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IDEAL–TRIANGULARIZABILITY AND COMMUTATORS OF CONSTANT SIGN

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Abstract. Let E be a Banach lattice with order continuous norm, and let A and B be positive compact operators such that the commutator $AB-BA$ is also positive. We prove that if A and B are ideal-triangularizable, then they are simultaneously ideal-triangularizable, or equivalently, the sum $A+B$ is ideal-triangularizable. We then show several related results for operators of constant sign (an operator T on E is of constant sign if either T or $-T$ is positive). In particular, we consider ideal-triangularizability for Lie sets of compact operators of constant sign (a set of operators is a Lie set whenever it is closed under taking commutators).

1. Introduction and preliminaries

As an application of Lomonosov's theorem [17] we can show that every commutative family of compact operators on a Banach space is triangularizable [20, Theorem 7.2.1]. We start this paper with Banach lattice analogs of this well-known result. The finite-dimensional setting has been already studied in [5]. For example, it was shown there that a commutative family of nonnegative matrices is ideal-triangularizable whenever every matrix in the family is ideal-triangularizable. This theorem follows easily from the first result in [5] asserting that ideal-triangularizable nonnegative matrices A and B are simultaneously ideal-triangularizable whenever the commutator $AB-BA$ is also a nonnegative matrix. In this paper we consider the infinite-dimensional setting. Section 2 is devoted to ideal-triangularizability of families of power-compact operators of constant sign, while Section 3 slightly improves a recent result [8, Theorem 4.5] for semigroups of positive compact operators. In Section 4 we restrict ourselves to Lie sets of compact operators. Although Lie algebras of compact operators were already considered by Wojtyński, Shulman, Turovskii, Kennedy, Radjavi and others in [25], [13], [15], [22], [23] and [14], it seems that Lie sets of compact operators were not considered so far in greater detail. In the rest of this section we recall some definitions and basic facts.

Let E be a Riesz space, and let E^+ denote the set of all positive vectors in E. The band

$$A^d := \{x \in E : \ |x| \wedge |a| = 0 \text{ for all } a \in A\}$$

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is called the *disjoint complement* of a set A in E. A band B of E is said to be a *projection band* if $E = B \oplus B^d$. The corresponding band projection onto B is denoted by P_B. It is well-known that every band of a Dedekind complete Riesz space is a projection band. A Banach lattice E is said to have an *order continuous norm* whenever every decreasing net $\{x_\alpha\}_\alpha$ with infimum 0 of vectors in E^+ converges to 0 in norm. It is well-known that every Banach lattice with order continuous norm is Dedekind complete, and that its closed ideals are bands.

An operator T on a Riesz space E is said to be of *constant sign* if either T or $-T$ is a positive operator. An operator T on E is said to be a *regular operator* if it can be written as a finite linear combination of positive operators. The vector space of all regular operators on E is denoted by $\mathscr{L}_r(E)$. It is well-known that $\mathscr{L}_r(E)$ is Dedekind complete whenever E is. Assume that E is a Banach lattice. It is well-known that every regular operator on E is necessarily bounded. A bounded operator T on E is *power-compact* if there exists a positive integer k such that T^k is a compact operator.

A nonzero vector $a \in E^+$ is an *atom* in a normed Riesz space E if $0 \leqslant x, y \leqslant a$ and $x \wedge y = 0$ imply either $x = 0$ or $y = 0$, or equivalently, if $0 \leqslant x \leqslant a$ implies $x = \lambda a$ for some $\lambda \geqslant 0$, i.e., the principal ideal B_a generated by a is one-dimensional. It turns out that B_a is a projection band [18]. The decomposition $E = B_a \oplus B_a^d$ implies that for an arbitrary (positive) vector $x \in E$ there exist a (positive) scalar λ_x and a (positive) vector $y_x \in B_a^d$ such that $x = \lambda_x a + y_x$. The linear functional $\varphi_a : E \to \mathbb{R}$ associated to the atom a is defined by $\varphi_a(x) = \lambda_x$. It is called a *coordinate functional associated with the atom a*.

Let E be a Dedekind complete Banach lattice. The *centre* $\mathscr{Z}(E)$ is the principal ideal in $\mathscr{L}_r(E)$ generated by the identity I, i.e.,

$$\mathscr{Z}(E) = \{T \in \mathscr{L}_r(E) : \ |T| \leqslant \lambda I \text{ for some } \lambda \geqslant 0\}.$$

Since $\mathscr{Z}(E)$ is also a band in a Dedekind complete Riesz space $\mathscr{L}_r(E)$, we have the following order direct sum decomposition $\mathscr{L}_r(E) = \mathscr{Z}(E) \oplus \mathscr{Z}(E)^d$. Let $\mathscr{P}$ be the band projection onto $\mathscr{Z}(E)$. The *atomic diagonal* (or just the *diagonal*) of a regular operator T on a Dedekind complete Banach lattice is the operator $\mathscr{D}(T) := P_A \mathscr{P}(T)$, where A denotes the band in E generated by all atoms of E. Let $\mathscr{A}$ be a maximal set of pairwise disjoint atoms in A. In [12], the second named author obtained the following description of the atomic diagonal of a positive operator T:

$$\mathscr{D}(T) = \sum_{a \in \mathscr{A}} P_a T P_a,$$

where the sum is the order limit of the net of operators $\sum_{a \in \mathscr{F}} P_a T P_a$ taken over all finite subsets $\mathscr{F}$ of $\mathscr{A}$. If $a \in A$ is an arbitrary atom, then $P_a T P_a = \varphi_a(Ta) P_a$ where P_a denotes the band projection onto the projection band B_a. Hence, we obtain the following identity that holds for a positive operator T:

$$\mathscr{D}(T) = \sum_{a \in \mathscr{A}} \varphi_a(Ta) P_a. \tag{1}$$

For the terminology and details not explained about Banach lattices and operators on them we refer the reader to [2] and [21].

We say that a chain $\mathscr{C}$ of closed subspaces of a Banach space X is a *complete* chain if it contains arbitrary intersections and closed linear spans of its members. If a closed subspace $\mathscr{M}$ is in a complete chain $\mathscr{C}$, then the *predecessor* $\mathscr{M}_-$ of $\mathscr{M}$ in $\mathscr{C}$ is defined as the closed linear span of all proper subspaces of $\mathscr{M}$ belonging to $\mathscr{C}$. Every maximal chain $\mathscr{C}$ of closed subspaces of X is complete and, for each subspace $\mathscr{M}$ in $\mathscr{C}$, the dimension of the space quotient space $\mathscr{M}/\mathscr{M}_-$ is at most one.

A family $\mathscr{F}$ of operators on a Banach space X is *reducible* if there exists a nontrivial closed subspace of X that is invariant under every operator from $\mathscr{F}$. Otherwise we say that $\mathscr{F}$ is *irreducible.* If there exists a maximal chain $\mathscr{C}$ of closed subspaces of X such that every subspace from the chain $\mathscr{C}$ is invariant under every operator from $\mathscr{F}$, then $\mathscr{F}$ is said to be *triangularizable*, and $\mathscr{C}$ is called a *triangularizing chain* for $\mathscr{F}$. A family $\mathscr{F}$ of operators on a Banach lattice E is said to be *ideal-reducible* if there exists a nontrivial closed ideal of E that is invariant under every operator from $\mathscr{F}$. Otherwise, we say that $\mathscr{F}$ is *ideal-irreducible*. A family $\mathscr{F}$ of operators on a Banach lattice is said to be *ideal-triangularizable* if it is triangularizable and at least one of (possibly many) triangularizing chains of $\mathscr{F}$ consists of closed ideals of E.

Let us recall [6, Theorem 6.7] that we will use several times. Throughout the paper, a semigroup of operators is just a family of operators closed under multiplication.

THEOREM 1.1. *Let E be a Banach lattice with order continuous norm. A semigroup $\mathscr{S}$ of positive compact operators on E is ideal-triangularizable if and only if every pair $\{S,T\}$ of operators in $\mathscr{S}$ is ideal-triangularizable.*

The preceding theorem does not hold for power-compact operators. In fact, there exists an irreducible semigroup of positive nilpotent operators on $L^2[0,1)$ but its every finite subset is ideal-triangularizable [9].

Let $\mathscr{I}$ and $\mathscr{J}$ be closed ideals in a Banach lattice E that are invariant under every operator from a given family $\mathscr{F}$ of operators on E. If $\mathscr{I} \subseteq \mathscr{J}$, then $\mathscr{F}$ induces a family $\widehat{\mathscr{F}}$ of operators on the quotient Banach lattice $\mathscr{J}/\mathscr{I}$ as follows. For each $T \in \mathscr{F}$, the operator $\widehat{T}$ is defined by $\widehat{T}(x+\mathscr{I}) = Tx+\mathscr{I}$. Any such family $\widehat{\mathscr{F}}$ is called a *family of ideal-quotients* of a family $\mathscr{F}$. A set $\mathscr{P}$ of properties is said to be *inherited by ideal-quotients* if every family of ideal-quotients of a family satisfying $\mathscr{P}$ also satisfies the same properties. Among well-known properties like compactness, quasinilpotence, positivity of operators and others, ideal-triangularizability is also inherited by ideal-quotients [11]. When $\mathscr{J}$ belongs to a complete chain of closed ideals and $\mathscr{I} = \mathscr{J}_-$, we denote the operator $\widehat{T}$ by $T_{\mathscr{J}}$.

The following lemma is a very important tool in obtaining ideal-triangularizing chains for families of positive operators on Banach lattices. The proof can be found in [4].

LEMMA 1.2. (The Ideal-triangularization Lemma) *Let $\mathscr{P}$ be the set of properties inherited by ideal-quotients. If every family of operators on a Banach lattice of dimension greater than one which satisfies $\mathscr{P}$ is ideal-reducible, then every such family is ideal-triangularizable.*

For a more detailed treatment on triangularizability we refer the reader to [20].

2. Families of operators

In [5] it is proved that ideal-triangularizable nonnegative matrices A and B are simultaneously ideal-triangularizable whenever the commutator $AB-BA$ is also a nonnegative matrix. An infinite-dimensional generalization of this theorem is the main result of this section (Theorem 2.3). We start with the following simple lemma.

LEMMA 2.1. *Let E be a Banach lattice, and let A and B be positive operators on E such that the commutator $AB-BA$ is of constant sign. Then the operators A and B are power-compact if and only if the operator $A+B$ is power-compact.*

Proof. If the operator $(A+B)^k$ is compact for some positive integer k, then $0 \leqslant A,B \leqslant A+B$ and [2, Theorem 5.13] imply that A^{3k} and B^{3k} are compact operators, so that the operators A and B are power-compact.

The converse statement follows from [7, Lemma 2.2]. □

The proof of the main result of this section is based on the following proposition.

PROPOSITION 2.2. *Let E be a Banach lattice with order continuous norm, and let A and B be positive power-compact operators on E such that the commutator $AB-BA$ is of constant sign. Suppose that A is a nonzero ideal-triangularizable operator and $A+B$ is an ideal-irreducible operator. Then A is a positive multiple of the identity operator I, i.e., there exists a number $\lambda>0$ such that $A=\lambda I$. Consequently, E is a finite-dimensional Banach lattice and the operator B is ideal-irreducible.*

Proof. The operator $C=A+B$ is power-compact by Lemma 2.1. Since C is ideal-irreducible, it is not quasinilpotent by [7, Theorem 1.3]. Without loss of generality we may assume $r(C)=1$. The Krein-Rutman theorem for positive power-compact operators (see e.g. [1, Exercise 7.1.9]) implies that there exists a nonzero positive linear functional φ on E such that $C^*\varphi=\varphi$. From this it follows that the absolute kernel

$$N(\varphi)=\{x\in E:\ \varphi(|x|)=0\}$$

of the functional φ is invariant under C. Since C is ideal-irreducible and φ is nonzero, we conclude that $N(\varphi)=\{0\}$, so that φ is strictly positive on E. By [21, Theorem V.5.2], the kernel of the operator $I-C$ is one-dimensional and it is spanned by some quasi-interior point $u\in E$. Obviously, $\varphi(u)>0$. Since C is power-compact, it is essentially nilpotent, which implies that 1 is not in the essential spectrum of C. By [1, Theorem 7.44], 1 is a pole of the resolvent of the operator C. Finally, [10, Proposition 2.6] implies that the kernel of the operator $I-C^*$ is spanned by the functional φ. Observe that

$$\varphi(Au-CAu)=\varphi(Au)-(C^*\varphi)(Au)=0.$$

We now consider only the case when $AB\geqslant BA$, as the other case can be treated similarly. Since

$$Au-CAu=(AC-CA)u=(AB-BA)u\geqslant 0$$

and φ is strictly positive, we conclude that $CAu = Au$. This implies that there exists $\lambda \geqslant 0$ such that $Au = \lambda u$. We necessarily have $\lambda > 0$, as $A \neq 0$ and u is a quasi-interior point. Since the functional

$$C^* A^* \varphi - A^* \varphi = (C^* A^* - A^* C^*)\varphi = (AC - CA)^* \varphi = (AB - BA)^* \varphi$$

is positive and u is a quasi-interior point in E, the equality

$$(C^* A^* \varphi - A^* \varphi)(u) = \varphi(ACu) - \varphi(Au) = 0$$

implies $A^* \varphi = C^* A^* \varphi$. Therefore, there exists $\mu \geqslant 0$ such that $A^* \varphi = \mu \varphi$. In fact, we have $\lambda = \mu$, since

$$\mu \varphi(u) = A^* \varphi(u) = \varphi(Au) = \lambda \varphi(u),$$

and $\varphi(u) \neq 0$.

Let $\mathscr{C}$ be an ideal-triangularizing chain for the operator A. By [16, Theorem 2.7], there exists a closed ideal $\mathscr{J} \in \mathscr{C}$ such that $\dim(\mathscr{J} / \mathscr{J}_-) = 1$ and λ is the diagonal coefficient of the operator A with respect to the chain $\mathscr{C}$ corresponding to the subspace $\mathscr{J}$. It follows that $\mathscr{J}_-$ is a maximal band in $\mathscr{J}$, and so, by [18, Theorem 26.7], there exists an atom $e \in \mathscr{J}$ such that $\mathscr{J} = \mathscr{J}_- \oplus \mathbb{R}e$. It should be noted that e is an atom in E as well. With respect to the decompositions $E = \mathscr{J}_- \oplus \mathbb{R}e \oplus \mathscr{J}^d$ and $E^* = \mathscr{J}_-^* \oplus \mathbb{R}\varphi_e \oplus (\mathscr{J}^d)^*$ the operators A and A^* can be decomposed as

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & \lambda & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \text{and} \qquad A^* = \begin{bmatrix} A_{11}^* & 0 & 0 \\ A_{12}^* & \lambda & 0 \\ A_{13}^* & A_{23}^* & A_{33}^* \end{bmatrix},$$

respectively. Let us write the vector u as $u = [u_1, u_2, u_3]^T$, where $u_1 \in \mathscr{J}_-$ and $u_3 \in \mathscr{J}^d$ are quasi-interior points of $\mathscr{J}_-$ and $\mathscr{J}^d$, respectively, and u_2 is a positive multiple of e. Similarly, we write $\varphi = [\varphi_1, \varphi_2, \varphi_3]^T$, where $\varphi_1 \in \mathscr{J}_-^*$ and $\varphi_3 \in (\mathscr{J}^d)^*$ are strictly positive functionals on $\mathscr{J}_-^*$ and $(\mathscr{J}^d)^*$, respectively, and φ_2 is a positive multiple of φ_e. The equality $Au = \lambda u$ implies

$$\begin{bmatrix} A_{11}u_1 + A_{12}u_2 + A_{13}u_3 \\ \lambda u_2 + A_{23}u_3 \\ A_{33}u_3 \end{bmatrix} = \begin{bmatrix} \lambda u_1 \\ \lambda u_2 \\ \lambda u_3 \end{bmatrix},$$

so that $A_{23}u_3 = 0$. Since u_3 is a quasi-interior point of $\mathscr{J}^d$, we have $A_{23} = 0$. Similarly, the equality $A^* \varphi = \lambda \varphi$ implies that $A_{12}^* \varphi_1 + \lambda \varphi_2 = \lambda \varphi_2$, so that $A_{12}^* \varphi_1 = 0$. Since the functional φ_1 is strictly positive on $\mathscr{J}_-$, we have $A_{12}^* = 0$ which gives us $A_{12} = 0$. Therefore, with respect to the decomposition $E = \mathscr{J}_- \oplus \mathbb{R}e \oplus \mathscr{J}^d$, the operator A has the form

$$A = \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & \lambda & 0 \\ 0 & 0 & A_{33} \end{bmatrix}. \tag{2}$$

Let $\mathscr{F}$ be the set of all atoms a of E such that there exists a closed ideal $\mathscr{I} \in \mathscr{C}$ satisfying $\mathscr{I}_- \oplus \mathbb{R}a = \mathscr{I}$, and the diagonal coefficient of the operator A with respect to the ideal $\mathscr{I}$ is λ. Let $\mathscr{B}$ be the band generated by all atoms in $\mathscr{F}$. We claim that $\mathscr{B} = E$. Suppose otherwise. By (2), we have

$$A = \begin{bmatrix} \lambda I & 0 \\ 0 & \tilde{A} \end{bmatrix}$$

with respect to the decomposition $E = \mathscr{B} \oplus \mathscr{B}^d$. Let us write the vector u as $u = [v_1, v_2]^T$, where $v_1 \in \mathscr{B}$ and $v_2 \in \mathscr{B}^d$ are quasi-interior points of $\mathscr{B}$ and $\mathscr{B}^d$, respectively. Similarly, we write $\varphi = [\phi_1, \phi_2]^T$, where $\phi_1 \in \mathscr{B}^*$ and $\phi_2 \in (\mathscr{B}^d)^*$ are strictly positive functionals on $\mathscr{B}$ and $\mathscr{B}^*$, respectively. Then we have $\tilde{A}v_2 = \lambda v_2$ and $\tilde{A}^*\phi_2 = \lambda\phi_2$. Now, [11, Proposition 2.3] and its proof imply that the operator $\tilde{A}$ is ideal-triangularizable on $\mathscr{B}^d$, and $\mathscr{C}' = \{\mathscr{J} \cap \mathscr{B}^d : \mathscr{J} \in \mathscr{C}\}$ is an ideal-triangularizing chain for $\tilde{A}$. By already proved, there exists a closed ideal $\mathscr{J}' \in \mathscr{C}'$ and an atom $f \in \mathscr{B}^d$ such that, with respect to the decomposition $\mathscr{B}^d = \mathscr{J}'_- \oplus \mathbb{R}f \oplus ((\mathscr{J}')^d \cap \mathscr{B}^d)$, the operator $\tilde{A}$ has the form

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & 0 & \tilde{A}_{13} \\ 0 & \lambda & 0 \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix}.$$

This is a contradiction with the definition of the band $\mathscr{B}$. Therefore, $\mathscr{B} = E$ and $A = \lambda I$. As A is power-compact, E has to be a finite-dimensional Banach lattice. Since $B = C - \lambda I$, the operator B is ideal-irreducible. $\square$

The following theorem is an infinite-dimensional extension of [5, Theorem 2.1].

THEOREM 2.3. *Let E be a Banach lattice with order continuous norm, and let A and B be positive power-compact operators such that the commutator $AB - BA$ is of constant sign. If A and B are ideal-triangularizable, then the pair $\{A, B\}$ is ideal-triangularizable, or equivalently, the sum $A + B$ is ideal-triangularizable.*

Proof. Assume that the sum $C = A + B$ is ideal-irreducible. Then $A \neq 0$, and so the operator B is ideal-irreducible by Proposition 2.2. This contradicts the ideal-triangularizability of B. Therefore, C is necessarily ideal-reducible. In view of Lemma 1.2 we now conclude that C is ideal-triangularizable, which is clearly equivalent to the ideal-triangularizability of the pair $\{A, B\}$, as $0 \leqslant A, B \leqslant C$. $\square$

The following example shows that in Theorem 2.3 we cannot omit the assumption that the operators A and B are power-compact.

EXAMPLE 2.4. *Let S and S^* be the forward and the backward shift on the Banach lattice l^2, respectively. Obviously each of them is ideal-triangularizable, and their commutator $S^*S - SS^*$ is a positive operator of rank one. An easy verification shows that the pair $\{S, S^*\}$ is ideal-irreducible.*

The following corollary is a generalization of Theorem 2.3 in the case of compact operators, and it can viewed as an order analog of [22, Corollary 4.16] (see also [24]).

COROLLARY 2.5. *Let E be a Banach lattice with order continuous norm, and let $\mathscr{F}$ and $\mathscr{G}$ be ideal-triangularizable families of positive compact operators on E. If $[A, B] \geqslant 0$ for all $A \in \mathscr{F}$ and $B \in \mathscr{G}$, then $\mathscr{F} \cup \mathscr{G}$ is ideal-triangularizable.*

Proof. Let $\mathscr{S}$ be the semigroup generated by the families $\mathscr{F}$ and $\mathscr{G}$. It suffices to prove that every pair of operators from $\mathscr{S}$ is ideal-triangularizable, as then we may apply Theorem 1.1.

Let us now choose operators S and T in $\mathscr{S}$. Then S and T are finite products of operators from $\mathscr{F}$ and $\mathscr{G}$. Suppose that $\{A_1,\ldots,A_n\} \subseteq \mathscr{F}$ and $\{B_1,\ldots,B_m\} \subseteq \mathscr{G}$ appear as factors in S and T. Ideal-triangularizability of $\mathscr{F}$ implies ideal-triangularizability of $A_1+\cdots+A_n$. Similarly, $B_1+\cdots+B_m$ is ideal-triangularizable as well. Theorem 2.3 and the following inequality

$$\sum_{i=1}^{n} A_i \sum_{j=1}^{m} B_j = \sum_{i=1}^{n}\sum_{j=1}^{m} A_i B_j \geqslant \sum_{i=1}^{n}\sum_{j=1}^{m} B_j A_i = \sum_{j=1}^{m} B_j \sum_{i=1}^{n} A_i$$

imply that the family $\{A_1,\ldots,A_n,B_1,\ldots,B_m\}$ is ideal-triangularizable, so that the pair $\{S,T\}$ is ideal-triangularizable which was needed to be proved. □

We now turn our attention to commutative families of ideal-triangularizable operators. In the special case of compact operators the following theorem is a simple consequence of Theorems 2.3 and 1.1.

THEOREM 2.6. *Let E be a Banach lattice with order continuous norm. Then every commutative family $\mathscr{F}$ of ideal-triangularizable power-compact positive operators on E is ideal-triangularizable.*

Proof. Let $\mathscr{S}$ be the semigroup generated by the family $\mathscr{F}$. Since $\mathscr{F}$ is commutative, $\mathscr{S}$ is commutative and it consists of power-compact operators. By Theorem 2.3 and a simple induction argument it is easy to see that every finite subset of $\mathscr{F}$ is ideal-triangularizable. It follows that every pair of operators from $\mathscr{S}$ is ideal-triangularizable.

We claim that $\mathscr{S}$ is ideal-reducible. If $\mathscr{S} = \{0\}$, then there is nothing to prove. Otherwise, there exists a nonzero power-compact positive operator A in $\mathscr{S}$. If A is nilpotent, then the absolute kernel $N(A) = \{x \in E : A|x| = 0\}$ is a nonzero closed ideal invariant under $\mathscr{S}$. So we may assume that A is not nilpotent. Hence, there exists a nonzero positive compact operator T in $\mathscr{S}$. Let $\mathscr{J}$ be the semigroup ideal in $\mathscr{S}$ generated by the operator T. Then $\mathscr{J}$ is ideal-triangularizable by Theorem 1.1, so that $\mathscr{S}$ is ideal-reducible by [6, Proposition 2.1].

Now, we apply the Ideal-triangularization Lemma to complete the proof. □

COROLLARY 2.7. *Let E be a Banach lattice with order continuous norm, and let $\mathscr{F}$ and $\mathscr{G}$ be commutative families of ideal-triangularizable power-compact positive operators on E. If $[A,B] \geqslant 0$ for all $A \in \mathscr{F}$ and $B \in \mathscr{G}$, then $\mathscr{F} \cup \mathscr{G}$ is ideal-triangularizable.*

Proof. By Theorem 2.6, $\mathscr{F}$ and $\mathscr{G}$ are ideal-triangularizable. Clearly, we may assume that $\mathscr{F} \neq \{0\}$ and $\mathscr{G} \neq \{0\}$. Let $\mathscr{S}$ be the semigroup generated by the families $\mathscr{F}$ and $\mathscr{G}$. As in the proof of Corollary 2.5 we show that every pair of operators from $\mathscr{S}$ is ideal-triangularizable; we also need the fact that a finite sum of commuting power-compact operators is power-compact as well. Let us prove that the family $\mathscr{F} \cup \mathscr{G}$ is ideal-reducible.

If $\mathscr{G}$ has a nonzero nilpotent operator B, then the absolute kernel $N(B)$ of B is a nontrivial closed ideal that is clearly invariant under every member of $\mathscr{G}$, and it is also invariant under any operator $A \in \mathscr{F}$, as $0 \leqslant B|Ax| \leqslant BA|x| \leqslant AB|x| = 0$ for all $x \in N(B)$. In a similar manner we show that if $\mathscr{F}$ contains a nonzero nilpotent operator A, then the closed ideal generated by the range of A is a nontrivial ideal that is invariant under every member of $\mathscr{F} \cup \mathscr{G}$.

Assume now that $\mathscr{F} \cup \mathscr{G}$ contains no nonzero nilpotent operator. Then the semigroup $\mathscr{S}$ contains a nonzero compact operator K. The semigroup ideal $\mathscr{J}$ of $\mathscr{S}$ generated by K is ideal-reducible by Theorem 1.1, and so the whole semigroup $\mathscr{S}$ is also ideal-reducible by [6, Proposition 2.1].

Finally, an application of the Ideal-triangularization Lemma completes the proof. $\square$

3. Semigroups of operators

Let $\mathscr{F}$ be a family of ideal-triangularizable nonnegative matrices. If $\mathscr{F}$ is ideal-triangularizable, then the diagonals of all commutators of matrices from $\mathscr{F}$ are zero. The converse statement does not hold, as it was shown in [8, Example 4.7]. However, if the given family is considered to be a semigroup, then the converse statement holds, as it was proved in [8, Corollary 4.6]. This is just a special case of a more general result [8, Theorem 4.5] which we recall here.

THEOREM 3.1. *Let E be a Banach lattice with order continuous norm, and let $\mathscr{S}$ be a semigroup of ideal-triangularizable positive compact operators. If $\mathscr{D}(AB) = \mathscr{D}(BA)$ for all A, B in $\mathscr{S}$, then $\mathscr{S}$ is ideal-triangularizable.*

In this section we slightly extend the preceding theorem to semigroups of ideal-triangularizable compact operators of constant sign with the property that the diagonals of commutators of the operators from the semigroup are also of constant sign. We need some facts on diagonals.

If T is a power-compact operator on a Banach space X, then, by the classical spectral theory, for each nonzero complex number λ the operator $\lambda - T$ has finite ascent k, i.e., k is the smallest positive integer such that $\ker((\lambda - T)^k) = \ker((\lambda - T)^{k+1})$. In this case the *(algebraic) multiplicity* $m(T, \lambda)$ of λ is the dimension of the subspace $\ker((\lambda - T)^k)$.

PROPOSITION 3.2. *Let A and B be positive power-compact ideal-triangularizable operators on a Banach lattice with order continuous norm. Suppose that $\mathscr{D}(B) \leqslant \mathscr{D}(A)$. If $m(A, \lambda) = m(B, \lambda)$ for every nonzero $\lambda \in \mathbb{C}$, then $\mathscr{D}(A) = \mathscr{D}(B)$.*

Proof. The equality $\sigma(A) \backslash \{0\} = \sigma(B) \backslash \{0\}$ implies that A and B are both either quasinilpotent or non-quasinilpotent. In the first case we have $\mathscr{D}(A) = \mathscr{D}(B) = 0$, by (1) and [8, Theorem 4.2].

Assume now that A and B are not quasinilpotent and $\mathscr{D}(A) \neq \mathscr{D}(B)$. Then there is an atom $e \in E$ such that $\varphi_e(Ae) > \varphi_e(Be)$. Let

$$\lambda = \max\{\varphi_e(Ae) : e \text{ is an atom of norm one in } E \text{ and } \varphi_e(Ae) > \varphi_e(Be)\}. \quad (3)$$

Note that this maximum actually exists by [8, Theorem 4.2] and the fact that A is a positive ideal-triangularizable power-compact operator.

Let us denote by $\mathscr{A}$ and $\mathscr{B}$ the set of all atoms $e \in E$ of norm one such that $\varphi_e(Ae) = \lambda$ and $\varphi_e(Be) = \lambda$, respectively. Since $m(A,\lambda) = m(B,\lambda)$, the cardinality of the set $\mathscr{A}$ is equal to the cardinality of the set $\mathscr{B}$, and both are finite. From (3) we obtain $\mathscr{A} \neq \mathscr{B}$ which implies that there exists some atom e of norm one in $\mathscr{B} \setminus \mathscr{A}$. Hence, $\varphi_e(Be) = \lambda < \varphi_e(Ae)$ which is in contradiction with maximality of λ. Therefore, we have $\mathscr{D}(A) = \mathscr{D}(B)$ and the proof is finished. $\square$

COROLLARY 3.3. *Let E be a Banach lattice with order continuous norm, and let A and B be positive operators on E such that AB and BA are ideal-triangularizable. If the diagonal of the commutator $AB-BA$ is of constant sign and one of the operators AB and BA is power-compact, then $\mathscr{D}(AB) = \mathscr{D}(BA)$.*

Proof. By Pietsch's principle of related operators [19, 3.3.3], both operators AB and BA are power-compact, and for every nonzero complex number λ we have that $m(AB,\lambda) = m(BA,\lambda)$. Now, Proposition 3.2 implies that $\mathscr{D}(AB) = \mathscr{D}(BA)$. $\square$

We now slightly extend Theorem 3.1.

THEOREM 3.4. *Let E be a Banach lattice with order continuous norm, and let $\mathscr{S}$ be a semigroup of ideal-triangularizable compact operators on E of constant sign. If the diagonal of the commutator of every pair of operators from $\mathscr{S}$ is of constant sign, then $\mathscr{S}$ is ideal-triangularizable.*

Proof. Define

$$\tilde{\mathscr{S}} = \{A \in \mathscr{S} : A \geqslant 0\} \cup \{-A : A \in \mathscr{S}, A \leqslant 0\}.$$

Then $\tilde{\mathscr{S}}$ is a semigroup of positive compact ideal-triangularizable operators, since for all $A,B \in \tilde{\mathscr{S}}$ we have either $AB \in \mathscr{S}$ or $-AB \in \mathscr{S}$. Furthermore, if A and B are in $\tilde{\mathscr{S}}$, then $\mathscr{D}(AB-BA)$ is of constant sign, and so $\mathscr{D}(AB) = \mathscr{D}(BA)$ by Corollary 3.3. Now we apply Theorem 3.1 to conclude that $\tilde{\mathscr{S}}$ (and so $\mathscr{S}$) is ideal-triangularizable. $\square$

4. Lie sets of operators

Let X be a Banach space and let $\mathscr{B}(X)$ be the Banach algebra of all continuous linear operators on X. A *Lie set* is a subset $\mathscr{M}$ of $\mathscr{B}(X)$ that is closed under the commutator $[A,B] = AB - BA$, i.e., $AB - BA$ is in $\mathscr{M}$ whenever A and B are in $\mathscr{M}$. A *Lie algebra* $\mathscr{L}$ is a Lie set that is also a subspace of $\mathscr{B}(X)$. Every operator $A \in \mathscr{L}$ defines a linear operator $\mathrm{ad}_{\mathscr{L}}(A) : \mathscr{L} \to \mathscr{L}$ by $\mathrm{ad}_{\mathscr{L}}(A)B = [A,B]$. A subspace $\mathscr{I}$ of $\mathscr{L}$ is said to be a *Lie ideal* if $[A,B] \in \mathscr{I}$ for all $A \in \mathscr{I}$ and $B \in \mathscr{L}$. If for every $A \in \mathscr{L}$ the operator $\mathrm{ad}_{\mathscr{L}}(A)$ is quasinilpotent, then $\mathscr{L}$ is said to be an *Engel Lie algebra*. An *Engel ideal* is a Lie ideal of $\mathscr{L}$ that is also an Engel Lie algebra on its own. For a Lie set $\mathscr{M}$, define the set $\mathscr{C}(\mathscr{M}) = \{[A,B] : A,B \in \mathscr{M}\}$, and let $[\mathscr{M},\mathscr{M}]$ denote the linear span of $\mathscr{C}(\mathscr{M})$.

Engel's theorem states that a Lie algebra of nilpotent endomorphisms on a finite-dimensional vector space is triangularizable. The assumptions of Engel's theorem can be relaxed. Namely, every Lie set $\mathscr{M}$ of endomorphisms of a finite-dimensional vector space is triangularizable if and only if the set $\mathscr{C}(\mathscr{M})$ consists of nilpotent operators (see [20, Corollary 1.7.8]). Wojtyński [25] posed a problem whether Engel's result could be extended to Lie algebras of quasinilpotent operators. He generalized Engel's result to Lie algebras of quasinilpotent Schatten operators on Hilbert spaces. Much later Shulman and Turovskii [22] (see also [23]) extended Wojtyński's result to Lie algebras of compact operators on Banach spaces.

THEOREM 4.1. [23, Corollary 4.23] *A Lie algebra of compact operators which contains a nonzero Engel ideal is reducible. In particular, every Engel Lie algebra of compact operators is triangularizable.*

In [5, Theorem 2.6] the authors proved that a collection $\mathscr{C}$ of ideal-triangularizable $n\times n$ matrices of constant sign is ideal-triangularizable if the Lie set generated by $\mathscr{C}$ consists of matrices of constant sign. Since a positive commutator of positive matrices is nilpotent [3], this result can be considered as a finite-dimensional lattice analog of the strengthened Engel's result [20, Corollary 1.7.8]. The main purpose of this section is to obtain an infinite-dimensional extension of [5, Theorem 2.6].

We proceed with the following result which is an extension (in the case of trace class operators) of [20, Theorem 7.3.4] to Lie sets. Let $\mathscr{B}^1(\mathscr{H})$ be the Banach space of all trace-class operators on a Hilbert space $\mathscr{H}$.

THEOREM 4.2. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{M}$ a Lie set in $\mathscr{B}^1(\mathscr{H})$. The following statements are equivalent:*

(a) $\mathscr{M}$ is triangularizable.

(b) Every pair of operators from $\mathscr{M}$ is triangularizable.

(c) For all A,B and C in $\mathscr{M}$ the operator $A(BC-CB)$ is quasinilpotent.

Proof. It is obvious that (a) implies (b). Let us now assume that (b) holds. Let A,B and C be arbitrary operators in $\mathscr{M}$. Since the pair $\{B,C\}$ is triangularizable, the operator $BC-CB$ is a quasinilpotent operator by [20, Theorem 7.2.6]. The operators A and $BC-CB$ are simultaneously triangularizable as well, since $BC-CB\in\mathscr{M}$. Therefore, the operator $A(BC-CB)$ is quasinilpotent operator by [20, Theorem 7.2.6] again, so that (c) holds.

For the proof that (c) implies (a) we start with the following observation. Since the properties of operators in the theorem are inherited by quotients, we will prove that $\mathscr{M}$ is reducible, as we may apply the Triangularization lemma [20, Lemma 7.1.11].

By Lidskii's theorem we have

$$\operatorname{tr}(ABC)-\operatorname{tr}(ACB)=\operatorname{tr}(A(BC-CB))=0$$

for all operators A,B and C from $\mathscr{M}$. Let $\mathscr{L}$ be the Lie algebra generated by the set $\mathscr{M}$. It should be noted that $\mathscr{L}$ is just the linear span of $\mathscr{M}$. Linearity of the trace implies that for all A,B and C from $\mathscr{L}$ we have

$$\operatorname{tr}(ABC) = \operatorname{tr}(ACB). \tag{4}$$

Let $\mathscr{L}_1$ be the closure of $\mathscr{L}$ in $\mathscr{B}^1(\mathscr{H})$ in the trace norm. Since the trace is continuous on $\mathscr{B}^1(\mathscr{H})$, we obtain that the equality (4) holds for all A,B and C from $\mathscr{L}_1$. If $\mathscr{L}_1$ is an Engel Lie algebra, then $\mathscr{L}_1$ is reducible by Theorem 4.1. Otherwise, [13, Lemma 4.2] implies that there exists a nonzero finite rank operator in $\mathscr{L}_1$. Let $\mathscr{F}_1$ be the Lie ideal of all finite rank operators in $\mathscr{L}_1$.

If the Lie ideal $\mathscr{F}_1$ of $\mathscr{L}_1$ satisfies $[\mathscr{F}_1,\mathscr{F}_1] = 0$, then $\mathscr{F}_1$ is a nonzero Engel ideal of $\mathscr{L}_1$, so that $\mathscr{L}_1$ is reducible by Theorem 4.1. Therefore we may assume that the Lie ideal $[\mathscr{F}_1,\mathscr{F}_1]$ of $\mathscr{L}_1$ is nonzero.

If the Lie ideal $[[\mathscr{F}_1,\mathscr{F}_1],[\mathscr{F}_1,\mathscr{F}_1]]$ of $\mathscr{L}_1$ is a zero ideal, then $[\mathscr{F}_1,\mathscr{F}_1]$ is a nonzero Engel ideal of $\mathscr{L}_1$, so that again $\mathscr{L}_1$ is reducible by Theorem 4.1. Finally we may assume that $[[\mathscr{F}_1,\mathscr{F}_1],[\mathscr{F}_1,\mathscr{F}_1]]$ is nonzero in $\mathscr{L}_1$.

Since the equality (4) holds for all A,B and C from $\mathscr{L}_1$, we now conclude that $\operatorname{tr}(AB) = 0$ for all operators $A \in \mathscr{F}_1$ and $B \in [\mathscr{F}_1,\mathscr{F}_1]$. Since $[\mathscr{F}_1,\mathscr{F}_1] \subseteq \mathscr{F}_1$, we therefore have $\operatorname{tr}(AB) = 0$ for all $A,B \in [\mathscr{F}_1,\mathscr{F}_1]$. [13, Theorem 4.6] implies that every operator in $[[\mathscr{F}_1,\mathscr{F}_1],[\mathscr{F}_1,\mathscr{F}_1]]$ is nilpotent, so that $[[\mathscr{F}_1,\mathscr{F}_1],[\mathscr{F}_1,\mathscr{F}_1]]$ is a nonzero Engel ideal of $\mathscr{L}_1$. Reducibility of $\mathscr{L}_1$ follows again from Theorem 4.1. □

It should be noted that Shulman and Turovskii [23, Corollary 5.17] proved that a Lie algebra $\mathscr{L}$ of compact operators is triangularizable if and only if every pair of operators from $\mathscr{L}$ is triangularizable. The preceding theorem relaxes the assumption that $\mathscr{L}$ is a Lie algebra and at the same time it requires more, namely, every operator needs to be a trace class operator.

COROLLARY 4.3. *Let E be a Banach lattice with order continuous norm and $\mathscr{M}$ a Lie set of compact operators on E of constant sign. Then $\mathscr{M}$ is triangularizable in both of the following cases:*

(a) $\mathscr{C}(\mathscr{M})$ consists of finite rank operators.

(b) E is a Hilbert lattice and $\mathscr{M} \subseteq \mathscr{B}^1(E)$.

If, in addition, every member of $\mathscr{M}$ is ideal-triangularizable, then $\mathscr{M}$ is ideal-triangularizable.

Proof. Assume that (a) holds. Let A and B be arbitrary operators in $\mathscr{M}$. Since A,B and $AB-BA$ are of constant sign, [3, Theorem 2.2] easily implies that the operator $AB-BA$ is quasinilpotent. Let $\mathscr{L}$ be the Lie algebra generated by $\mathscr{M}$, that is, $\mathscr{L}$ is the linear span of $\mathscr{M}$. It suffices to show that $\mathscr{L}$ is triangularizable.

The set $\mathscr{C}(\mathscr{M})$ is a Lie subset of finite rank nilpotent operators in $\mathscr{M}$. Suppose that A is an arbitrary operator in $\mathscr{C}(\mathscr{M})$. Since A is nilpotent, the operator $\operatorname{ad}(A)$ is a nilpotent operator on $\mathscr{B}(E)$ which implies that the restriction $\operatorname{ad}_{\mathscr{L}}(A)$ of $\operatorname{ad}(A)$ to the Lie algebra $\mathscr{L}$ is nilpotent as well. [15, Lemma 4.1] implies that the Lie ideal $[\mathscr{L},\mathscr{L}]$ of $\mathscr{L}$ consists of nilpotent finite rank operators. We conclude from [15, Theorem 4.7(8)] that $\mathscr{L}$ is triangularizable.

Assume now that (b) holds. Let A and B be arbitrary operators in $\mathscr{M}$. Without any loss of generality we may assume that A, B and $[A,B]$ are positive operators on E. By Zorn's lemma there exists a maximal chain $\mathscr{C}$ of closed ideals invariant under the operator $A+B$. It should be clear that every ideal in $\mathscr{C}$ is invariant under both A and B. The maximality of the chain $\mathscr{C}$ implies that the operator $(A+B)_{\mathscr{J}} = A_{\mathscr{J}} + B_{\mathscr{J}}$ on $\mathscr{J}/\mathscr{J}_-$ is ideal-irreducible. Since we have $[A_{\mathscr{J}},(A+B)_{\mathscr{J}}] = [A_{\mathscr{J}},B_{\mathscr{J}}] \geqslant 0$, [3, Theorem 2.2] implies $[A_{\mathscr{J}},B_{\mathscr{J}}] = 0$.

Let p be an arbitrary polynomial in two non-commuting variables. We claim that the operator $p(A,B)(AB-BA)$ is quasinilpotent. The block triangular form of Ringrose's theorem [20, Theorem 7.2.7] implies that $\sigma(p(A,B)(AB-BA)) \cup \{0\}$ is equal to the union

$$\bigcup_{\mathscr{J}\in\mathscr{C},\,\mathscr{J}\neq\mathscr{J}_-} \sigma(p(A_{\mathscr{J}},B_{\mathscr{J}})(A_{\mathscr{J}}B_{\mathscr{J}} - B_{\mathscr{J}}A_{\mathscr{J}})) \cup \{0\},$$

so that we actually have $\sigma(p(A,B)(AB-BA)) = \{0\}$, since $A_{\mathscr{J}}$ and $B_{\mathscr{J}}$ commute on $\mathscr{J}/\mathscr{J}_-$. An infinite-dimensional version of McCoy's theorem [20, Theorem 7.3.3] implies that A and B are simultaneously triangularizable. An application of Theorem 4.2 implies that $\mathscr{M}$ is triangularizable.

It remains to show that $\mathscr{M}$ is even ideal-triangularizable if every member of $\mathscr{M}$ is ideal-triangularizable. If $\mathscr{M}$ is commutative, then $\mathscr{M}$ is ideal-triangularizable by Theorem 2.6. So, we may assume that $\mathscr{M}$ is not commutative. Then $\mathscr{M}$ contains a nonzero positive commutator C of operators from $\mathscr{M}$. Let $\mathscr{S}$ be the semigroup generated by the set of all positive operators in $\mathscr{M}$. Triangularizability of $\mathscr{M}$ implies that $\mathscr{S}$ is triangularizable as well (and every triangularizing chain for $\mathscr{M}$ is a triangularizing chain for $\mathscr{S}$). Let $\mathscr{J}$ be the semigroup ideal in $\mathscr{S}$ generated by C. Triangularizability of $\mathscr{S}$ and [20, Theorem 7.2.6] imply that $\mathscr{J}$ consists of quasinilpotent operators. Now, [4, Theorem 4.5] implies that $\mathscr{J}$ is ideal-reducible, so that $\mathscr{S}$ is ideal-reducible by [6, Proposition 2.1]. We finish the proof by applying the Ideal-triangularization lemma. $\square$

We do not know whether Theorem 4.2 holds for Lie sets of compact operators. We conclude the paper by mentioning that Theorem 4.2 and Corollary 4.3 still hold if we enlarge $\mathscr{M}$ by adding multiples of the identity operator to operators in $\mathscr{M}$.

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