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ON THE SINGULAR VECTORS OF THE GENERALIZED LYAPUNOV OPERATOR

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Abstract. In this paper, we study the largest and the smallest singular vectors of the generalized Lyapunov operator. For real matrices A,B with order n, we prove that $\max_{\|X\|_F=1}\|AXB^T+BXA^T\|_F$ is achieved by a symmetric matrix for $n\leqslant 3$ and give a counterexample for order $n=4$. We also prove that $\min_{\|X\|_F=1}\|AXB^T+BXA^T\|_F$ is achieved by a symmetric matrix for $n\leqslant 2$ and give a counterexample for order $n=3$. It is shown that the minimizer is symmetric, if the minimum is zero, or if the real parts of the eigenvalues of $A-\lambda B$ are of one sign.

1. Introduction

Lyapunov equations plays a fundamental role in control theory and the stability analysis of linear systems [1, 17, 23, 24]. They are also widely used in other fields of pure and applied mathematics. This has motivated a continuous interest to both the theory and numerical treatment of Lyapunov operators and equations [16, 21, 2, 6]. In the past years, the generalized Lyapunov equations, the generalized continuous-time Lyapunov equation (GCLE)

$$L_c(X)=AXB^T+BXA^T=-Q \tag{1}$$

and the generalized discrete-time Lyapunov equation (GDLE)

$$L_d(X)=AXA^T-BXB^T=-Q, \tag{2}$$

where A,B and Q are given real $n\times n$ matrices, have received considerable interest [3, 4, 13, 25, 26, 27, 28].

As shown in [12], the solvability of Equations (1) and (2) can be described in terms of the generalized eigenstructure of the matrix pair (A,B). A matrix pencil $xA-yB$ is called regular if $\det(xA-yB)\neq 0$ for some pair of complex numbers (x,y). The pair $(x,y)\neq(0,0)$ is called a generalized eigenvalue if $\det(xA-yB)=0$. For the more conventional form of a matrix pencil $A-\lambda B$, the eigenvalues are given by $\lambda=y/x$ with $\lambda=\infty$ when $x=0$. The regular matrix pencil $A-\lambda B$ is called *c-stable* if all

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finite eigenvalues of $A-\lambda B$ lie in the open left half-plane, and *d-stable* if all finite eigenvalues of $A-\lambda B$ lie inside the unit circle.

The norm of a generalized continuous-time Lyapunov operator (GCLO) and a generalized discrete-time Lyapunov operator (GDLO) are defined as

$$\|L_c\| = \max_{\|X\|_F=1} \|AXB^T + BXA^T\|_F \tag{3}$$

and

$$\|L_d\| = \max_{\|X\|_F=1} \|AXA^T - BXB^T\|_F, \tag{4}$$

respectively, where $\|\cdot\|_F$ denotes the Frobenius norm [18, 5]. The separation of the GCLO and the GDLO are defined as

$$sep_c(A,B) = \min_{\|X\|_F=1} \|AXB^T + BXA^T\|_F \tag{5}$$

and

$$sep_d(A,B) = \min_{\|X\|_F=1} \|AXA^T - BXB^T\|_F, \tag{6}$$

respectively. References for separation and condition estimation can be found in [6, 8, 14, 19, 20], for example.

The conjecture that the minimizer X for the separation of the continuous-time Lyapunov operator, i.e.,

$$\min_{\|X\|_F=1} \|AX + XA^T\|_F,$$

is symmetric was discussed in [7]. In general the conjecture is false, but it is true in many cases (e.g. A is normal or stable). The conjecture means that the smallest singular "vectors" of $L(X) = AX + XA^T$ can be symmetric. The largest singular "vectors" was also studied in [7]. The authors of [7] "proved" that the maximizer of

$$\max_{\|X\|_F=1} \|AX + XA^T\|_F,$$

is symmetric. But there is an error in their proof(see [9]). Thus, the problem that whether the maximizer is symmetric or not is still open. In [10], the maximizer being symmetric was proved for normal matrices, and in [11] a method of numeric testing for the problem was presented. For non-negative, non-positive, and tridiagonal matrices, [15] gives an affirmative answer to the conjecture.

In this paper, we consider the largest and the smallest singular "vectors" of the generalized Lyapunov operator. As shown in the following equation

$$AXA^T - BXB^T = \frac{1}{2}(CXD^T + DXC^T),$$

where $C = A+B, D = A-B$, we will discuss Equations (3) and (5) instead of Equations (4) and (6) in some cases. In Section 2, we show that the maximizer of Equation (3) and Equation (4) is symmetric for the matrices A,B with order $n \leqslant 3$. We also give a counterexample for A,B with oder 4. If there exist orthogonal matrices U and V

such that UAV is a tridiagonal matrix and UBV is a diagonal matrix, then Equation (3) and Equation (4) is achieved by a symmetric matrix. In Section 3, we discuss the separation of the GCLO and the GDLO, i.e., Equation (5) and Equation (6). We show that the minimizer is achieved by a symmetric matrix for A,B with oder 2. We give a counterexample for order 3. It is shown that the minimizer is symmetric, if the minimum is zero, or if the real parts of the eigenvalues of $A-\lambda B$ are of one sign.

2. The largest singular vectors of the GLO

Before proving the main results, we need several lemmas. The following lemma is simply extend the Lemma 1 in [7].

LEMMA 2.1. *Suppose $A,B \in \mathbf{R}^{n\times n}$. Then (3), (4), (5) and (6) are achieved by either a symmetric matrix or a skew-symmetric matrix.*

LEMMA 2.2. *For $A,B \in \mathbf{R}^{n\times n}$,*

$$\max_{X\neq 0} \frac{\|AXB^T+BXA^T\|_F}{\|X\|_F}$$

is achieved by a symmetric matrix X if and only if for every skew-symmetric real matrix P, there exists a symmetric real matrix Q such that $\|Q\|_F=\|P\|_F$ and

$$tr((A(Q+P)B^T+B(Q+P)A^T)^2)\geqslant 0.$$

Proof. By Lemma 2.1, we know that (3) is achieved by a symmetric matrix X if and only if for every skew-symmetric real matrix P, there exists a symmetric real matrix Q such that

$$\frac{\|L_c(Q)\|_F}{\|Q\|_F}\geqslant\frac{\|L_c(P)\|_F}{\|P\|_F}.$$

Obviously, it is equivalent to that for every skew-symmetric real matrix P, there exists a symmetric real matrix Q such that $\|Q\|_F=\|P\|_F$ and

$$\|L_c(Q)\|_F\geqslant\|L_c(P)\|_F.$$

Note that

$$\begin{aligned}
&tr(L_c(Q+P)^2)=tr((L_c(Q)+L_c(P))^2)\\
=&tr(L_c(Q)^2+L_c(Q)L_c(P)+L_c(P)L_c(Q)+L_c(P)^2)\\
=&tr((L_c(Q))^TL_c(Q))-tr((L_c(P))^TL_c(P))+tr(L_c(Q)L_c(P))-tr(L_c(Q)L_c(P))^T\\
=&\|L_c(Q)\|_F^2-\|L_c(P)\|_F^2,
\end{aligned}$$

Then $\|L_c(Q)\|_F^2\geqslant\|L_c(P)\|_F^2$ if and only if $tr((L_c(Q+P))^2)\geqslant 0$. The proof is complete. □

The following theorem shows that (3) can be achieved by a symmetric matrix for the case that A is orthogonally similar to a tridiagonal matrix.

THEOREM 2.3. *Suppose there exist orthogonal matrices U and V such that UAV is a tridiagonal matrix and UBV is a diagonal matrix. Then (3) is achieved by a symmetric matrix X.*

Proof. The restrictions of L_c on the symmetric matrix set S_n and the skew-symmetric matrix set K_n are denoted by L_c^S and L_c^K respectively. By the unitary invariance property of Frobenius norm, we have

$$\|L_c\| = \max_{\|X\|_F=1} \|(UAV)V^TXV(UBV)^T + (UBV)V^TXV(UAV)^T\|_F,$$

where U and V are orthogonal matrices. Thus without loss of generality, we assume that A is a tridiagonal matrix, say

$$A = \begin{bmatrix} a_1 & b_1 & 0 & \cdots & 0 & 0 \\ c_1 & a_2 & b_2 & \cdots & 0 & 0 \\ 0 & c_2 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & b_{n-1} \\ 0 & 0 & 0 & \cdots & c_{n-1} & a_n \end{bmatrix},$$

and $B = diag(d_1, d_2, \cdots, d_n)$. Suppose $P \in K_n$, say

$$P = \begin{bmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ -x_{12} & 0 & x_{23} & \cdots & x_{2n} \\ -x_{13} & -x_{23} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -x_{1n} & -x_{2n} & -x_{3n} & \cdots & 0 \end{bmatrix}.$$

Take

$$Q = \begin{bmatrix} 0 & x_{12} & x_{13} & \cdots & x_{1n} \\ x_{12} & 0 & x_{23} & \cdots & x_{2n} \\ x_{13} & x_{23} & 0 & \cdots & x_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & x_{3n} & \cdots & 0 \end{bmatrix}.$$

Obviously, $Q \in S_n$ and $\|Q\|_F = \|P\|_F$. By calculation, we have

$$L_c(Q+P) = \begin{bmatrix} 2b_1d_1x_{12} & * & \cdots & * \\ 0 & (2c_1x_{12} + 2b_2x_{23})d_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 2c_{n-1}d_nx_{n-1\ n} \end{bmatrix}.$$

Hence

$$tr((L_c(Q+P))^2) = 4(b_1d_1x_{12})^2 + 4\sum_{i=2}^{n-1}((c_{i-1}d_ix_{i-1\ i} + b_id_ix_{i\ i+1}))^2 + 4(c_{n-1}d_nx_{n-1\ n})^2 \geqslant 0,$$

By Lemma 2.2, we have $\|L_c\| = \|L_c^S\|$. $\square$

In the discrete case, we also have the following theorem, which is an analogue to Theorem 2.3.

THEOREM 2.4. *Suppose there exist orthogonal matrices U and V such that UAV and UBV are upper bidiagonal matrices. Then (4) is achieved by a symmetric matrix X.*

The following theorem shows that the largest singular vector is symmetric for all real matrices of order $n \leqslant 3$.

THEOREM 2.5. *Suppose $A, B \in \mathbf{R}^{n\times n}$. If $n \leqslant 3$, then (3) is achieved by a symmetric matrix X.*

Proof. Note that

$$\|L_c\| = \max_{\|X\|_F=1} \|(UAV)V^T XV(UBV)^T + (UBV)V^T XV(UAV)^T\|_F$$

for any orthogonal matrices U and V.

Suppose $n=2$. By *Singular Value Decomposition*, there are orthogonal matrices U and V such that UBV is a diagonal matrix. It follows from Theorem 2.3 that the largest singular vector is symmetric.

Suppose $n=3$. For a real skew-symmetric matrix $K \neq 0$ with order 3, there exists an orthogonal matrix $U \in \mathbf{R}^{3\times 3}$ such that

$$UKU^T = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix},$$

where $K_1 \in \mathbf{R}^{2\times 2}$ is a skew-symmetric matrix. By *Singular Value Decomposition*, there exist orthogonal matrices $U_1 \in \mathbf{R}^{2\times 2}$ and $V \in \mathbf{R}^{3\times 3}$ such that

$$V(BU^T)\begin{bmatrix} U_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix},$$

where $B_{11} = \begin{bmatrix} s_1 & 0 \\ 0 & s_2 \end{bmatrix}$. Suppose

$$VAU^T\begin{bmatrix} U_1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

where $A_{11} \in \mathbf{R}^{2\times 2}$. By the unitary invariance property of the Frobenius norm, we have

$$\begin{aligned}
&\frac{\|AKB^T + BKA^T\|_F^2}{\|K\|_F^2} \\
&= \frac{\|(VAU^T)UKU^T(VBU^T)^T + (VBU^T)UKU^T(VAU^T)^T\|_F^2}{\|UKU^T\|_F^2} \\
&= \frac{\left\| \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} U_1^T K_1 U_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11}^T & 0 \\ B_{12}^T & B_{22}^T \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix} \begin{bmatrix} U_1^T K_1 U_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & A_{22}^T \end{bmatrix} \right\|_F^2}{\|U_1^T K_1 U_1\|_F^2}
\end{aligned}$$

Note that $U_1^T K_1 U_1$ is a skew-symmetric matrix. Hence it suffices to prove that for real matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \ B = \begin{bmatrix} b_{11} & 0 & b_{13} \\ 0 & b_{22} & b_{23} \\ 0 & 0 & b_{33} \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

there exists a symmetric matrix $S \in \mathbf{R}^{3\times 3}$ such that

$$\frac{\|ASB^T + BSA^T\|_F}{\|S\|_F} \geqslant \frac{\|AKB^T + BKA^T\|_F}{\|K\|_F}.$$

By a similar argument as in the proof of Lemma 2.2, it suffices to prove that there exists a symmetric matrix $S \in \mathbf{R}^{3\times 3}$ such that $\|S\|_F = \|K\|_F$ and

$$tr(A(S+K)B^T + B(S+K)A^T)^2 \geqslant 0.$$

Take

$$S = \begin{bmatrix} 0 & \lambda & 0 \\ \lambda & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously $\|S\|_F = \|K\|_F$. By calculation, we have

$$A(S+K)B^T + B(S+K)A^T = \begin{bmatrix} 2\lambda a_{12}b_{11} & 2\lambda(a_{11}b_{22}+a_{22}b_{11}) & 2\lambda a_{32}b_{11} \\ 0 & 2\lambda a_{21}b_{22} & 0 \\ 0 & 2\lambda a_{31}b_{22} & 0 \end{bmatrix}.$$

Hence

$$tr(A(S+K)B^T + B(S+K)A^T)^2 = 4\lambda^2\left[(a_{12}b_{11})^2 + (a_{21}b_{22})^2\right] \geqslant 0.$$

The proof is complete. $\square$

The above theorem shows the correctness of the following corollary in discrete case.

COROLLARY 2.6. *Suppose* $A, B \in \mathbf{R}^{n\times n}$. *If* $n \leqslant 3$, *then*

$$\max_{X\neq 0} \frac{\|AXA^T - BXB^T\|_F}{\|X\|_F}$$

is achieved by a symmetric matrix X.

Proof. Note that

$$AXA^T - BXB^T = \frac{1}{2}(CXD^T + DXC^T),$$

where $C = A + B$ and $D = A - B$. Thus the result follows from Theorem 2.5. $\square$

In general, for real matrices A and B,

$$\max_{X\neq 0} \frac{\|AXA^T - BXB^T\|_F}{\|X\|_F} \tag{7}$$

can not always be achieved by a symmetric matrix. The following example shows that the maximizer should be skew-symmetric for some A, B in the case $n = 4$. By the proof of Corollary 2.6, we know that the generalized continuous case is similar.

EXAMPLE 2.7. Suppose

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The maximum in (7) is 4.259367. It is achieved by the real skew-symmetric matrix

$$X \approx \begin{bmatrix} 0 & 0.063318 & -0.249461 & -0.153498 \\ -0.063318 & 0 & -0.586480 & -0.249461 \\ 0.249461 & 0.586480 & 0 & -0.063318 \\ 0.153498 & 0.249461 & 0.063318 & 0 \end{bmatrix}.$$

But the largest quotient in (7) that can be achieved from a real symmetric matrix X is approximately 4.257641. The singular values and "vectors" were computed using the Kronecker product matrix $(A \otimes A - B \otimes B)$ for the generalized discrete-time Lyapunov operator.

3. The separation of the generalized Lyapunov operators

The smallest singular vectors were investigated in continuous-time case [7]. In this section, we discuss the smallest singular vectors of the discrete-time Lyapunov operator, the GCLO and the GDLO with similar techniques as in [7].

3.1. The separation of the discrete-time Lyapunov operator

We consider the discrete case in this subsection. The separation of the discrete-time Lyapunov operator is achieved by a symmetric matrix for the 2×2 real matrices.

THEOREM 3.1. *Suppose $A \in \mathbf{R}^{2\times 2}$. Then*

$$\min_{X\neq 0} \frac{\|AXA^T - X\|_F}{\|X\|_F} \tag{8}$$

is achieved by a symmetric matrix.

Proof. In the proof of Theorem 5 in [7], it is shown that for a real matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

there exists an orthogonal matrix $U \in \mathbf{R}^{2\times 2}$ such that

$$UAU^T = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{11} \end{bmatrix}.$$

By the unitary invariance property of Frobenius norm, we suppose

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{11} \end{bmatrix}.$$

Suppose the skew-symmetric matrix $K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Take

$$S = \begin{cases} diag(a_{12}, -a_{21}), & \text{if } a_{12}^2 + a_{21}^2 \neq 0, \\ I_2, & \text{if } a_{12} = a_{21} = 0. \end{cases}$$

By calculation, we have

$$\frac{\|ASA^T - S\|_F}{\|S\|_F} = |a_{11}^2 - a_{12}a_{21} - 1| = \frac{\|AKA^T - K\|_F}{\|K\|_F}.$$

By Lemma 2.1, we know that (8) is achieved by either a symmetric matrix or skew-symmetric matrix. It follows from the identity above that (8) can be achieved by a symmetric matrix. □

By contrast with the relatively brief proof of Theorem 3.1, we will give a different proof for general case (see Theorem 3.11).

The separation (8) can also be achieved by a symmetric matrix for the case A being a normal matrix. An analogue to Corollary 2 in [7], we have the following lemma, which can be used to prove the normal case.

LEMMA 3.2. *If (8) is minimized by a complex matrix $X \in \mathbf{C}^{n\times n}$, then it is minimized by $\bar{X}$, $X+\bar{X}$, $X-\bar{X}$, $X+X^*$ and $X-X^*$ (whenever these are nonzero), where $\bar{X}$ denotes the conjugate matrix of X and X^* denotes the conjugate transpose matrix of X.*

THEOREM 3.3. *If A is a normal real matrix, then (8) can be minimized by a symmetric matrix.*

Proof. For a normal matrix A, there exists a unitary matrix $U \in C^{n\times n}$ such that $UAU^* = diag(a_1, \cdots, a_n)$. By the unitary invariance property of Frobenius norm, we have

$$\frac{\|AXA^T - X\|_F}{\|X\|_F} = \frac{\|(UAU^*)UXU^*(UAU^*)^* - UXU^*\|_F}{\|UXU^*\|_F}. \tag{9}$$

Suppose $UXU^* = (y_{ij})$. Then we have

$$\frac{\|AXA^T - X\|_F}{\|X\|_F} = \frac{\sqrt{\sum|(a_i\bar{a}_j - 1)y_{ij}|^2}}{\|UXU^*\|_F}. \tag{10}$$

Suppose $|a_k\bar{a}_l - 1| = \min|a_i\bar{a}_j - 1|$. So

$$\frac{\sqrt{\sum|(a_i\bar{a}_j - 1)y_{ij}|^2}}{\|UXU^*\|_F} \geqslant \frac{\sqrt{|(a_k\bar{a}_l - 1)|^2\sum|y_{ij}|^2}}{\sqrt{\sum|y_{ij}|^2}} = |(a_k\bar{a}_l - 1)|.$$

Then the (10) is minimized by setting $UXU^* = (y_{ij})$, where

$$y_{ij} = \begin{cases} y, & \text{if } i = k \text{ and } j = l, \\ 0, & \text{if } i \neq k \text{ and } j \neq l, \end{cases}$$

and y is chosen so that the real part of $X + X^*$ is nonzero. By Lemma 3.2, we have that $X + X^*$ minimizes (9). Thus the real part of $X + X^*$ also minimizes (9). □

The following example shows that the minimizer of (8) should be a skew-symmetric matrix for some matrix A with order $n = 3$.

EXAMPLE 3.4. Suppose

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

The minimum in (8) is 0.554958. It is achieved by the real skew-symmetric matrix

$$X \approx \begin{bmatrix} 0 & -0.417907 & -0.231920 \\ 0.417907 & 0 & 0.521121 \\ 0.231920 & -0.521121 & 0 \end{bmatrix}.$$

But the smallest value in (8) that can be achieved from a real symmetric matrix X is approximately 0.556442. Similar to Example 2.7, the singular values and "vectors" were computed using the Kronecker product matrix $(A \otimes A - I)$ for the discrete-time Lyapunov operator.

3.2. The separation of the generalized Lyapunov operator

In this subsection, we study the separation of the GCLO and the GDLO, i.e., (5) and (6). The following lemma shows the condition for unique solvability of the GCLE and the GDLE.

LEMMA 3.5. ([12]) *Let $A - \lambda B$ be a regular pencil.*

(1) The GCLE (1) has a unique solution if and only if all eigenvalues of $A - \lambda B$ are finite and $\lambda_i + \lambda_j \neq 0$ for any two eigenvalues λ_i and λ_j of $A - \lambda B$.

(2) The GDLE (2) has a unique solution if and only if $\lambda_i\lambda_j \neq 1$ for any two eigenvalues λ_i and λ_j of $A - \lambda B$(under the convention $0 \cdot \infty = 1$).

The classical result about the positive definite solution of the stable Lyapunov equation remains valid for the generalized equations [25].

LEMMA 3.6. ([25]) *Let B be nonsingular and Q be positive definite (semidefinite).*

(1) If $Re(\lambda_i) < 0$ for all eigenvalues λ_i of $A-\lambda B$, then the solution matrix X of the GCLE (1) is positive definite (semidefinite).

(2) If $|\lambda_i| < 1$ for all eigenvalues λ_i of $A-\lambda B$, then the solution matrix X of the GDLE (2) is positive definite (semidefinite).

THEOREM 3.7. *Suppose $A, B \in \mathbf{R}^{n\times n}$, and*

$$sep_c(A,B) = \min_{X\neq 0} \frac{\|AXB^T + BXA^T\|_F}{\|X\|_F}. \tag{11}$$

(1) The separation $sep_c(A,B) \neq 0$ if and only if $A-\lambda B$ is regular, all eigenvalues of $A-\lambda B$ are finite and $\lambda_i + \lambda_j \neq 0$ for any two eigenvalues λ_i and λ_j of $A-\lambda B$.

(2) If $sep_c(A,B) = 0$, then (11) can be achieved by a symmetric matrix.

Proof. (1) Note that $sep_c(A,B) \neq 0$ if and only if the equation

$$AXB^T + BXA^T = 0 \tag{12}$$

has a unique solution $X = 0$. If A (or B) is singular, there exists nonzero $x \in \mathbf{R}^n$ such that $Ax = 0$ (or $Bx = 0$).Thus Equation (12) has nonzero solution $X = xx^T$. So $sep_c(A,B) \neq 0$ if and only if A and B is nonsingular and Equation (12) has unique solution. By Lemma 3.5 which was proved in [12], the proof is complete.

(2) We prove the statement (2) by studying two cases.

Case 1. If A (or B) is singular, there exists $x \in \mathbf{R}^n$ such that $Ax = 0$ (or $Bx = 0$). So we have

$$Axx^TB^T + Bxx^TA^T = 0.$$

Thus (11) is achieved by a symmetric matrix xx^T.

Case 2. If A and B are nonsingular, then

$$\min_{X\neq 0} \frac{\|AXB^T + BXA^T\|_F}{\|X\|_F} = \min_{X\neq 0} \frac{\|(AB^{-1})BXB^T + BXB^T(AB^{-1})^T\|_F}{\|BXB^T\|_F} \cdot \frac{\|BXB^T\|_F}{\|X\|_F}.$$

Hence

$$\min_{X\neq 0} \frac{\|(AB^{-1})Y + Y^T(AB^{-1})^T\|_F}{\|Y\|_F} = 0, \tag{13}$$

where $Y = BXB^T$. By Theorem 3 in [7], we have that there exists a symmetric matrix Y_0 such that (13) holds. Thus (5) is minimized by $B^{-1}Y_0B^{-T}$. $\square$

In the following of this section, we use the notation $A \geqslant 0$ to express the fact that a Hermitian matrix A is a positive semidefinite. If A and B are Hermitian matrices, then we say $A \geqslant B$ if and only if $A - B \geqslant 0$.

LEMMA 3.8. ([7]) *If X and Y are Hermitian matrices such that $X \geqslant Y \geqslant -X$, then $\|X\|_F \geqslant \|Y\|_F$.*

The following theorem shows that the separation of the GCLO and the GDLO can be achieved by a symmetric matrix for regular stable matrix pencil.

THEOREM 3.9. *Suppose $A-\lambda B$ is a regular pencil.*
(1) If all eigenvalues λ_i of $A-\lambda B$ are finite and $Re(\lambda_i)<0$, then

$$sep_c(A,B)=\min_{X\neq 0}\frac{\|AXB^T+BXA^T\|_F}{\|X\|_F}$$

can be achieved by a symmetric matrix.
(2) If all eigenvalues λ_i of $A-\lambda B$ are finite and $|\lambda_i|<1$, then

$$sep_d(A,B)=\min_{X\neq 0}\frac{\|AXA^T-BXB^T\|_F}{\|X\|_F}$$

can be achieved by a symmetric matrix.

Proof. We only prove the continuous-time case, the discrete-time case being similar. By Lemma 2.1, for any skew-symmetric matrix K, it suffices to construct a symmetric matrix S such that

$$\frac{\|ASB^T+BSA^T\|_F}{\|S\|_F}\leqslant\frac{\|AKB^T+BKA^T\|_F}{\|K\|_F}.$$

Suppose $M=AKB^T+BKA^T$, where K is a skew-symmetric matrix. Obviously, M is a skew-symmetric matrix. So there exists an orthogonal matrix $U\in\mathbf{R}^{n\times n}$ such that

$$UMU^T=diag(M_1,\cdots,M_r,\cdots,M_s),$$

where $M_1=\cdots=M_r=0$ and $M_j=\begin{bmatrix}0&\lambda_j\\-\lambda_j&0\end{bmatrix}$, $j=r+1,\cdots,s$.

Take a real matrix R such that

$$URU^T=diag(R_1,\cdots,R_r,\cdots,R_s),$$

where $R_j=M_j$ for $j=1,\cdots,r$ and $R_j=\begin{bmatrix}|\lambda_j|&0\\0&|\lambda_j|\end{bmatrix}$ for $j=r+1,\cdots,s$. Note that

$$R_j\geqslant\mathbf{i}M_j\geqslant -R_j,$$

where $\mathbf{i}^2=-1$ and $j=1,\cdots,s$. Hence

$$R\geqslant\mathbf{i}M\geqslant -R. \tag{14}$$

By Lemma 3.5, there exists a symmetric matrix S satisfying $ASB^T+BSA^T=R$. So we have

$$A(S-\mathbf{i}K)B^T+B(S-\mathbf{i}K)A^T=R-\mathbf{i}M,$$

and

$$A(\mathbf{i}K+S)B^T + B(\mathbf{i}K+S)A^T = \mathbf{i}M + R.$$

By Lemma 3.6 and Equation (14), we have $S \geqslant \mathbf{i}K \geqslant -S$. Lemma 3.8 shows that $\|S\| \geqslant \|\mathbf{i}K\| = \|K\|$. Hence

$$\begin{aligned}\frac{\|ASB^T + BSA^T\|_F}{\|S\|_F} &= \frac{\|R\|_F}{\|S\|_F} = \frac{\|URU^T\|_F}{\|S\|_F} = \frac{\|UMU^T\|_F}{\|S\|_F} = \frac{\|M\|_F}{\|S\|_F} \\ &= \frac{\|AKB^T + BKA^T\|_F}{\|S\|_F} \leqslant \frac{\|AKB^T + BKA^T\|_F}{\|K\|_F}. \quad \square\end{aligned}$$

The proof of Theorem 3.9 shows that the result is also true under the alternative condition for $Re(\lambda_i) > 0$ in continuous-time case and for $|\lambda_i| > 1$ in discrete-time case.

In general the separation of the GCLO and the GDLO can not always be minimized by a symmetric matrix. In the case $n = 3$ a counterexample can be derived from Example 3.4. The following lemma can be used to study the separation of the GCLO and the GDLO with order 2.

LEMMA 3.10. *Suppose* $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Then for any* $A \in \mathbf{R}^{2\times 2}$, *the separation of the GDLO*

$$\min_{X\neq 0} \frac{\|AXA^T - BXB^T\|_F}{\|X\|_F} \tag{15}$$

can be minimized by a symmetric matrix.

Proof. By Lemma 2.1, (15) can be achieved by either a symmetric or a skew-symmetric matrix. Thus for

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

it suffices to construct a symmetric matrix S such that

$$\frac{\|ASA^T - BSB^T\|_F}{\|S\|_F} = \frac{\|AKA^T - BKB^T\|_F}{\|K\|_F}.$$

In fact, if take

$$S = \begin{bmatrix} 0 & a_{11}a_{22} - a_{12}a_{21} - a_{22}^2 - a_{12}^2 \\ a_{11}a_{22} - a_{12}a_{21} - a_{22}^2 - a_{12}^2 & 2(a_{11}a_{12} + a_{21}a_{22}) \end{bmatrix}$$

when $a_{11}a_{12} + a_{21}a_{22} \neq 0$, and $S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ when $a_{11}a_{12} + a_{21}a_{22} = 0$, then we have

$$\frac{\|ASA^T - BSB^T\|_F}{\|S\|_F} = |a_{11}a_{22} - a_{12}a_{21}| = \frac{\|AKA^T - BKB^T\|_F}{\|K\|_F}. \quad \square$$

Finally we point out that the separation of the GCLO and the GDLO can be minimized by a symmetric matrix for A, B with order $n = 2$. Here we only show the general continuous case.

THEOREM 3.11. *For any $A,B \in \mathbf{R}^{2\times 2}$, the separation of the GCLO*

$$\min_{X\neq 0} \frac{\|AXB^T + BXA^T\|_F}{\|X\|_F} \tag{16}$$

can be achieved by a symmetric matrix.

Proof. If A or B is singular, then $\min_{\|X\|_F=1} \|AXB^T + BXA^T\|_F = 0$. The results follows from Thereom 3.7. Thus we can suppose both A and B are nonsingular.

If all the eigenvalues λ_i of $A-\lambda B$ are imaginary, then $\lambda_1 = a+b\mathbf{i}, \lambda_2 = a-b\mathbf{i}$. In the case $a=0$, the result follows from Thereom 3.7. In the case $a\neq 0$, the result follows from Theorem 3.9.

Suppose all the eigenvalues λ_i of $A-\lambda B$ are real. If $\lambda_1\lambda_2 > 0$, the result follows from Theorem 3.9. Now consider the case that all the eigenvalues λ_i of $A-\lambda B$ are real and $\lambda_1\lambda_2 < 0$. Note that

$$AXB^T + BXA^T = \frac{1}{2\lambda_1}(CXC^T - DXD^T),$$

where $C = A+\lambda_1 B$, $D = A-\lambda_1 B$. So D has rank 1. By the unitary invariance property of Frobenius norm, there exist orthogonal matrices $U,V \in \mathbf{R}^{2\times 2}$ such that $UDV = \begin{bmatrix} d & 0 \\ 0 & 0 \end{bmatrix}$, where $d\in \mathbf{R}$ and $d\neq 0$. The result follows from Lemma 3.10. $\square$

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