let $\mathscr{B}$ be the unit ball of $B(K)$ equipped with the $*$-strong operator topology. Since K is separable, $\mathscr{B}$ is metrizable by setting $d(S,T) = \sum_{j=1}^\infty 2^{-j}(\|(S-T)e_j\| + \|(S^*-T^*)e_j\|)$ for any $S,T \in \mathscr{B}$, where $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis for K. Then the metric space $(\mathscr{B},d)$ is complete and separable. For each $1 \leqslant i,j \leqslant n$, let $\mathscr{B}_{ij} = \mathscr{B}$. Take $\mathscr{C} = \prod_{1 \leqslant i,j \leqslant n} \mathscr{B}_{ij}$ equipped with the product topology. It follows that $\mathscr{C}$ is a complete separable metric space.

By the choices of $\{U_s\}$, we know that the maps $s \to U_s a_r'(s) U_s^*$, and $s \to U_s a_{gh}(s) U_s^*$ from X to $B(K)$ are measurable for any $r \in \mathbb{N}$ and any $g,h = 1,2,\ldots,m$. By Lemma 14.3.1 in [16], there exists a Borel μ-null subset N_1 of X such that the maps

$$(s,b) \to b U_s a_r'(s) U_s^*, \tag{4.1}$$

$$(s,b) \to U_s a_r'(s) U_s^* b, \tag{4.2}$$

$$(s,b) \to b U_s a_{gh}(s) U_s^*, \tag{4.3}$$

$$(s,b) \to U_s a_{gh}(s) U_s^* b \tag{4.4}$$

are Borel measurable from $(X \setminus N_1) \times \mathscr{B}$ to $B(K)$ for any $r \in \mathbb{N}$ and any $g,h = 1,2,\ldots,m$. Since ρ is a faithful normal tracial state, by Lemma 2.2, we may assume that, for every $s \in X$, there exists a positive, faithful, normal, tracial functional ρ_s on $\mathscr{M}_s$ such that $\rho(a) = \int_X \rho_s(a(s)) d\mu$ for any $a \in \mathscr{M}$. By Proposition 2.5, there is a Borel μ-null subset N_2 of X such that, for each $j \in \{1,2,\ldots,k\}$, the map

$$(s,b) \to \rho_s((U_s^* b U_s a_j(s) - a_j(s) U_s^* b U_s)^*(U_s^* b U_s a_j(s) - a_j(s) U_s^* b U_s)) \tag{4.5}$$

is Borel measurable from $(X \setminus N_2) \times \mathscr{B}$ to $\mathbb{C}$. From the fact that each ρ_s is a positive, faithful, normal, tracial functional on $\mathscr{M}_s$, it follows that ρ_s is a positive scalar multiple of the unique trace τ_s on $\mathscr{M}_s$ for each s in $X \setminus N_2$.

Let $N = N_1 \cup N_2$. *Let η be the collection of all these elements*

$$(p, E_{11}, E_{12}, \ldots, E_{nn}) \in (X \setminus N) \times \mathscr{C}$$

such that

(i) for all $i_1, i_2, i_3, i_4 \in \{1, 2, \ldots, n\}$,

$$E_{i_1 i_2} = E^*_{i_2 i_1} \quad and \quad E_{i_1 i_2} E_{i_3 i_4} = \delta_{i_2 i_3} E_{i_1 i_4}; \tag{4.6}$$

(ii)

$$E_{11} + E_{22} + \ldots + E_{nn} = I; \tag{4.7}$$

(iii) for all $i_1, i_2 \in \{1, 2, \ldots, n\}$ and $r \in \mathbb{N}$,

$$E_{i_1 i_2} U_s a'_r(s) U^*_s = U_s a'_r(s) U^*_s E_{i_1 i_2}; \tag{4.8}$$

(iv) for all $i_1, i_2 \in \{1, 2, \ldots, n\}$ and $g, h \in \{1, 2, \ldots, m\}$,

$$E_{i_1 i_2} U_s a_{gh}(s) U^*_s = U_s a_{gh}(s) U^*_s E_{i_1 i_2}; \tag{4.9}$$

(v) for all i in $\{1, 2, \ldots, n\}$ and j in $\{1, 2, \ldots, k\}$,

$$\rho_s((U^*_s E_{ii} U_s a_j(s) - a_j(s) U^*_s E_{ii} U_s)^* (U^*_s E_{ii} U_s a_j(s) - a_j(s) U^*_s E_{ii} U_s)) < \varepsilon.; \tag{4.10}$$

We have the following claim.

Claim 4.1.1. *The set η is analytic.*

Proof of Claim 4.1.1. The maps

$$(E_{11}, E_{12}, \ldots, E_{nn}) \to E_{i_1 i_2},$$
$$(E_{11}, E_{12}, \ldots, E_{nn}) \to E^*_{i_2 i_1},$$
$$(E_{11}, E_{12}, \ldots, E_{nn}) \to E_{i_1 i_2} E_{i_3 i_4},$$
$$(E_{11}, E_{12}, \ldots, E_{nn}) \to E_{11} + E_{22} + \ldots + E_{nn}$$

are continuous from $\mathscr{C}$ (with the product topology) to $\mathscr{B}$ (with the $*$-strong operator topology). Therefore, we obtain that the maps

$$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to E_{i_1 i_2},$$
$$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to E^*_{i_2 i_1},$$
$$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to E_{i_1 i_2} E_{i_3 i_4},$$
$$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to E_{11} + E_{22} + \ldots + E_{nn}$$

are Borel measurable from $(X \setminus N) \times \mathscr{C}$ to $\mathscr{B}$ for all $1 \leqslant i_1, i_2, i_3, i_4 \leqslant n$. From the fact that the maps (4.1), (4.2), (4.3) and (4.4) are Borel measurable from $(X \setminus N_1) \times \mathscr{B}$ to $B(K)$ and the map (4.5) is Borel measurable from $(X \setminus N_2) \times \mathscr{B}$ to $\mathbb{C}$, it follows that the following maps

$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to E_{i_1 i_2} U_s a'_r(s) U^*_s$,
$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to U_s a'_r(s) U^*_s E_{i_1 i_2}$,
$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to E_{i_1 i_2} U_s a_{gh}(s) U^*_s$,
$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to U_s a_{gh}(s) U^*_s E_{i_1 i_2}$,
$(s, E_{11}, E_{12}, \ldots, E_{nn}) \to \rho_s((U^*_s E_{ii} U_s a_j(s) - a_j(s) U^*_s E_{ii} U_s)^* (U^*_s E_{ii} U_s a_j(s) - a_j(s) U^*_s E_{ii} U_s))$

are Borel measurable when restricted to $(X \setminus N) \times \mathscr{C}$ for all $1 \leqslant i_1, i_2, i \leqslant n, 1 \leqslant g, h \leqslant m, r \in \mathbb{N}$, and $1 \leqslant j \leqslant k$. Therefore η is a Borel set. Thus η is analytic by Theorem 14.3.5 in [16]. This completes the proof of Claim 4.1.1. $\square$

Claim 4.1.2. *Let π be the projection of $X \times \mathscr{C}$ onto X. Then $\pi(\eta) = X \setminus N$.*

Proof of Claim 4.1.2. Notice that an element $(s, E_{11}, E_{12}, \ldots, E_{nn})$ in $(X \setminus N) \times \mathscr{C}$ satisfies conditions (i) and (ii) if and only if $\{E_{i_1 i_2}\}_{i_1, i_2 = 1}^n$ is a system of matrix units for a matrix algebra which is isomorphic to $\mathscr{M}_n(\mathbb{C})$. Condition (iii) is equivalent to that $U_s^* E_{i_1 i_2} U_s \in \mathscr{M}_s$. Condition (iv) is equivalent to that $U_s^* E_{i_1 i_2} U_s \in \mathscr{A}_s'$.

By assumption, for each $s \in X$, $\mathscr{M}_s$ and $\mathscr{N}_s$ are type II$_1$ factors with Property Γ and $\mathscr{M}_s \cong \mathscr{A}_s \otimes \mathscr{N}_s$. Thus $\mathscr{A}_s' \cap \mathscr{M}_s = \mathscr{N}_s$. It follows from the argument in the preceding paragraph that, for each $s \in X$, there exists a system of matrix units $\{U_s^* E_{11} U_s, U_s^* E_{12} U_s, \ldots, U_s^* E_{nn} U_s\}$ in $\mathscr{N}_s$ such that $(s, E_{11}, E_{12}, \ldots, E_{nn})$ satisfies conditions (i), (ii), (iii), (iv) and (v). Therefore the image of η under π is exactly $X \setminus N$. This completes the proof of Claim 4.1.2. $\square$

Continue the proof of Lemma 4.1. By Claim 4.1.1 and Claim 4.1.2, η is analytic and $\pi(\eta) = X \setminus N$. By the measure-selection principle (Theorem 14.3.6 in [16]), there is a measurable mapping

$$s \to (E_{11,s}, E_{12,s}, \ldots, E_{nn,s})$$

from $X \setminus N$ to $\mathscr{C}$ such that, for $s \in X \setminus N$ almost everywhere, $(s, E_{11,s}, E_{12,s}, \ldots, E_{nn,s})$ satisfies conditions (i), (ii), (iii), (iv) and (v) (see (4.6), (4.7), (4.8), (4.9), and (4.10)). Defining $E_{i_1 i_2, s} = 0$ for $s \in N, 1 \leqslant i_1, i_2 \leqslant n$, we get *a measurable map*

$$s \to (E_{11,s}, E_{12,s}, \ldots, E_{nn,s}) \tag{4.11}$$

from X to $\mathscr{C}$ such that, for $s \in X$ almost everywhere, $(s, E_{11,s}, E_{12,s}, \ldots, E_{nn,s})$ satisfies conditions (i), (ii), (iii), (iv) and (v) (see (4.6), (4.7), (4.8), (4.9), and (4.10)).

From (4.11), for any $1 \leqslant i_1, i_2 \leqslant n$ and any two vectors $x, y \in H$, it follows

$$\langle U_s^* E_{i_1 i_2, s} U_s x(s), y(s) \rangle = \langle E_{i_1 i_2, s} U_s x(s), U_s y_s \rangle,$$

and the map $s \to \langle U_s^* E_{i_1 i_2, s} U_s x(s), y(s) \rangle$ are measurable. Since

$$|\langle U_s^* E_{i_1 i_2, s} U_s x(s), y(s) \rangle| \leqslant \|x(s)\| \|y(s)\|,$$

we know $s \to \langle U_s E_{i_1 i_2, s} U_s x(s), y(s) \rangle$ is integrable. By Definition 14.1.1 in [16], it follows that

$$U_s^* E_{i_1 i_2, s} U_s x(s) = (p_{i_1 i_2} x)(s) \tag{4.12}$$

almost everywhere for some $p_{i_1 i_2} x \in H$. From (4.12), we have that

$$p_{i_1 i_2}(s) = U_s^* E_{i_1 i_2, s} U_s \tag{4.13}$$

for almost every $s \in X$ and thus $p_{i_1 i_2} \in \mathscr{M}$. By condition (iv),

$$U_s^* E_{i_1 i_2, s} U_s \in \mathscr{A}_s'.$$

Hence

$$p_{i_1 i_2} \in \mathscr{A}' \cap \mathscr{M} = \mathscr{N}. \tag{4.14}$$

Since conditions (i) and (ii) together imply that $\{E_{i_1 i_2,s}\}^n_{i_1,i_2=1}$ is a system of matrix units, by (4.13), we obtain that $p_{11}(s), p_{22}(s), \ldots, p_{nn}(s)$ are n orthogonal equivalent projections in $\mathscr{M}_s$ with sum I_s almost everywhere. Therefore (4.14) implies that $p_{11}, p_{22}, \ldots, p_{nn}$ are n orthogonal equivalent projections in $\mathscr{N}$ with sum I. For each $i \in \{1, 2, \ldots, n\}$, let $p_i = p_{ii}$. From condition (v), we conclude that $p_1, p_2, \ldots, p_n$ is a family of mutually orthogonal equivalent projections in $\mathscr{N}$ with sum I satisfying, $\forall i = 1, 2, \ldots, n, \forall j = 1, 2, \ldots, k$,

$$\rho_s((p_i(s)a_j(s) - a_j(s)p_i(s))^*(p_i(s)a_j(s) - a_j(s)p_i(s))) < \varepsilon \text{ for } s \in X \text{ almost everywhere.}$$

This ends the proof of the lemma. □

A slight modification of the proof in Lemma 4.1 gives us the next corollary.

COROLLARY 4.2. *Let $\mathscr{M}$ be a type II$_1$ von Neumann algebra acting on a separable Hilbert space H. Let $m \in \mathbb{N}$ and $\mathscr{A}$ be a unital subalgebra of $\mathscr{M}$ such that $\mathscr{A} \cong M_m(\mathbb{C})$. Let $\mathscr{N} = \mathscr{A}' \cap \mathscr{M}$. Assume that $\mathscr{M} = \int_X \oplus \mathscr{M}_s d\mu$ and $H = \int_X \oplus H_s d\mu$ are the direct integral decompositions relative to the center $\mathscr{Z}$ of $\mathscr{M}$. Assume that $\mathscr{M}$ has Property Γ. Then, $\forall a_1, a_2, \ldots, a_k \in \mathscr{M}$, $\forall n \in \mathbb{N}$ and $\forall \varepsilon > 0$, there exist a μ-null subset X_0 of X and a family of mutually orthogonal equivalent projections $\{p_1, p_2, \ldots, p_n\}$ in $\mathscr{N}$ with sum I such that*

$$\|p_i(s)a_j(s) - a_j(s)p_i(s)\|_{2,s} < \varepsilon, \qquad \forall i = 1, 2, \ldots, n, \forall j = 1, 2, \ldots, k \text{ and } s \in X \setminus X_0,$$

where $\|\cdot\|_{2,s}$ is the 2-norm induced by the unique trace τ_s on $\mathscr{M}_s$.

In [20], Popa proved that if $\mathscr{A}$ is a type II$_1$ factor with separable predual, then there is a hyperfinite subfactor $\mathscr{B}$ of $\mathscr{A}$ such that $\mathscr{B}' \cap \mathscr{A} = \mathbb{C}I$. The following lemma is essentially Theorem 8 in [25]. The proof presented here is based on the direct integral theory for von Neumann algebras and is different from the one in [25].

LEMMA 4.3. ([25]) *If $\mathscr{M}$ is a type II$_1$ von Neumann algebra acting on a separable Hilbert space H, then there is a hyperfinite type II$_1$ subfactor $\mathscr{R}$ of $\mathscr{M}$ such that $\mathscr{R}' \cap \mathscr{M} = \mathscr{Z}$, where $\mathscr{Z}$ is the center of $\mathscr{M}$.*

Proof. By Lemma 2.1, $\mathscr{M}$ can be decomposed (relative to its center) as a direct integral $\int_X \oplus \mathscr{M}_s d\mu$ over a locally compact complete separable metric measure space (X, μ) and $\mathscr{M}_s$ is a type II$_1$ factor almost everywhere. In the following we assume that $\mathscr{M}_s$ is a type II$_1$ factor for every $s \in X$.

By Remark 2.4, we can obtain a separable Hilbert space K and a family of unitaries $\{U_s : H_s \to K; s \in X\}$ such that the maps $s \to U_s x(s)$ and $s \to U_s a(s) U_s^*$ are measurable for any $x \in H$ and any decomposable $a \in B(H)$. Let $\mathscr{B}$ be the unit ball of

$B(K)$ with the $*$-strong operator topology. We observe that $\mathscr{B}$ is metrizable by setting $d(S,T) = \sum_{j=1}^{\infty} 2^{-j}(\|(S-T)e_j\| + \|(S^*-T^*)e_j\|)$ for any $S, T \in \mathscr{B}$, where $\{e_j : j \in \mathbb{N}\}$ is an orthonormal basis of K. Moreover, $(\mathscr{B}, d)$ is a complete separable metric space. Let $\mathscr{C} = \mathscr{B} \times \mathscr{B}$ equipped with the product topology. It follows that $\mathscr{C}$ is a complete separable metric space.

Let $\{a'_j : j \in \mathbb{N}\}$ be a SOT dense subset of the unit ball $(\mathscr{M}')_1$. By Lemma 14.1.24, we may assume that $(\mathscr{M}')_s = (\mathscr{M}_s)'$ for every $s \in X$ and we use the notation $\mathscr{M}'_s$ for both. By Remark 3.10, we may assume further that $\{a'_j(s) : j \in \mathbb{N}\}$ is SOT dense in $(\mathscr{M}'_s)_1$ for every $s \in X$. Let $\{y_j : j \in \mathbb{N}\}$ be a countable dense subset in H. By Lemma 14.1.3 in [16], the Hilbert space generated by $\{y_j(s) : j \in \mathbb{N}\}$ is H_s for almost every $s \in X$. Replacing $\{y_j : j \in \mathbb{N}\}$ by the set of all finite rational-linear combinations of vectors in $\{y_j : j \in \mathbb{N}\}$ if necessary, in the following we assume that $\{y_j(s) : j \in \mathbb{N}\}$ is dense in H_s for every $s \in X$.

Fix an irrational number $\theta \in (0,1)$. We denote by (s, W, V) an element in $X \times \mathscr{B} \times \mathscr{B} = X \times \mathscr{C}$.

The maps $W \to WW^*$, $W \to W^*W$, $V \to VV^*$, $V \to V^*V$ are $*$-SOT continuous from $\mathscr{B}$ to $\mathscr{B}$. The maps $(W,V) \to WV, (W,V) \to e^{2\pi i\theta}VW$ are continuous from $\mathscr{C}$ with the product topology to $\mathscr{B}$ with the $*$-strong operator topology. Therefore the maps

$$(s, W, V) \to WW^*, \tag{4.15}$$

$$(s, W, V) \to W^*W, \tag{4.16}$$

$$(s, W, V) \to VV^*, \tag{4.17}$$

$$(s, W, V) \to V^*V, \tag{4.18}$$

$$(s, W, V) \to WV, \tag{4.19}$$

$$(s, W, V) \to e^{2\pi i\theta}VW \tag{4.20}$$

are Borel measurable from $X \times \mathscr{C}$ to $\mathscr{B}$. By Remark 2.4, the maps

$$s \to U_s a'_j(s) U_s^*$$

from X to $B(K)$ and

$$s \to U_s y_j(s)$$

from X to K are all measurable for each $j \in \mathbb{N}$.

Let

$$\mathbb{Q}\langle X, Y, Z_1, Z_2, \ldots \rangle$$

be the collection of all $*$-polynomials in intermediate variables $X, Y, Z_1, Z_2, \ldots$ with rational coefficients. It is a countable set. By Lemma 14.3.1 in [16], there exists a Borel μ-null subset N of X such that, $\forall j_1, j_2 \in \mathbb{N}$, $\forall f \in \mathbb{Q}\langle X, Y, Z_1, Z_2, \ldots \rangle$, the maps

$$(s, W, V) \to WU_s a'_j(s) U_s^*, \tag{4.21}$$

$$(s, W, V) \to U_s a'_j(s) U_s^* W, \tag{4.22}$$

$$(s, W, V) \to VU_s a'_j(s) U_s^*, \tag{4.23}$$

$$(s, W, V) \to U_s a'_j(s) U_s^* V \tag{4.24}$$

are Borel measurable from $(X \setminus N) \times \mathscr{C}$ to $\mathscr{B}$ and the map

$$(s, W, V) \rightarrow \|f(W, V, \{U_s a'_j(s) U_s^* : j \in \mathbb{N}\}) U_s y_{j_1}(s) - U_s y_{j_2(s)}\| \qquad (4.25)$$

is Borel meaurable from $(X \setminus N) \times \mathscr{C}$ to $\mathbb{C}$.

Now we introduce the set η as follows.

Let η be the collection of all these elements $(s, W, V) \in (X \setminus N) \times \mathscr{C}$ satisfying

  (i) $WW^* = W^*W = VV^* = V^*V = I$, where I is the identity in $B(K)$;

  (ii) $WU_s a'_j(s) U_s^* = U_s a'_j(s) U_s^* W$ and $VU_s a'_j(s) U_s^* = U_s a'_j(s) U_s^* V$ for every $j \in \mathbb{N}$;

  (iii) $WV = e^{2\pi i \theta} VW$;

  (iv) for all $N, j_1, j_2 \in \mathbb{N}$, there exists an f in $\mathbb{Q}\langle X, Y, Z_1, Z_2, \ldots \rangle$ such that

$$\|f(W, V, \{U_s a'_j(s) U_s^* : j \in \mathbb{N}\}) U_s y_{j_1}(s) - U_s y_{j_2}(s)\| < 1/N.$$

Claim 4.3.1. *The set η is analytic.*

Proof of Claim 4.3.1. Since the maps (4.15)–(4.25) are all Borel measurable when restricted to $(X \setminus N) \times \mathscr{C}$, η is a Borel set. By Theorem 14.3.5 in [16], η is analytic. This completes the proof of Claim 4.3.1. $\square$

Claim 4.3.2. *Let π be the projection of $X \times \mathscr{C}$ onto X. Then $\pi(\eta) = X \setminus N$.*

Proof of Claim 4.3.2. We observe that an element (s, W, V) satisfies conditions (i), (ii) and (iii) if and only if $U_s^* WU_s$ and $U_s^* VU_s$ are two unitaries in $\mathscr{M}_s$ such that $(U_s^* WU_s)(U_s^* VU_s) = e^{2\pi i \theta}(U_s^* VU_s)(U_s^* WU_s)$. Since $\{y_j(s) : j \in \mathbb{N}\}$ is dense in H_s for every $s \in X$, condition (iv) is equivalent to the condition that the von Neumann algebra generated by $\{U_s^* WU_s, U_s^* VU_s\} \cup \{a'_j(s) : j \in \mathbb{N}\}$ is $B(H_s)$.

For each $s \in X$, $\mathscr{M}_s$ is a type II$_1$ factor with separable predual. By Popa's result in [20], there exists a type II$_1$ hyperfinite subfactor $\mathscr{R}^{(s)}$ of $\mathscr{M}_s$ such that $(\mathscr{R}^{(s)})' \cap \mathscr{M}_s = \mathbb{C}I_s$. Notice a hyperfinite II$_1$ factor always contains an irrational rotation C*-algebra as a SOT dense subalgebra. Combining with the argument in the prededing paragraph, we know that there exist two unitaries $U_s^* WU_s$ and $U_s^* VU_s$ in $\mathscr{R}^{(s)}$ (where W, V are unitaries in $\mathscr{B}$) such that they generate $\mathscr{R}^{(s)}$ as a von Neumann algebra and (s, W, V) satisfies conditions (i), (ii) and (iii). The condition $(\mathscr{R}^{(s)})' \cap \mathscr{M}_s = \mathbb{C}I_s$ is equivalent to the condition that the von Neumann algebra generated by $\mathscr{R}^{(s)} \cup \mathscr{M}_s'$ is $B(H_s)$. Since $U_s^* WU_s$ and $U_s^* VU_s$ generate $\mathscr{R}^{(s)}$ as a von Neumann algebra and $\{a'_j(s) : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball of $\mathscr{M}_s'$, the von Neumann algebra $W^*(U_s^* WU_s, U_s^* VU_s, \{a'_j(s) : j \in \mathbb{N}\})$ generated by $U_s^* WU_s, U_s^* VU_s$ and $\{a'_j(s) : j \in \mathbb{N}\}$ is $B(H_s)$. Hence, from the argument in the preceding paragraph, it follows that (s, W, V) satisfies satisfy condition (iv). Therefore the image of η under π is $X \setminus N$. This completes the proof of Claim 4.3.2. $\square$

Continue the proof of Lemma 4.3. By Claim 4.3.1 and Claim 4.3.2, we know that η is analytic and $\pi(\eta) = X \setminus N$. By the measure-selection principle (Theorem 14.3.6

in [16]), there is a measurable map $s \to (W_s, V_s)$ from $X \setminus N$ to $\mathscr{C}$ such that (s, W_s, V_s) satisfies condition (i), (ii), (iii) and (iv) for $s \in X \setminus N$ almost everywhere. Defining $W_s = V_s = 0$ for any $s \in N$, we get *a measurable map*

$$s \to (W_s, V_s) \tag{4.26}$$

from X to $\mathscr{C}$ such that (s, W_s, V_s) satisfies condition (i), (ii), (iii) and (iv) for $s \in X$ almost everywhere.

For any two vectors $x, y \in H$, we have

$$\langle U_s^* W_s U_s x(s), y(s) \rangle = \langle W_s U_s x(s), U_s y(s) \rangle. \tag{4.27}$$

Combining (4.27) with (4.26), we know the map

$$s \to \langle U_s^* W_s U_s x(s), y(s) \rangle$$

from X to $\mathbb{C}$ is measurable. Since

$$|\langle U_s^* W_s U_s x(s), y(s) \rangle| \leqslant \|x(s)\| \|y(s)\|,$$

we obtain that

$$s \to \langle U_s^* W_s U_s x(s), y(s) \rangle$$

is integrable. By Definition 14.1.1 in [16], it follows that

$$U_s^* W_s U_s x(s) = (\bar{W} x)(s)$$

almost everywhere for some $\bar{W} x \in H$. Therefore

$$\bar{W}(s) = U_s^* W_s U_s \tag{4.28}$$

for almost every $s \in X$. Since conditions (i) and (ii) imply that $U_s^* W_s U_s$ is a unitary in $\mathcal{M}_s$, we obtain from equation (4.28) that $\bar{W}$ is a unitary in $\mathcal{M}$. Similarly we can find another unitary $\bar{V}$ in $\mathcal{M}$ such that

$$\bar{V}(s) = U_s^* V_s U_s \tag{4.29}$$

for almost every $s \in X$ and thus, from condition (iii),

$$\bar{W}(s)\bar{V}(s) = e^{2\pi i \theta} \bar{V}(s)\bar{W}(s)$$

for almost every $s \in X$. Therefore

$$\bar{W}\bar{V} = e^{2\pi i \theta} \bar{V}\bar{W}. \tag{4.30}$$

Let $\mathscr{R}^{(s)}$ be the von Neumann subalgebra generated by $U_s^* W_s U_s$ and $U_s^* V_s U_s$ in $\mathcal{M}_s$. From condition (iv), we know that $(\mathscr{R}^{(s)})' \cap M_s = \mathbb{C} I_s$ for $s \in X$ almost everywhere.

Let $\mathscr{R}$ be a von Neumann subalgebra of $\mathscr{M}$ generated by two unitaries $\bar{W}, \bar{V}$. From (4.28), (4.29) and (4.30), it follows that $\mathscr{R}$ is a hyperfinite type II$_1$ factor and $\mathscr{R}_s = \mathscr{R}^{(s)}$ for almost every $s \in X$.

To complete the proof, we just need to show that $\mathscr{R}' \cap \mathscr{M} = \mathscr{Z}$. Suppose $a \in \mathscr{R}' \cap \mathscr{M}$. Then $a(s) \in \mathscr{R}'_s \cap \mathscr{M}_s$ for almost every $s \in X$. Since $(\mathscr{R}^{(s)})' \cap \mathscr{M}_s = \mathbb{C}I_s$ and $\mathscr{R}_s = \mathscr{R}^{(s)}$ for almost every $s \in X$, $a(s) = c_s I_s$ for almost every $s \in X$ and thus $a \in \mathscr{Z}$. Hence $\mathscr{R}' \cap \mathscr{M} = \mathscr{Z}$. $\square$

The following result is a generalization of Theorem 5.4 in [4] in the setting of von Neumman algebras. The proof follows the similar line as the one used in Theorem 5.4 in [4].

THEOREM 4.4. *Let $\mathscr{M}$ be a type II$_1$ von Neumann algebra with separable predual and $\mathscr{Z}$ be the center of $\mathscr{M}$. Let ρ be a faithful normal tracial state on $\mathscr{M}$ and $\|\cdot\|_2$ be the 2-norm on $\mathscr{M}$ induced by ρ. If $\mathscr{M}$ has Property Γ, then there exists a hyperfinite type II$_1$ subfactor $\mathscr{R}$ of $\mathscr{M}$ such that*

(I) $\mathscr{R} \cap \mathscr{M}' = \mathscr{Z}$;

(II) *for any $n \in \mathbb{N}$, any elements $a_1, a_2, \ldots, a_k$ in $\mathscr{M}$, there exists a countable collection of projections $\{p_{1t}, p_{2t}, \ldots, p_{nt} : t \in \mathbb{N}\}$ in $\mathscr{R}$ such that*

   (i) *for each $t \in \mathbb{N}$, $p_{1t}, p_{2t}, \ldots, p_{nt}$ are n orthogonal equivalent projections in $\mathscr{R}$ with sum I;*

   (ii) $\lim_{t \to \infty} \|p_{it} a_j - a_j p_{it}\|_2 = 0$ *for any $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, k$.*

Proof. Since $\mathscr{M}$ has separable predual, by Proposition A.2.1 in [12], there is a faithful normal representation π of $\mathscr{M}$ on a separable Hilbert space. Replacing $\mathscr{M}$ by $\pi(\mathscr{M})$ and ρ by $\rho \circ \pi^{-1}$, we may assume that $\mathscr{M}$ is acting on a separable Hilbert space H.

By Lemma 2.1, there are direct integral decompositions $\mathscr{M} = \int_X \bigoplus \mathscr{M}_s d\mu$ and $H = \int_X H_s d\mu$ of $(\mathscr{M}, H)$ relative to $\mathscr{Z}$ over (X, μ), where $\mathscr{M}_s$ is a type II$_1$ factor for almost every $s \in X$. We assume that every $\mathscr{M}_s$ is a type II$_1$ factor. Notice that ρ is a faithful, normal, tracial state on $\mathscr{M}$. From Lemma 2.2, we might assume there is a positive, faithful, normal, tracial linear functional ρ_s on $\mathscr{M}_s$ for every $s \in X$ such that

$$\rho(a) = \int_X \rho_s(a(s)) d\mu, \qquad \forall a \in \mathscr{M}.$$

Let $\{\phi_i : i \in \mathbb{N}\}$ be a sequence of normal states on $\mathscr{M}$ that is norm dense in the set of all normal states on $\mathscr{M}$. Let $\{b_j : j \in \mathbb{N}\}$ be a sequence of elements that is *SOT* dense in the unit ball $(\mathscr{M})_1$ of $\mathscr{M}$. By Remark 3.10, we may assume that $\{b_j(s) : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball $(\mathscr{M}_s)_1$ of $\mathscr{M}_s$ for every $s \in X$.

Let τ be the unique center-valued trace on $\mathscr{M}$ such that $\tau(a) = a$ for all $a \in \mathscr{Z}$ (see Theorem 8.2.8 in [16]).

We will show that *there is an increasing sequence $\{\mathscr{A}_t : t \in \mathbb{N}\}$ of full matricial algebras in $\mathscr{M}$ satisfying, for all $t \in \mathbb{N}$,*

(a) *there exists a μ-null subset N_t of X such that, for each $1 \leqslant l \leqslant t$, there exist l equivalent orthogonal projections $p_1, p_2, \ldots, p_l$ in $\mathscr{A}_t$ with sum I satisfying*

$$\rho_s((p_i(s)b_j(s) - b_j(s)p_i(s))^*(p_i(s)b_j(s) - b_j(s)p_i(s))) < 1/t$$

for any $i = 1, 2, \ldots, l$; $j = 1, 2, \ldots, t$; $s \in X \setminus N_t$;

(b) *let $\mathscr{U}_t$ be the unitary group of $\mathscr{A}_t$ and $d\mu_t$ be the normalized Haar measure on $\mathscr{U}_t$. Then for any $i, j = 1, 2, \ldots, t$,*

$$|\phi_i(\int_{\mathscr{U}_t} ub_ju^* d\mu_t - \tau(b_j))| < 1/t.$$

First, we observe that conditions (a) and (b) are satisfied by letting $\mathscr{A}_1 = \mathbb{C}1$ and $p_1 = I$. Now suppose $\mathscr{A}_{t-1}$ has been constructed. Take $\mathscr{N}_1 = \mathscr{A}'_{t-1} \cap \mathscr{M}$. By Lemma 11.4.11 in [16], $\mathscr{M} \cong \mathscr{A}_{t-1} \overline{\otimes} \mathscr{N}_1$.

Next, in order to construct $\mathscr{A}_t$, we will apply Lemma 4.1 $t-1$ times. At the first time, applying Lemma 4.1 to $\mathscr{A}_{t-1}$ and the set $\{b_1, b_2, \ldots, b_t\}$, we obtain two equivalent orthogonal projections $p_{1,1}, p_{2,1}$ in $\mathscr{N}_1$ with sum I and a μ-null subset $N_{t,1}$ of X such that

$$\rho_s((p_{i,1}(s)b_j(s) - b_j(s)p_{i,1}(s))^*(p_{i,1}(s)b_j(s) - b_j(s)p_{i,1}(s))) < 1/t \qquad (4.31)$$

for any $i = 1, 2$; $j = 1, 2, \ldots, t$, $s \in X \setminus N_{t,1}$. Note that $p_{1,1}, p_{2,1}$ are two equivalent orthogonal projections in $\mathscr{N}_1$ with sum I. There is a unital subalgebra $\mathscr{B}_{t,1}$ of $\mathscr{N}_1$ such that $\mathscr{B}_{t,1} \cong M_2(\mathbb{C})$ and $p_{1,1}, p_{2,1} \in \mathscr{B}_{t,1}$. Take

$$\mathscr{A}_{t,1} = \mathscr{A}_{t-1} \overline{\otimes} \mathscr{B}_{t,1}. \qquad (4.32)$$

Now suppose that $\mathscr{A}_{t,l-1}$ have been constructed for some $2 \leqslant l \leqslant t-1$. By applying Lemma 4.1 to $\mathscr{A}_{t,l-1}$ and $\{b_1, b_2, \ldots, b_t\}$, we can find $l+1$ equivalent orthogonal projections $p_{1,l}, p_{2,l}, \ldots, p_{l+1,l}$ in $\mathscr{A}'_{t,l-1} \cap \mathscr{M}$ with sum I and a μ-null subset $N_{t,l}$ of X such that

$$\rho_s((p_{i,l}(s)b_j(s) - b_j(s)p_{i,l}(s))^*(p_{i,l}(s)b_j(s) - b_j(s)p_{i,l}(s))) < 1/t \qquad (4.33)$$

for any $i = 1, 2, \ldots, l+1$, $j = 1, 2, \ldots, t$, and $s \in X \setminus N_{t,l}$. Again there is a unital subalgebra $\mathscr{B}_{t,l}$ of $\mathscr{A}'_{t,l-1} \cap \mathscr{M}$ such that $\mathscr{B}_{t,l} \cong M_{l+1}(\mathbb{C})$ and $p_{1,l}, p_{2,l}, \ldots, p_{l+1,l} \in \mathscr{B}_{t,l}$. Take

$$\mathscr{A}_{t,l} = \mathscr{A}_{t,l-1} \overline{\otimes} \mathscr{B}_{t,l}. \qquad (4.34)$$

Now we let

$$\mathscr{B}_t = \mathscr{A}_{t,t-1} \qquad (4.35)$$

and

$$N_t = \cup_{l=1}^{t-1} N_{t,l}. \qquad (4.36)$$

Then $\mu(N_t) = 0$. By (4.31), (4.32), (4.33), (4.34), (4.35) and (4.36), $\mathscr{B}_t$ contains sets of projections satisfying condition (a).

Let $\mathscr{N} = \mathscr{B}_t' \cap \mathscr{M}$. By Lemma 11.4.11 in [16], we know that $\mathscr{M} \cong \mathscr{B}_t \overline{\otimes} \mathscr{N}$. By the arguments in Section 11.2 in [16], $\mathscr{N}$ is a type II$_1$ von Neumann algebra, and therefore, by Lemma 4.3, there is a hyperfinite subfactor $\mathscr{S}$ of $\mathscr{N}$ such that $\mathscr{S}' \cap \mathscr{N} = \mathscr{Z}_{\mathscr{N}}$, where $\mathscr{Z}_{\mathscr{N}}$ is the center of $\mathscr{N}$. Hence $(\mathscr{B}_t \overline{\otimes} \mathscr{S})' \cap \mathscr{M} = \mathbb{C}I \otimes \mathscr{Z}_{\mathscr{N}} = \mathscr{Z}$. Since $\mathscr{S}$ is a hyperfinite type II$_1$ factor, there exists an increasing sequence $\{\mathscr{F}_r : r \in \mathbb{N}\}$ of matrix subalgebras of $\mathscr{S}$ whose union is ultraweakly dense in $\mathscr{S}$ and thus $\cup_{r \in \mathbb{N}} \mathscr{B}_t \overline{\otimes} \mathscr{F}_r$ is ultraweakly dense in $\mathscr{B}_t \overline{\otimes} \mathscr{S}$. Let $\mathscr{V}_r$ be the unitary group of $\mathscr{B}_t \overline{\otimes} \mathscr{F}_r$ with normalized Haar measure dv_r. Since $(\mathscr{B}_t \overline{\otimes} \mathscr{S})' \cap \mathscr{M} = \mathscr{Z}$ and τ is a center-valued trace on $\mathscr{M}$ such that $\tau(a) = a$ for all $a \in \mathscr{Z}$, Lemma 5.4.4 in [22] shows that $\tau(a) = \lim_{r \to \infty} \int_{\mathscr{V}_r} vav^* dv_r$ ultraweakly for all $a \in \mathscr{M}$. Since each ϕ_i is normal, there exists r large enough such that

$$|\phi_i(\int_{\mathscr{V}_r} vb_j v^* dv_r - \tau(b_j))| < 1/t, \quad \forall i, j = 1, 2, \ldots, t. \tag{4.37}$$

Now we let

$$\mathscr{A}_t = \mathscr{B}_t \overline{\otimes} \mathscr{F}_r.$$

Then $\mathscr{A}_t$ satisfies both conditions (a) and (b). The construction is finished.

Let $\mathscr{R} \subset \mathscr{M}$ be the ultraweak closure of $\cup_{t \in \mathbb{N}} \mathscr{A}_t$. It follows that $\mathscr{R}$ is a finite von Neumann algebra containing an ultraweakly dense matricial C*-algebra. By Corollary 12.1.3 in [16], $\mathscr{R}$ is a hyperfinite type II$_1$ subfactor of $\mathscr{M}$.

Now fix $n \in \mathbb{N}$, $\varepsilon > 0$ and elements $a_1, a_2, \ldots, a_k$ in $\mathscr{M}$. We may first assume that $\|a_l\| \leqslant 1$ for any $1 \leqslant l \leqslant k$. Since $\{b_j : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball of $\mathscr{M}$, there exist elements $b_{j_1}, b_{j_2}, \ldots, b_{j_k}$ such that

$$\|a_l - b_{j_l}\|_2 < \varepsilon/3 \tag{4.38}$$

for any $1 \leqslant l \leqslant k$. For each integer $t > \max\{n, j_1, j_2, \ldots, j_k\}$, by condition (a), there exist a μ-null subset N_t of X and a set of n orthogonal equivalent projections $\{p_{1t}, p_{2t}, \ldots, p_{nt}\}$ in $\mathscr{A}_t$ such that

$$\rho_s((p_{it}(s)b_{j_l}(s) - b_{j_l}(s)p_{it}(s))^*(p_{it}(s)b_{j_l}(s) - b_{j_l}(s)p_{it}(s))) < 1/t \tag{4.39}$$

for all $i \in \{1, 2, \ldots, n\}$, $l \in \{1, 2, \ldots, k\}$, and $s \in X \setminus N_t$.

Take $N = \cup_{t \in \mathbb{N}} N_t$. Then $\mu(N) = 0$ and inequality (4.39) implies

$$\lim_{t \to \infty} \rho_s((p_{it}(s)b_{j_l}(s) - b_{j_l}(s)p_{it}(s))^*(p_{it}(s)b_{j_l}(s) - b_{j_l}(s)p_{it}(s))) = 0 \tag{4.40}$$

for all $i \in \{1, 2, \ldots, n\}$, $l \in \{1, 2, \ldots, k\}$, and $s \in X \setminus N$. For any fixed $i \in \{1, 2, \ldots, n\}$, $l \in \{1, 2, \ldots, k\}$, define function $f_t : X \to \mathbb{C}$ such that

$$f_t(s) = \rho_s((p_{it}(s)b_{j_l}(s) - b_{j_l}(s)p_{it}(s))^*(p_{it}(s)b_{j_l}(s) - b_{j_l}(s)p_{it}(s))).$$

Then $|f_t(s)| \leqslant \rho_s(4I_s)$ for almost every $s \in X$. Since

$$\int_X \rho_s(4I_s) d\mu = \rho(4I) = 4,$$

by the Dominated Convergence Theorem, (4.40) gives

$$\lim_{t\to\infty}\rho((p_{it}b_{j_l}-b_{j_l}p_{it})^*(p_{it}b_{j_l}-b_{j_l}p_{it}))=0$$

for all $i \in \{1,2,\ldots,n\}$, $l \in \{1,2,\ldots,k\}$. Hence there exists $t_0 \in \mathbb{N}$ such that

$$\|p_{it}b_{j_l}-b_{j_l}p_{it}\|_2 < \varepsilon/3 \tag{4.41}$$

for all $i \in \{1,2,\ldots,n\}$, $l \in \{1,2,\ldots,k\}$ and $t > t_0$.

Therefore for any $t > t_0$, it follows from (4.38) and (4.41) that

$$\begin{aligned}
\|p_{it}a_l - a_l p_{it}\|_2 &\leqslant \|p_{it}b_{j_l}-b_{j_l}p_{it}\|_2 + \|p_{it}(a_l-b_{j_l})-(a_l-b_{j_l})p_{it}\|_2 \\
&\leqslant \|p_{it}b_{j_l}-b_{j_l}p_{it}\|_2 + 2\|a_l - b_{j_l}\|_2 \\
&< \varepsilon
\end{aligned}$$

for all $i \in \{1,2,\ldots,n\}$, $l \in \{1,2,\ldots,k\}$. Hence $\lim_{t\to\infty}\|p_{it}a_l - a_l p_{it}\|_2 = 0$ for all $i \in \{1,2,\ldots,n\}$, $l \in \{1,2,\ldots,k\}$.

It remains to show that $\mathscr{R}' \cap \mathscr{M} = \mathscr{Z}$. Suppose that $a \in \mathscr{R}' \cap \mathscr{M}$ and $\|a\| = 1$. Since the sequence $\{b_j : j \in \mathbb{N}\}$ is *SOT* dense in the unit ball of $\mathscr{M}$, we can choose a subsequence $\{b_{j_l} : l \in \mathbb{N}\}$ that converges to a in the strong operator topology. Therefore this subsequence converges to a ultraweakly. By the fact that τ is ultraweakly continuous, $\lim_{l\to\infty}\tau(b_{j_l}) = \tau(a)$ ultraweakly. Since $a \in \mathscr{R}'$, for each $i \in \mathbb{N}$,

$$\begin{aligned}
|\phi_i(\int_{\mathscr{U}_{j_l}}ub_{j_l}u^*d\mu_{j_l}-a)| &= |\phi_i(\int_{\mathscr{U}_{j_l}}u(b_{j_l}-a)u^*d\mu_{j_l})| \\
&\leqslant (\phi_i((b_{j_l}-a)^*(b_{j_l}-a)))^{1/2} \\
&\to 0.
\end{aligned}$$

From the fact that the sequence $\{\phi_i : i \in \mathbb{N}\}$ is norm dense in the set of normal states on $\mathscr{M}$, we get that $\int_{\mathscr{U}_{j_l}}ub_{j_l}u^*d\mu_{j_l}$ converges to a ultraweakly. By condition (b), $\int_{\mathscr{U}_{j_l}}ub_{j_l}u^*d\mu_{j_l}$ converges to $\tau(a)$ ultraweakly. Therefore $a = \tau(a)$ and thus $a \in \mathscr{Z}$. Hence $\mathscr{R}' \cap \mathscr{M} = \mathscr{Z}$. The proof is complete. $\square$

5. Necessary inequalities

Suppose $\mathscr{M}$ is a von Neumann algebra and $\mathscr{N}$ is a von Neumann subalgebra of $\mathscr{M}$. A map $\phi : \mathscr{M}^k \to B(H)$ is called $\mathscr{N}$-multimodular if, for any $s \in \mathscr{N}$ and any $a_1, a_2, \ldots, a_k \in \mathscr{M}$,

$$s\phi(a_1, a_2, \ldots, a_k) = \phi(sa_1, a_2, \ldots, a_k),$$

$$\phi(a_1, a_2, \ldots, a_k)s = \phi(a_1, a_2, \ldots, a_k s),$$

$$\phi(a_1, a_2, \ldots, a_i s, a_{i+1}, \ldots, a_k) = \phi(a_1, a_2, \ldots, a_i, sa_{i+1}, \ldots, a_k).$$

For any $n \in \mathbb{N}$, the n-fold amplication $\phi^{(n)} : (M_n(\mathcal{M}))^k \to M_n(\mathcal{M})$ of a bounded map $\phi : \mathcal{M}^k \to \mathcal{M}$ is defined in [6] and [7] as follows: for elements $(a_{ij}^{(1)}), (a_{ij}^{(2)}), \ldots, (a_{ij}^{(k)})$ in $M_n(\mathcal{M})$, the (i,j) entry of $\phi^{(n)}((a_{ij}^{(1)}), (a_{ij}^{(2)}), \ldots, (a_{ij}^{(k)}))$ is

$$\sum_{1 \leqslant j_1, j_2, \ldots, j_{n-1} \leqslant n} \phi(a_{ij_1}^{(1)}, a_{j_1 j_2}^{(2)}, \ldots, a_{j_{n-2} j_{n-1}}^{(n-1)}, a_{j_{n-1} j}^{(n)}).$$

A bounded map ϕ is said to be completely bounded if $\sup_{n \in \mathbb{N}} \{\|\phi^{(n)}\| : n \in \mathbb{N}\} < \infty$. When ϕ is completely bounded, we denote $\|\phi\|_{cb} = \sup_{n \in \mathbb{N}} \{\|\phi^{(n)}\| : n \in \mathbb{N}\}$.

Let $\{e_{ij}\}_{i,j=1}^n$ be the standard matrix units for $M_n(\mathbb{C})$. Then

$$\|\phi^{(n)}(e_{11} a_1 e_{11}, e_{11} a_2 e_{11}, \ldots, e_{11} a_k e_{11})\| \leqslant \|\phi\| \|a_1\| \ldots \|a_k\|$$

for any $a_1, a_2, \ldots, a_k$ in $M_n(\mathcal{M})$.

If $\mathcal{M}$ is a type II$_1$ von Neumann algebra and n is a positive integer, $M_n(\mathcal{M})$ is also a type II$_1$ von Neumann algebra. In the rest of this section, we let τ_n be the center-valued trace on $M_n(\mathcal{M})$ such that $\tau_n(a) = a$ for any a in the center of $M_n(\mathcal{M})$ (see Theorem 8.2.8 in [16]). Let

$$\gamma_n(a) = (\|a\|^2 + n\|\tau_n(a^* a)\|)^{1/2} \tag{5.1}$$

for each $a \in M_n(\mathcal{M})$.

Replacing tr_n by τ_n in the proof of Lemma 3.1 in [4], we can obtain the next lemma directly.

LEMMA 5.1. *Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra acting on a Hilbert space H. Suppose $\mathcal{R}$ is a hyperfinite type II$_1$ subfactor of $\mathcal{M}$ such that $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$, the center of $\mathcal{M}$. Let θ be a positive number and n be a positive integer. If $\psi : M_n(\mathcal{M}) \times M_n(\mathcal{M}) \to B(H^n)$ is a normal bilinear map satisfying*

$$\psi(ac, b) = \psi(a, cb), a, b \in M_n(\mathcal{M}), c \in M_n(\mathcal{R})$$

and

$$\|\psi(a e_{11}, e_{11} b)\| \leqslant \theta \|a\| \|b\|, a, b \in M_n(\mathcal{M}),$$

then

$$\|\psi(a, b)\| \leqslant \theta \gamma_n(a) \gamma_n(b)$$

for any $a, b \in M_n(\mathcal{M})$

If Lemma 3.1 in [4] is replaced by the preceding Lemma 5.1, the proof of Theorem 3.3 in [4] gives us the following result.

LEMMA 5.2. *Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra acting on a Hilbert space H. Suppose $\mathcal{M}$ has a hyperfinite subfactor $\mathcal{R}$ such that $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$, the*

center of $\mathcal{M}$. Fix $k \in \mathbb{N}$. If $\phi : \mathcal{M}^k \to B(H)$ is a k-linear $\mathcal{N}$-multimodular normal map, then

$$\|\phi^{(n)}(a_1, a_2, \ldots, a_k)\| \leqslant 2^{k/2}\|\phi\|\gamma_n(a_1)\gamma_n(a_2)\ldots\gamma_n(a_k)$$

for all $a_1, a_2, \ldots, a_k \in M_n(\mathcal{M})$ and $n \in \mathbb{N}$.

COROLLARY 5.3. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra and $\mathcal{Z}$ the center of $\mathcal{M}$. Suppose $\mathcal{R}$ is a hyperfinite type II_1 subfactor of $\mathcal{M}$ such that $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$, the center of $\mathcal{M}$. Let $n, k \in \mathbb{N}$. Suppose $p_1, p_2, \ldots, p_n$ are n orthogonal equivalent projections in $M_n(\mathcal{M})$ with sum I and $\phi : \mathcal{M}^k \to B(H)$ is a k-linear $\mathcal{R}$-multimodular normal map. Then*

$$\|\phi^{(n)}(a_1 p_j, a_2 p_j, \ldots, a_k p_j)\| \leqslant 2^k\|\phi\|\|a_1\|\|a_2\|\ldots\|a_k\|$$

for any $j = 1, 2, \ldots, n$ and any $a_1, a_2, \ldots, a_k \in M_n(\mathcal{M})$.

Proof. By Lemma 5.2, for any $j = 1, 2, \ldots, n$,

$$\|\phi^{(n)}(a_1 p_j, a_2 p_j, \ldots, a_k p_j)\| \leqslant 2^{k/2}\|\phi\|\gamma_n(a_1 p_j)\gamma_n(a_2 p_j)\ldots\gamma_n(a_k p_j). \qquad (5.2)$$

Since $p_1, p_2, \ldots, p_n$ are orthogonal equivalent projections with sum I, $\tau_n(p_j) = \frac{1}{n}I$ for each j. Then for any $1 \leqslant i \leqslant k$,

$$
\begin{aligned}
\gamma_n(a_i p_j) &= (\|a_i p_j\|^2 + n\|\tau_n(p_j a_i^* a_i p_j)\|)^{1/2} \\
&\leqslant (\|a_i\|)^2 + n\|a_i\|^2\|\tau_n(p_j)\|)^{1/2} \\
&= \sqrt{2}\|a_i\|.
\end{aligned}
$$

Therefore (5.2) gives

$$\|\phi^{(n)}(a_1 p_j, a_2 p_j, \ldots, a_k p_j)\| \leqslant 2^k\|\phi\|\|a_1\|\|a_2\|\ldots\|a_k\|$$

for any $j = 1, 2, \ldots, n$ and any $a_1, a_2, \ldots, a_k \in M_n(\mathcal{M})$. $\quad\square$

6. Hochschild cohomology of type II_1 von Neumann algebras with separable predual and property Γ

Let us recall some notations from [4]. Let S_k, $k \geqslant 2$, be the set of nonempty subsets of $\{1, 2, \ldots, k\}$. Suppose $\phi : \mathcal{M}^k \to B(H)$ is a k-linear map, p is a projection in $\mathcal{M}$ and $\sigma \in S_k$.

Define $\phi_{\sigma, p} : \mathcal{M}^k \to B(H)$ by

$$\phi_{\sigma, p}(a_1, \ldots, a_k) = \phi(b_1, b_2, \ldots, b_k),$$

where $b_i = pa_i - a_i p$ for $i \in \sigma$ and $b_i = a_i$ otherwise.

Denote by $l(\sigma)$ the least integer in σ. Define $\phi_{\sigma,p,i} : \mathcal{M}^k \to B(H)$ by changing the i-th variable in $\phi_{\sigma,p}$ from a_i to $pa_i - a_ip$, $1 \leqslant i < l(\sigma)$, and replacing $pa_i - a_ip$ by $p(pa_i - a_ip)$ if $i = l(\sigma)$.

The following is Lemma 6.1 in [4].

LEMMA 6.1. ([4]) *Let p be a projection in a von Neumann algebra $\mathcal{M}$. Let $\mathscr{C}_k, k \geqslant 2$, be the set of k-linear maps $\phi : \mathcal{M}^k \to B(H)$ satisfying*

$$p\phi(a_1, a_2, \ldots, a_k) = \phi(pa_1, a_2, \ldots, a_k) \tag{6.1}$$

and

$$\phi(a_1, \ldots, a_ip, a_{i+1}, \ldots, a_k) = \phi(a_1, \ldots, a_i, pa_{i+1}, \ldots, a_k) \tag{6.2}$$

for any $a_1, a_2, \ldots, a_k \in \mathcal{M}$ and $1 \leqslant i \leqslant k-1$. Then if $\phi \in \mathscr{C}_k$,

$$p\phi(a_1, a_2, \ldots, a_k) - p\phi(a_1 p, \ldots, a_k p) = \sum_{\sigma \in S_k} (-1)^{|\sigma|+1} p\phi_{\sigma,p}(a_1, \ldots, a_k).$$

Moreover, for each $\sigma \in S_k$,

$$p\phi_{\sigma,p}(a_1, \ldots, a_k) = \sum_{i=1}^{l(\sigma)} \phi_{\sigma,p,i}(a_1, a_2, \ldots, a_k).$$

Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra with separable predual. Suppose ρ is a faithful normal tracial state on $\mathcal{M}$. Then by Lemma 3.3, the 2-norm induced by ρ gives the same topology as the strong operator topology on bounded subsets of $\mathcal{M}$. The unit ball $(\mathcal{M})_1$ is a metric space under this 2-norm. Using a similar argument as Section 4 in [4], we can get the joint continuity of ϕ on $(\mathcal{M})_1 \times (\mathcal{M})_1 \times \ldots \times (\mathcal{M})_1$ in the 2-norm induced by ρ. Therefore we have the following lemma.

LEMMA 6.2. *Let $\mathcal{M}$ be a type II$_1$ von Neumann algebra with separable predual and $\phi : \mathcal{M}^k \to B(H)$ be a bounded k-linear separately normal map. Let ρ be a faithful normal tracial state on $\mathcal{M}$. Suppose $\{p_t : t \in \mathbb{N}\}$ is a sequence of projections in $\mathcal{M}$ satisfying (6.1), (6.2) and*

$$\lim_{t \to \infty} \|p_t a - a p_t\|_2 = 0$$

for any $a \in \mathcal{M}$, where $\|\cdot\|_2$ is the 2-norm induced by ρ. Then for any $a_1, a_2, \ldots, a_k \in \mathcal{M}$, each $\sigma \in S_k$, each integer $i \leqslant l(\sigma)$ and each pair of unit vectors $x, y \in H$,

$$\lim_{t \to \infty} \langle \phi_{\sigma,p_t,i}(a_1, a_2, \ldots, a_k)x, y \rangle = 0,$$

and

$$\lim_{t \to \infty} \langle p_t \phi_{\sigma,p_t}(a_1, \ldots, a_k)x, y \rangle = 0.$$

Proof. The proof is similar to the one of Lemma 6.2 in [4] and is skipped here. □

Now we have the following result.

THEOREM 6.3. *Let $\mathcal{M}$ be a type II_1 von Neumann algebra with separable pre-dual and $\mathcal{Z}$ be the center of $\mathcal{M}$. Let ρ be a faithful normal tracial state on $\mathcal{M}$ and $\|\cdot\|_2$ be the 2-norm induced by ρ on $\mathcal{M}$. Suppose $\mathcal{R}$ is a hyperfinite type II_1 subfactor of $\mathcal{M}$ such that*

(I) $\mathcal{R}' \cap \mathcal{M} = \mathcal{Z}$;

(II) *for any $n \in \mathbb{N}$, any elements $a_1, a_2, \ldots, a_m$ in $\mathcal{M}$, there exists a countable collection of projections $\{p_{1t}, p_{2t}, \ldots, p_{nt} : t \in \mathbb{N}\}$ in $\mathcal{R}$ such that*

    (i) *for each $t \in \mathbb{N}$, $p_{1t}, p_{2t}, \ldots, p_{nt}$ are mutually orthogonal equivalent projections in $\mathcal{R}$ with sum I, the identity in $\mathcal{M}$;*

    (ii) $\lim_{t \to \infty} \|p_{it} a_l - a_l p_{it}\|_2 = 0$ *for any $i = 1, 2, \ldots, n$, $l = 1, 2, \ldots, m$.*

Then a bounded k-linear $\mathcal{R}$-multimodular separately normal map $\phi : \mathcal{M}^k \to B(H)$ is completely bounded and $\|\phi\|_{cb} \leqslant 2^k \|\phi\|$.

Proof. The proof is similar to the one for Theorem 6.3 in [4] and is sketched here for the purpose of completeness.

Fix $n \in \mathbb{N}$ and k elements $b_1, b_2, \ldots, b_k \in M_n(\mathcal{M})$.

By condition (II), we can find a family of projections $\{q_{it} : 1 \leqslant i \leqslant n; t \in \mathbb{N}\}$ in $\mathcal{R}$ such that

(a) for each $t \in \mathbb{N}$, $q_{1t}, \ldots, q_{nt}$ are n orthogonal equivalent projections in $\mathcal{R}$ with sum I;

(b) $\lim_{t \to \infty} \|q_{it} a - a q_{it}\|_2 = 0$ for any $a \in \mathcal{M}, 1 \leqslant i \leqslant n$.

Let $q'_{it} = I_n \otimes q_{it} \in M_n(\mathcal{R})$ for each i and t. We obtain that

(a') for each $t \in \mathbb{N}$, $q'_{1t}, \ldots, q'_{nt}$ are n orthogonal equivalent projections in $M_n(\mathcal{R})$ with sum $I_n \otimes I$;

(b') $\lim_{t \to \infty} \|q'_{it} b - b q'_{it}\|_2 = 0$ for any $b \in M_n(\mathcal{M}), 1 \leqslant i \leqslant n$.

Since ϕ is an $\mathcal{R}$-multimodular map, $\phi^{(n)}$ is an $M_n(\mathcal{R})$-multimodular map. Assume that $\mathcal{M}$ acts on a Hilbert space H. For any two unit vectors x, y in H^n and any $t \in \mathbb{N}$, by Lemma 6.1,

$$\langle \phi^{(n)}(b_1, \ldots, b_k) x, y \rangle$$
$$= \langle \sum_{i=1}^{n} q'_{it} \phi^{(n)}(b_1, \ldots, b_k) x, y \rangle$$
$$= \langle \sum_{i=1}^{n} \sum_{\sigma \in S_k} (-1)^{|\sigma|+1} q'_{it} \phi^{(n)}_{\sigma, q'_{it}}(b_1, \ldots, b_k) x, y \rangle + \langle \sum_{i=1}^{n} q'_{it} \phi^{(n)}(b_1 q'_{it}, \ldots, b_k q'_{it}) x, y \rangle$$
$$= \langle \sum_{i=1}^{n} \sum_{\sigma \in S_k} (-1)^{|\sigma|+1} q'_{it} \phi^{(n)}_{\sigma, q'_{it}}(b_1, \ldots, b_k) x, y \rangle + \langle \sum_{i=1}^{n} q'_{it} \phi^{(n)}(b_1 q'_{it}, \ldots, b_k q'_{it}) q'_{it} x, y \rangle.$$

Therefore

$$\langle \phi^{(n)}(b_1,\ldots,b_k)x,y \rangle - \langle \sum_{i=1}^{n} \sum_{\sigma \in S_k} (-1)^{|\sigma|+1} q'_{it} \phi^{(n)}_{\sigma,q'_{it}}(b_1,\ldots,b_k)x,y \rangle$$

$$= \langle \sum_{i=1}^{n} q'_{it} \phi^{(n)}(b_1 q'_{it},\ldots,b_k q'_{it}) q'_{it} x, y \rangle. \tag{6.3}$$

Since $\{q'_{1t},\ldots,q'_{nt}\}$ is a set of n orthogonal projections for each $t \in \mathbb{N}$, by Corollary 5.3,

$$\| \sum_{i=1}^{n} q'_{it} \phi^{(n)}(b_1 q'_{it},\ldots,b_k q'_{it}) q'_{it} \| \leqslant \max_{1 \leqslant i \leqslant n} \| q'_{it} \phi^{(n)}(b_1 q'_{it},\ldots,b_k q'_{it}) q'_{it} \|$$

$$\leqslant 2^k \|\phi\| \|b_1\| \ldots \|b_k\|. \tag{6.4}$$

By Lemma 6.2, condition (b') implies

$$\lim_{t \to \infty} \langle q'_{it} \phi^{(n)}_{\sigma,q'_{it}}(b_1,\ldots,b_k)x,y \rangle = 0 \tag{6.5}$$

for each $1 \leqslant i \leqslant n$ and $\sigma \in S_k$.

Letting $t \to \infty$ for both sides of (6.3), it follows from inequality (6.4) and equation (6.5) that

$$\langle \phi^{(n)}(b_1,\ldots,b_k)x,y \rangle \leqslant 2^k \|\phi\| \|b_1\| \ldots \|b_k\|.$$

Since n,x,y were arbitrarily chosen, $\|\phi\|_{cb} \leqslant 2^k \|\phi\|$. $\quad\square$

The following is the main result of the paper.

THEOREM 6.4. *If $\mathcal{M}$ is a type II$_1$ von Neumann algebra with separable predual and Property Γ, then the Hochschild cohomology group*

$$H^k(\mathcal{M},\mathcal{M}) = 0, \qquad \forall\, k \geqslant 2.$$

Proof. By Theorem 4.4, there is a hyperfinite type II$_1$ subfactor $\mathcal{R}$ of $\mathcal{M}$ satisfying conditions (I) and (II) in Theorem 6.3.

Now consider the cohomology groups $H^k(\mathcal{M},\mathcal{M})$. By Theorem 3.1.1 in [22], it suffices to consider a k-linear $\mathcal{R}$-multimodular separately normal cocycle ϕ. Theorem 5.3 shows that such cocycles are completely bounded. By Theorem 4.3.1 in [22], completely bounded Hochschild cohomology groups are trivial. It follows that ϕ is a coboundary, whence $H^k(\mathcal{M},\mathcal{M}) = 0, k \geqslant 2$. $\quad\square$

The next result in [2] follows directly from Theorem 6.4 and Example 3.9.

COROLLARY 6.5. *Suppose that $\mathcal{M}_1$ is a type II$_1$ von Neumann algebra with separable predual and $\mathcal{M}_2$ is a type II$_1$ factor with separable predual. If $\mathcal{M}_2$ has Property Γ, then the Hochschild cohomology group*

$$H^k(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathcal{M}_1 \otimes \mathcal{M}_2) = 0, \qquad k \geqslant 2.$$

In particular, if $\mathscr{M}$ is a type II_1 von Neumann algebra with separable predual satisfying $\mathscr{M} \cong \mathscr{M} \otimes \mathscr{R}$, where $\mathscr{R}$ is the hyperfinite II_1 factor, then

$$H^k(\mathscr{M}, \mathscr{M}) = 0, \qquad \forall\, k \geqslant 2.$$

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