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# THE JORDAN ALGEBRAIC STRUCTURE OF THE CIRCULAR CONE

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*Abstract.* In this paper, we study and analyze the algebraic structure of the circular cone. We establish a new efficient spectral decomposition, set up the Jordan algebra associated with the circular cone, and prove that this algebra forms a Euclidean Jordan algebra with a certain inner product. We then show that the cone of squares of this Euclidean Jordan algebra is indeed the circular cone itself. The circular cones form a much more general class than the second-order cones, so we generalize some important algebraic properties in the Euclidean Jordan algebra of the second-order cones to the Euclidean Jordan algebra of the circular cones.

## 1. Introduction

The most well-known examples of symmetric cones are the nonnegative orthant cone in the space of real numbers $\mathbb{R}$, the second-order cone in the real vector space $\mathbb{R}^n$, and the cone of positive semidefinite matrices in the space of real symmetric matrices. The symmetric optimization problems associated with these three symmetric cones are respectively linear programming, second-order cone programming [1], and semidefinite programming [2]. In this paper, we study and analyze the algebraic structure of the circular cone as a symmetric cone perceived to be associated with the so called circular programming [3]. The applications of the circular cones lie in various real-world engineering problems, for example the optimal grasping manipulation problems for multi-fingered robots [4].

Let $\theta \in \left(0, \frac{\pi}{2}\right)$ and $\|\cdot\|$ denote the standard Euclidean norm. The *circular cone* [5, 6] of dimension $n$ is defined as

$$\mathscr{L}^n_\theta := \left\{ \begin{bmatrix} x_0 \\ \bar{\boldsymbol{x}} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 \tan\theta \geqslant \|\bar{\boldsymbol{x}}\| \right\}. \tag{1}$$

It is popularly known (see for example [5, 6]) that the dual of the circular cone (1), denoted by $\mathscr{L}^{n\star}_\theta$, is the circular cone

$$\mathscr{L}^n_{\frac{\pi}{2}-\theta} = \left\{ \begin{bmatrix} x_0 \\ \bar{\boldsymbol{x}} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 \cot\theta \geqslant \|\bar{\boldsymbol{x}}\| \right\}. \tag{2}$$

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An important special case is that in which $\theta = \frac{\pi}{4}$. In this case, the circular cones (1) and (2) reduce to the well-known second-order cone $\mathscr{Q}^n$ given by

$$\mathscr{Q}^n := \left\{ \begin{bmatrix} x_0 \\ \bar{\boldsymbol{x}} \end{bmatrix} \in \mathbb{R} \times \mathbb{R}^{n-1} : x_0 \geqslant \|\bar{\boldsymbol{x}}\| \right\} = \mathscr{L}^n_{\frac{\pi}{4}}.$$

All recent papers on the circular cone (see for example [5,6,3,7]) have treated this cone as a non-symmetric cone. In this paper, we study and analyze the circular cone (1) as a symmetric cone in the real vector space $\mathbb{R}^n$. We also establish a new spectral decomposition associated with the circular cone. We will see that this new spectral decomposition is much more efficient than the spectral decomposition established by Zhou and Chen in [5]. We use this new spectral decomposition to form the Euclidean Jordan algebra associated with the circular cone under a certain inner product. We also show that the cone of squares of this Euclidean Jordan algebra is indeed the circular cone itself. Some important algebraic properties of the Euclidean Jordan algebra associated with the second-order cone are generalized to the Euclidean Jordan algebra associated with the circular cone.

This paper is organized as follows: In Section 2, we review the definition of symmetric cones and give the necessary foundation from the theory of Euclidean Jordan algebra that is needed for our subsequent development. Section 3 views the circular cone (1) is a symmetric cone. In Section 4, we establish a new spectral decomposition associated with the circular cone that successfully (unlike the spectral decomposition established in [5]) allows us to set up its Jordan algebra. In Section 5, we show that the Jordan algebra established in Section 4 forms a Euclidean Jordan algebra, we then show that the cone of squares of this Euclidean Jordan algebra is the circular cone. In Section 6, we generalize important properties of the theory of Euclidean Jordan algebra associated with the second-order cones to the theory of Euclidean Jordan algebra associated with the circular cones. Section 7 contains some concluding remarks and future work.

## 2. Background and fundamentals

As we mentioned earlier, we review in this section the definition of a symmetric cone and some notions from the theory of the Jordan algebras. The text of Faraut and Korányi [8] covers the foundations of this theory.

### 2.1. Symmetric cones

A cone is said to be *closed* iff it is closed under the taking of sequential limits, *convex* iff it is closed under taking convex combinations, *pointed* iff it does not contain two opposite nonzero vectors (so the origin is an extreme point), *solid* iff it has a nonempty interior, and *regular* iff it is a closed, convex, pointed, solid cone. Clearly, the circular cone is regular.

Let $\mathscr{V}$ be a finite-dimensional Euclidean vector space over $\mathbb{R}$ with inner product "$\bullet$". A regular $\mathscr{K} \subset \mathscr{V}$ is said to be *self-dual* if it coincides with its dual cone $\mathscr{K}^*$, i.e.,

$$\mathscr{K} = \mathscr{K}^* := \{\boldsymbol{s} \in \mathscr{V} : \boldsymbol{x} \bullet \boldsymbol{s} \geqslant 0, \ \forall \boldsymbol{x} \in \mathscr{K}\}.$$

By $GL(n,\mathbb{R})$ we mean the general linear group of degree $n$ over $\mathbb{R}$ (i.e., the set of $n \times n$ invertible matrices with entries from $\mathbb{R}$, together with the operation of ordinary matrix multiplication). For a regular cone $\mathscr{K} \subset \mathscr{V}$, we denote by $\operatorname{Int}(\mathscr{K})$ the interior of $\mathscr{K}$, by $\operatorname{Bd}(\mathscr{K})$ the boundary of $\mathscr{K}$ and by $\operatorname{Aut}(\mathscr{K})$ the automorphism group of $\mathscr{K}$, i.e., $\operatorname{Aut}(\mathscr{K}) := \{\varphi \in GL(n,\mathbb{R}) : \varphi(\mathscr{K}) = \mathscr{K}\}$.

DEFINITION 1. Let $\mathscr{V}$ be a finite-dimensional real Euclidean space. A regular $\mathscr{K} \subset \mathscr{V}$ is said to be homogeneous if for each $\boldsymbol{u}, \boldsymbol{v} \in \operatorname{Int}(\mathscr{K})$, there exists an invertible linear map $F : \mathscr{V} \longrightarrow \mathscr{V}$ such that

1. $F(\mathscr{K}) = \mathscr{K}$, i.e., $F$ is an automorphism of $\mathscr{K}$, and
2. $F(\boldsymbol{u}) = \boldsymbol{v}$.

In other words, $\operatorname{Aut}(\mathscr{K})$ acts transitively on the interior of $\mathscr{K}$.

A regular $\mathscr{K}$ is said to be *symmetric* if it is both self-dual and homogeneous.

We end this part by introducing some notations that will be used in the sequel. We use $\mathbb{R}$ to denote the field of real numbers. For each vector $\boldsymbol{x} \in \mathbb{R}^n$ whose first entry is indexed with $0$, we write $\overline{\boldsymbol{x}}$ for the subvector consisting of entries $1$ through $n-1$ (therefore $\boldsymbol{x} = (x_0; \overline{\boldsymbol{x}}) \in \mathbb{R} \times \mathbb{R}^{n-1}$), and we let $\hat{\boldsymbol{x}} = \frac{\overline{\boldsymbol{x}}}{\|\overline{\boldsymbol{x}}\|}$ if $\hat{\boldsymbol{x}} \neq \boldsymbol{0}$ and $\hat{\boldsymbol{x}}$ be any vector $\boldsymbol{w} \in \mathbb{R}^{n-1}$ satisfying $\|\boldsymbol{w}\| = 1$ if $\overline{\boldsymbol{x}} = \boldsymbol{0}$. We let $\mathscr{E}^n$ denote the $n$ dimensional real vector space $\mathbb{R} \times \mathbb{R}^{n-1}$ whose elements $\boldsymbol{x}$ are indexed with $0$.

We define the *circular identity matrix* and the *circular reflection matrix* as follows:

$$J_\theta := \begin{bmatrix} 1 & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & \cot^2\theta \, I_{n-1} \end{bmatrix} \text{ and } R_\theta := \begin{bmatrix} 1 & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & -\cot^2\theta \, I_{n-1} \end{bmatrix}. \tag{3}$$

Throughout this paper, we will use the circular identity matrix $J_\theta$ as a generalization of the identity matrix $I_n$ (where $I_n = J_{\frac{\pi}{4}}$), and the circular reflection matrix $R_\theta$ as a generalization of the reflection matrix:

$$R := \begin{bmatrix} 1 & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & -I_{n-1} \end{bmatrix} = R_{\frac{\pi}{4}}. \tag{4}$$

The *circular inner product* of the two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{E}^n$ is defined as follows (see also [7]):

$$\boldsymbol{x} \bullet \boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle_\theta := x_0 y_0 + \cot^2\theta \, \overline{\boldsymbol{x}}^\mathsf{T} \overline{\boldsymbol{y}} = \boldsymbol{x}^\mathsf{T} J_\theta \boldsymbol{y} \tag{5}$$

Clearly, the circular inner product $\langle \cdot, \cdot \rangle_\theta$ is a generalization of the standard inner product $\langle \cdot, \cdot \rangle$ defined as

$$\boldsymbol{x} \bullet \boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle := \boldsymbol{x}^\mathsf{T} \boldsymbol{y} = x_0 y_0 + \overline{\boldsymbol{x}}^\mathsf{T} \overline{\boldsymbol{y}} = \boldsymbol{x}^\mathsf{T} I_n \boldsymbol{y}. \tag{6}$$

## 2.2. Euclidean Jordan algebras

Our presentation in this section follows that of [9, Section 2].

DEFINITION 2. A finite-dimensional vector space $\mathscr{J}$ over $\mathbb{R}$ is called an algebra over $\mathbb{R}$ if a bilinear map $\circ : \mathscr{J} \times \mathscr{J} \longrightarrow \mathscr{J}$ exists.

Let $\boldsymbol{x}$ be an element in an algebra $\mathscr{J}$, then we define $\boldsymbol{x}^{(n)}$ recursively by $\boldsymbol{x}^{(n)} := \boldsymbol{x} \circ \boldsymbol{x}^{(n-1)}$ for $n \geqslant 2$.

DEFINITION 3. Let $\mathscr{J}$ be a finite-dimensional $\mathbb{R}$ algebra with a bilinear product $\circ : \mathscr{J} \times \mathscr{J} \longrightarrow \mathscr{J}$. Then $(\mathscr{J}, \circ)$ is called a Jordan algebra if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{J}$

1. $\boldsymbol{x} \circ \boldsymbol{y} = \boldsymbol{y} \circ \boldsymbol{x}$ (commutativity);
2. $\boldsymbol{x} \circ (\boldsymbol{x}^{(2)} \circ \boldsymbol{y}) = \boldsymbol{x}^{(2)} \circ (\boldsymbol{x} \circ \boldsymbol{y})$ (Jordan's axiom).

The product $\boldsymbol{x} \circ \boldsymbol{y}$ between two elements $\boldsymbol{x}$ and $\boldsymbol{y}$ of a Jordan algebra $(\mathscr{J}, \circ)$ is called the *Jordan multiplication* between $\boldsymbol{x}$ and $\boldsymbol{y}$. A Jordan algebra $(\mathscr{J}, \circ)$ has an *identity element* if there exists a (necessarily unique) element $\boldsymbol{e} \in \mathscr{J}$ such that $\boldsymbol{x} \circ \boldsymbol{e} = \boldsymbol{e} \circ \boldsymbol{x} = \boldsymbol{x}$ for all $\boldsymbol{x} \in \mathscr{J}$. A Jordan algebra $(\mathscr{J}, \circ)$ is not necessarily associative, that is, $\boldsymbol{x} \circ (\boldsymbol{y} \circ \boldsymbol{z}) = (\boldsymbol{x} \circ \boldsymbol{y}) \circ \boldsymbol{z}$ may not hold in general. However, it is power associative, i.e., $\boldsymbol{x}^{(p)} \circ \boldsymbol{x}^{(q)} = \boldsymbol{x}^{(p+q)}$ for all integers $p, q \geqslant 1$.

EXAMPLE 1. It can be verified that the space $\mathscr{E}^n$ with the Jordan multiplication

$$\boldsymbol{x} \circ \boldsymbol{y} = \begin{bmatrix} \boldsymbol{x}^{\mathsf{T}} \boldsymbol{y} \\ x_0 \overline{y} + y_0 \overline{x} \end{bmatrix} \tag{7}$$

forms a Jordan algebra with the identity vector

$$\boldsymbol{e} := \begin{bmatrix} 1 \\ \boldsymbol{0} \end{bmatrix}. \tag{8}$$

DEFINITION 4. A Jordan algebra $\mathscr{J}$ is called Euclidean if there exists an inner product $\langle \cdot, \cdot \rangle$ on $(\mathscr{J}, \circ)$ such that for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathscr{J}$

1. $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0 \ \forall \, \boldsymbol{x} \neq \boldsymbol{0}$ (positive definiteness);
2. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ (symmetry);
3. $\langle \boldsymbol{x}, \boldsymbol{y} \circ \boldsymbol{z} \rangle = \langle \boldsymbol{x} \circ \boldsymbol{y}, \boldsymbol{z} \rangle$ (associativity).

That is, $\mathscr{J}$ admits a positive definite, symmetric, quadratic form which is also associative.

EXAMPLE 2. It is easy to verify that the space $\mathscr{E}^n$, with the Jordan multiplication "$\circ$" defined in (7), is a Euclidean Jordan algebra with the standard inner product $\langle \cdot, \cdot \rangle$ defined in (6).

DEFINITION 5. Let $\mathscr{J}$ be a Jordan algebra. Then

1. for $\boldsymbol{x} \in \mathscr{J}$, $\deg(\boldsymbol{x}) := \min\left\{r > 0 : \{\boldsymbol{e}, \boldsymbol{x}, \boldsymbol{x}^{(2)}, \ldots, \boldsymbol{x}^{(r)}\} \text{ is linearly dependent}\right\}$ is called the degree of $\boldsymbol{x}$;
2. $\operatorname{rank}(\mathscr{J}) := \max_{\boldsymbol{x} \in \mathscr{J}} \deg(\boldsymbol{x})$ is called the rank of $\mathscr{J}$.

A *spectral decomposition* is a decomposition of $\boldsymbol{x}$ into idempotents together with the eigenvalues. Recall that two elements $\boldsymbol{c}_1, \boldsymbol{c}_2 \in \mathscr{J}$ are said to be *orthogonal* if $\boldsymbol{c}_1 \circ \boldsymbol{c}_2 = \boldsymbol{0}$. A set of elements of $\mathscr{J}$ is orthogonal if all its elements are mutually orthogonal to each other. An element $\boldsymbol{c} \in \mathscr{J}$ is said to be an *idempotent* if $\boldsymbol{c}^2 = \boldsymbol{c}$. An idempotent is *primitive* if it is non-zero and cannot be written as a sum of two (necessarily orthogonal) non-zero idempotents.

DEFINITION 6. Let $\mathscr{J}$ be a Jordan algebra. Then a subset $\{\boldsymbol{c}_1, \boldsymbol{c}_2, \ldots, \boldsymbol{c}_r\}$ of $\mathscr{J}$ is called:

1. a complete system of orthogonal idempotents if it is an orthogonal set of idempotents where $\boldsymbol{c}_1 + \boldsymbol{c}_2 + \cdots + \boldsymbol{c}_r = \boldsymbol{e}$;
2. a Jordan frame if it is a complete system of orthogonal primitive idempotents.

EXAMPLE 3. Let $\boldsymbol{x}$ be a vector in the algebra of the second-order cone $\mathscr{Q}^n$, the space $\mathscr{E}^n$, and define

$$\boldsymbol{c}_1(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} 1 \\ \hat{\boldsymbol{x}} \end{bmatrix} \text{ and } \boldsymbol{c}_2(\boldsymbol{x}) = \frac{1}{2}\begin{bmatrix} 1 \\ -\hat{\boldsymbol{x}} \end{bmatrix}.$$

It is easy to see that the set $\{\boldsymbol{c}_1(\boldsymbol{x}), \boldsymbol{c}_2(\boldsymbol{x})\}$ is a Jordan frame in $\mathscr{E}^n$.

THEOREM 1. (Spectral decomposition, [8, Theorem 7]) *Let $\mathscr{J}$ be a Jordan algebra with rank $r$. Then for $\boldsymbol{x} \in \mathscr{J}$ there exist real numbers $\lambda_1(\boldsymbol{x}), \lambda_2(\boldsymbol{x}), \ldots, \lambda_r(\boldsymbol{x})$ (called the eigenvalues of $\boldsymbol{x}$), and a Jordan frame $\boldsymbol{c}_1(\boldsymbol{x}), \boldsymbol{c}_2(\boldsymbol{x}), \ldots, \boldsymbol{c}_r(\boldsymbol{x})$ such that*

$$\boldsymbol{x} = \lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) + \cdots + \lambda_r(\boldsymbol{x})\boldsymbol{c}_r(\boldsymbol{x}). \tag{9}$$

SPECTRAL DECOMPOSITION 1. Let $\boldsymbol{x}$ be a vector in the algebra of the second-order cone $\mathscr{Q}^n$, the space $\mathscr{E}^n$. The spectral decomposition of $\boldsymbol{x}$ associated with $\mathscr{Q}^n$ is obtained as follows:

$$\boldsymbol{x} = \underbrace{(x_0 + \|\overline{\boldsymbol{x}}\|)}_{\lambda_1(\boldsymbol{x})} \underbrace{\left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ \hat{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{c}_1(\boldsymbol{x})} + \underbrace{(x_0 - \|\overline{\boldsymbol{x}}\|)}_{\lambda_2(\boldsymbol{x})} \underbrace{\left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ -\hat{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{c}_2(\boldsymbol{x})}. \tag{10}$$

DEFINITION 7. Let $\boldsymbol{x}$ be an element in a rank-$r$ Jordan algebra $\mathscr{J}$ with the spectral decomposition given in (9). Then

1. $\operatorname{trace}(\boldsymbol{x}) := \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) + \cdots + \lambda_r(\boldsymbol{x})$ is the trace of $\boldsymbol{x}$ in $\mathscr{J}$;

2. $\det(\boldsymbol{x}) := \lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})\cdots\lambda_r(\boldsymbol{x})$ is the determinant of $\boldsymbol{x}$ in $\mathscr{J}$.

EXAMPLE 4. Let $\boldsymbol{x}$ be an element in $\mathscr{E}^n$ with the spectral decomposition given in (10). Then the eigenvalues of $\boldsymbol{x}$ are $\lambda_{1,2}(\boldsymbol{x}) = x_0 \pm \|\overline{\boldsymbol{x}}\|$. We also have that

$$\operatorname{trace}(\boldsymbol{x}) = 2x_0, \text{ and } \det(\boldsymbol{x}) = x_0^2 - \|\overline{\boldsymbol{x}}\|^2. \tag{11}$$

Observe also that $\lambda_1(\boldsymbol{x}) = \lambda_2(\boldsymbol{x})$ if and only if $\overline{\boldsymbol{x}} = \boldsymbol{0}$, and therefore $\boldsymbol{x}$ is a multiple of the identity. Thus, every $\boldsymbol{x} \in \mathscr{E}^n - \{\alpha \boldsymbol{e} : \alpha \in \mathbb{R}\}$ has degree 2. This implies that $\operatorname{rank}(\mathscr{E}^n) = 2$, which is independent of the dimension of the underlying vector space.

Now, for $\boldsymbol{x} \in \mathscr{J}$ having the spectral decomposition given in (10), we can define $\boldsymbol{x}^2$ as

$$\boldsymbol{x}^2 := (\lambda_1(\boldsymbol{x}))^2\boldsymbol{c}_1(\boldsymbol{x}) + (\lambda_2(\boldsymbol{x}))^2\boldsymbol{c}_2(\boldsymbol{x}) + \cdots + (\lambda_r(\boldsymbol{x}))^2\boldsymbol{c}_r(\boldsymbol{x}).$$

One can easily see that $\boldsymbol{x}^2 = \boldsymbol{x}^{(2)} = \boldsymbol{x} \circ \boldsymbol{x}$. More generally, $\boldsymbol{x}^p = \boldsymbol{x}^{(p)}$ for any integer $p \geqslant 1$. We also have the following definition.

DEFINITION 8. Let $\boldsymbol{x}$ be an element of a Jordan algebra $\mathscr{J}$ with a spectral decomposition given in (9). If $\det(\boldsymbol{x}) \neq 0$ (i.e., all $\lambda_i(\boldsymbol{x}) \neq 0$), then we say that $\boldsymbol{x}$ is invertible. In this case, the inverse of $\boldsymbol{x}$ is defined by

$$\boldsymbol{x}^{-1} := \frac{1}{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \frac{1}{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x}) + \cdots + \frac{1}{\lambda_r(\boldsymbol{x})}\boldsymbol{c}_r(\boldsymbol{x}).$$

More generally, if $f$ is any real valued continuous function, then it is also possible to extend the above definition to define $f(\boldsymbol{x})$ as

$$f(\boldsymbol{x}) := f(\lambda_1(\boldsymbol{x}))\boldsymbol{c}_1(\boldsymbol{x}) + f(\lambda_2(\boldsymbol{x}))\boldsymbol{c}_2(\boldsymbol{x}) + \cdots + f(\lambda_r(\boldsymbol{x}))\boldsymbol{c}_r(\boldsymbol{x}).$$

EXAMPLE 5. Let $\boldsymbol{x}$ be an invertible vector in the algebra $\mathscr{E}^n$. One can see that

$$\boldsymbol{x}^{-1} = \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_0 \\ -\overline{x} \end{bmatrix} = \frac{R}{\det(\boldsymbol{x})}\,\boldsymbol{x}, \tag{12}$$

where $R$ is the reflection matrix defined in (4).

We now define the cone of squares of a Euclidean Jordan algebra.

DEFINITION 9. If $\mathscr{J}$ is a Euclidean Jordan algebra then its cone of squares is the set

$$\mathscr{K}_{\mathscr{J}} := \{\boldsymbol{x}^2 : \boldsymbol{x} \in \mathscr{J}\}.$$

The following fundamental result gives a one-to-one correspondence between Euclidean Jordan algebras and symmetric cones.

THEOREM 2. (Jordan algebraic characterization of symmetric cones, [8]) *A regular cone $\mathscr{K}$ is symmetric iff $\mathscr{K} = \mathscr{K}_{\mathscr{J}}$ for some Euclidean Jordan algebra $\mathscr{J}$.*

EXAMPLE 6. The cone of squares of $(\mathscr{E}^n, \circ)$, with "$\circ$" defined in (7), is the second-order cone $\mathscr{Q}^n$.

We now define three well-known maps: two linear maps from $\mathscr{J}$ into itself, namely $L(\boldsymbol{x})$ and $Q_{\boldsymbol{x}}$, and one quadratic map from $\mathscr{J} \times \mathscr{J}$ into $\mathscr{J}$, namely $Q_{\boldsymbol{x},\boldsymbol{y}}$. These important maps play a crucial role in the development of the interior point methods for conic programming.

DEFINITION 10. Let $\boldsymbol{x}$ and $\boldsymbol{z}$ be elements in a rank-$r$ Jordan algebra $\mathscr{J}$. Then

1. The linear map $L(\boldsymbol{x}) : \mathscr{J} \longrightarrow \mathscr{J}$ is defined by

$$L(\boldsymbol{x})\boldsymbol{y} := \boldsymbol{x} \circ \boldsymbol{y},$$

   for all $\boldsymbol{y} \in \mathscr{J}$.

2. The quadratic operator $Q_{\boldsymbol{x},\boldsymbol{z}} : \mathscr{J} \times \mathscr{J} \longrightarrow \mathscr{J}$ is defined by

$$Q_{\boldsymbol{x},\boldsymbol{z}} := L(\boldsymbol{x})L(\boldsymbol{z}) + L(\boldsymbol{z})L(\boldsymbol{x}) - L(\boldsymbol{x} \circ \boldsymbol{z}).$$

3. The quadratic representation of $\boldsymbol{x}$, $Q_{\boldsymbol{x}} : \mathscr{J} \longrightarrow \mathscr{J}$, is defined by

$$Q_{\boldsymbol{x}} := 2L(\boldsymbol{x})^2 - L(\boldsymbol{x}^2) = Q_{\boldsymbol{x},\boldsymbol{x}}.$$

Note that $L(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}$ and $L(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2$. Note also that $L(\boldsymbol{e}) = Q_{\boldsymbol{e}} = I$, $\operatorname{trace}(\boldsymbol{e}) = r$, $\det(\boldsymbol{e}) = 1$ (since all the eigenvalues of $\boldsymbol{e}$ are equal to one).

EXAMPLE 7. From (7), the explicit formula of the $L(\cdot)$ operator for the algebra of the second-order cone, the space $\mathscr{E}^n$, can be immediately given by

$$L(\boldsymbol{x}) = \operatorname{Arw}(\boldsymbol{x}) = \begin{bmatrix} x_0 & \overline{x}^{\mathsf{T}} \\ \overline{x} & x_0 I \end{bmatrix}. \tag{13}$$

Here $\operatorname{Arw}(\boldsymbol{x})$ is the arrow-shaped matrix associated with the vector $\boldsymbol{x} \in \mathscr{E}^n$. Hence, quadratic operator for the algebra of the second-order cone is given by

$$\begin{aligned} Q_{\boldsymbol{x},\boldsymbol{z}} &= \operatorname{Arw}(\boldsymbol{x})\operatorname{Arw}(\boldsymbol{z}) + \operatorname{Arw}(\boldsymbol{z})\operatorname{Arw}(\boldsymbol{x}) - \operatorname{Arw}(\boldsymbol{x} \circ \boldsymbol{z}) \\ &= \begin{bmatrix} \boldsymbol{x}^{\mathsf{T}}\boldsymbol{z} & (x_0\overline{z}^{\mathsf{T}} + z_0\overline{x}^{\mathsf{T}}) \\ x_0\overline{z} + z_0\overline{x} & (\overline{x}\overline{z}^{\mathsf{T}} + \overline{z}\overline{x}^{\mathsf{T}})(x_0 z_0 - \overline{x}^{\mathsf{T}}\overline{z}) I_{n-1} \end{bmatrix}. \end{aligned}$$

We can also easily verify that the quadratic representation for the algebra of the second-order cone is given by

$$Q_{\boldsymbol{x}} = 2\operatorname{Arw}^2(\boldsymbol{x}) - \operatorname{Arw}(\boldsymbol{x}^2) = \begin{bmatrix} \|\boldsymbol{x}\|^2 & 2x_0\overline{x}^{\mathsf{T}} \\ 2x_0\overline{x} & \det(\boldsymbol{x})I + 2\overline{x}\overline{x}^{\mathsf{T}} \end{bmatrix} = 2\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}} - \det(\boldsymbol{x})R. \tag{14}$$

## 3. Symmetrization of the circular cone

The circular cone given in (1) can be rewritten as

$$\mathscr{L}_\theta^n = \{\boldsymbol{x} \in \mathscr{E}^n : x_0 \geqslant \cot\theta \|\overline{\boldsymbol{x}}\|\},$$

and the circular cone given in (2) can be rewritten as

$$\mathscr{L}^n_{\frac{\pi}{2}-\theta} = \{\boldsymbol{x} \in \mathscr{E}^n : x_0 \geqslant \tan\theta \|\overline{\boldsymbol{x}}\|\}.$$

It can be shown that, under the standard inner product $\langle\cdot,\cdot\rangle$ defined in (6), the dual of the circular cone $\mathscr{L}_\theta^n$ is the circular cone $\mathscr{L}^n_{\frac{\pi}{2}-\theta}$. In fact, in our subsequent development, we will see that the inner product that should be considered in dealing with the algebraic and analytic operations associated with the circular cones is indeed the circular inner product $\langle\cdot,\cdot\rangle_\theta$ defined in (5) and not the standard inner product $\langle\cdot,\cdot\rangle$. In this part, we prove that the circular cone, with the circular inner product $\langle\cdot,\cdot\rangle_\theta$, is self-dual and homogeneous, and hence symmetric.

LEMMA 1. (See also [7, Section 1]) The circular cone $\mathscr{L}_\theta^n$ is self-dual.

*Proof.* First, we show that $\mathscr{L}_\theta^n \subseteq \mathscr{L}_\theta^{n\star}$. Let $\boldsymbol{x} = (x_0;\overline{\boldsymbol{x}}) \in \mathscr{L}_\theta^n$, we show that $\boldsymbol{x} \in \mathscr{L}_\theta^{n\star}$ by verifying that $\langle \boldsymbol{x},\boldsymbol{y}\rangle_\theta \geqslant 0$ for any $\boldsymbol{y} \in \mathscr{L}_\theta^n$. So let $\boldsymbol{y} = (y_0;\overline{\boldsymbol{y}}) \in \mathscr{L}_\theta^n$. Then

$$\langle \boldsymbol{x},\boldsymbol{y}\rangle_\theta = x_0\, y_0 + \cot^2\theta \overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}} \geqslant \cot^2\theta\,(\|\overline{\boldsymbol{x}}\|\,\|\overline{\boldsymbol{y}}\| + \overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}}) \geqslant \cot^2\theta\,(|\overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}}| + \overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}}) \geqslant 0,$$

where the first inequality follows from the fact that $\boldsymbol{x},\boldsymbol{y} \in \mathscr{L}_\theta^n$, and the second follows from Hölder's inequality. Thus, $\mathscr{L}_\theta^n \subseteq \mathscr{L}_\theta^{n\star}$.

Now, we prove that $\mathscr{L}_\theta^{n\star} \subseteq \mathscr{L}_\theta^n$. Let $\boldsymbol{y} = (y_0;\overline{\boldsymbol{y}}) \in \mathscr{L}_\theta^{n\star}$, and consider $\boldsymbol{x} := (\cot\theta\|\overline{\boldsymbol{y}}\|; -\overline{\boldsymbol{y}}) \in \mathscr{L}_\theta^n$. Then we obtain

$$\begin{aligned} 0 \leqslant \langle \boldsymbol{x},\boldsymbol{y}\rangle_\theta &= \cot\theta\, y_0\|\overline{\boldsymbol{y}}\| - \cot^2\theta\, \overline{\boldsymbol{y}}^{\mathsf{T}}\overline{\boldsymbol{y}} = \cot\theta\, y_0\|\overline{\boldsymbol{y}}\| - \cot^2\theta\, \|\overline{\boldsymbol{y}}\|^2 \\ &= \|\overline{\boldsymbol{y}}\|\cot\theta\,(y_0 - \cot\theta\,\|\overline{\boldsymbol{y}}\|). \end{aligned}$$

This implies that $y_0 \geqslant \cot\theta\|\overline{\boldsymbol{y}}\|$. Hence, $\boldsymbol{y} \in \mathscr{L}_\theta^n$. This demonstrates that $\mathscr{L}_\theta^n$ is self dual. The proof is complete. □

LEMMA 2. The circular cone $\mathscr{L}_\theta^n$ is homogeneous.

*Proof.* Note that the circular cone $\mathscr{L}_\theta^n$ can be rewritten as

$$\mathscr{L}_\theta^n = \{\boldsymbol{x} \in \mathscr{E}^n : \boldsymbol{x}^{\mathsf{T}} R_\theta \boldsymbol{x} \geqslant 0\},$$

where $R_\theta$ is the circular reflection matrix defined in (3). Define

$$\mathscr{G}_\theta := \{M \in \mathbb{R}^{n\times n} : M^{\mathsf{T}} R_\theta M = R_\theta\}.$$

It is not hard to prove that $\mathscr{G}_\theta$ is a group. Observe that each element of the group $\mathscr{G}_\theta$ maps $\mathscr{L}^n_\theta$ onto itself (because, for every $M \in \mathscr{G}_\theta$, we have that $(M\boldsymbol{x})^\mathsf{T} R_\theta (M\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T} R_\theta \boldsymbol{x}$), and so does the direct product $\mathscr{H}_\theta := [0,1] \times \mathscr{G}_\theta$. It now remains to show that the group $\mathscr{H}_\theta$ acts transitively on the interior of $\mathscr{L}^n_\theta$. To do so, it is enough to show that, for any $\boldsymbol{x} \in \text{Int}(\mathscr{L}^n_\theta)$, there exists an element in $\mathscr{H}_\theta$ that maps $\boldsymbol{e}$ to $\boldsymbol{x}$.

Since $\boldsymbol{x}^\mathsf{T} R_\theta \boldsymbol{x} > 0$, we may write $\boldsymbol{x}$ as $\boldsymbol{x} = \lambda \cot^2\theta \boldsymbol{y}$ with $\lambda = \sqrt{\boldsymbol{x}^\mathsf{T} R_\theta \boldsymbol{x}}$ and $\boldsymbol{y} \in \mathscr{E}^n$. Moreover, by [10, Theorem 3.20], there exists a reflector matrix $P$ (accordingly, $P^2 = I_{n-1}$) such that

$$P \begin{bmatrix} \mathbf{0} \\ r \end{bmatrix} = \cot\theta \overline{\boldsymbol{y}},$$

with $|r| = \cot\theta \|\overline{\boldsymbol{y}}\|$. We then have

$$y_0^2 - r^2 = y_0^2 - \cot^2\theta \|\overline{\boldsymbol{y}}\|^2 = \boldsymbol{y}^\mathsf{T} J_\theta \boldsymbol{y} = \frac{1}{\lambda^2} \boldsymbol{x}^\mathsf{T} J_\theta \boldsymbol{x} = 1.$$

Therefore, there exists $t \geqslant 0$ such that $y_0 = \cosh t$ and $r = \sinh t$. Now, we define

$$\hat{P} := \begin{bmatrix} 1 & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & P \end{bmatrix} \text{ and } H_t(\theta) := \begin{bmatrix} \cot^2\theta \cosh t & \mathbf{0}^\mathsf{T} & \cot\theta \sinh t \\ \mathbf{0} & I_{n-2} & \mathbf{0} \\ \cot\theta \sinh t & \mathbf{0}^\mathsf{T} & \cosh t \end{bmatrix}.$$

Since $\hat{P}, H_t(\theta) \in \mathscr{G}_\theta$, we have that $\hat{P} H_t(\theta) \in \mathscr{G}_\theta$, and therefore $\lambda \hat{P} H_t(\theta) \in \mathscr{H}_\theta$. The result follows by observing that

$$\lambda \hat{P} H_t(\theta) \boldsymbol{e} = \lambda \cot^2\theta\, \boldsymbol{y} = \boldsymbol{x}.$$

Thus, the circular cone $\mathscr{L}^n_\theta$ is homogeneous. The proof is complete. □

COROLLARY 1. The circular cone $\mathscr{L}^n_\theta$ is a symmetric cone.

Corollary 1 means that, in view of Theorem 2, the circular cone $\mathscr{L}^n_\theta$ is the cone of squares of some Euclidean Jordan algebra. In the next section, we study the Jordan algebraic structure of the circular cone in order to be able to form such a Euclidean Jordan algebra. We show Corollary 1 by proving the following two lemmas.

## 4. A new spectral decomposition associated with the circular cone

In this section, we at first use the spectral decomposition established by Zhou and Chen in [5] to study the algebraic structure of the circular cones (1) in order to be able to possibly establish a Jordan multiplication "$\circ$" associated with the algebra $\mathscr{E}^n$ that generalizes the Jordan multiplication given in (7). When we succeed, we will use this Jordan multiplication to establish the Jordan algebra associated with the circular cone.

The following spectral decomposition is previously established by Zhou and Chen [5]. It can be viewed as a generalization of Spectral Decomposition 1 associated with $\mathscr{Q}^n$ given in (10).

SPECTRAL DECOMPOSITION 2. Let $\boldsymbol{x}$ be a vector in the algebra of the circular cone $\mathscr{L}_\theta^n$, the space $\mathscr{E}^n$. A spectral decomposition of $\boldsymbol{x}$ can be obtained as follows:

$$\boldsymbol{x} = \underbrace{(x_0 + \tan\theta\|\bar{\boldsymbol{x}}\|)}_{\lambda_1(\boldsymbol{x})} \underbrace{\begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\hat{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{c}_1(\boldsymbol{x})} + \underbrace{(x_0 - \cot\theta\|\bar{\boldsymbol{x}}\|)}_{\lambda_2(\boldsymbol{x})} \underbrace{\begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\hat{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{c}_2(\boldsymbol{x})}.$$

Under Spectral Decomposition 2, we can generalize the notions and concepts given in the examples of Section 2. For instance

$$\begin{aligned} \operatorname{trace}(\boldsymbol{x}) &= 2x_0 + (\tan\theta - \cot\theta)\|\bar{\boldsymbol{x}}\|, \\ \det(\boldsymbol{x}) &= x_0^2 + (\tan\theta - \cot\theta)x_0\|\bar{\boldsymbol{x}}\| - \|\bar{\boldsymbol{x}}\|^2. \end{aligned} \tag{15}$$

Clearly the definitions of $\operatorname{trace}(\boldsymbol{x})$ and $\det(\boldsymbol{x})$ in (15) generalize the definitions of $\operatorname{trace}(\boldsymbol{x})$ and $\det(\boldsymbol{x})$ given in (11). We have $\boldsymbol{c}_1 + \boldsymbol{c}_2 = \boldsymbol{e}$ (the identity vector defined in (8)), $\operatorname{trace}(\boldsymbol{e}) = 2$ and $\det(\boldsymbol{e}) = 1$. We also have

$\langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle = 0$ (orthogonality with respect to the standard inner product),
$\boldsymbol{c}_1 \in \operatorname{Bd}(\mathscr{L}_\theta^n)$, while $\boldsymbol{c}_2 \in \mathscr{L}_\theta^n \iff \theta \in [\frac{\pi}{4}, \frac{\pi}{2})$,
$\lambda_1(\boldsymbol{c}_1) = 1$, $\lambda_2(\boldsymbol{c}_1) = 0$, while $\lambda_1(\boldsymbol{c}_2) = 2\sin^2\theta$, and $\lambda_2(\boldsymbol{c}_2) = \sin^2\theta - \cos^2\theta$,
$\boldsymbol{c}_1 = S_\theta \boldsymbol{c}_2$, and $\boldsymbol{c}_2 = S_\theta^{-1}\boldsymbol{c}_1$,

where $S_\theta$ is the matrix defined by

$$S_\theta := \begin{bmatrix} \cot^2\theta & \mathbf{0}^\mathsf{T} \\ \mathbf{0} & -I_{n-1} \end{bmatrix}.$$

Observe that

$$\begin{aligned} \boldsymbol{x}^{-1} &= \frac{1}{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \frac{1}{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x}) \\ &= \frac{1}{x_0 + \|\bar{\boldsymbol{x}}\|\tan\theta} \begin{bmatrix} \cos^2\theta \\ (\sin\theta\cos\theta)\hat{\boldsymbol{x}} \end{bmatrix} + \frac{1}{x_0 - \|\bar{\boldsymbol{x}}\|\cot\theta} \begin{bmatrix} \sin^2\theta \\ -(\sin\theta\cos\theta)\hat{\boldsymbol{x}} \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{\cos^2\theta}{x_0 + \|\bar{\boldsymbol{x}}\|\tan\theta} + \dfrac{\sin^2\theta}{x_0 - \|\bar{\boldsymbol{x}}\|\cot\theta} \\ \dfrac{(\sin\theta\cos\theta)\hat{\boldsymbol{x}}}{x_0 + \|\bar{\boldsymbol{x}}\|\tan\theta} + \dfrac{-(\sin\theta\cos\theta)\hat{\boldsymbol{x}}}{x_0 - \|\bar{\boldsymbol{x}}\|\cot\theta} \end{bmatrix} \\ &= \frac{1}{(x_0 + \|\bar{\boldsymbol{x}}\|\tan\theta)(x_0 - \|\bar{\boldsymbol{x}}\|\cot\theta)} \begin{bmatrix} x_0 + \|\bar{\boldsymbol{x}}\|(\tan\theta - \cot\theta) \\ -\bar{\boldsymbol{x}} \end{bmatrix} \\ &= \frac{1}{\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})} \begin{bmatrix} \lambda_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x}) - x_0 \\ -\bar{\boldsymbol{x}} \end{bmatrix}. \end{aligned}$$

Thus, the definition of the inverse of $\boldsymbol{x}$ is given by

$$\boldsymbol{x}^{-1} = \frac{1}{\det(\boldsymbol{x})} \begin{bmatrix} \operatorname{trace}(\boldsymbol{x}) - x_0 \\ -\bar{\boldsymbol{x}} \end{bmatrix},$$

which generalizes the definition of $\boldsymbol{x}^{-1}$ given in (12).

The arrow-shaped matrix $\operatorname{Arw}(\boldsymbol{x})$ associated with $\boldsymbol{x}$ in the algebra of $\mathscr{L}_\theta^n$, which is obtained by applying

$$\begin{aligned}\operatorname{Arw}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2 &= (\lambda_1(\boldsymbol{x}))^2\boldsymbol{c}_1(\boldsymbol{x}) + (\lambda_2(\boldsymbol{x}))^2\boldsymbol{c}_2(\boldsymbol{x})\\ &= \begin{bmatrix} \boldsymbol{x}^{\mathsf{T}}\boldsymbol{x} \\ 2x_0\overline{\boldsymbol{x}} + (\tan\theta - \cot\theta)\|\overline{\boldsymbol{x}}\|\overline{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} x_0^2 + \overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{x}} \\ \operatorname{trace}(\boldsymbol{x})\overline{\boldsymbol{x}} \end{bmatrix},\end{aligned}$$

is the matrix

$$\operatorname{Arw}(\boldsymbol{x}) = \begin{bmatrix} x_0 & \overline{\boldsymbol{x}}^{\mathsf{T}} \\ \overline{\boldsymbol{x}} & (\operatorname{trace}(\boldsymbol{x}) - x_0)I_{n-1} \end{bmatrix}, \tag{16}$$

Clearly, the definition of the arrow-shaped matrix (16) associated with $\mathscr{L}_\theta^n$ generalizes the definition of the arrow-shaped matrix (13) associated with $\mathscr{Q}^n$.

Now, we give our own definition of an important notion associated with the algebra of the circular cones, namely the binary operation "$\circ$" which generalizes the binary operation associated with the algebra of the second-order cone $\mathscr{Q}^n$ defined in (7).

DEFINITION 11. The Jordan multiplication $\boldsymbol{x}\circ\boldsymbol{y}$ between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in the algebra of the circular cone, the space $\mathscr{E}^n$, is defined by

$$\boldsymbol{x}\circ\boldsymbol{y} := \frac{1}{2}(\operatorname{Arw}(\boldsymbol{x})\boldsymbol{y} + \operatorname{Arw}(\boldsymbol{y})\boldsymbol{x}).$$

Using Definition 11 and the arrow-shaped matrix defined in (16) we conclude that, under Spectral Decomposition 2, the binary operation "$\circ$" between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by

$$\boldsymbol{x}\circ\boldsymbol{y} = \frac{1}{2}\begin{bmatrix} 2\boldsymbol{x}^{\mathsf{T}}\boldsymbol{y} \\ \operatorname{trace}(\boldsymbol{x})\overline{\boldsymbol{y}} + \operatorname{trace}(\boldsymbol{y})\overline{\boldsymbol{x}} \end{bmatrix}, \tag{17}$$

which generalizes the Jordan multiplication (7) associated with $\mathscr{Q}^n$.

The identity vector is $\boldsymbol{e} = \boldsymbol{c}_1(\boldsymbol{x}) + \boldsymbol{c}_2(\boldsymbol{x})$, which is the same identity vector defined in (8). Observe that $\operatorname{trace}(\boldsymbol{e}) = 2, \det(\boldsymbol{e}) = 1$ (since all the eigenvalues of $\boldsymbol{e}$ are equal to one), $\operatorname{Arw}(\boldsymbol{e}) = I_n$, $\operatorname{Arw}(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}$, and $\operatorname{Arw}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2$. One can easily see that

$$\begin{aligned}&\boldsymbol{x}\circ\boldsymbol{y} = \boldsymbol{y}\circ\boldsymbol{x} \ \text{(commutativity)},\\ &\boldsymbol{x}\circ\boldsymbol{e} = \boldsymbol{x},\\ &\boldsymbol{x}^2 = \boldsymbol{x}^{(2)} = \boldsymbol{x}\circ\boldsymbol{x},\\ &\boldsymbol{x}\circ\boldsymbol{x}^{-1} = \boldsymbol{e}.\end{aligned}$$

The Spectral Decomposition 2 has some unfavorable features making it difficult to use. For instance, the property that $\boldsymbol{x}^{(2)} = \boldsymbol{x}^2$ (that is, $\boldsymbol{x}\circ\boldsymbol{x} = (\lambda_1(\boldsymbol{x}))^2\boldsymbol{c}_1(\boldsymbol{x}) + (\lambda_2(\boldsymbol{x}))^2\boldsymbol{c}_2(\boldsymbol{x})$) cannot be generalized. So, $\boldsymbol{x}^p = \boldsymbol{x}^{(p)}$ is not true for integer $p > 2$. For instance, simple but long and straightforward computations show that

$$\begin{aligned}\boldsymbol{x}^3 &= (\lambda_1(\boldsymbol{x}))^3\boldsymbol{c}_1(\boldsymbol{x}) + (\lambda_2(\boldsymbol{x}))^3\boldsymbol{c}_2(\boldsymbol{x})\\ &= \begin{bmatrix} x_0^3 + 3x_0\|\overline{\boldsymbol{x}}\|^2 + (\tan\theta - \cot\theta)\|\overline{\boldsymbol{x}}\|^3 \\ 3x_0^2\overline{\boldsymbol{x}} + 3(\tan\theta - \cot\theta)x_0\|\overline{\boldsymbol{x}}\|\overline{\boldsymbol{x}} + (\tan^3\theta + \cot^3\theta)\cos\theta\sin\theta\|\overline{\boldsymbol{x}}\|^2\overline{\boldsymbol{x}} \end{bmatrix},\end{aligned}$$

while

$$
\begin{aligned}
\boldsymbol{x}^{(3)} &= \boldsymbol{x} \circ \boldsymbol{x}^2 \\
&= \begin{bmatrix} x_0^3 + 3x_0\|\overline{\boldsymbol{x}}\|^2 + (\tan\theta - \cot\theta)\|\overline{\boldsymbol{x}}\|^3 \\ 3x_0^2\overline{\boldsymbol{x}} + 3(\tan\theta - \cot\theta)x_0\|\overline{\boldsymbol{x}}\|\overline{\boldsymbol{x}} + \tan^2\theta\,\|\overline{\boldsymbol{x}}\|^2\overline{\boldsymbol{x}} \end{bmatrix}.
\end{aligned}
$$

The equality $\boldsymbol{x}^p = \boldsymbol{x}^{(p)}$ holds only when $\theta = \frac{\pi}{4}$. In fact $\boldsymbol{x}^{(p)} \circ \boldsymbol{x}^{(q)} \neq \boldsymbol{x}^{(p+q)}$ for integers $p, q > 1$, and hence the algebra $(\mathscr{E}^n, \circ)$ is not power associative when Spectral Decomposition 2 and the operation "$\circ$" defined in (17) are used. Furthermore, Jordan's axiom $\boldsymbol{x} \circ (\boldsymbol{x}^{(2)} \circ \boldsymbol{y}) = \boldsymbol{x}^{(2)} \circ (\boldsymbol{x} \circ \boldsymbol{y})$ is not satisfied when the definition of "$\circ$" in (17) is used. So, we conclude the following corollary.

COROLLARY 2. *Under Spectral Decomposition 2, the algebra $(\mathscr{E}^n, \circ)$ is not a Jordan algebra.*

Corollaries 2 motivates us to develop and find out another spectral decomposition in $\mathscr{E}^n$ that makes $(\mathscr{E}^n, \circ)$ a Jordan algebra. Observe that, under Spectral Decomposition 2, we have

$$\boldsymbol{c}_1^2 = \boldsymbol{c}_1, \text{ but } \boldsymbol{c}_2^2 \neq \boldsymbol{c}_2, \text{ and } \boldsymbol{c}_1 \circ \boldsymbol{c}_2 \neq \boldsymbol{0}.$$

This means that $\boldsymbol{c}_2$ is not idempotent and the vectors $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ are not orthogonal. Therefore, the set $\{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ is not a Jordan frame. So, we are looking for a spectral decomposition in which the set $\{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ is a Jordan frame, that is

$$\boldsymbol{c}_1^2 = \boldsymbol{c}_1, \boldsymbol{c}_2^2 = \boldsymbol{c}_2, \boldsymbol{c}_1 \circ \boldsymbol{c}_2 = \boldsymbol{0}, \text{ and } \boldsymbol{c}_1 + \boldsymbol{c}_2 = \boldsymbol{e},$$

and the distributive law holds as well. Under such a spectral decomposition, we can use the induction to prove that $\boldsymbol{x}^{(p)} = \boldsymbol{x}^p$ for any nonnegative integer $p$ as follows [1]: The assertion is clearly true for $p = 0$ ($\boldsymbol{x}^{(0)} = \boldsymbol{e} = \boldsymbol{c}_1 + \boldsymbol{c}_2 = \boldsymbol{x}^0$). Now, we assume that the assertion is true for $k$ (i.e., $\boldsymbol{x}^{(k)} = \boldsymbol{x}^k$), then we have

$$
\begin{aligned}
&\boldsymbol{x}^{(k+1)} \\
&= \boldsymbol{x} \circ \boldsymbol{x}^{(k)} \\
&= \boldsymbol{x} \circ \boldsymbol{x}^k \\
&= (\lambda_1(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})) \circ \left(\lambda_1^k(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) + \lambda_2^k(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x})\right) \\
&= \lambda_1^{k+1}(\boldsymbol{x})\boldsymbol{c}_1^2(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\boldsymbol{c}_2^2(\boldsymbol{x}) + \lambda_1(\boldsymbol{x})\lambda_2^k(\boldsymbol{x})\boldsymbol{c}_1(\boldsymbol{x}) \circ \boldsymbol{c}_2(\boldsymbol{x}) + \lambda_1^k(\boldsymbol{x})\lambda_2(\boldsymbol{x})\boldsymbol{c}_2(\boldsymbol{x}) \circ \boldsymbol{c}_1(\boldsymbol{x}) \\
&= \lambda_1^{k+1}(\boldsymbol{x})\boldsymbol{c}_1^2(\boldsymbol{x}) + \lambda_2^{k+1}(\boldsymbol{x})\boldsymbol{c}_2^2(\boldsymbol{x}) \\
&= \boldsymbol{x}^{k+1}.
\end{aligned}
$$

Similarly, we can prove that $\boldsymbol{x}^{(p)} \circ \boldsymbol{x}^{(q)} = \boldsymbol{x}^{(p+q)}$ for nonnegative integers, which means that the algebra $(\mathscr{E}^n, \circ)$ would be power associative.

We are ready to introduce a new efficient spectral decompositions in $\mathscr{E}^n$ associated with the circular cone, which can also be viewed as another generalization of Spectral Decomposition 1 associated with $\mathscr{Q}^n$ given in (10). This new spectral decomposition is previously unseen and is non-redundant and different, in terms of its characteristics and efficiency, from Spectral Decomposition 2. Under this spectral decomposition, we will see that the set $\{\boldsymbol{c}_1, \boldsymbol{c}_2\}$ is a Jordan frame and $(\mathscr{E}^n, \circ)$ is not only a Jordan algebra (in which the distributive law and the power associativity law hold), but also a Euclidean Jordan algebra. Therefore, this new spectral decomposition in $\mathscr{E}^n$ is expected to be the successful for underlying the analysis of the interior point methods for circular programming that extends the analysis of the interior point methods for second-order cone programming.

SPECTRAL DECOMPOSITION 3. Let $\boldsymbol{x}$ be a vector in the algebra of the circular cone $\mathscr{L}_\theta^n$, the space $\mathscr{E}^n$. The circular spectral decomposition of $\boldsymbol{x}$ is obtained as follows:

$$\boldsymbol{x} = \underbrace{(x_0 + \cot\theta\|\bar{\boldsymbol{x}}\|)}_{\lambda_1(\boldsymbol{x})} \underbrace{\left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ \tan\theta\,\hat{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{c}_1(\boldsymbol{x})} + \underbrace{(x_0 - \cot\theta\|\bar{\boldsymbol{x}}\|)}_{\lambda_2(\boldsymbol{x})} \underbrace{\left(\frac{1}{2}\right)\begin{bmatrix} 1 \\ -\tan\theta\,\hat{\boldsymbol{x}} \end{bmatrix}}_{\boldsymbol{c}_2(\boldsymbol{x})}.$$

Under Spectral Decomposition 3, we have

$$\operatorname{trace}(\boldsymbol{x}) = 2x_0, \text{ and } \det(\boldsymbol{x}) = x_0^2 - \cot^2\theta\|\bar{\boldsymbol{x}}\|^2. \tag{18}$$

Note that the definition of $\operatorname{trace}(\boldsymbol{x})$ in (18) is exactly the definition of $\operatorname{trace}(\boldsymbol{x})$ given in (11), and the definition of $\det(\boldsymbol{x})$ in (18) generalizes the definition of $\det(\boldsymbol{x})$ given in (11). We have $\boldsymbol{c}_1 + \boldsymbol{c}_2 = \boldsymbol{e}$ (the identity vector defined in (8)), $\operatorname{trace}(\boldsymbol{e}) = 2$ and $\det(\boldsymbol{e}) = 1$. We also have

$\langle \boldsymbol{c}_1, \boldsymbol{c}_2 \rangle_\theta = 0$ (orthogonality with respect to the circular inner product),
$\boldsymbol{c}_1, \boldsymbol{c}_2 \in \operatorname{Bd}(\mathscr{L}_\theta^n)$,
$\lambda_1(\boldsymbol{c}_1) = \lambda_1(\boldsymbol{c}_2) = 1$, and $\lambda_2(\boldsymbol{c}_1) = \lambda_2(\boldsymbol{c}_2) = 0$,
$\boldsymbol{c}_1 = R\boldsymbol{c}_2$, and $\boldsymbol{c}_2 = R\boldsymbol{c}_1$,

where $R$ is the reflection matrix defined in (4).

With a little calculation, we obtain

$$\boldsymbol{x}^{-1} = \frac{1}{\lambda_1(\boldsymbol{x})}\boldsymbol{c}_1(\boldsymbol{x}) + \frac{1}{\lambda_2(\boldsymbol{x})}\boldsymbol{c}_2(\boldsymbol{x}) = \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_0 \\ -\bar{\boldsymbol{x}} \end{bmatrix} = \frac{R}{\det(\boldsymbol{x})}\,\boldsymbol{x}, \tag{19}$$

which is exactly the definition of $\boldsymbol{x}^{-1}$ given in (12) associated with $\mathscr{Q}^n$.

The arrow-shaped matrix $\operatorname{Arw}(\boldsymbol{x})$ associated with $\boldsymbol{x}$ in the algebra of $\mathscr{L}_\theta^n$, which is obtained by applying $\operatorname{Arw}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2 = (\lambda_1(\boldsymbol{x}))^2\boldsymbol{c}_1(\boldsymbol{x}) + (\lambda_2(\boldsymbol{x}))^2\boldsymbol{c}_2(\boldsymbol{x})$, is the matrix

$$\operatorname{Arw}(\boldsymbol{x}) = \begin{bmatrix} x_0 & \cot^2\theta\,\bar{\boldsymbol{x}}^{\mathsf{T}} \\ \bar{\boldsymbol{x}} & x_0 I_{n-1} \end{bmatrix}, \tag{20}$$

Note that the definition of the arrow-shaped matrix (20) associated with $\mathscr{L}_\theta^n$ generalizes the definition of the arrow-shaped matrix (14) associated with $\mathscr{Q}^n$. Note also that $\boldsymbol{x} \in \mathscr{L}_\theta^n$ if and only if the matrix $\operatorname{Arw}(\boldsymbol{x})$ is positive semidefinite. Therefore, circular programming is a special case of semidefinite programming [2] and it includes second-order cone programming as a special case.

Using Definition 11 and the arrow-shaped matrix defined in (20) we conclude that, under Spectral Decomposition 3, the Jordan multiplication between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ is given by

$$\boldsymbol{x} \circ \boldsymbol{y} = \begin{bmatrix} x_0 y_0 + \cot^2\theta\, \overline{\boldsymbol{x}}^{\mathsf{T}} \overline{\boldsymbol{y}} \\ x_0 \overline{\boldsymbol{y}} + y_0 \overline{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}^{\mathsf{T}} J_\theta \boldsymbol{y} \\ x_0 \overline{\boldsymbol{y}} + y_0 \overline{\boldsymbol{x}} \end{bmatrix}, \tag{21}$$

where $J_\theta$ is the circular identity matrix defined in (3). This generalizes the Jordan multiplication (7) associated with $\mathscr{Q}^n$.

Observe that $\boldsymbol{c}_1^2 = \boldsymbol{c}_1, \boldsymbol{c}_2^2 = \boldsymbol{c}_2, \boldsymbol{c}_1 \circ \boldsymbol{c}_2 = \boldsymbol{0}, \mathrm{Arw}(\boldsymbol{e}) = I_n, \mathrm{Arw}(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}$, and $\mathrm{Arw}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2$. One can easily see that

$$\begin{aligned}
&\boldsymbol{x} \circ \boldsymbol{e} = \boldsymbol{x}, \\
&\boldsymbol{x} \circ \boldsymbol{x}^{-1} = \boldsymbol{e}, \\
&\boldsymbol{x}^p = \boldsymbol{x}^{(p)} \ \text{ for any nonnegative integer } p, \\
&\boldsymbol{x}^{(p)} \circ \boldsymbol{x}^{(q)} = \boldsymbol{x}^{(p+q)} \ \text{ for any nonnegative integer } p, q \geqslant 1.
\end{aligned}$$

So, under Spectral Decomposition 3, the algebra $(\mathscr{E}^n, \circ)$ is power associative. In fact, we can also see that

$$\begin{aligned}
&\boldsymbol{x} \circ \boldsymbol{y} = \boldsymbol{y} \circ \boldsymbol{x} \ \text{ (commutativity)}, \\
&\boldsymbol{x} \circ (\boldsymbol{x}^2 \circ \boldsymbol{y}) = \boldsymbol{x}^2 \circ (\boldsymbol{x} \circ \boldsymbol{y}) \ \text{ (Jordan's axiom)}.
\end{aligned}$$

Therefore, we conclude the following corollary.

COROLLARY 3. *Under Spectral Decomposition 3, the algebra $(\mathscr{E}^n, \circ)$ is a Jordan algebra.*

We end this section with some inequalities associated with the circular cones and second-order cone. Under Spectral Decomposition 1 in $\mathscr{E}^n$ (that is, in the framework of second-order cone), Chen [11] established the following inequalities and equalities: For $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{Q}^n$, we have

(1) $\det(\boldsymbol{e} + \boldsymbol{x})^{1/2} \geqslant 1 + \det(\boldsymbol{x})$;

(2) $\det(\boldsymbol{x} + \boldsymbol{y}) \geqslant \det(\boldsymbol{x}) + \det(\boldsymbol{y})$;

(3) $\det(\alpha\boldsymbol{x} + (1-\alpha)\boldsymbol{y}) \geqslant \alpha^2 \det(\boldsymbol{x}) + (1-\alpha)^2 \det(\boldsymbol{y}), \forall \alpha \in [0,1]$;

(4) $\det(\boldsymbol{e} + \boldsymbol{x} + \boldsymbol{y}) \leqslant \det(\boldsymbol{e} + \boldsymbol{x}) \det(\boldsymbol{e} + \boldsymbol{y})$;

(5) If $\boldsymbol{x} - \boldsymbol{y} \in \mathscr{Q}^n$, then $\det(\boldsymbol{x}) \geqslant \det(\boldsymbol{y})$, $\mathrm{trace}(\boldsymbol{x}) \geqslant \mathrm{trace}(\boldsymbol{y})$, and $\lambda_i(\boldsymbol{x}) \geqslant \lambda_i(\boldsymbol{y})$ for $i = 1, 2$;

(6) $\mathrm{trace}(\boldsymbol{x} + \boldsymbol{y}) = \mathrm{trace}(\boldsymbol{x}) + \mathrm{trace}(\boldsymbol{y})$ and $\det(\gamma\boldsymbol{x}) = \gamma^2 \det(\boldsymbol{x})$ for all $\gamma \in \mathbb{R}$.

For $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{L}_\theta^n$ having Spectral Decomposition 2 in $\mathscr{E}^n$, the authors in [6, Section 3] proved that the only inequality holds (independent of the angle $\theta$) is inequality (1). They also showed inequalities (2)-(5) hold dependent on the angle $\theta$, and inequality (6) fails no matter what value of the angle is chosen.

For $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{L}_\theta^n$ having Spectral Decomposition 3 in $\mathscr{E}^n$, we can prove that *all* the inequalities (1)-(6) hold (independent of the angle $\theta$). We omit the proof of these results since it is a simple extension from that in the framework of second-order cone in [11].

## 5. Forming a Euclidean Jordan algebra of the circular cone

In Section 4, we have seen that Spectral Decomposition 3 is the successful spectral decomposition in $\mathscr{E}^n$ used to establish a Jordan algebra from its corresponding Jordan multiplication, and therefore it is expected to be the most powerful spectral decomposition used to carry out the analysis on circular cones. From now on, we work under Spectral Decomposition 3 and its corresponding Jordan multiplication "$\circ$" defined in (21).

Note that the inner product considered with the algebraic and analytic operations associated to the second-order cone $\mathscr{Q}^n$ is the standard inner product $\langle \cdot, \cdot \rangle$, for which using (7) we have

$$\boldsymbol{x} \bullet \boldsymbol{y} = \frac{1}{2}\operatorname{trace}(\boldsymbol{x} \circ \boldsymbol{y}) = \boldsymbol{x}^{\mathsf{T}} \boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle.$$

Likewise, it is more convenient to define the inner product that should be considered with the algebraic and analytic operations associated to the circular cone $\mathscr{L}_\theta^n$ by the circular inner product $\langle \cdot, \cdot \rangle_\theta$, as using (21) we have

$$\boldsymbol{x} \bullet \boldsymbol{y} = \frac{1}{2}\operatorname{trace}(\boldsymbol{x} \circ \boldsymbol{y}) = \boldsymbol{x}^{\mathsf{T}} J_\theta \boldsymbol{y} = \langle \boldsymbol{x}, \boldsymbol{y} \rangle_\theta.$$

We have the following theorem.

THEOREM 3. *The Jordan algebra $(\mathscr{E}^n, \circ)$ is a Euclidean Jordan algebra with the circular inner product $\langle \cdot, \cdot \rangle_\theta$ defined in (5).*

*Proof.* Consider the circular inner product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_\theta = \boldsymbol{x}^{\mathsf{T}} J_\theta \boldsymbol{y} = x_0 y_0 + \cot^2\theta\, \overline{\boldsymbol{x}}^{\mathsf{T}} \overline{\boldsymbol{y}}$. It is straightforward to show that $\langle \boldsymbol{x}, \boldsymbol{x} \rangle_\theta > 0$ for all $\boldsymbol{x} \neq \boldsymbol{0}$, $\langle \boldsymbol{x}, \boldsymbol{y} \rangle_\theta = \langle \boldsymbol{y}, \boldsymbol{x} \rangle_\theta$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{E}^n$, and that $\langle \boldsymbol{x}, \boldsymbol{y} \circ \boldsymbol{z} \rangle_\theta = \langle \boldsymbol{x} \circ \boldsymbol{y}, \boldsymbol{z} \rangle_\theta$ for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathscr{E}^n$. Therefore, the circular inner product $\langle \cdot, \cdot \rangle_\theta$ is positive definite, symmetric and associative. Thus, according to Definition 4, the Jordan algebra $(\mathscr{E}^n, \circ)$ is Euclidean. The proof is complete. $\square$

Being Euclidean, Theorem 2 tells us that another way to prove Corollary 1 is proving the following important theorem which characterizes the circular cone.

THEOREM 4. *The cone of squares of the Euclidean Jordan algebra $(\mathscr{E}^n, \circ)$ is the circular cone $\mathscr{L}_\theta^n$.*

*Proof.* We need to show that $\mathscr{L}_\theta^n = \mathscr{K}_\theta^n$ where $\mathscr{K}_\theta^n$ is the cone of squares:

$$\mathscr{K}_\theta^n := \{\boldsymbol{x}^2 : \boldsymbol{x} \in \mathscr{E}^n\} = \left\{ \begin{bmatrix} x_0^2 + \cot^2\theta\, \|\overline{\boldsymbol{x}}\|^2 \\ 2x_0 \overline{\boldsymbol{x}} \end{bmatrix} : \boldsymbol{x} \in \mathscr{E}^n \right\}.$$

Let $\boldsymbol{x} \in \mathscr{K}_\theta^n$, then there exists $\boldsymbol{y} \in \mathscr{E}^n$ such that

$$\boldsymbol{x} = \begin{bmatrix} y_0^2 + \cot^2\theta \, \|\overline{\boldsymbol{y}}\|^2 \\ 2y_0\overline{\boldsymbol{y}} \end{bmatrix}.$$

It follows that

$$x_0 = y_0^2 + \cot^2\theta \, \|\overline{\boldsymbol{y}}\|^2 \geqslant 2\cot\theta \, y_0\|\overline{\boldsymbol{y}}\| = \cot\theta \, \|\overline{\boldsymbol{x}}\|,$$

where the inequality follows by observing that $(y_0 - \cot\theta\|\overline{\boldsymbol{y}}\|)^2 \geqslant 0$. This means that $\boldsymbol{x} \in \mathscr{L}_\theta^n$ and hence $\mathscr{K}_\theta^n \subseteq \mathscr{L}_\theta^n$.

Now, we prove that $\mathscr{L}_\theta^n \subseteq \mathscr{K}_\theta^n$. Let $\boldsymbol{x} \in \mathscr{L}_\theta^n$. We need $\boldsymbol{x} = \boldsymbol{y}^2$ for some $\boldsymbol{y} \in \mathscr{E}^n$. Equivalently, we need to show that the system of $n$ equations

$$\begin{aligned} x_0 &= y_0^2 + \cot^2\theta \, \|\overline{\boldsymbol{y}}\|^2, \\ x_i &= 2y_0y_i; \; i = 1,2,\ldots,n-1, \end{aligned} \tag{22}$$

has at least one real solution.

Assuming at first that $y_0 \neq 0$, then we have $y_i = \dfrac{x_i}{2y_0}$. Substituting $y_i$ in the $0^{\text{th}}$ equation of (22) we get the following quartic equation

$$4y_0^4 - 4x_0y_0^2 + \cot^2\theta\|\overline{\boldsymbol{x}}\|^2 = 0.$$

This equation has up to four solutions, namely

$$y_0 = \pm\sqrt{\frac{x_0 \pm \sqrt{x_0^2 - \cot^2\theta\|\overline{\boldsymbol{x}}\|^2}}{2}}.$$

Since $x_0 \geqslant \cot\theta\|\overline{\boldsymbol{x}}\|$, all these four solutions are real. Note that elements $\boldsymbol{x} \in \mathrm{Bd}(\mathscr{L}_\theta^n)$ (where $x_0 = \cot\theta\|\overline{\boldsymbol{x}}\|$) have only two square roots, one of which is in $\mathrm{Bd}(\mathscr{L}_\theta^n)$. Elements $\boldsymbol{x} \in \mathrm{Int}(\mathscr{L}_\theta^n)$ have four square roots, except for multiples of the identity (wherever $\overline{\boldsymbol{x}} = \boldsymbol{0}$). In such a case if $y_0 = 0$, then $y_i$ can be arbitrarily chosen, as long as $\|\overline{\boldsymbol{y}}\| = \tan\theta$ which gives $\boldsymbol{x} = \boldsymbol{e}$. The identity has infinitely many square roots (assuming $n > 2$) for which two of them are $\pm\boldsymbol{e}$, and all others are of the form $(0;\boldsymbol{q})$ with $\|\boldsymbol{q}\| = \tan\theta$. Thus, every $\boldsymbol{x} \in \mathscr{L}_\theta^n$ has a unique square root in $\mathscr{L}_\theta^n$. This completes the proof. $\square$

The following definition extends the definitions in Example 7.

DEFINITION 12. Consider the Euclidean Jordan algebra $(\mathscr{E}^n, \circ)$, we have the following:

1. The linear map $L : \mathscr{E}^n \longrightarrow \mathscr{E}^n$ associated with the circular cone $\mathscr{L}_\theta^n$ is given by

$$L(\boldsymbol{x}) = \mathrm{Arw}(\boldsymbol{x}) = \begin{bmatrix} x_0 & \cot^2\theta \, \overline{\boldsymbol{x}}^{\mathsf{T}} \\ \overline{\boldsymbol{x}} & x_0 I_{n-1} \end{bmatrix}.$$

2. The quadratic map $Q_{\boldsymbol{x},\boldsymbol{z}} : \mathscr{E}^n \times \mathscr{E}^n \longrightarrow \mathscr{E}^n$ associated with the circular cone $\mathscr{L}^n_\theta$ is given by

$$\begin{aligned} Q_{\boldsymbol{x},\boldsymbol{z}} &= \operatorname{Arw}(\boldsymbol{x})\operatorname{Arw}(\boldsymbol{z}) + \operatorname{Arw}(\boldsymbol{z})\operatorname{Arw}(\boldsymbol{x}) - \operatorname{Arw}(\boldsymbol{x}\circ\boldsymbol{z}) \\ &= \begin{bmatrix} x_0 z_0 + \cot^2\theta\,\overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{z}} & \cot^2\theta\,(x_0\overline{z}^{\mathsf{T}} + z_0\overline{x}^{\mathsf{T}}) \\ x_0\overline{z} + z_0\overline{x} & x_0 z_0 I_{n-1} + \cot^2\theta\,(\overline{x}\overline{z}^{\mathsf{T}} + \overline{z}\overline{x}^{\mathsf{T}} - \overline{x}^{\mathsf{T}}\overline{z} I_{n-1}) \end{bmatrix}. \end{aligned} \tag{23}$$

3. The linear map $Q_{\boldsymbol{x}} : \mathscr{E}^n \longrightarrow \mathscr{E}^n$ associated with the circular cone $\mathscr{L}^n_\theta$ is given by

$$\begin{aligned} Q_{\boldsymbol{x}} &= 2\operatorname{Arw}^2(\boldsymbol{x}) - \operatorname{Arw}(\boldsymbol{x}^2) \\ &= \begin{bmatrix} x_0^2 + \cot^2\theta\,\|\overline{\boldsymbol{x}}\|^2 & 2\cot^2\theta\, x_0\overline{x}^{\mathsf{T}} \\ 2x_0\overline{x} & \det(\boldsymbol{x}) I_{n-1} + 2\cot^2\theta\,\overline{x}\overline{x}^{\mathsf{T}} \end{bmatrix}. \end{aligned} \tag{24}$$

Note that $\operatorname{Arw}(\boldsymbol{x})\boldsymbol{e} = \boldsymbol{x}, \operatorname{Arw}(\boldsymbol{x})\boldsymbol{x} = \boldsymbol{x}^2, Q_{\boldsymbol{x}}\boldsymbol{e} = \boldsymbol{x}^2, Q_{\boldsymbol{x}}\boldsymbol{x}^{-1} = \boldsymbol{x}$, and $\operatorname{Arw}(\boldsymbol{e}) = Q_{\boldsymbol{e}} = I_n$.

The new spectral decomposition established in this paper allows us to generalize many crucial theorems of the theory of Euclidean Jordan algebra of the second-order cone to the theory of Euclidean Jordan algebra of the circular cones. It also allows to compute explicit formulas for the derivatives of important expressions such as the gradient $\nabla_{\boldsymbol{x}}(\ln\det(\boldsymbol{x}))$ and the Hessian $\nabla^2_{\boldsymbol{xx}}(\ln\det(\boldsymbol{x}))$. This is the substance of the next section.

## 6. Some algebraic properties of the circular cone

In this section, we generalize some results associated with the algebra of the second-order cone to the algebra of the circular cone. These results are important for developing interior point algorithms for circular programming. We start with the following lemma where its second item generalizes item 8 of [1, Theorem 8].

LEMMA 3. *Let $\boldsymbol{x},\boldsymbol{y} \in \mathscr{E}^n$. Then*

1. $Q_{\boldsymbol{x}} = 2\boldsymbol{x}\boldsymbol{x}^{\mathsf{T}} J_\theta - \det(\boldsymbol{x})R$.

2. $\det(Q_{\boldsymbol{x}}\boldsymbol{y}) = \det^2(\boldsymbol{x})\det(\boldsymbol{y})$.

*Proof.* It is not hard to prove item 1. To prove item 2, let $\boldsymbol{z} = Q_{\boldsymbol{x}}\boldsymbol{y}, \alpha = \boldsymbol{x}^{\mathsf{T}} J_\theta \boldsymbol{y}$ and $\gamma = \det(\boldsymbol{x})$. From item 1, we have $\boldsymbol{z} = 2\alpha\boldsymbol{x} - \gamma R\boldsymbol{y}$. Accordingly, we get

$$z_0^2 = (2\alpha x_0 - \gamma y_0)^2 = 4\alpha^2 x_0^2 - 4\alpha\gamma x_0 y_0 + \gamma^2 y_0^2,$$

and

$$\|\overline{\boldsymbol{z}}\|^2 = \|2\alpha\overline{\boldsymbol{x}} + \gamma\overline{\boldsymbol{y}}\|^2 = 4\alpha^2\|\overline{\boldsymbol{x}}\|^2 + 4\alpha\gamma\overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}} + \gamma^2\|\overline{\boldsymbol{y}}\|^2.$$

It follows that

$$\begin{aligned}
\det(Q_{\boldsymbol{x}}\boldsymbol{y}) &= \det(\boldsymbol{z})\\
&= z_0^2 - \cot^2\theta\|\overline{\boldsymbol{z}}\|^2\\
&= 4\alpha^2 x_0^2 - 4\alpha\gamma x_0 y_0 + \gamma^2 y_0^2 - \cot^2\theta(4\alpha^2\|\overline{\boldsymbol{x}}\|^2 + 4\alpha\gamma\overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}} + \gamma^2\|\overline{\boldsymbol{y}}\|^2)\\
&= 4\alpha^2(x_0^2 - \cot^2\theta\|\overline{\boldsymbol{x}}\|^2) - 4\alpha\gamma(x_0 y_0 + \cot^2\theta\overline{\boldsymbol{x}}^{\mathsf{T}}\overline{\boldsymbol{y}}) + \gamma^2(y_0^2 - \cot^2\theta\|\overline{\boldsymbol{y}}\|^2)\\
&= 4\alpha^2\gamma - 4\alpha^2\gamma + \gamma^2\det(\boldsymbol{y})\\
&= \det{}^2(\boldsymbol{x})\det(\boldsymbol{y}). \quad \square
\end{aligned}$$

The following theorem will play an important role in the development of the primal-dual interior point methods for the circular programming (see for example [9, Section 3]). This theorem generalizes [1, Theorem 9].

THEOREM 5. *Let* $\boldsymbol{p} \in \mathscr{E}^n$ *be invertible. Then* $Q_{\boldsymbol{p}}(\mathscr{L}_\theta^n) = \mathscr{L}_\theta^n$.

*Proof.* We first prove that $Q_{\boldsymbol{p}}(\mathscr{L}_\theta^n) \subseteq \mathscr{L}_\theta^n$. Let $\boldsymbol{x} \in \mathscr{L}_\theta^n$ and $\boldsymbol{y} = Q_{\boldsymbol{p}}\boldsymbol{x}$. By Lemma 3, we have $\det(\boldsymbol{y}) \geqslant 0$. Therefore, either $\boldsymbol{y} \in \mathscr{L}_\theta^n$ or $\boldsymbol{y} \in -\mathscr{L}_\theta^n$. That is, either $\lambda_{1,2}(\boldsymbol{y}) \geqslant 0$ or $\lambda_{1,2}(\boldsymbol{y}) \leqslant 0$. Note that $2y_0 = \operatorname{trace}(\boldsymbol{y}) = \lambda_1(\boldsymbol{y}) + \lambda_2(\boldsymbol{y})$. So, to show that $\boldsymbol{y} \in \mathscr{L}_\theta^n$ (equivalently, $\lambda_{1,2}(\boldsymbol{y}) \geqslant 0$), it suffices to show that $y_0 \geqslant 0$. By using the fact that $x_0 \geqslant \cot\theta\|\overline{\boldsymbol{x}}\|$, and then applying the Cauchy-Schwarz inequality to $|\boldsymbol{p}^{\mathsf{T}}\boldsymbol{x}|$ we get

$$\begin{aligned}
y_0 &= 2(p_0x_0 + \cot^2\theta\overline{\boldsymbol{p}}^{\mathsf{T}}\overline{\boldsymbol{x}})p_0 - (p_0^2 - \cot^2\theta\|\boldsymbol{p}\|^2)x_0\\
&= p_0^2x_0 + \cot^2\theta\|\overline{\boldsymbol{p}}\|^2x_0 + \cot^2\theta p_0(\overline{\boldsymbol{p}}^{\mathsf{T}}\overline{\boldsymbol{x}})\\
&= x_0(p_0^2 + \cot^2\theta\|\overline{\boldsymbol{p}}\|^2) + \cot^2\theta p_0(\overline{\boldsymbol{p}}^{\mathsf{T}}\overline{\boldsymbol{x}})\\
&\geqslant \cot\theta\|\overline{\boldsymbol{x}}\|(p_0^2 + \cot^2\theta\|\overline{\boldsymbol{p}}\|^2) + \cot\theta^2 p_0(\overline{\boldsymbol{p}}^{\mathsf{T}}\overline{\boldsymbol{x}})\\
&\geqslant \cot\theta\|\overline{\boldsymbol{x}}\|(p_0^2 + \cot^2\theta\|\overline{\boldsymbol{p}}\|^2) - \cot^2\theta|p_0|\|\overline{\boldsymbol{p}}\|\|\overline{\boldsymbol{x}}\|\\
&= \cot\theta\|\overline{\boldsymbol{x}}\|(p_0 + \cot\theta\|\overline{\boldsymbol{p}}\|)^2\\
&\geqslant 0.
\end{aligned}$$

Thus $Q_{\boldsymbol{p}}(\mathscr{L}_\theta^n) \subseteq \mathscr{L}_\theta^n$. Also $\boldsymbol{p}^{-1}$ is invertible, therefore $Q_{\boldsymbol{p}^{-1}}\boldsymbol{x} \in \mathscr{L}_\theta^n$ for each $\boldsymbol{x} \in \mathscr{L}_\theta^n$. It follows that for every $\boldsymbol{x} \in \mathscr{L}_\theta^n$, since $\boldsymbol{x} = Q_{\boldsymbol{p}}Q_{\boldsymbol{p}^{-1}}\boldsymbol{x}$, $\boldsymbol{x}$ is the image of $Q_{\boldsymbol{p}^{-1}}\boldsymbol{x}$; that is $\mathscr{L}_\theta^n \subseteq Q_{\boldsymbol{p}}(\mathscr{L}_\theta^n)$. The result is established. $\square$

We end this section with the following results which are key ingredients in designing barrier and penalty algorithms for circular programming. The first two items in following theorem generalize item 6 of [1, Theorem 8], and the last item generalizes the same result in [8, Chapter 10] but our setting is much more general.

THEOREM 6. *Let* $\boldsymbol{x}, \boldsymbol{u} \in \mathscr{E}^n$. *Then*

1. *The gradient* $\nabla_{\boldsymbol{x}} \ln\det\boldsymbol{x} = 2J_\theta\boldsymbol{x}^{-1}$, *provided that* $\det(\boldsymbol{x})$ *is positive (so* $\boldsymbol{x}$ *is invertible).*

2. *The Hessian* $\nabla^2_{\boldsymbol{xx}} \ln\det \boldsymbol{x} = -2J_\theta Q_{\boldsymbol{x}^{-1}}$. *Hence the gradient* $\nabla_{\boldsymbol{x}} \boldsymbol{x}^{-1} = -Q_{\boldsymbol{x}^{-1}}$, *provided that* $\boldsymbol{x}$ *is invertible.*

3. *The gradient* $\nabla_{\boldsymbol{x}} Q_{\boldsymbol{x}}[\boldsymbol{u}] = 2\, Q_{\boldsymbol{x},\boldsymbol{u}}$.

*Proof.* To prove item 1, we let

$$f(\boldsymbol{x}) = \ln\det(\boldsymbol{x}) = \ln(\lambda_1(\boldsymbol{x})\lambda_2(\boldsymbol{x})) = \ln\left(x_0^2 - \cot^2\theta\|\overline{\boldsymbol{x}}\|^2\right).$$

Since $\det(\boldsymbol{x})$ is positive, we have

$$\begin{aligned}
\nabla_{\boldsymbol{x}} f(\boldsymbol{x}) &= \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} 2x_0 \\ -2\cot^2\theta\,\overline{\boldsymbol{x}} \end{bmatrix} \\
&= \frac{2}{\det(\boldsymbol{x})}\begin{bmatrix} 1 & \boldsymbol{0}^\mathsf{T} \\ \boldsymbol{0} & \cot^2\theta\, I_{n-1} \end{bmatrix}\begin{bmatrix} x_0 \\ -\overline{\boldsymbol{x}} \end{bmatrix} \\
&= 2J_\theta \boldsymbol{x}^{-1},
\end{aligned}$$

where the last equality follows from (19). The result in item 1 is established. To prove item 2, we let

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{x}^{-1} = \frac{1}{\det(\boldsymbol{x})}\begin{bmatrix} x_0 \\ -\overline{\boldsymbol{x}} \end{bmatrix} = \begin{bmatrix} \dfrac{x_0}{x_0^2 - \cot^2\theta|\overline{\boldsymbol{x}}\|^2} \\ \dfrac{-1}{x_0^2 - \cot^2\theta|\overline{\boldsymbol{x}}\|^2}\,\overline{\boldsymbol{x}} \end{bmatrix}.$$

Then, we have

$$\nabla_{\boldsymbol{x}} \boldsymbol{g}(\boldsymbol{x}) = \frac{1}{(\det(\boldsymbol{x}))^2}\begin{bmatrix} -x_0^2 - \cot^2\theta & \cot^2\theta\, x_0\overline{\boldsymbol{x}} \\ 2x_0\overline{\boldsymbol{x}} & -\left(\det(\boldsymbol{x})I_{n-1} + \cot^2\theta\,\overline{\boldsymbol{x}}\overline{\boldsymbol{x}}^\mathsf{T}\right) \end{bmatrix} = -Q_{\boldsymbol{x}^{-1}},$$

where the last equality follows from the definition of $Q.$ in (24) applied on $\boldsymbol{x}^{-1}$. The result in item 2 is established. To prove item 3, we let

$$\begin{aligned}
\boldsymbol{h}(\boldsymbol{x},\boldsymbol{u}) &= Q_{\boldsymbol{x}}[\boldsymbol{u}] \\
&= \begin{bmatrix} x_0^2 + \cot^2\theta\,\|\overline{\boldsymbol{x}}\|^2 & 2\cot^2\theta\, x_0\overline{x}^\mathsf{T} \\ 2x_0\overline{x} & \det(\boldsymbol{x})I_{n-1} + 2\cot^2\theta\,\overline{x}\overline{x}^\mathsf{T} \end{bmatrix}\begin{bmatrix} u_0 \\ \overline{u} \end{bmatrix} \\
&= \begin{bmatrix} x_0^2 u_0 + \cot^2\theta\,\|\overline{\boldsymbol{x}}\|^2 u_0 + 2\cot^2\theta\, x_0\overline{x}^\mathsf{T}\overline{u} \\ 2x_0u_0\overline{x} + \det(\boldsymbol{x})\overline{u} + 2\cot^2\theta\,\overline{x}\overline{x}^\mathsf{T}\overline{u} \end{bmatrix}.
\end{aligned}$$

Then,

$$\nabla_{\boldsymbol{x}} \boldsymbol{h}(\boldsymbol{x},\boldsymbol{u}) = \begin{bmatrix} x_0u_0 + \cot^2\theta\overline{\boldsymbol{x}}^\mathsf{T}\overline{\boldsymbol{u}} & \cot^2\theta(x_0\overline{u}^\mathsf{T} + u_0\overline{x}^\mathsf{T}) \\ x_0\overline{u} + u_0\overline{x} & x_0u_0I_{n-1} + \cot^2\theta(\overline{x}\overline{u}^\mathsf{T} + \overline{u}\overline{x}^\mathsf{T} - \overline{x}^\mathsf{T}\overline{u}I_{n-1}) \end{bmatrix} = 2\,Q_{\boldsymbol{x},\boldsymbol{u}},$$

where the last equality follows from the definition of $Q_{\cdot,\cdot}$ in (23). This establishes the result in item 3. The proof is complete. $\square$

## 7. Conclusions

In this paper, we have considered the circular inner product for the circular case, have established its new efficient spectral decomposition (Spectral Decomposition 3) associated with the circular cone, have set up the corresponding Jordan algebra, and have formed the Euclidean Jordan algebra associated with this cone. We have also generalized many important properties of the Euclidean Jordan algebra associated with the second-order cone to the Euclidean Jordan algebra associated with circular cone.

By looking at all known symmetric cones and their corresponding inner products, we see that each symmetric cone must be associated with a certain inner product that forms its Euclidean Jordan algebra and re-builds this symmetric cone as the cone of squares of resulting Euclidean Jordan algebra. For instance: For the nonnegative orthant cone in the algebra $(\mathbb{R}, \circ)$, the scalar inner product $x \bullet y = xy$ is considered between real numbers $x$ and $y$. For the second-order cone in the algebra $(\mathscr{E}^n, \circ)$, the standard inner product $\boldsymbol{x} \bullet \boldsymbol{y} = \boldsymbol{x}^{\mathsf{T}} \boldsymbol{y}$ is considered between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. For the cone of positive semidefinite matrices in the algebra of real symmetric matrices, the Frobenius inner product $X \bullet Y = \text{trace}(X^{\mathsf{T}} Y)$ is considered between real symmetric matrices $X$ and $Y$. Similarly, the circular cone in the algebra $(\mathscr{E}^n, \circ, \theta)$, the circular inner product $\boldsymbol{x} \bullet \boldsymbol{y} = \boldsymbol{x}^{\mathsf{T}} J_\theta \boldsymbol{y}$ must be considered between two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$. Note that all these inner product are unified as (see [9]) $\boldsymbol{x} \bullet \boldsymbol{y} = \frac{1}{2}\text{trace}(\boldsymbol{x} \circ \boldsymbol{y})$, where $\boldsymbol{x}$ and $\boldsymbol{y}$ are two elements in the corresponding Jordan algebra $(\mathscr{J}, \circ)$.

In [7, Section 1], the authors treated the circular cone as a non-symmetric cone by adopting the standard inner product in their work in order to be able to exploit many results of the Euclidean Jordan algebra of the second-order cones, which are previously developed in the literature. This paper views the circular cone as a new paradigm of symmetric cones by adopting the circular inner product. As a consequence, the particular Euclidean Jordan algebra associated with the circular cone has been independently developed in this work.

We believe that the results of this paper are crucial in designing and analyzing interior-point algorithms for all circular programming problems. Our current research is using the new spectral decomposition and the machinery of Euclidean Jordan algebra associated with the circular cone (1) established in this paper to develop primal-dual path following interior point algorithms for the circular programming problems.

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