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ON THE SPECTRAL SINGULARITIES AND SPECTRALITY OF THE HILL OPERATOR

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Abstract. First we study the spectral singularity at infinity and investigate the connections of the spectral singularities and the spectrality of the Hill operator. Then we consider the spectral expansion when there is not the spectral singularity at infinity.

1. Introduction

In this paper we investigate the one dimensional Schrödinger operator $L(q)$ generated in $L_2(-\infty,\infty)$ by the differential expression

$$l(y)=-y^{''}(x)+q(x)y(x), \tag{1}$$

where q is 1-periodic, Lebesgue integrable on $[0,1]$ and complex-valued potential. Without loss of generality, we assume that the integral of q over $[0,1]$ is 0. It is well-known [1,5-9] that the spectrum $\sigma(L)$ of the operator L is the union of the spectra $\sigma(L_t)$ of the operators $L_t(q)$ for $t\in(-\pi,\pi]$ generated in $L_2[0,1]$ by (1) and the boundary conditions

$$y(1)=e^{it}y(0),\ y^{'}(1)=e^{it}y^{'}(0). \tag{2}$$

The eigenvalues of L_t are the roots of the characteristic equation

$$F(\lambda)=2\cos t, \tag{3}$$

where $F(\lambda)=:\varphi^{'}(1,\lambda)+\theta(1,\lambda)$ is the Hill discriminant $\theta(x,\ \lambda)$ and $\varphi(x,\lambda)$ are the solutions of the equation $l(y)=\lambda y$ satisfying the following initial conditions

$$\theta(0,\lambda)=\varphi^{'}(0,\lambda)=1,\quad \theta^{'}(0,\lambda)=\varphi(0,\lambda)=0.$$

In this paper we study the spectral singularity of $L(q)$ at infinity, investigate the connections of the spectral singularities and the spectrality of $L(q)$ and consider the spectral expansion of $L(q)$ when there is not the spectral singularity at infinity. Note that the spectral singularities of the operator $L(q)$ are the points of its spectrum in neighborhoods of which the projections of $L(q)$ are not uniformly bounded (see [9] and

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[10]). McGarvey [6] proved that $L(q)$ is a spectral operator if and only if the projections of the operators $L_t(q)$ are bounded uniformly with respect to t in $(-\pi,\pi]$. Tkachenko proved in [9] that the non-self-adjoint operator L can be reduced to triangular form if all eigenvalues of the operators L_t for $t\in(-\pi,\pi]$ are simple. Gesztezy and Tkachenko [3,4] proved two versions of a criterion for the Hill operator $L(q)$ with $q\in L_2[0,1]$ to be a spectral operator of scalar type, in sense of Dunford, one analytic and one geometric. The analytic version was stated in term of the solutions of Hill's equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems. In paper [12, 13] we found the conditions on the potential q such that $L(q)$ has no spectral singularity at infinity and it is an asymptotically spectral operator.

Now let us recall the precise definition of the spectral singularities and asymptotic spectrality of $L(q)$. Following [4, 10], we define the projections and the spectral singularities of L as follows. By Definition 2.4 of [4], a closed arc

$$\gamma=:\{z\in\mathbb{C}:z=\lambda(t),t\in[\alpha,\beta]\} \tag{4}$$

with $\lambda(t)$ analytic in an open neighborhood of $[\alpha,\beta]$ and $F(\lambda(t))=2\cos t$,

$$\frac{\partial F(\lambda(t))}{\partial\lambda}\neq 0,\ \forall t\in[\alpha,\beta],\ \lambda^{'}(t)\neq 0,\ \forall t\in(\alpha,\beta)$$

is called a regular spectral arc of $L(q)$. The projection $P(\gamma)$ corresponding to the regular spectral arc γ is defined by

$$P(\gamma)f=\frac{1}{2\pi}\int_{\gamma}(\Phi_+(x,\lambda)F_-(\lambda,f)+\Phi_-(x,\lambda)F_+(\lambda,f))\frac{\varphi(1,\lambda)}{p(\lambda)}d\lambda, \tag{5}$$

where

$$\Phi_\pm(x,\lambda)=\theta(x,\lambda)+(\varphi(1,\lambda))^{-1}(e^{\pm it}-\theta(1,\lambda))\varphi(x,\lambda)$$

are the Floquet solution and

$$F_\pm(\lambda,f)=\int_{\mathbb{R}}f(x)\Phi_\pm(x,\lambda)dx,\ p(\lambda)=\sqrt{4-F^2(\lambda)}.$$

DEFINITION 1. We say that $\lambda\in\sigma(L(q))$ is a spectral singularity of $L(q)$ if for all $\varepsilon>0$ there exists a sequence $\{\gamma_n\}$ of the regular spectral arcs $\gamma_n\subset\{z\in\mathbb{C}:\mid z-\lambda\mid<\varepsilon\}$ such that

$$\lim_{n\to\infty}\parallel P(\gamma_n)\parallel=\infty. \tag{6}$$

In the similar way, we defined in [12] the spectral singularity at infinity.

DEFINITION 2. We say that the operator L has a spectral singularity at infinity if there exists a sequence $\{\gamma_n\}$ of the regular spectral arcs such that $d(0,\gamma_n)\to\infty$ as $n\to\infty$ and (6) holds, where $d(0,\gamma_n)$ is the distance from the point $(0,0)$ to the arc γ_n.

The asymptotic spectrality of the operator $L(q)$ was defined in [12] as follows. Let $e(t,\gamma)$ be the spectral projection defined by contour integration of the resolvent of $L_t(q)$, where $\gamma \in R$ and R is the ring consisting of all sets which are the finite union of the half closed rectangles. In [6] it was proved Theorem 3.5 (for the differential operators of arbitrary order with periodic coefficients rather than for $L(q)$) which can be written in the form:

$L(q)$ is a spectral operator if and only if

$$\sup_{\gamma\in R}(ess \sup_{t\in(-\pi,\pi]} \| e(t,\gamma) \|) < \infty, \tag{7}$$

where $ess\sup$ (essential supremum) is the infimum of the essential upper bounds, and an essential upper bound of f is a number A such that $f(t) \leqslant A$ for almost all $t \in (-\pi,\pi]$. According to this theorem, in [12] we gave the following definition of the asymptotic spectrality.

DEFINITION 3. The operator $L(q)$ is said to be an asymptotically spectral operator if there exists a positive constant C such that

$$\sup_{\gamma\in R(C)} (ess \sup_{t\in(-\pi,\pi]} \| e(t,\gamma) \|) < \infty,$$

where $R(C)$ is the ring consisting of all sets which are the finite union of the half closed rectangles lying in $\{\lambda \in \mathbb{C} :| \lambda |> C\}$.

In [12] and [13] we obtained the following results about asymptotic spectrality under the following conditions.

CONDITION 1. *Let $q \in W_1^p[0,1]$, $q^{(k)}(0) = q^{(k)}(1)$, for $k = 0,1,...,s-1$ and $q^{(s)}(0) \neq q^{(s)}(1)$ for some $s \leqslant p$. Suppose that $q_n \sim q_{-n}$, $| q_n |> cn^{-s-1}$ and at least one of the following inequalities*

$$\operatorname{Re} q_n q_{-n} \geqslant 0, \ \ | \operatorname{Im} q_n q_{-n} |\geqslant \varepsilon \mid q_n q_{-n} \mid$$

hold for some $c > 0$ and $\varepsilon > 0$ and for large values of n, where q_n is the Fourier coefficient of the potential q and $q_n \sim q_{-n}$ means that $q_n = O(q_{-n})$ and $q_{-n} = O(q_n)$ as $n \to \infty$.

CONDITION 2. *Suppose that*

$$q(x) = ae^{-i2\pi x} + be^{i2\pi x}, \tag{8}$$

$$| a |=| b |, \ \inf_{q,p\in\mathbb{N}}\{| q\alpha-(2p-1) |\} \neq 0, \tag{9}$$

where a and b are the complex numbers and $\alpha = \pi^{-1}\arg(ab)$.

In [12] we proved that if Condition 1 holds then $L(q)$ is an asymptotically spectral operator and has no spectral singularity at infinity. It was proven in [13] that the operator $L(q)$ with the potential (8) is an asymptotically spectral operator and has no spectral singularity at infinity if and only if (9) holds.

2. Spectrum, spectral singularity and spectrality

First, let us discuss the spectrum of $L(q)$ by using some results of [11, 12] about the uniform with respect to t in $(-\pi,\pi]$ asymptotic formulae for eigenvalues of the operator $L_t(q)$. Note that, the formula $f(k,t)=O(h(k))$ as $k\to\infty$ is said to be uniform with respect to t in a set I if there exist positive constants M and N, independent of t, such that

$$\mid f(k,t))\mid < M\mid h(k)\mid$$

for all $t\in I$ and $\mid k\mid\geqslant N$.

In the case $q=0$ the eigenvalues and eigenfunctions of $L_t(q)$ are $(2\pi n+t)^2$ and $e^{i(2\pi n+t)x}$ for $n\in\mathbb{Z}$ respectively. In [11] we proved that the large eigenvalues of the operator $L_t(q)$ for $t\neq 0,\pi$ consist of the sequence $\{\lambda_n(t):\mid n\mid\gg 1\}$ satisfying

$$\lambda_n(t)=(2\pi n+t)^2+O\left(\frac{\ln|n|}{n}\right) \tag{10}$$

as $n\to\infty$. For any fixed $\rho\in(0,\frac{\pi}{2})$, asymptotic formula (10) is uniform with respect to t in $[\rho,\pi-\rho]$. There exists a positive number $N(\rho)$, independent of t, such that the eigenvalues $\lambda_n(t)$ for $t\in[\rho,\pi-\rho]$ and $\mid n\mid > N(\rho)$ are simple and hence are the analytic function in some neighborhood of $[\rho,\pi-\rho]$.

In the paper [12] we proved that there exists a positive integer $N(0)$ such that the disk

$$U(n,t,\rho)=:\{\lambda\in\mathbb{C}:\left|\lambda-(2\pi n+t)^2\right|\leqslant 15\pi n\rho\} \tag{11}$$

for $t\in[0,\rho]$, where $15\pi\rho<1$, and $n>N(0)$ contains two eigenvalues (counting with multiplicities) denoted by $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ and these eigenvalues can be chosen as continuous functions of t on the interval $[0,\rho]$. In addition to these eigenvalues, the operator $L_t(q)$ for $t\in[0,\rho]$ has only $2N(0)+1$ eigenvalues denoted by $\lambda_k(t)$ for $k=0,\pm1,\pm2,...,\pm N$ (see Remark 2.1 of [12]). Similarly, there exists a positive integer $N(\pi)$ such that the disk $U(n,t,\rho)$ for $t\in[\pi-\rho,\pi]$ and $n>N(\pi)$ contains two eigenvalues (counting with multiplicities) denoted again by $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ that are continuous functions of t on the interval $[\pi-\rho,\pi]$. In addition to these eigenvalues, the operator $L_t(q)$ for $t\in[\pi-\rho,\pi]$ has only $2N(\pi)$ eigenvalues. Thus for $n>N$, where $N=\max\{N(\rho),N(0),N(\pi)\}$, the eigenvalues $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ are continuous on $[0,\rho]\cup[\pi-\rho,\pi]$. Moreover, for $\mid n\mid>N$ the eigenvalue $\lambda_n(t)$, defined by (10), is analytic function in the neighborhood of $[\rho,\pi-\rho]$. On the other hand, by (10) there exist only two eigenvalues $\lambda_{-n}(\rho)$ and $\lambda_n(\rho)$ of the operator $L_\rho(q)$ lying in the disk $U(n,\rho,\rho)$. Therefore these 2 eigenvalues coincides with the eigenvalues $\lambda_{n,1}(\rho)$ and $\lambda_{n,2}(\rho)$. By (10) $\mathrm{Re}(\lambda_{-n}(\rho))<\mathrm{Re}(\lambda_n(\rho))$. Let $\lambda_{n,2}(\rho)$ be the eigenvalue whose real part is larger. Then

$$\lambda_{n,1}(\rho)=\lambda_{-n}(\rho),\ \lambda_{n,2}(t)=\lambda_n(\rho),\ \forall n>N. \tag{12}$$

In the same way we obtain that

$$\lambda_{n,1}(\pi-\rho)=\lambda_n(\pi-\rho),\ \lambda_{n,2}(\pi-\rho)=\lambda_{-(n+1)}(\pi-\rho),\ \forall n>N \tag{13}$$

if $\lambda_{n,2}(\pi-\rho))$ is the eigenvalue whose real part is larger. Let Γ_{-n} be the union of the following continuous curves $\{\lambda_{n-1,2}(t) : t \in [\pi-\rho,\pi]\}$, $\{\lambda_{-n}(t) : t \in [\rho,\pi-\rho]\}$ and $\{\lambda_{n,1}(t) : t \in [0,\rho]\}$. By (12) and (13) these curves are connected and Γ_{-n} is a continuous curve. Similarly, the curve Γ_n which is the union of the curves $\{\lambda_{n,2}(t) : t \in [0,\rho]\}$, $\{\lambda_n(t) : t \in [\rho,\pi-\rho]\}$ and $\{\lambda_{n,1}(t) : t \in [\pi-\rho,\pi]\}$ is a continuous curve. For $t \in [0,\rho]$ redenote $\lambda_{n,1}(t)$ by $\lambda_{-n}(t)$ and $\lambda_{n,2}(t)$ by $\lambda_n(t)$, where $n > N$. In the same way we put $\lambda_n(t) =: \lambda_{n,1}(t)$, $\lambda_{-(n+1)}(t) =: \lambda_{n,2}(t)$ for $t \in [\pi-\rho,\pi]$ and $n > N$. In this notation we have

$$\Gamma_n = \{\lambda_n(t) : t \in [0,\pi]\}. \tag{14}$$

The eigenvalues of $L_{-t}(q)$ coincides with the eigenvalues of $L_t(q)$, because they are roots of equation (3) and $\cos(-t) = \cos t$. We define the eigenvalue $\lambda_n(-t)$ of $L_{-t}(q)$ by $\lambda_n(-t) = \lambda_n(t)$ for all $t \in (0,\pi)$. Then $\lambda_n(t)$ for $|n| > N$ is an continuous function on $(-\pi,\pi]$, $\Gamma_n = \{\lambda_n(t) : t \in (-\pi,\pi]\}$ and

$$\sigma(L(q)) = \bigcup_{t\in(-\pi,\pi]} \sigma(L_t(q)) \supset \bigcup_{|n|>N} \Gamma_n.$$

Thus the spectrum $\sigma(L)$ of the operator L contains the continuous curves Γ_n for $|n| > N$. The remaining part of $\sigma(L)$ consist of finite simple analytic arcs $\gamma_1,\gamma_2,...,\gamma_m$ whose endpoints are the eigenvalues of $L_0(q)$ and $L_\pi(q)$ and the roots of the equations $\frac{dF(\lambda)}{d\lambda} = 0$ lying in the spectrum (see [9]). On the other hand, the above arguments show that $\gamma_1 \cup \gamma_2 \cup ... \cup \gamma_m$ is the union of $2N+1$ eigenvalues (counting multiplicity and denoted by $\lambda_n(t)$ for $n = 0,\pm1,...,\pm N$) of $L_t(q)$ for $t \in (-\pi,\pi]$. Moreover, if $\lambda_n(t)$ is a root of (3) of multiplicity k, then it is the end point of k curves $\gamma_{n_1},\gamma_{n_2},...,\gamma_{n_k}$, that is, these k curves are joined by $\lambda_n(t)$. Therefore one can numerate the eigenvalues so that

$$\gamma_1 \cup \gamma_2 \cup ... \cup \gamma_m = \cup_{|n|\leqslant N} \Gamma_n \tag{15}$$

where $\Gamma_n = \{\lambda_n(t) : t \in (-\pi,\pi]\}$ for $|n| \leqslant N$ are continuous curves. Thus we have

$$\sigma(L(q)) = \bigcup_{n\in\mathbb{Z}} \Gamma_n. \tag{16}$$

Using (10), (11) and the definition of $\lambda_n(t)$ one can readily see that

$$\left|\lambda_n(t) - (2\pi k \pm t)^2\right| \geqslant |(n-k)|\,|n+k| \tag{17}$$

for $k \neq \pm n, \pm(n+1)$ and $t \in (-\pi,\pi]$, where $|n| > N$. In [11], [12] to write the asymptotic formulas for the eigenfunction $\Psi_{n,t}(x)$ corresponding to the eigenvalue $\lambda_n(t)$ we used the following relations

$$(\lambda_n(t) - (2\pi k + t)^2)(\Psi_{n,t}, e^{i(2\pi k+t)x}) = (q\Psi_{n,t}, e^{i(2\pi k+t)x}), \tag{18}$$

$$\left|(q\Psi_{n,t}, e^{i(2\pi k+t)x})\right| < 3M \tag{19}$$

for $t \in (-\pi,\pi]$, $|n| > N$ and $k \in \mathbb{Z}$, where

$$M = \sup_{n\in\mathbb{Z}} |q_n|, \; q_n = \int_0^1 q(x) e^{-i2\pi nx} dx.$$

From (17)–(19) we obtain the following, uniform with respect to t in $(-\pi,\pi]$, asymptotic formulas
$$\sum_{k\in\mathbb{Z}\backslash\{\pm n,\pm(n+1)\}}\left|(\Psi_{n,t},e^{i(2\pi k+t)x})\right|^2=O(n^{-2})$$
and
$$\sum_{k\in\mathbb{Z}\backslash\{\pm n,\pm(n+1)\}}\left|(\Psi_{n,t},e^{i(2\pi k+t)x})\right|=O\left(\frac{\ln n}{n}\right)$$
Therefore $\Psi_{n,t}(x)$ has an expansion of the form
$$\Psi_{n,t}(x)=\sum_{k\in\{\pm n,\pm(n+1)\}}u_{n,k}(t)e^{i(2\pi k+t)x}+h_{n,t}(x), \tag{20}$$
where
$$\|h_{n,t}\|=O(n^{-1}),\quad \sup_{x\in[0,1],\,t\in(-\pi,\pi]}\mid h_{n,t}(x)\mid=O\left(\frac{\ln|n|}{n}\right),\ (h_{n,t},e^{i(2\pi k+t)x})=0 \tag{21}$$
for $k\in\{\pm n,\pm(n+1)\}$ and
$$u_{n,k}(t)=(\Psi_{n,t},e^{i(2\pi k+t)x}). \tag{22}$$
Let $\{\chi_{n,t}:n\in\mathbb{Z}\}$ be biorthogonal to $\{\Psi_{n,t}:n\in\mathbb{Z}\}$ and $\Psi^*_{n,t}(x)$ be the normalized eigenfunction of $(L_t(q))^*$ corresponding to $\overline{\lambda_n(t)}$. Since the boundary condition (2) is self-adjoint we have $(L_t(q))^*=L_t(\overline{q})$. Therefore, we have
$$\Psi^*_{n,t}(x)=\sum_{k\in\{\pm n,\pm(n+1)\}}u^*_{n,k}(t)e^{i(2\pi k+t)x}+h^*_{n,t}(x), \tag{23}$$
where $u^*_{n,k}(t)=(\Psi^*_{n,t},e^{i(2\pi k+t)x})$ and
$$\left\|h^*_{n,t}\right\|=O(n^{-1}),\quad \sup_{x\in[0,1],\,t\in(-\pi,\pi]}\mid h^*_{n,t}(x)\mid=O\left(\frac{\ln|n|}{n}\right),\ (h^*_{n,t},e^{i(2\pi k+t)x})=0 \tag{24}$$
for $k\in\{\pm n,\pm(n+1)\}$. Formulas (21) and (24) are uniform with respect to t in $(-\pi,\pi]$.

Introduce the functions
$$\alpha_n(t)=(\Psi_{n,t},\Psi^*_{n,t})_{(0,1)},\ \chi_{n,t}(x)=\frac{1}{\overline{\alpha_n(t)}}\Psi^*_{n,t}(x), \tag{25}$$
where $(.,.)_{(a,b)}$ denotes the inner product in $L_2(a,b)$. One can easily verify that
$$\Psi_{n,t}(x)=\frac{\Phi_+(x,\lambda_n(t))}{\|\Phi_+(\cdot,\lambda_n(t))\|},\ \chi_{n,t}(x))=\frac{1}{\overline{\alpha_n(t)}}\frac{\overline{\Phi_-(x,\lambda_n(t))}}{\left\|\overline{\Phi_-(\cdot,\lambda_n(t))}\right\|}, \tag{26}$$
$$\Psi_{n,t}(x+1)=e^{it}\Psi_{n,t}(x),\ \Psi^*_{n,t}(x+1)=e^{it}\Psi^*_{n,t}(x),\ \chi_{n,t}(x+1)=e^{it}\chi_{n,t}(x), \tag{27}$$

where Φ_+ and Φ_- are defined in (5). Using these formulas and the equalities

$$d\lambda = -p(\lambda)\left(\frac{dF}{d\lambda}\right)^{-1} dt, \ \frac{dF(\lambda_n(t))}{d\lambda} = -\varphi(1,\lambda_n(t))(\Phi_+(\cdot,\lambda_n(t)),\overline{\Phi_-(\cdot,\lambda_n(t))}) \tag{28}$$

and changing the variable λ to the variable t in the integral (5) we get

$$P(\gamma)f = \frac{1}{2\pi}\int_{\delta}(f,\chi_{n,t})_{\mathbb{R}}\Psi_{n,t}dt, \tag{29}$$

where $\delta = \{t \in (-\pi,\pi] : \lambda_n(t) \in \gamma\}$ and $\gamma \subset \Gamma_n$.

It is very natural that the projection of the operator $L(q)$ is connected with the projection

$$e(t,C) =: P_t(\lambda_n(t))f = \frac{1}{2\pi i}\int_C (L_t - \lambda I)^{-1} f dt \tag{30}$$

of the operator $L_t(q)$, where C is a closed contour enclosing $\lambda_n(t)$ but no other eigenvalues of $L_t(q)$. In this paper we use the following proposition which was proved in [10].

PROPOSITION 1. *Let $\gamma \subset \Gamma_n$ be regular spectral arc and $\delta = \{t : \lambda_n(t) \in \gamma\}$. Then the operators $P(\gamma)$ and $P_t(\lambda_n(t))$ for $t \in \delta$ are projections and*

$$\| P_t(\lambda_n(t)) \| = | \alpha_n(t) |^{-1}, \tag{31}$$

$$\| P(\gamma) \| = \sup_{t\in\delta} \| P_t(\lambda_n(t)) \| = \sup_{t\in\delta} | \alpha_n(t) |^{-1} . \tag{32}$$

Let us give the brief proof of (31) and (32). Using the well-known equality $P_t(\lambda_n(t)) = (f,\chi_{n,t})\Psi_{n,t}$ and (25) we obtain that P_t is a projection and

$$\| P_t(\lambda_n(t))f \| \leqslant | \alpha_n(t) |^{-1} \|f\| \quad \& \ \| P_t(\lambda_n(t))\Psi_{n,t} \| = | \alpha_n(t) |^{-1}$$

which yield (31) and show that the operators P_t for $t \in \delta$ are uniformly bounded, since $|\alpha_n|$ is continuous and nonzero in the compact δ. Moreover, one can easily verify that

$$P(\gamma)f = \lim_{N_i\to\infty} \sum_{j=-N_1}^{N_2} \sum_{k=-N_3}^{N_4} \frac{1}{2\pi} T_j^* \int_0^{2\pi} e^{i(j-k)t} P_t T_k f dt,$$

where $P_t = 0$ for $t \notin \delta$, $T_k f(x) = f(x+k)$ if $x \in [0,1)$ and $T_k f(x) = 0$ if $x \notin [0,1)$. Therefore it follows from Theorem 5.13 of [5] that $P(\gamma)$ is a projection and (32) holds.

PROPOSITION 2. *A number $\lambda \in \sigma(L_t(q))$ for $t \in (0,\pi)$ is a spectral singularity of $L(q)$ if and only if it is a multiple eigenvalue of $L_t(q)$. Moreover, Γ_n does not contain the spectral singularities if and only if there exists $\beta > 0$ such that $| \alpha_n(t) |^{-1} < \beta$ for all $t \in (0,\pi)$.*

Proof. It is well-known [1] that for $t \neq 0, \pi$ the operator L_t cannot have two eigenfunctions corresponding to one eigenvalue λ . Therefore $\lambda_n(t)$ is a multiple eigenvalue of $L_t(q)$ if and only if there exists an associated function corresponding to $\Psi_{n,t}$ that occurs if and only if $\alpha_n(t) = 0$. On the other hand, one can readily see that $|\alpha_n(t)|$ is continuous on $(0,\pi)$. Thus the proof follows from (32) and Definition 1. $\square$

Now we are ready to prove the main results of this chapter.

THEOREM 1. *The following statements are equivalent*

(a) *The operator $L(q)$ has no spectral singularity at infinity.*

(b) *$L(q)$ is an asymptotically spectral operator.*

(c) *There exists N such that the following are satisfied: (i) The operator $L(q)$ has no spectral singularity on Γ_n for $| n |> N$ and hence $L(q)$ may have at most finitely many spectral singularities. (ii) For $|n| > N$ and $t \in (0,\pi)$ the eigenvalues $\lambda_n(t)$ are simple. (iii) The algebraic multiplicity of $\lambda_n(0)$ and $\lambda_n(\pi)$ for $|n| > N$ are equal to their geometric multiplicities, that is, there are not associated functions corresponding to those eigenvalues. $(iiii)$ There exists a constant d such that*

$$| \alpha_n(t) |^{-1} < d \tag{33}$$

for all $| n |> N$ and $t \in (-\pi,0) \cup (0,\pi)$.

Proof. First let us show that (c) implies (a). From (32) and (33) it follows that $\| P(\gamma) \| < d$ for all regular spectral arcs $\gamma \in \Gamma_n$ whenever $| n |> N$. Therefore, by Definition 2, (a) holds.

Now we prove that (a) implies (c). Suppose that (a) holds. If (i) does not hold then there exist a sequence of pairs $\{(n_k,t_k)\}$ such that $|n_k| \to \infty$ and $\lambda_{n_k}(t_k)$ for $k = 1,2,...,$ are spectral singulatities of $L(q)$. Then by Definition 1 one can choose a sequence of regular arc $\gamma_{n_k} \subset \Gamma_{n_k}$ such that

$$\| P(\gamma_{n_k}) \| > k \tag{34}$$

which contradicts (a) (see Definition 2). To complete the proof of (i) it remains to note that the spectral singularities of $L(q)$ are contained in the set

$$\left\{ \lambda : \frac{dF(\lambda)}{d\lambda} = 0,\ \lambda \in \sigma(L(q)) \right\} \tag{35}$$

and the entire function $\frac{dF(\lambda)}{d\lambda}$ has at most finite number of roots on the compact set

$$\cup_{|n| \leqslant N} \Gamma_n.$$

If (ii) does not hold then there exist a sequence of pairs $\{(n_k,t_k)\}$ such that $t_k \in (0,\pi)$ and $\lambda_{n_k}(t_k)$ is a multiple eigenvalue. Then by Proposition 2 the numbers $\lambda_{n_k}(t_k)$ for $k = 1,2,...,$ are spectral singulatities of $L(q)$ which contradicts (i).

Since for $|n| > N$ the multiplicity of the eigenvalue λ_n is not greater than 2, if (iii) does not hold then there exist infinitely many eigenvalues to which corresponds one eigenfunction and one associated function. Therefore arguing as in the proof of the Proposition 2 we obtain that there exist infinitely many spectral singulatities which again contradicts (i).

If $(iiii)$ does not hold then there exist a sequence of pairs $\{(n_k,t_k)\}$, where $|n_k| \to \infty$ and $t_k \in (-\pi,0)\cup(0,\pi)$, such that

$$\left|\alpha_{n_k}(t_k)\right|^{-1} > k.$$

Moreover, by (ii) the eigenvalues $\lambda_{n_k}(t_k)$ for $|n_k| > N$ are simple. Therefore using the continuity of $|\alpha_{n_k}(t)|$ at t_k and (32) we obtain that there exists a sequence of regular arc $\gamma_{n_k} \subset \Gamma_{n_k}$ such that $\| P(\gamma_{n_k}) \| \to \infty$ which contradicts (a) (see Definition 2).

Now we prove that (a) and (b) are equivalent. If (a) does not hold then it is clear that (b) also does not holds. Suppose that (a) holds. Then (33) holds too. Let C be a positive constant such that if $\lambda_n(t) \in \{\lambda \in \mathbb{C} : |\lambda| > C\}$, then $|n| > N$ for all $t \in (-\pi,\pi]$, where N is defined in (c). If $\gamma \in R(C)$, then γ encloses finite number of the simple eigenvalues of $L_t(q)$ for $t \in (0,\pi)$. Thus, there exists a finite subset $J(t,\gamma)$ of $\{n \in \mathbb{Z}: |n| > N\}$ such that the eigenvalue $\lambda_k(t)$ lies inside γ if and only if $k \in J(t,\gamma)$. It is well-known that the simple eigenvalues are the simple poles of the Green function of $L_t(q)$ and the projection $e(t,\gamma)$ has the form

$$e(t,\gamma)f = \sum_{n\in J(t,\gamma)} \frac{1}{\alpha_n(t)} (f,\Psi^*_{n,t})_{(0,1)}\Psi_{n,t}. \tag{36}$$

Therefore the proof of the theorem follows from the following lemma $\square$

LEMMA 1. *If (33) holds then there exists a positive constant D such that*

$$\| \sum_{n\in J} \frac{1}{\alpha_n(t)} (f,\Psi^*_{n,t})\Psi_{n,t} \|^2 < D \| f \|^2 \tag{37}$$

for all $t \in (-\pi,0)\cup(0,\pi)$ and for all subset J of $\{n \in \mathbb{Z} : |n| > N\}$.

Proof. First let us prove that there exists a positive constant c such that

$$\| \sum_{n\in J} \frac{1}{\alpha_n(t)} (f,\Psi^*_{n,t})\Psi_{n,t} \|^2 < c \sum_{n:|n|>N} |(f,\Psi^*_{n,t})|^2 \tag{38}$$

for all $t \in (-\pi,0)\cup(0,\pi)$. By (20) we have

$$\| \sum_{n\in J} \frac{1}{\alpha_n(t)} (f,\Psi^*_{n,t})\Psi_{n,t} \|^2 \leqslant 2S_1 + 2S_2^2 \tag{39}$$

where

$$S_1 = \| \sum_{n\in J} \frac{1}{\alpha_n(t)} (f,\Psi^*_{n,t})(\sum_{k\in\{\pm n,\pm(n+1)\}} u_{n,k}(t)e^{i(2\pi k+t)x}) \|^2,$$

$$S_2 = \| \sum_{n\in J} \frac{1}{\alpha_n(t)} (f, \Psi^*_{n,t}) h_{n,t} \|$$

Since $\left\{ e^{i(2\pi n+t)x} : n \in \mathbb{Z} \right\}$ is an orthonormal basis and $\left| u_{n,k}(t) \right| \leqslant 1$ (see (22)) using (33) and the Bessel inequality one can easily verify that

$$S_1 \leqslant 16d^2 \sum_{n:|n|>N} |(f, \Psi^*_{n,t})|^2 \tag{40}$$

It follows from (33) and (21) that

$$S_2 < c_1 \sum_{n:|n|>N} |(f, \Psi^*_{n,t})| \frac{1}{|n|}$$

for some constant c_1. Now using the Schwarz inequality for l_2 we obtain

$$S_2 < c_1 \left(\sum_{n:|n|>N} |(f, \Psi^*_{n,t})|^2 \right)^{1/2} \left(\sum_{n:|n|>N} \frac{1}{n^2} \right)^{1/2} \leqslant \frac{c_1}{\sqrt{N}} \left(\sum_{n:|n|>N} |(f, \Psi^*_{n,t})|^2 \right)^{1/2}.$$

Thus (38) follows from (39) and (40). Therefore to prove (37) it is enough to show that there exists a positive constant c_2 such that

$$\sum_{n:|n|>N} |(f, \Psi^*_{n,t})|^2 \leqslant c_2 \| f \|^2 . \tag{41}$$

It can be proved arguing as in the proof of (38) and using (23) instead of (20) □

THEOREM 2. *The operator $L(q)$ is a spectral operator if and only if it has no spectral singularities at $\sigma(L(q))$ and at infinity.*

Proof. If $L(q)$ is a spectral operator, then by (7), (30) and Proposition 1 there exists a positive constant c_3 such that $\| P(\gamma) \| < c_3$ for all regular spectral arcs γ. Therefore, by Definition 1 and Definition 2, the operator $L(q)$ has no spectral singularities at $\sigma(L(q))$ and at infinity. If $L(q)$ has no spectral singularities at $\sigma(L(q))$ and at infinity then (33) holds for all $n \in \mathbb{Z}$ and $t \in (-\pi, 0) \cup (0, \pi)$. Using this and arguing as in the proof of the implication $(a) \Longrightarrow (b)$ of Theorem 1, we obtain that $L(q)$ is a spectral operator □

DEFINITION 4. The component Γ_n, defined by (14), of the spectrum $\sigma(L(q))$ of the operator $L(q)$ is said to be separated if $\Gamma_n \cap \Gamma_m = \emptyset$ for all $m \neq n$. Thus all component Γ_n of the spectrum $\sigma(L(q))$ are separated if and only if all eigenvalues of the operators L_t for $t \in (-\pi, \pi]$ are simple.

COROLLARY 1. *The Mathieu operator $L(2a\cos x)$, where a is a complex number, is a spectral operator if and only if*

$$\inf_{q,p\in\mathbb{N}} \{ | 2q\alpha - (2p-1) | \} \neq 0 \tag{42}$$

and all eigenvalues of the operators L_t for $t \in (-\pi, \pi]$ are simple.

Proof. Using the result of [13] mentioned in the end of introduction which states that $L(2a\cos x)$ has no spectral singularities at infinity if and only if (42) holds, we prove the corollary as follows. If (42) holds and all eigenvalues are simple then L has no spectral singularity at infinity and in spectrum and hence, by Theorem 2, L is a spectral operator. Now suppose that L is a spectral operator. Then, by Theorem 2, L has no spectral singularities at infinity and hence (42) holds. It remains to prove that all eigenvalues are simple. By Proposition 2 the eigenvalues $\lambda_n(t)$ for $t\in(0,\pi)$ and for all n are simple, since L has no spectral singularity in spectrum (see Theorem 2). Now to complete the proof of the corollary it remains to prove that the eigenvalues $\lambda_n(0)$ and $\lambda_n(\pi)$ for $n\in\mathbb{Z}$ are simple. It is well-known that, the geometric multiplicity of $\lambda_n(0)$ and $\lambda_n(\pi)$ for all n is 1 (see p. 34–35 of [1]. Note that in [1] it was proved for the real a. However, the proof pass through for the complex a.) Therefore if λ_n is a multiple eigenvalue then there corresponds one eigenfunction and at least one associated function. Then, arguing as in the proof of Proposition 2 we obtain that λ_n is a spectral singularity which contradicts the spectrality of L. $\square$

3. On the spectral expansion of the asymptotically spectral Hill operator

In this section we examine the spectral expansion theorem in the case when $L(q)$ has no spectral singularity at infinite by using Section 2 and the results of [2] where it was proved that for every $f\in L_2(-\infty,\infty)$ there exists $f_t(x)$ such that

$$f(x)=\frac{1}{2\pi}\int_0^{2\pi} f_t(x)dt, \tag{43}$$

$$f_t(x)=\sum_{k=-\infty}^{\infty} f(x+k)e^{-ikt},\ \int_{-\infty}^{\infty}|f(x)|^2\,dx=\frac{1}{2\pi}\int_0^{2\pi}\int_0^1 |f_t(x)|^2\,dxdt \tag{44}$$

and the followings hold

$$f_t(x+1)=e^{it}f_t(x),\ (f,\chi_{k,t})_{(-\infty,\infty)}.=(f_t,\chi_{k,t})_{(0,1)}=\frac{1}{\alpha_k(t)}(f_t,\Psi_{k,t}^*)=:a_k(t). \tag{45}$$

In this case, by Theorem 1(c) and Proposition 2, the roots of $\frac{dF(\lambda)}{d\lambda}=0$ lying in the set $\{\lambda_n(t): t\in(0,\pi),\ n\in\mathbb{Z}\}$ is finite. Denote these roots (if exists) by $\lambda_1,\lambda_2,...\lambda_m$. Let $t_1,t_2,...t_m$ be a point of $(0,\pi)$ such that $\lambda_k\in\sigma(L_{t_k})$. Introduce the set $E=(-\pi,\pi)\backslash\{0,\pm t_1,\pm t_2,...\pm t_m\}$. By the definition of E, if $t\in E$, then the eigenvalues $\lambda_k(t)$, for all $k\in\mathbb{Z}$, are simple and the system $\left\{\Psi_{k,t}(x): k\in\mathbb{Z}\right\}$ of eigenfunctions of L_t forms a Riesz basis in $L_2[0,1]$, since if $t\neq 0,\pi$ then the system of the eigenfunctions and associated functions of $L_t(q)$ with potential $q\in L_1[0,1]$ forms Riesz basis of $L_2[0,1]$ (see [11]). Therefore

$$f_t(x)=\sum_{k=-\infty}^{\infty} a_k(t)\Psi_{k,t}(x), \tag{46}$$

where the series converges in the norm of $L_2(0,1)$. This with (43) implies that

$$f(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f_t(x)dt=\frac{1}{2\pi}\int_{E} f_t(x)dt=\frac{1}{2\pi}\int_{E}\sum_{k=-\infty}^{\infty} a_k(t)\Psi_{k,t}(x)dt \tag{47}$$

Since in this section it is assumed that $L(q)$ has no spectral singularity at infinite, by Theorem 1 there are at most finite number of spectral singularities denoted by $\lambda_1,\lambda_2,...\lambda_s$. Here $s\geqslant m$, since by Proposition 2, $\lambda_1,\lambda_2,...\lambda_m$ are spectral singularities. Let S be the set of integers such that Γ_n contains spectral singularities for $n\in S$. Note that S is a finite subset of $\mathbb{Z}$. By Proposition 2 and (45), if $n\notin S$ then

$$\int_0^{2\pi}\int_0^1 \left|a_k(t)\Psi_{k,t}(x)\right|^2 dxdt\leqslant \beta^2\int_0^{2\pi}\int_0^1 |f_t(x)|^2 dxdt<\infty \tag{48}$$

Therefore by Fubini theorem

$$\int_E a_k(t)\Psi_{k,t}(x)dt \tag{49}$$

exists for almost all x when $k\notin S$. Similarly by (33) the integral (49) exists for $|k|>N$.

THEOREM 3. *If the operator $L(q)$ has no spectral singularity at infinity, then every function $f\in L_2(-\infty,\infty)$ has the spectral decomposition*

$$f(x)=\frac{1}{2\pi}\int_E\sum_{k\in S} a_k(t)\Psi_{k,t}(x)dt+\sum_{k\in\mathbb{Z}\setminus S}\frac{1}{2\pi}\int_E a_k(t)\Psi_{k,t}(x)dt, \tag{50}$$

where the series in (50) converges in the norm of $L_2(a,b)$ for every $a,b\in\mathbb{R}$.

Proof. First let us consider the series

$$\sum_{k>N} a_k(t)\Psi_{k,t}(x), \tag{51}$$

where N is defined in Theorem 1 and $t\in E$. Let $R_n(x,t)$ be remainder of (51)

$$R_n(x,t)=\sum_{k>n} a_k(t)\Psi_{k,t}(x),$$

where $n>N$. Since the series (51) converges in the norm of $L_2(0,1)$ by (45) and (27) we have

$$R_n(x+1,t)=e^{it}R_n(x,t),\ R_n(.,t)\in L_2(-m,m) \tag{52}$$

for $t\in E$ and for all $m\in\mathbb{N}$. Repeating the proof of (38) and using (52) we obtain

$$\| R_n(.,t)\|^2_{(-m,m)}\leqslant 2mc_4\sum_{k:|k|>n} |\,(f_t,\Psi^*_{k,t})_{(0,1)}\,|^2 \tag{53}$$

for some constant c_4, where $\| f \|_{(-m,m)}$ is the $L_2(-m,m)$ norm of f. On the other hand, it follows from (33), (23) and (24) that

$$\sum_{k:|k|>n} |\,(f_t,\Psi^*_{k,t})_{(0,1)}\,|^2 \leqslant c_5 \sum_{k:|k|>n} |\,(f_t,e^{i(2\pi k+t)x})_{(0,1)}\,|^2 + c_5 \| f_t \|^2_{(0,1)}\, n^{-1} \tag{54}$$

for some constant c_5. Now using the Parseval equality for $L(q)$ when $q=0$ (see [2]) we obtain

$$\sum_{k\in\mathbb{Z}} \frac{1}{2\pi} \int_E |\,(f_t,e^{i(2\pi k+t)x})_{(0,1)}\,|^2\, dt = \int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{1}{2\pi} \int_E \| f_t \|^2_{(0,1)}\, dt$$

Therefore by (54) and (53) the following integral exists and

$$I_n =: \int_E \int_{(-m,m)} |\, R_n(x,t)\,|^2\, dxdt \to 0 \tag{55}$$

as $n\to\infty$. Thus by Fubini theorem $R_n(x,t)$ is integrable with respect to t for almost all x.

Now using the inequality

$$\left| \int_E f(t)dt \right|^2 \leqslant 2\pi \int_E |f(t)|^2\, dt,$$

Fubini theorem and then (55), we obtain

$$\| \int_E \sum_{k>n} a_k(t)\Psi_{k,t}dt \|^2_{(-m,m)} \leqslant 2\pi \int_{(-m,m)} \int_E |\sum_{k>n} a_k(t)\Psi_{k,t}(x)\,|^2\, dtdx = I_n \to 0 \tag{56}$$

as $n\to\infty$. Hence the series (51) is integrable and

$$\int_E (\sum_{k>N} a_k(t)\Psi_{k,t}(x))dt = \sum_{k>N} \int_E a_k(t)\Psi_{k,t}(x)dt, \tag{57}$$

where the last series converges in the norm of $L_2(-m,m)$ for every $m\in\mathbb{N}$. In the same way we prove that

$$\int_E (\sum_{k<-N} a_k(t)\Psi_{k,t}(x))dt = \sum_{k<-N} \int_E a_k(t)\Psi_{k,t}(x)dt. \tag{58}$$

Therefore using (57), (58) and (47) and taking into account that the integral in (49) exists for $k\notin S$ we obtain that the first integral in (50) exists and (50) holds $\square$

Now changing the variable to λ in (50) by using (28) and taking into account that $\lambda_n(-t)=\lambda_n(t)$, $\Gamma_n=\lim_{\varepsilon\to 0}\Gamma_n(\varepsilon)$, where $\Gamma_n(\varepsilon)=\{\lambda=\lambda_n(t): t\in(\varepsilon,\pi-\varepsilon)\}$, and

$$\lim_{\varepsilon\to 0} \int_{(-\varepsilon,\varepsilon)\cup(\pi-\varepsilon,\pi+\varepsilon)} \left(\sum_{k\in S} a_k(t)\Psi_{k,t}(x) \right) dt = 0$$

by the absolute continuity of the integral, we obtain

THEOREM 4. *If the operator $L(q)$ has no spectral singularity at infinity, then every function $f \in L_2(-\infty,\infty)$ has the spectral decomposition*

$$f(x) = \frac{1}{\pi} p.v. \left(\int\limits_{\gamma(s)} (\phi(x,\lambda)) \frac{1}{p(\lambda)} d\lambda \right) + \frac{1}{\pi} \sum_{k \in \mathbb{Z} \backslash S} \left(\int\limits_{\Gamma_k} (\phi(x,\lambda)) \frac{1}{p(\lambda)} d\lambda \right), \tag{59}$$

where

$$\phi(x,\lambda) = \theta^{'}(1,\lambda)h(\lambda)\varphi(x,\lambda) + \frac{1}{2}(\theta(1,\lambda) - \varphi^{'}(1,\lambda))(h(\lambda)\theta(x,\lambda) + g(\lambda)\varphi(x,\lambda)) - \varphi(1,\lambda)g(\lambda)\theta(x,\lambda),$$

$$h(\lambda) = \int\limits_{-\infty}^{\infty} \varphi(x,\lambda) f(x) dx, \; g(\lambda) = \int\limits_{-\infty}^{\infty} \theta(x,\lambda) f(x) dx, \; p(\lambda) = \sqrt{4 - F^2(\lambda)},$$

$\gamma(s) =: \bigcup_{k \in S} \Gamma_k$ is the part of the spectrum that contains the spectral singularities and p.v. means that the integral over $\gamma(s)$ is the limit of the integral over $\bigcup_{k \in S} \Gamma_k(\varepsilon)$ as $\varepsilon \to 0$. The series in (59) converges in the norm of $L_2(a,b)$ for every $a,b \in \mathbb{R}$.

The results of [12] and [13] mentioned in the end of the introduction with Theorem 3 and Theorem 4 imply

COROLLARY 2. *If the potential q satisfies one of the Condition 1 and Condition 2 then the spectral expansions of $L(q)$ have the forms (50) and (59).*

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