CRAWFORD NUMBERS OF COMPANION MATRICES

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Abstract. The (generalized) Crawford number $C(A)$ of an n-by-n complex matrix A is, by definition, the distance from the origin to the boundary of the numerical range $W(A)$ of A. If A is a companion matrix

$$
\begin{bmatrix}
0 & 1 & & & & \\
 & 0 & 1 & & & \\
 & & \cdot & \cdot & & \\
 & & & \cdot & \cdot & \\
 & & & & \cdot & \cdot \\
 & & & & 0 & 1 \\
-a_n & -a_{n-1} & \cdots & & -a_2 & -a_1
\end{bmatrix},
$$

then it is easily seen that $C(A) \geqslant \cos(\pi/n)$. The main purpose of this paper is to determine when the equality $C(A) = \cos(\pi/n)$ holds. A sufficient condition for this is that the boundary of $W(A)$ contains a point λ for which the subspace of $\mathbb{C}^n$ spanned by the vectors x with $\langle Ax, x \rangle = \lambda \|x\|^2$ has dimension 2, while a necessary condition is $\sum_{j=0}^{n-2} a_{n-j} e^{(n-j)i\theta} \sin\big((j+1)\pi/n\big) = \sin(\pi/n)$ for some real θ. Examples are given showing that in general these conditions are not simultaneously necessary and sufficient. We then prove that they are if A is (unitarily) reducible. We also establish a lower bound for the numerical radius $w(A)$ of A: $w(A) \geqslant \cos(\pi/(n+1))$, and show that the equality holds if and only if A is equal to the n-by-n Jordan block.

1. Introduction

Let A be an n-by-n complex matrix. The *Crawford number $c(A)$* (resp., *generalized Crawford number $C(A)$*) of A is, by definition, the distance from the origin to the numerical range (resp., the boundary of the numerical range) of A. Recall that the *numerical range* of A is the subset $W(A) \equiv \{\langle Ax, x \rangle : x \in \mathbb{C}^n, \|x\| = 1\}$ of the complex plane, where $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ denote the standard inner product and its associated norm in $\mathbb{C}^n$, respectively. It is known that $W(A)$ is a nonempty compact convex subset of the plane and contains all the eigenvalues of A. For its other properties, the reader may consult [11, Chapter 1]. The Crawford number $c(A)$ of A was first considered in [3]

and named specifically in [14, p. 74]. The generalized $C(A)$ appeared first in [2, p. 66], where it was called the inner numerical radius of A. Basic properties of both Crawford numbers can be found in [15, Proposition 1.1].

In this paper, we are concerned with the Crawford numbers of companion matrices. A *companion matrix* A is one of the form

$$\begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \cdot & \cdot & & \\ & & & \cdot & \cdot & \\ & & & & 0 & 1 \\ -a_n & -a_{n-1} & \cdots & & -a_2 & -a_1 \end{bmatrix}. \tag{1}$$

It is known that the characteristic and minimal polynomials of such an A are both equal to $z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$. Moreover, since 0 is in $W(A)$, we obviously have $c(A) = 0$. The main purpose of this paper is to estimate the value of $C(A)$.

In Section 2 below, we first consider upper bounds of $C(A)$ for a general matrix A. This is done via two other parameters of A. For an n-by-n matrix A, its *numerical radius* and *numerical inradius* are given by $w(A) \equiv \max\{|z| : z \in W(A)\}$ and $R(A) \equiv \max\{r \geqslant 0 : \{z \in \mathbb{C} : |z-a| \leqslant r\} \subseteq W(A) \text{ for some } a \in W(A)\}$, respectively. In Proposition 2.1, we show that $C(A) \leqslant \min\{w(A), \|A\| \cos(\pi/(n+1))\}$ and determine when the equality holds. This is achieved by way of $R(A)$ via [6, Theorem 4.5].

Section 3 deals with lower bounds of $C(A)$ and $w(A)$ for A a companion matrix of the form (1). A lower bound of $C(A)$ is easy to obtain: $C(A) \geqslant \cos(\pi/n)$ (cf. Proposition 3.1). The main concern is to determine when the equality holds. One sufficient condition for $C(A) = \cos(\pi/n)$ is that, for some λ in $\partial W(A)$, the subspace of $\mathbb{C}^n$ spanned by the vectors x satisfying $\langle Ax, x \rangle = \lambda \|x\|^2$ is of dimension 2 (cf. Proposition 3.2). In particular, this is the case if $\partial W(A)$ contains a line segment (cf. Corollary 3.4). An example is given showing that the converse does not hold even for 3-by-3 companion matrices (cf. Example 3.5). On the other hand, we also obtain a necessary condition for $C(A) = \cos(\pi/n)$: the existence of some real θ such that $\sum_{j=0}^{n-2} a_{n-j} e^{(n-j)i\theta} \sin((j+1)\pi/n) = \sin(\pi/n)$ and $\text{Re}\left(\sum_{j=1}^{n-1} a_{n-j} e^{(n-j)i\theta} \sin(j\pi/n)\right) \geqslant 0$ (cf. Proposition 3.6). Again, this is not sufficient for $n = 3$ by Example 3.10. However, we can strengthen it to a complete characterization of $C(A) = \cos(\pi/n)$ in the cases $n = 3$ and 4 (cf. Proposition 3.8). We conclude this section with a lower bound of $w(A)$: $w(A) \geqslant \cos(\pi/(n+1))$, and show that the equality holds if and only if A is equal to J_n, the n-by-n Jordan block

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$$

(cf. Theorem 3.11).

Finally, in Section 4, we completely characterize those n-by-n (unitarily) reducible companion matrices A which satisfy $C(A) = \cos(\pi/n)$. Among other things, we show that this is the case if and only if the sufficient (resp., necessary) condition in Proposition 3.2 (resp., Proposition 3.6) is satisfied (cf. Theorem 4.1).

An n-by-n matrix A is (*unitarily*) *reducible* if it is unitarily similar to the direct sum of two other matrices; it is (*unitarily*) *irreducible* if otherwise. We use $\mathrm{Re}\,A$ to denote the *real part* $(A + A^*)/2$ of A and $\sigma(A)$ the set of eigenvalues of A. The n-by-n identity matrix is I_n. Our general reference for properties of matrices is [10].

2. Upper bound for $C(A)$

We start with a minimax expression for $C(A)$. It is from [2, Theorem 2.1].

PROPOSITION 2.1. *For any n-by-n matrix A, we have*

$$C(A) = |\min_{\theta \in \mathbb{R}} \max \sigma (\mathrm{Re}\,(e^{i\theta}A))|.$$

Moreover, if 0 is in $W(A)$, then

$$C(A) = \min_{\theta \in \mathbb{R}} \max \sigma (\mathrm{Re}\,(e^{i\theta}A)).$$

The next proposition gives upper bounds of $C(A)$ and conditions for their attainment.

PROPOSITION 2.2. *Let A be an n-by-n matrix with $0 \in W(A)$. Then*

(a) $C(A) \leqslant R(A) \leqslant \min\{w(A), \|A\| \cos(\pi/(n+1))\}$,

(b) $C(A) = w(A)$ *if and only if $W(A)$ is a circular disc centered at the origin, and*

(c) $C(A) = \|A\| \cos(\pi/(n+1))$ *if and only if A is unitarily similar to $\|A\|J_n$.*

Proof. The inequality $R(A) \leqslant \|A\| \cos(\pi/(n+1))$ in (a) and the assertion on the equality in (c) were proven in [6, Theorem 4.5]. Other assertions are obvious. □

In view of (a) above, we may wonder when $C(A)$ and $R(A)$ are equal to each other. If A is a 2-by-2 companion matrix, then this is the case if and only if either A is normal or the trace of A is zero. This can be seen via the easily verified fact that the largest circular disc contained in an elliptic disc is the one centered at the center of the latter with radius half of the length of its minor axis. Unfortunately, for companion matrices of size 3, this is no longer the case. For example, if

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\sqrt{3}i & 4 & (\sqrt{3}/4)i \end{bmatrix},$$

then A is a 3-by-3 nonnormal irreducible companion matrix with a nonzero trace, whose numerical range is an elliptic disc with foci ± 2 and semi-minor axis of length

$\sqrt{13}/2$ (cf. [7, Example 2.1]). Hence $C(A) = R(A) = \sqrt{13}/2$. The next example shows that, conversely, a 3-by-3 irreducible companion matrix A with trace zero may still have $C(A) < R(A)$.

EXAMPLE 2.3. Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & 0 \end{bmatrix}.$$

Then A is a 3-by-3 nonnormal companion matrix with trace zero. Note that A is irreducible by [7, Theorem 1.1]. Since the eigenvalues of $\mathrm{Re}\,A$ can be computed to be $-1/2$, $-1/2$, 1, and $W(A)$ contains the numerical range $W(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix})$ $(= \{z \in \mathbb{C} : |z| \leqslant 1/2\})$ of the leading principal submatrix $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ of A, we obtain from Proposition 2.1 that $C(A) = 1/2$. On the other hand, for the unitary matrix

$$U = \frac{1}{\sqrt{6}} \begin{bmatrix} -\sqrt{2} & \sqrt{3} & 1 \\ -\sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & 2 \end{bmatrix},$$

we have

$$U^*AU = \begin{bmatrix} 1 & \sqrt{2/3} & -\sqrt{2} \\ -\sqrt{2/3} & -1/2 & -1/\sqrt{12} \\ \sqrt{2} & 1/\sqrt{12} & -1/2 \end{bmatrix}.$$

Its leading submatrix $A' \equiv \begin{bmatrix} 1 & \sqrt{2/3} \\ -\sqrt{2/3} & -1/2 \end{bmatrix}$ has numerical range $W(A')$ equal to the elliptic disc with foci $(1 \pm \sqrt{5/3}i)/4$ and semi-minor axis of length $3/4$. We infer from $W(A) \supseteq W(A')$ that $R(A) \geqslant R(A') = 3/4 > 1/2 = C(A)$.

3. Lower bound for $C(A)$

We start with a simple observation, which leads to a lower bound for $C(A)$ of a companion matrix A.

PROPOSITION 3.1. *If A is an n-by-n companion matrix, then $C(A) \geqslant \cos(\pi/n)$.*

Proof. Let A be of the form (1). Since J_{n-1} is a submatrix of A, its numerical range $W(J_{n-1})$ $(= \{z \in \mathbb{C} : |z| \leqslant \cos(\pi/n)\}$ by [12]) is contained in $W(A)$. It follows that $C(A) \geqslant \cos(\pi/n)$ as asserted. □

The remaining problem is to determine when $C(A)$ equals $\cos(\pi/n)$ (for an n-by-n companion matrix A). In the following, we will give some sufficient/necessary conditions for the equality to hold. We start with a sufficient one.

PROPOSITION 3.2. *Let A be an n-by-n companion matrix. If there is a point λ in the boundary of $W(A)$ such that $\bigvee\{x \in \mathbb{C}^n : \langle Ax, x \rangle = \lambda \|x\|^2\}$ is of dimension 2, then $C(A) = \cos(\pi/n)$.*

For an n-by-n matrix A and a point λ in $W(A)$, we use M_λ to denote the subspace $\bigvee \{x \in \mathbb{C}^n : \langle Ax, x \rangle = \lambda \|x\|^2\}$ of $\mathbb{C}^n$ spanned by the vectors x satisfying $\langle Ax, x \rangle = \lambda \|x\|^2$. Basic properties of M_λ were given in [4, Theorem 1]. In particular, if λ is in $\partial W(A)$, let L be a supporting line of $W(A)$ which passes through λ and, if λ is an extreme point of $W(A)$, satisfies $L \cap \partial W(A) = \{\lambda\}$. Then [4, Theorem 1 (i) and (ii)] says that

$$
M_\lambda = \begin{cases} \{x \in \mathbb{C}^n : \langle Ax, x \rangle = \lambda \|x\|^2\} & \text{if } \lambda \text{ is an extreme point of } W(A), \\ \bigcup_{\lambda' \in L \cap \partial W(A)} \{x \in \mathbb{C}^n : \langle Ax, x \rangle = \lambda' \|x\|^2\} & \text{if otherwise.} \end{cases} \tag{2}
$$

Moreover, let R be a ray from the origin which is perpendicular to L, and let $\theta \in [0, 2\pi)$ be the angle from the positive x-axis to R. Then $\mathrm{Re}\,(e^{-i\theta}\lambda)$ is the maximum eigenvalue of $\mathrm{Re}\,(e^{-i\theta}A)$ and $M_\lambda = \ker \mathrm{Re}\,(e^{-i\theta}(A - \lambda I_n))$.

The next lemma says that, for a companion matrix A and a point λ in $\partial W(A)$, the associated subspace M_λ can only have dimension 1 or 2.

LEMMA 3.3. *If A is an n-by-n companion matrix, then $\dim \ker \mathrm{Re}\,(A - \lambda I_n) \leqslant 2$ for any λ in $\mathbb{C}$. In particular, if λ is in $\partial W(A)$, then $\dim M_\lambda \leqslant 2$.*

Proof. Assume that $\dim \ker \mathrm{Re}\,(A - \lambda I_n) \geqslant 3$ for some λ in $\mathbb{C}$. Since $\mathrm{Re}\,J_{n-1}$ is a principal submatrix of $\mathrm{Re}\,A$, their eigenvalues interlace (cf. [10, Theorem 4.3.17]). Our assumption implies that $\mathrm{Re}\,\lambda$ is an eigenvalue of $\mathrm{Re}\,J_{n-1}$ with multiplicity at least 2. This contradicts the known fact that $\mathrm{Re}\,J_{n-1}$ has only simple eigenvalues (cf. [9, p. 373]). Thus $\dim \ker \mathrm{Re}\,(A - \lambda I_n) \leqslant 2$ for all λ as asserted.

The second assertion follows from $M_\lambda = \ker \mathrm{Re}\,(e^{-i\theta}(A - \lambda I_n))$ for some θ in $[0, 2\pi)$, the fact that $e^{-i\theta}A$ is unitarily similar to a companion matrix (cf. [7, Lemma 2.8]), and our first assertion. $\square$

Proof of Proposition 3.2. Let $K = \mathbb{C}^{n-1} \oplus \{0\}$. Since

$$
\dim(M_\lambda \cap K) = \dim M_\lambda + \dim K - \dim(M_\lambda + K) \geqslant 2 + (n-1) - n = 1,
$$

there is a unit vector x in $M_\lambda \cap K$. From $x \in M_\lambda$, we have $\langle Ax, x \rangle = \lambda'$ for some λ' in $\partial W(A)$, where $\lambda' = \lambda$ if λ is an extreme point of $W(A)$, and λ' is some point on the line segment of $\partial W(A)$ which passes through λ if otherwise (cf. (2)). But from $x \in K$, we also have $\lambda' \in W(J_{n-1})$ $(= \{z \in \mathbb{C} : |z| \leqslant \cos(\pi/n)\})$. This shows that λ' is in $\partial W(A) \cap \partial W(J_{n-1})$, and therefore $C(A) = \cos(\pi/n)$ as asserted. $\square$

COROLLARY 3.4. *If A is an n-by-n companion matrix such that $\partial W(A)$ has a line segment, then $C(A) = \cos(\pi/n)$.*

Proof. If λ is any point on the line segment of $\partial W(A)$, then M_λ is of dimension at least 2 by (2). The assertion then follows from Proposition 3.2. $\square$

Note that the converses of the assertions in Proposition 3.2 and Corollary 3.4 are both false as shown by the next example.

EXAMPLE 3.5. Let A be the 3-by-3 companion matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & -1 \end{bmatrix}.$$

By Proposition 3.1, we have $C(A) \geqslant 1/2$. For any θ in $[0, 2\pi)$, the matrix

$$\mathrm{Re}\,(e^{i\theta}A) = \begin{bmatrix} 0 & e^{i\theta}/2 & -e^{-i\theta}/2 \\ e^{-i\theta}/2 & 0 & e^{i\theta}/2 \\ -e^{i\theta}/2 & e^{-i\theta}/2 & -\mathrm{Re}\,e^{i\theta} \end{bmatrix}$$

has characteristic polynomial

$$p_\theta(z) = \det\left(zI_3 - \mathrm{Re}\,(e^{i\theta}A)\right) = z^3 + \cos\theta \cdot z^2 - \frac{3}{4}z + \cos\theta(\cos^2\theta - 1).$$

Hence

$$p_\theta\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{4}\cos\theta - \frac{3}{8} + \cos^3\theta - \cos\theta = (\cos\theta - 1)\left(\cos\theta + \frac{1}{2}\right)^2.$$

Thus $p_\theta(1/2) = 0$ if and only if $\cos\theta = 1$ or $-1/2$, which is the case if and only if $\theta = 0$, $2\pi/3$ or $4\pi/3$. Since

$$p_0(z) = z^3 + z^2 - \frac{3}{4}z = z\left(z - \frac{1}{2}\right)\left(z + \frac{3}{2}\right),$$

which shows that $1/2$ is the largest eigenvalue of $\mathrm{Re}\,A$ with multiplicity 1, we obtain $C(A) = 1/2$. On the other hand, since

$$p_{2\pi/3}(z) = z^3 - \frac{1}{2}z^2 - \frac{3}{4}z + \frac{3}{8} = \left(z - \frac{1}{2}\right)\left(z^2 - \frac{3}{4}\right),$$

the three eigenvalues of $\mathrm{Re}\,(e^{2\pi i/3}A)$ are $-\sqrt{3}/2 < 1/2 < \sqrt{3}/2$. In particular, this implies that $(1/2)e^{-2\pi i/3}$ is not in $\partial W(A)$. Indeed, since $\sqrt{3}/2$ is an eigenvalue of $\mathrm{Re}\,(e^{2\pi i/3}A)$, there exists $z_0 \in W(e^{2\pi i/3}A)$ such that $\mathrm{Re}\,(z_0) = \sqrt{3}/2 > 1/2$. As $W(e^{2\pi i/3}A)$ contains the closed disc $W(J_2) = \{z \in \mathbb{C} : |z| \leqslant 1/2\}$, $W(e^{2\pi i/3}A)$ contains the convex hull of $\{z_0\} \cup W(J_2)$ which has $1/2$ as an interior point. Thus, $W(A) = e^{-2\pi i/3}W(e^{2\pi i/3}A)$ contains $(1/2)e^{-2\pi i/3}$ as an interior point. Moreover, since A is a real matrix, $W(A)$ is symmetric about the real axis, we infer that $(1/2)e^{-4\pi i/3}$ is not in $\partial W(A)$. Thus $\partial W(A) \cap \partial W(J_2)$ consists of the single point $1/2$. Thus if λ is any point in $\partial W(A)$ with $\dim M_\lambda = 2$, then we infer from the proof of Proposition 3.2 and (2) that $M_\lambda = M_{\lambda'}$ for some λ' in $\partial W(A) \cap \partial W(J_2)$ $(= \{1/2\})$. Therefore, $\lambda' = 1/2$ and $\dim\ker\mathrm{Re}\,A = \dim M_\lambda = \dim M_{\lambda'} = 2$, which contradicts what has been proven above. Hence $\dim M_\lambda = 1$ for all λ in $\partial W(A)$.

Next we consider a necessary condition for $C(A) = \cos(\pi/n)$, which is in terms of the entries of the n-by-n companion matrix A. This has partially appeared before in [8, Lemma 3].

PROPOSITION 3.6. *Let A be an n-by-n companion matrix of the form* (1)*. If $C(A) = \cos(\pi/n)$, then there is a real θ such that*

$$\sum_{j=0}^{n-2} a_{n-j} e^{(n-j)i\theta} \sin \frac{(j+1)\pi}{n} = \sin \frac{\pi}{n} \tag{3}$$

and

$$\mathrm{Re} \left(\sum_{j=1}^{n-1} a_{n-j} e^{(n-j)i\theta} \sin \frac{j\pi}{n} \right) \geqslant 0. \tag{4}$$

Proof. By Proposition 2.1, there is a real θ such that $\cos(\pi/n) = \max \sigma \left(\mathrm{Re}\,(e^{i\theta}A) \right)$. Let

$$x = \sqrt{\frac{2}{n}} \left[e^{-i\theta} \sin \frac{\pi}{n} \quad e^{-2i\theta} \sin \frac{2\pi}{n} \quad \cdots \quad e^{-(n-1)i\theta} \sin \frac{(n-1)\pi}{n} \right]^T$$

in $\mathbb{C}^{n-1}$ and $y = \begin{bmatrix} x \\ 0 \end{bmatrix}$ in $\mathbb{C}^n$. Then $\cos(\pi/n) = \max W \left(\mathrm{Re}\,(e^{i\theta}A) \right)$ and x is a unit vector satisfying $\left(\mathrm{Re}\,(e^{i\theta} J_{n-1}) \right) x = \left(\cos(\pi/n) \right) x$ (cf. [9, p. 373]). Hence

$$\left\langle \left(\mathrm{Re}\,(e^{i\theta}A) \right) y, y \right\rangle = \left\langle \left(\mathrm{Re}\,(e^{i\theta} J_{n-1}) \right) x, x \right\rangle = \left\langle \left(\cos \frac{\pi}{n} \right) x, x \right\rangle = \cos \frac{\pi}{n}.$$

We infer from $\left\langle \left((\cos(\pi/n)) I_n - \mathrm{Re}\,(e^{i\theta}A) \right) y, y \right\rangle = 0$ and $\left(\cos(\pi/n) \right) I_n - \mathrm{Re}\,(e^{i\theta}A) \geqslant 0$ that $\mathrm{Re}\,(e^{i\theta}A) y = \cos(\pi/n) y$. It follows that the nth component of the vector $\mathrm{Re}\,(e^{i\theta}A) y$ equals 0, which yields the equality in (3).

To prove (4), let

$$A_1 = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_{n-1} & \cdots & -a_2 & -a_1 \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ -a_{n-1} & \cdots & -a_1 \end{bmatrix}.$$

Since $\mathrm{Re}\,(e^{i\theta}A_1) = \mathrm{Re}\,(e^{i\theta} J_{n-1}) + \mathrm{Re}\,(e^{i\theta}A_2)$, we have

$$\cos \frac{\pi}{n} = \max W \left(\mathrm{Re}\,(e^{i\theta}A) \right) \geqslant \left\langle \left(\mathrm{Re}\,(e^{i\theta}A_1) \right) x, x \right\rangle$$
$$= \left\langle \left(\mathrm{Re}\,(e^{i\theta} J_{n-1}) \right) x, x \right\rangle + \left\langle \left(\mathrm{Re}\,(e^{i\theta}A_2) \right) x, x \right\rangle$$
$$= \cos \frac{\pi}{n} + \left\langle \left(\mathrm{Re}\,(e^{i\theta}A_2) \right) x, x \right\rangle,$$

which yields that $\mathrm{Re}\,\langle e^{i\theta}A_2 x, x \rangle \leqslant 0$. Plugging in the matrix A_2 and the vector x, we obtain the inequality in (4). $\square$

Note that condition (3) involves only $a_n, \ldots, a_2$ and condition (4) $a_{n-1}, \ldots, a_1$. In a sense, they complement each other. Unfortunately, even put together they are still not sufficient. An example of a 3-by-3 companion matrix to this effect is given in Example 3.10.

For a 2-by-2 companion matrix A, condition (3) is necessary and sufficient for $C(A) = \cos(\pi/2)$.

PROPOSITION 3.7. *Let* $A = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}$. *Then* $C(A) = 0$ *if and only if* $|a_2| = 1$.

Proof. Let a and b be eigenvalues of A. Then $a_2 = ab$ and $a_1 = -(a+b)$. Hence $A = \begin{bmatrix} 0 & 1 \\ -ab & a+b \end{bmatrix}$ is unitarily similar to $\begin{bmatrix} a & |1+a\overline{b}| \\ 0 & b \end{bmatrix}$, whose numerical range $W(A)$ is the elliptic disc with foci a and b and length of minor axis $|1+a\overline{b}|$. Note that $C(A) = 0$ if and only if 0 is on the ellipse $\partial W(A)$, which is in turn equivalent to $|a| + |b| = 2\big((|a-b|/2)^2 + (|1+a\overline{b}|/2)^2\big)^{1/2}$ or to $2\big(|ab| + \mathrm{Re}\,(a\overline{b})\big) = |1+a\overline{b}|^2$. A simple computation shows that this is the same as $(1 - |ab|)^2 = 0$ or $|a_2| = |ab| = 1$ as required. $\quad\square$

The next proposition shows that condition (3) can be strengthened to characterizations of 3-by-3 and 4-by-4 companion matrices A with $C(A)$ equal to $1/2$ and $\sqrt{2}/2$, respectively.

PROPOSITION 3.8. (a) *Let* A *be a* 3*-by-*3 *companion matrix of the form* (1). *Then* $C(A) = 1/2$ *if and only if there is a real* θ *such that* $a_3 e^{3i\theta} + a_2 e^{2i\theta} = 1$ *and* $|a_3|^2 \leqslant 1 + 2\mathrm{Re}\,(a_1 e^{i\theta})$.

(b) *Let* A *be a* 4*-by-*4 *companion matrix of the form* (1). *Then* $C(A) = \sqrt{2}/2$ *if and only if there is a real* θ *such that* $a_4 e^{4i\theta} + \sqrt{2}a_3 e^{3i\theta} + a_2 e^{2i\theta} = 1$,

$$-\frac{3\sqrt{2}}{4} - \frac{b}{2} \leqslant \mathrm{Re}\Delta, \quad and \quad \mathrm{Re}\Delta \pm \sqrt{3}\mathrm{Im}\Delta \leqslant \frac{3\sqrt{2}}{2} + b, \tag{5}$$

where

$$\Delta = \left(\frac{1}{2}\big(\Delta_2 + \sqrt{\Delta_2^2 - 4\Delta_1^3}\big)\right)^{1/3}, \quad \Delta_1 = b^2 - 3c, \quad \Delta_2 = 2b^3 - 9bc + 27d, \tag{6}$$

$$b = \frac{\sqrt{2}}{2} + \mathrm{Re}\,(a_1 e^{i\theta}), \quad c = \frac{\sqrt{2}}{2}\mathrm{Re}\,(a_1 e^{i\theta}) - \frac{1}{4}(|a_2 e^{2i\theta} - 1|^2 + |a_3|^2 + |a_4|^2),$$

and

$$d = -\frac{1}{4}\mathrm{Re}\,\big(a_3(\overline{a}_2 e^{i\theta} + \overline{a}_4 e^{-i\theta} - 1)\big) - \frac{\sqrt{2}}{8}(|a_2 e^{2i\theta} - 1|^2 + |a_3|^2 + |a_4|^2).$$

(*Here* $(\cdot)^{1/3}$ *denotes the cubic root with the smallest argument.*)

Proof. (a) If $C(A) = 1/2$, then, by Proposition 2.1, there is a real θ for which $\max \sigma\big(\mathrm{Re}\,(e^{i\theta}A)\big) = 1/2$. Since $e^{i\theta}A$ is unitarily similar to the companion matrix

$$A' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3' & -a_2' & -a_1' \end{bmatrix},$$

where $a_j' = a_j e^{ji\theta}$ for $1 \leqslant j \leqslant 3$, and $C(A') = C(e^{i\theta}A) = C(A) = 1/2$, we may assume from the outset that $\theta = 0$. Thus $1/2$ is in $\sigma(\mathrm{Re}\,A)$. Note that

$$\mathrm{Re}\,A = \begin{bmatrix} 0 & 1/2 & -\overline{a}_3/2 \\ 1/2 & 0 & (-\overline{a}_2 + 1)/2 \\ -a_3/2 & (-a_2 + 1)/2 & -\mathrm{Re}\,a_1 \end{bmatrix}$$

with characteristic polynomial

$$p(z) \equiv \det(zI_3 - \mathrm{Re}\,A)$$
$$= z^3 + (\mathrm{Re}\,a_1)z^2 - \frac{1}{4}(|a_2 - 1|^2 + |a_3|^2 + 1)z - \frac{1}{4}\mathrm{Re}\left(a_3(\bar{a}_2 - 1) + a_1\right).$$

Hence

$$0 = p\left(\frac{1}{2}\right) = \frac{1}{8} + \frac{1}{4}\mathrm{Re}\,a_1 - \frac{1}{8}(|a_2 - 1|^2 + |a_3|^2 + 1) - \frac{1}{4}\mathrm{Re}\left(a_3(\bar{a}_2 - 1) + a_1\right)$$
$$= -\frac{1}{8}(|a_2 - 1|^2 + |a_3|^2) - \frac{1}{4}\mathrm{Re}\left(a_3(\bar{a}_2 - 1)\right),$$

which yields that $|(a_2 - 1) + a_3|^2 = 0$ or $a_2 + a_3 = 1$. Thus

$$p(z) = \left(z - \frac{1}{2}\right)\left(z^2 + \left(\frac{1}{2} + \mathrm{Re}\,a_1\right)z - \frac{1}{2}(|a_3|^2 - \mathrm{Re}\,a_1)\right),$$

and the eigenvalues of $\mathrm{Re}\,A$ are $1/2$ and

$$t_{\pm} \equiv \frac{1}{2}\left(-\frac{1}{2} - \mathrm{Re}\,a_1 \pm \left((\frac{1}{2} - \mathrm{Re}\,a_1)^2 + 2|a_3|^2\right)^{1/2}\right).$$

We derive from $t_+ \leqslant 1/2$ that $\left(((1/2) - \mathrm{Re}\,a_1)^2 + 2|a_3|^2\right)^{1/2} \leqslant (3/2) + \mathrm{Re}\,a_1$, which, after a simple computation, yields $|a_3|^2 \leqslant 1 + 2\mathrm{Re}\,a_1$.

Conversely, if $a_2 + a_3 = 1$ and $|a_3|^2 \leqslant 1 + 2\mathrm{Re}\,a_1$, then we reverse the above arguments to obtain $p(1/2) = 0$ and $((1/2) - \mathrm{Re}\,a_1)^2 + 2|a_3|^2 \leqslant ((3/2) + \mathrm{Re}\,a_1)^2$. The latter yields that $t_- \leqslant t_+ \leqslant 1/2$. Thus $\max \sigma(\mathrm{Re}\,A) = 1/2$. It follows from Propositions 2.1 and 3.1 that $C(A) = 1/2$.

(b) If $C(A) = \sqrt{2}/2$, then, as in (a), we may assume that $\theta = 0$ and $\max \sigma(\mathrm{Re}\,A) = \sqrt{2}/2$. The characteristic polynomial of $\mathrm{Re}\,A$ can be computed to be

$$p(z) = z^4 + (\mathrm{Re}\,a_1)z^3 - \frac{1}{4}(|a_2 - 1|^2 + |a_3|^2 + |a_4|^2 + 2)z^2$$
$$- \frac{1}{4}\mathrm{Re}\,(2a_1 - a_3 + a_3\bar{a}_2 + a_3\bar{a}_4)z + \frac{1}{16}|a_2 - a_4 - 1|^2.$$

Since $p(\sqrt{2}/2) = 0$, we obtain, after some computations, that $a_4 + \sqrt{2}a_3 + a_2 = 1$ and $p(z) = (z - \sqrt{2}/2)q(z)$, where $q(z) = z^3 + bz^2 + cz + d$,

$$b = \frac{\sqrt{2}}{2} + \mathrm{Re}\,a_1, \quad c = \frac{\sqrt{2}}{2}\mathrm{Re}\,a_1 - \frac{1}{4}(|a_2 - 1|^2 + |a_3|^2 + |a_4|^2),$$

and

$$d = -\frac{1}{4}\mathrm{Re}\left(a_3(\bar{a}_2 + \bar{a}_4 - 1)\right) - \frac{\sqrt{2}}{8}(|a_2 - 1|^2 + |a_3|^2 + |a_4|^2).$$

The zeros of $q(z)$ are given by Cardano's formula: $z_1 = -(b + \Delta + \Delta')/3$, $z_2 = -(b + \omega\Delta + \omega^2\Delta')/3$ and $z_3 = -(b + \omega^2\Delta + \omega\Delta')/3$, where $\omega = e^{2\pi i/3}$, Δ is as in (6) and

$\Delta' = \left((\Delta_2 - \sqrt{\Delta_2^2 - 4\Delta_1^3})/2\right)^{1/3}$. If $\Delta_2^2 > 4\Delta_1^3$, then both Δ and Δ' are real. On the other hand, the z_j's, being the eigenvalues of $\mathrm{Re}\,A$, are also real. In particular, we have

$$\omega\Delta + \omega^2\Delta' = \overline{\omega}\Delta + \overline{\omega}^2\Delta' = \omega^2\Delta + \omega\Delta',$$

and, therefore, $(1 - \omega)(\Delta - \Delta') = 0$. Hence $\Delta = \Delta'$ or $\Delta_2^2 = 4\Delta_1^3$, contradicting our assumption. Thus we must have $\Delta_2^2 \leqslant 4\Delta_1^3$. In this case, Δ and Δ' are conjugates to each other. Hence $z_1, z_2, z_3 \leqslant \sqrt{2}/2$ if and only if

$$\min\{\mathrm{Re}\,\Delta, \mathrm{Re}\,(\omega\Delta), \mathrm{Re}\,(\omega^2\Delta)\} \leqslant -\frac{3\sqrt{2}}{4} - \frac{b}{2},$$

which is in turn equivalent to the inequalities in (5).

Conversely, if $a_4 + \sqrt{2}a_3 + a_2 = 1$ and (5) holds, then we can reverse the above arguments to obtain $p(\sqrt{2}/2) = 0$. Moreover, the inequalities in (5) guarantee that $\sqrt{2}/2$ is the largest zero of $p(z)$. Thus $\max\sigma(\mathrm{Re}\,A) = \sqrt{2}/2$ and $C(A) = \sqrt{2}/2$ follows from Propositions 2.1 and 3.1. $\square$

Note that, in Proposition 3.6, condition (3) is equivalent to the fact that $\cos(\pi/n)$ is a zero of the characteristic polynomial of $\mathrm{Re}\,(e^{i\theta}A)$. Some extra inequalities such as the ones in Proposition 3.8 (a) and (b) are needed to guarantee that $\cos(\pi/n)$ is its largest zero. Admittedly, the ones in Proposition 3.8 (b) are unwieldy, but nevertheless they do give a characterization for $C(A) = \sqrt{2}/2$. For general values of n, we don't expect tractable inequalities can be derived.

For $n = 3$, the inequality in Proposition 3.8 (a) can be replaced by the noninvertibility of the companion matrix A.

PROPOSITION 3.9. *Let A be a 3-by-3 noninvertible companion matrix of the form* (1). *Then $C(A) = 1/2$ if and only if $a_3e^{3i\theta} + a_2e^{2i\theta} = 1$ for some real θ.*

Proof. In view of Proposition 3.6, we need only prove that $a_3e^{3i\theta} + a_2e^{2i\theta} = 1$ for some real θ implies $C(A) = 1/2$. Since A is noninvertible, we have $a_3 = 0$ and hence $a_2e^{2i\theta} = 1$. If $1 + 2\mathrm{Re}\,(a_1e^{i\theta}) \geqslant 0$, then $C(A) = 1/2$ by Proposition 3.8 (a). Otherwise, for the case of $1 + 2\mathrm{Re}\,(a_1e^{i\theta}) < 0$, consider the matrix $-e^{i\theta}A$, which is unitarily similar to

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_2' & -a_1' \end{bmatrix},$$

where $a_2' = a_2e^{2i\theta}$ and $a_1' = -a_1e^{i\theta}$. Hence $a_2' = 1$ and $1 + 2\mathrm{Re}\,a_1' > 2 > 0$. We conclude from Proposition 3.8 (a) that in this case we also have $C(A) = 1/2$. $\square$

The next example shows that the noninvertibility of A in the preceding proposition is essential.

EXAMPLE 3.10. If $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -3 & -1 \end{bmatrix}$, then A is invertible with $a_3 = -2$, $a_2 = 3$ and $a_1 = 1$. We have $a_3 + a_2 = 1$, but

$$|a_3|^2 = 4 > 1 + 2\cos\theta = 1 + 2\mathrm{Re}\,(a_1 e^{i\theta})$$

for all real θ. Hence $C(A) > 1/2$ by Propositions 3.1 and 3.8 (a). Note also that, in this example, we have $\mathrm{Re}\,(a_2 + a_1) = 4 > 0$. Thus conditions (3) and (4) of Proposition 3.6 together do not guarantee that $C(A) = 1/2$.

We conclude this section with a lower bound for the numerical radius of a companion matrix.

THEOREM 3.11. *If A is an n-by-n companion matrix of the form* (1), *then $w(A) \geqslant \cos(\pi/(n+1))$. Moreover, the equality holds if and only if $A = J_n$.*

Proof. Assume that $w(A) \leqslant \cos(\pi/(n+1)) \equiv r$. Then the matrix $rI_n - \mathrm{Re}\,(e^{i\theta}A)$ is positive semidefinite for all real θ. Note that $e^{i\theta}A$ is unitarily similar to the companion matrix

$$A_\theta \equiv \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_n e^{ni\theta} & \cdots & -a_2 e^{2i\theta} & -a_1 e^{i\theta} \end{bmatrix},$$

and if $x = [x_1 \ \ldots \ x_n]^T$ in $\mathbb{C}^n$, where $x_j = \sin(j\pi/(n+1))$ for $1 \leqslant j \leqslant n$, then $(\mathrm{Re}\,J_n)x = rx$ (cf. [9, p. 373]). Hence we have

$$0 \leqslant \langle (rI_n - \mathrm{Re}\,A_\theta)x, x \rangle = \langle (rI_n - \mathrm{Re}\,J_n)x, x \rangle + \langle (\mathrm{Re}\,(J_n - A_\theta))x, x \rangle$$

$$= 0 + \frac{1}{2}\left\langle \begin{bmatrix} 0 & \cdots & 0 & \overline{a}_n e^{-ni\theta} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & \overline{a}_2 e^{-2i\theta} \\ a_n e^{ni\theta} & \cdots & a_2 e^{2i\theta} & a_1 e^{i\theta} + \overline{a}_1 e^{-i\theta} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} \right\rangle$$

$$= \frac{1}{2}\sum_{j=1}^{n}(\overline{a}_j e^{-ji\theta}x_n x_j + a_j e^{ji\theta}x_{n-j+1}x_n) \equiv \frac{1}{2}p(e^{i\theta})$$

for all real θ. This shows that the trigonometric polynomial $p(e^{i\theta})$ assumes only nonnegative values. Thus the Riesz–Fejér theorem yields that $p(e^{i\theta}) = |q(e^{i\theta})|^2$ for some polynomial $q(z) = b_0 z^n + b_1 z^{n-1} + \cdots + b_{n-1}z + b_n$ of degree at most n (cf. [13, p. 77, Problem 40]). The equality of their constant terms results in $|b_0|^2 + \cdots + |b_n|^2 = 0$. Hence $b_0 = \cdots = b_n = 0$ and thus $p(e^{i\theta}) = |q(e^{i\theta})|^2 = 0$ for all real θ. Therefore, all the coefficients of p are zero and so $a_j = 0$ for all j. This shows that $w(A) \geqslant w(J_n) = \cos(\pi/(n+1))$ and the equality here implies that $A = J_n$. Since the converse of the latter is trivial, this completes the proof. $\square$

4. Reducible companion matrix

In this section, we derive necessary and sufficient conditions for an n-by-n reducible companion matrix to have Crawford number equal to $\cos(\pi/n)$. Recall that an n-by-n companion matrix is reducible if and only if its eigenvalues are of the form $a\omega_1,\ldots,a\omega_p,(1/\overline{a})\omega_{p+1},\cdots,(1/\overline{a})\omega_n$, where $a \neq 0$, $1 \leqslant p \leqslant n-1$, and the ω_j's are distinct with $\{\omega_1,\ldots,\omega_n\} = \{e^{2\pi i j/n} : 0 \leqslant j \leqslant n-1\}$. In this case, let $\lambda_j = a\omega_j$ if $1 \leqslant j \leqslant p$, and $(1/\overline{a})\omega_j$ if $p+1 \leqslant j \leqslant n$, let $u_j = [1 \; \lambda_j \; \ldots \; \lambda_j^{n-1}]^T$ be an eigenvector of A associated with λ_j for all j, and let $M = \bigvee\{u_j : 1 \leqslant j \leqslant p\}$ and $N = \bigvee\{u_j : p+1 \leqslant j \leqslant n\}$. Then M and N are subspaces of $\mathbb{C}^n$ orthogonal to each other satisfying $AM \subseteq M$ and $AN \subseteq N$. If A_1 and A_2 are the restrictions of A to M and N, respectively, then A is unitarily similar to $A_1 \oplus A_2$ with $\sigma(A_1) = \{a\omega_1,\ldots,a\omega_p\}$ and $\sigma(A_2) = \{(1/\overline{a})\omega_{p+1},\cdots,(1/\overline{a})\omega_n\}$ (cf. proof of [7, Theorem 1.1]).

The following theorem is the main result of this section. In particular, it shows that the converses of the assertions in Propositions 3.2 and 3.6 are true for reducible companion matrices.

THEOREM 4.1. *Let A be an n-by-n reducible companion matrix of the form* (1), *which is unitarily similar to $A_1 \oplus A_2$ with $\sigma(A_1) = \{a\omega_1,\ldots,a\omega_p\}$ and $\sigma(A_2) = \{(1/\overline{a})\omega_{p+1},\cdots,(1/\overline{a})\omega_n\}$ as above, where $|a| \leqslant 1$. Then the following are equivalent:*

(a) $C(A) = \cos(\pi/n)$,

(b) $w(A_1) \geqslant \cos(\pi/n)$,

(c) *there is a point λ in $\partial W(A)$ such that $\dim M_\lambda = 2$, and*

(d) *there is a real θ such that $\sum_{j=1}^{n-2} a_{n-j} e^{(n-j)i\theta} \sin\left((j+1)\pi/n\right) = \sin(\pi/n)$.*

We start the proof with the following two lemmas, which relate the vector $x = [x_1 \; \ldots \; x_n]^T$, where $x_j = \sin(j\pi/n)$ for $1 \leqslant j \leqslant n$, to the direct summands A_1 and A_2 of A.

LEMMA 4.2. *Let*

$$A = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ a & & & 0 \end{bmatrix}$$

be an n-by-n matrix, where $a \neq 0, -1$, and $x = [x_1 \; \ldots \; x_n]^T$, where $x_j = \sin(j\pi/n)$, $1 \leqslant j \leqslant n$. Then x is a cyclic vector for A, that is, the vectors $x, Ax, \ldots, A^{n-1}x$ span $\mathbb{C}^n$.

Proof. It suffices to show that $x, Ax, \ldots, A^{n-1}x$ are linearly independent. Assuming that $\sum_{\ell=1}^n b_\ell A^{\ell-1}x = 0$, we need to check $b_\ell = 0$ for all ℓ. This will be done by

induction from n to 1. Indeed, from the zeroness of the jth component of $\sum_\ell b_\ell A^{\ell-1}x$, we obtain, as $x_n = 0$,

$$\Big(\sum_{k=1}^{n-j} b_k \sin \frac{(k+j-1)\pi}{n}\Big) + a\Big(\sum_{k=1}^{j-1} b_{k+n-j+1} \sin \frac{k\pi}{n}\Big) = 0, \quad 1 \leqslant j \leqslant n. \qquad (7)$$

Letting $j = 1$, we have

$$\begin{aligned}
0 &= \sum_{k=1}^{n-1} b_k \sin \frac{k\pi}{n} = \sum_{k=1}^{n-1} b_k \sin \frac{k\pi}{n} \cos \frac{\pi}{n} \\
&= \frac{1}{2} \sum_{k=1}^{n-1} b_k \Big(\sin \frac{(k+1)\pi}{n} + \sin \frac{(k-1)\pi}{n} \Big) \\
&= \frac{1}{2} \Big(\sum_{k=1}^{n-2} b_k \sin \frac{(k+1)\pi}{n} + \sum_{k=2}^{n-1} b_k \sin \frac{(k-1)\pi}{n} \Big) \\
&= \frac{1}{2} \Big(-ab_n \sin \frac{\pi}{n} - b_n \sin \frac{(n-1)\pi}{n} \Big),
\end{aligned}$$

where the last equality is a consequence of $j = 2$ and $j = n$ of (7) (for the latter, we also need $a \neq 0$). It follows that $b_n + ab_n = 0$. Since $a \neq -1$, we obtain $b_n = 0$.

Next assume that $b_n = b_{n-1} = \cdots = b_{n-j_0+2} = 0$ for some j_0, $2 \leqslant j_0 \leqslant n$. We proceed to show that $b_{n-j_0+1} = 0$. Indeed, from $j = j_0$ of (7) we have

$$\begin{aligned}
0 &= \sum_{k=1}^{n-j_0} b_k \sin \frac{(k+j_0-1)\pi}{n} = \sum_{k=1}^{n-j_0} b_k \sin \frac{(k+j_0-1)\pi}{n} \cos \frac{\pi}{n} \\
&= \frac{1}{2} \sum_{k=1}^{n-j_0} b_k \Big(\sin \frac{(k+j_0)\pi}{n} + \sin \frac{(k+j_0-2)\pi}{n} \Big) \\
&= \frac{1}{2} \Big(\sum_{k=1}^{n-j_0-1} b_k \sin \frac{(k+j_0)\pi}{n} + \sum_{k=1}^{n-j_0} b_k \sin \frac{(k+j_0-2)\pi}{n} \Big). \qquad (8)
\end{aligned}$$

Plugging

$$\Big(\sum_{k=1}^{n-j_0+1} b_k \sin \frac{(k+j_0)\pi}{n} \Big) + ab_{n-j_0+1} \sin \frac{\pi}{n} = 0$$

and

$$\sum_{k=1}^{n-j_0+1} b_k \sin \frac{(k+j_0-2)\pi}{n} = 0,$$

the equalities of (7) for $j = j_0 + 1$ and $j_0 - 1$, respectively, into (8), we obtain $ab_{n-j_0+1} \sin(\pi/n) + b_{n-j_0+1} \sin((n-1)\pi/n) = 0$. Thus $(a+1)b_{n-j_0+1} = 0$ or $b_{n-j_0+1} = 0$ (as $a \neq -1$). Hence, by induction, we have $b_\ell = 0$ for all ℓ. $\quad\square$

LEMMA 4.3. *Let A be an n-by-n reducible companion matrix with eigenvalues $a\omega_1, \ldots, a\omega_p, (1/\bar{a})\omega_{p+1}, \cdots, (1/\bar{a})\omega_n$, where $0 < |a| < 1$, $1 \leqslant p \leqslant n-1$, and the ω_j's are distinct with $\{\omega_1, \ldots, \omega_n\} = \{e^{2\pi i j/n} : 0 \leqslant j \leqslant n-1\}$. Let $x = [x_1 \ \ldots \ x_n]^T$, where $x_j = \sin(j\pi/n)$, $1 \leqslant j \leqslant n$, and M and N be as before. Then x is not in M nor in N.*

Proof. Let A'_1 be the n-by-n matrix

$$\begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ a^n & & & 0 \end{bmatrix}.$$

It is easily seen that $A'_1 M \subseteq M$. If x is in M, then $x, A'_1 x, \ldots, A'^{n-1}_1 x$ are all in M, which implies, by Lemma 4.2, that $M = \mathbb{C}^n$. This contradicts the fact that $\dim M = p \leqslant n-1$. Thus x is not in M. Similarly, if

$$A'_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ (1/\overline{a})^n & & & 0 \end{bmatrix},$$

then $A'_2 N \subseteq N$ and we can argue as above that x is not in N. $\square$

The next lemma is an easy consequence of results from [5].

LEMMA 4.4. *Let A be an n-by-n companion matrix unitarily similar to $A_1 \oplus A_2$ with $\sigma(A_1) = \{a\omega_1, \ldots, a\omega_p\}$ and $\sigma(A_2) = \{(1/\overline{a})\omega_{p+1}, \cdots, (1/\overline{a})\omega_n\}$, where $|a| \leqslant 1$, $1 \leqslant p \leqslant n-1$, and the ω_j's are distinct such that $\{\omega_1, \ldots, \omega_n\} = \{e^{2\pi i j/n} : 0 \leqslant j \leqslant n-1\}$. Then the following are equivalent:*

(a) $\partial W(A)$ *has a line segment,*

(b) *neither $W(A_1)$ nor $W(A_2)$ is contained in the other, and*

(c) *either $|a| = 1$ or $|a| < 1$ and $w(A_1) > \cos(\pi/n)$.*

Proof. (a) $\Rightarrow$ (b). If $W(A_1) \subseteq W(A_2)$, then $W(A) = W(A_2)$. Since A_2 is of class S^{-1}_{n-p}, that is, A_2 is an $(n-p)$-by-$(n-p)$ matrix satisfying $\sigma(A_2) \subseteq \{z \in \mathbb{C} : |z| > 1\}$ and $\operatorname{rank}(I_{n-p} - A^*_2 A_2) = 1$, the boundary of $W(A_2)$ contains no line segment (cf. [5, Theorem 2.5 (4)]). Thus $\partial W(A)$ has no line segment, which contradicts (a). Hence we must have $W(A_1) \not\subseteq W(A_2)$. Similarly, since A_1 is of class S_p ($\|A_1\| \leqslant 1$, $\sigma(A_1) \subseteq \{z \in \mathbb{C} : |z| < 1\}$ and $\operatorname{rank}(I_p - A^*_1 A_1) = 1$), $\partial W(A_1)$ contains no line segment (cf. [6, Lemma 2.2]). Thus we infer as above that $W(A_2) \not\subseteq W(A_1)$.

(b) $\Rightarrow$ (c). If $|a| < 1$, then $w(A_1) > \cos(\pi/n)$ follows from [5, Theorem 2.11].

(c) $\Rightarrow$ (a). If $n = 2$, then our assumption implies that $W(A)$ is itself a line segment. Hence we may assume that $n \geqslant 3$. In this case, if $|a| = 1$, then the eigenvalues of A are $a\omega^j$, $0 \leqslant j \leqslant n-1$, where $\omega = e^{2\pi i/n}$. Hence [7, Corollary 1.2] implies that A is unitary and thus (a) holds. On the other hand, if $|a| < 1$ and $w(A_1) > \cos(\pi/n)$, then $W(A_1) \not\subseteq W(A_2)$ by [5, Theorem 2.11]. Also, if $W(A_2) \subseteq W(A_1)$, then $\sigma(A_2) \subseteq$

$W(A_1) \subseteq \{z \in \mathbb{C} : |z| < 1\}$, which leads to a contradiction. Hence $W(A_2) \not\subseteq W(A_1)$ and (a) follows. $\quad \square$

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. If $|a| = 1$, then conditions (a), (b), (c) and (d) are all satisfied (cf. Proposition 3.7 for $n = 2$ and [7, Corollary 1.2] for $n \geqslant 3$). Hence in the following we may assume that $0 < |a| < 1$. Let x in $\mathbb{C}^n$ and the subspaces M and N of $\mathbb{C}^n$, be defined as before.

(a) $\Rightarrow$ (b). Assume that $C(A) = \cos(\pi/n)$ and $w(A_1) < \cos(\pi/n)$. By Proposition 2.1, there is a real θ such that $\cos(\pi/n) = \max \sigma(\mathrm{Re}(e^{i\theta}A))$. As before, we may assume that $\theta = 0$ and thus $W(A)$ is contained in the closed half-plane $L \equiv \{z \in \mathbb{C} : \mathrm{Re}\, z \leqslant \cos(\pi/n)\}$. Since x is not in M and N by Lemma 4.3, there are nonzero vectors u and v in M and N, respectively, with $\|u\|^2 + \|v\|^2 = 1$ such that $x/\|x\| = u + v$. Note that u and v (resp., $(\mathrm{Re}\,A)u$ and v, and u and $(\mathrm{Re}\,A)v$) are orthogonal to each other. Hence, by the proof of Proposition 3.6, we have

$$\cos\frac{\pi}{n} = \left\langle (\mathrm{Re}\,A)\frac{x}{\|x\|}, \frac{x}{\|x\|} \right\rangle = \langle (\mathrm{Re}\,A)(u+v), u+v \rangle$$
$$= \langle (\mathrm{Re}\,A)u, u \rangle + \langle (\mathrm{Re}\,A)v, v \rangle$$
$$= \|u\|^2 \mathrm{Re}\left\langle A_1\frac{u}{\|u\|}, \frac{u}{\|u\|} \right\rangle + \|v\|^2 \mathrm{Re}\left\langle A\frac{v}{\|v\|}, \frac{v}{\|v\|} \right\rangle.$$

Since $w(A_1) < \cos(\pi/n)$, we have $\mathrm{Re}\,\langle A_1(u/\|u\|), u/\|u\|\rangle < \cos(\pi/n)$. On the other hand, $W(A) \subseteq L$ implies that $\mathrm{Re}\,\langle A(v/\|v\|), v/\|v\|\rangle \leqslant \cos(\pi/n)$. These two together yield a contradiction to the above convex combination for $\cos(\pi/n)$. Thus we must have $w(A_1) \geqslant \cos(\pi/n)$.

(b) $\Rightarrow$ (a). If $w(A_1) > \cos(\pi/n)$, then $\partial W(A)$ has a line segment by Lemma 4.4. Hence (a) holds by Corollary 3.4.

Next assume that $w(A_1) = \cos(\pi/n)$. After a rotation, we may further assume that $\cos(\pi/n)$ is in $W(A_1)$. Since $W(A) = W(A_2) \supseteq W(J_{n-1}) = \{z \in \mathbb{C} : |z| \leqslant \cos(\pi/n)\}$ by [5, Theorem 2.11], to prove (a), we need only show that $\mathrm{Re}\,A \leqslant (\cos(\pi/n))I_n$. Assuming the contrary that $\mathrm{Re}\,A \not\leqslant (\cos(\pi/n))I_n$, we would also have $\mathrm{Re}\,A_2 \not\leqslant (\cos(\pi/n))I_{n-p}$ since $W(\mathrm{Re}\,A_2) = \mathrm{Re}\,W(A_2) = \mathrm{Re}\,W(A) = W(\mathrm{Re}\,A)$. Let $v = [v_1 \ \ldots \ v_n]^T$ be a unit vector in N such that $\mathrm{Re}\,\langle Av, v \rangle > \cos(\pi/n)$. This shows that $\langle Av, v \rangle$ is not in $W(J_{n-1})$, and thus $v_n \neq 0$. On the other hand, since $\cos(\pi/n)$ is in $W(A_1)$, there is a unit vector $u = [u_1 \ \ldots \ u_n]^T$ in M such that $\langle Au, u \rangle = \cos(\pi/n)$. We now check that $u_n \neq 0$. Indeed, if $u_n = 0$, then $\cos(\pi/n) = \langle Au, u \rangle = \langle J_{n-1}u', u' \rangle$, where $u' = [u_1 \ \ldots \ u_{n-1}]^T$ is a unit vector in $\mathbb{C}^{n-1}$. Thus u' is a multiple of $x' \equiv [x_1 \ \ldots \ x_{n-1}]^T$, where $x_j = \sin(j\pi/n)$, $1 \leqslant j \leqslant n-1$ (cf. [9, Proposition 1 (3)]), and hence u is a multiple of x. The latter would imply that x is in M, which contradicts Lemma 4.3. Hence $u_n \neq 0$. Let $s = u_n/(|u_n|^2 + |v_n|^2)^{1/2}$, $t = -v_n/(|u_n|^2 + |v_n|^2)^{1/2}$ and $w = sv + tu$. Since $|s|^2 + |t|^2 = 1$ and u and v are unit vectors in M and N, respectively, which are orthogonal to each other, it follows that w is also a unit vector. Its nth component $sv_n + tu_n$ is easily seen to be 0, which shows that w is in $\mathbb{C}^{n-1} \oplus \{0\}$. Therefore, $\langle Aw, w \rangle$ is in $W(J_{n-1})$ and hence $\cos(\pi/n) \geqslant \mathrm{Re}\,\langle Aw, w \rangle = \mathrm{Re}\,\langle A(sv + tu), sv + tu \rangle$. Since Au and Av are in M

and N, respectively, Au and v are orthogonal to each other and so are Av and u. Thus

$$
\begin{aligned}
\operatorname{Re}\langle A(sv+tu), sv+tu \rangle &= \operatorname{Re}\left(|s|^2 \langle Av, v \rangle + |t|^2 \langle Au, u \rangle\right) \\
&= |s|^2 \operatorname{Re}\langle Av, v \rangle + |t|^2 \operatorname{Re}\langle Au, u \rangle \\
&> |s|^2 \cos\frac{\pi}{n} + |t|^2 \cos\frac{\pi}{n} \\
&= \cos\frac{\pi}{n},
\end{aligned}
$$

where the strict inequality is a consequence of the nonzeroness of s. This leads to a contradiction. We conclude that $\operatorname{Re} A \leqslant \left(\cos(\pi/n)\right)I_n$, which proves (a).

(b) $\Rightarrow$ (c). If $w(A_1) > \cos(\pi/n)$, then $\partial W(A)$ has a line segment by Lemma 4.4. Hence (c) holds by [4, Theorem 1 (ii)] and Lemma 3.3. On the other hand, if $w(A_1) = \cos(\pi/n)$, then [5, Theorem 2.11] implies that $W(A) = W(A_2)$. Since $C(A) = \cos(\pi/n)$ by (a) (which has been shown to be equivalent to (b)), there is a point $\lambda = \left(\cos(\pi/n)\right)e^{i\theta}$ (θ real) in $\partial W(A) \cap \partial W(J_{n-1})$. After the rotation around the origin by the angle $-\theta$, we may assume that $\theta = 0$. Letting $x' = [x_1 \ \ldots \ x_{n-1}]^T$ in $\mathbb{C}^{n-1}$, we have $\langle Ax, x \rangle = \langle J_{n-1}x', x' \rangle = \lambda$ by [9, Proposition 1 (3)]. On the other hand, as λ is also in $\partial W(A_2)$, there is a unit vector v in N such that $\langle A_2 v, v \rangle = \lambda$. Therefore, both x and v are in M_λ. Since x is not in N by Lemma 4.3, they are linearly independent. This shows that $\dim M_\lambda \geqslant 2$. Thus (c) follows from Lemma 3.3.

Since (c) $\Rightarrow$ (a) and (a) $\Rightarrow$ (d) have been proven in Propositions 3.2 and 3.6, respectively, to complete the proof we need only show that (d) $\Rightarrow$ (b).

(d) $\Rightarrow$ (b). Condition (d) (with $\theta = 0$) implies that the vector x is such that $(\operatorname{Re} A)x = \left(\cos(\pi/n)\right)x$. Hence $\cos(\pi/n)$ is in $\sigma(\operatorname{Re} A) = \sigma(\operatorname{Re} A_1) \cup \sigma(\operatorname{Re} A_2)$. We now show that $\cos(\pi/n)$ is always in $\sigma(\operatorname{Re} A_1)$. Indeed, assume that it is in $\sigma(\operatorname{Re} A_2)$, but not in $\sigma(\operatorname{Re} A_1)$. Let v be a unit vector in N such that $(\operatorname{Re} A_2)v = \left(\cos(\pi/n)\right)v$. We also have $(\operatorname{Re} A)v = \left(\cos(\pi/n)\right)v$. As x is not in N by Lemma 4.3, the vectors x and v are linearly independent. Thus $\dim\ker\left(\operatorname{Re} A - (\cos(\pi/n))I_n\right) \geqslant 2$. But

$$
\begin{aligned}
&\dim\ker\left(\operatorname{Re} A - \left(\cos\frac{\pi}{n}\right)I_n\right) \\
&= \dim\ker\left(\operatorname{Re} A_1 - \left(\cos\frac{\pi}{n}\right)I_p\right) + \dim\ker\left(\operatorname{Re} A_2 - \left(\cos\frac{\pi}{n}\right)I_{n-p}\right) \\
&= 0 + 1 = 1
\end{aligned}
$$

by [1, Theorem 2.8], which leads to a contradiction. Thus $\cos(\pi/n)$ is in $\sigma(\operatorname{Re} A_1)$ or in $W(\operatorname{Re} A_1)$ ($= \operatorname{Re} W(A_1)$), and hence $w(A_1) \geqslant \cos(\pi/n)$. $\quad\square$

The following are some consequences of Theorem 4.1.

COROLLARY 4.5. *Under the same assumptions of Theorem* 4.1, *if* $|a| \geqslant \cos(\pi/n)$, *then* $C(A) = \cos(\pi/n)$.

Proof. Since $\sigma(A_1) \subseteq W(A_1)$, the assumption $|a| \geqslant \cos(\pi/n)$ implies that $w(A_1) \geqslant \cos(\pi/n)$. Theorem 4.1 then yields $C(A) = \cos(\pi/n)$. $\quad\square$

COROLLARY 4.6. *Under the same assumptions of Theorem 4.1, if $\sigma(A_1)$ is a singleton, then $C(A) = \cos(\pi/n)$ if and only if $|a| \geqslant \cos(\pi/n)$.*

Proof. In this case, we have $w(A_1) = |a|$. The assertion follows from Theorem 4.1 immediately. $\square$

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