DIFFUSIVE SYSTEMS AND WEIGHTED HANKEL OPERATORS

AOLO BASHAR ABUSAKSAKA AND JONATHAN R. PARTINGTON

(*Communicated by B. Jacob*)

Abstract. We consider diffusive systems, regarded as input/output systems with a kernel given as the Fourier–Borel transform of a measure in the left half-plane. Associated with these are a family of weighted Hankel integral operators, and we provide conditions for them to be bounded, Hilbert–Schmidt or nuclear, thereby generalizing results of Widom, Howland and others.

1. Introduction

In this paper we explore various operator-theoretic properties associated with linear time-invariant systems, beginning with the comparatively simple property of BIBO stability and then considering properties of weighted integral operators, including Hankel operators used in H^∞ approximation (see, e.g. [5]) and the Glover operator used in L^2 approximation [6].

The systems we consider will have impulse responses expressible as Laplace transforms of measures, and thus may be discussed using the language of diffusive systems in the sense of Montsény. In [11], diffusive systems are defined as SISO linear time-invariant convolution systems of the form

$$y(t) = \int_0^t h(t-\tau)u(\tau)\,d\tau,$$

where the impulse reponse h is the Laplace transform of a signed measure (or more generally a distribution) μ defined on $(0,\infty)$; i.e.,

$$h(t) = \int_0^\infty e^{-\xi t}\,d\mu(\xi) \qquad (t \geqslant 0).$$

The associated transfer function is the Stieltjes transform of μ, given by the formula

$$G(s) = (\mathscr{L}h)(s) = \int_0^\infty e^{-st}h(t)\,dt = \int_0^\infty \frac{d\mu(\xi)}{s+\xi}$$

for $s \in \mathbb{C}_+$, the open right half-plane. As explained in [11] a diffusive system with measure μ can be realized in terms of the heat equation

$$\Psi_t(x,t) = \Psi_{xx}(x,t) + \delta(x)u(t)$$

with $\Psi(x,0) = 0$ $(x \in \mathbb{R})$, and

$$y(t) = \int_{-\infty}^{\infty} 4\pi^2 x \Psi(x,t) \, d\mu(4\pi^2 x^2).$$

Some advantages of diffusive representations are that we may represent causal convolutions as classical input/output dynamical systems. This allows the use of a range of PDE techniques. Diffusive systems are also appropriate for modelling long-memory systems, fractional integrators, etc.

More recently, in the book [12] and the tutorial article [3], the notion of a diffusive system has been generalized. The starting point is now a mapping $\gamma \in W^{1,\infty}(J;\mathbb{C})$, the classical Sobolev space of absolutely continuous functions with γ, γ' bounded; here J is a subset of $\mathbb{R}$, defining a closed (possibly at ∞) contour lying in a sector in the left-hand complex half-plane $\mathbb{C}_-$; in this case we have the expression

$$h(t) = \frac{1}{2\pi i} \int_\gamma e^{tp} G(p) \, dp = \int_J e^{\gamma(\xi)t} \mu(\xi) \, d\xi,$$

where $G = \mathscr{L}h$ is the transfer function, and $\mu(\xi) = \dfrac{\gamma'(\xi)}{2\pi i} G(\gamma(\xi))$.

In this note we shall work with a more convenient definition, which is also slightly more general. We take an arbitrary σ-finite Borel measure μ on $\mathbb{C}_-$ satisfying the condition

$$\int_{\mathbb{C}_-} e^{ts} \, d|\mu|(s) < \infty \qquad \text{for all} \quad t > 0. \tag{1}$$

This enables us to define h directly as the Fourier–Borel transform of μ, namely,

$$h(t) = \int_{\mathbb{C}_-} e^{tp} \, d\mu(p), \tag{2}$$

in which case we also have the Stieltjes transform formula

$$G(s) = \int_{\mathbb{C}_-} \frac{d\mu(p)}{s - p} \qquad \text{for} \quad s \in \mathbb{C}_+.$$

Since the functionals $f \mapsto f^{(k)}(a)$ can be expressed using Cauchy integrals for any $a \in \mathbb{C}_-$ and $k = 0,1,2,\ldots$ we see that the impulse responses $t^k e^{-at}$, and hence all finite-dimensional stable systems, can be represented in this way using measures μ (rather than requiring distributions).

REMARK 1.1. Yet more general definitions in terms of holomorphic distributions can be found in the thesis [1]. For if we let $\mathscr{X}$ denote the Fréchet space of analytic functions $f : \mathbb{C}_+ \to \mathbb{C}$ satisfying the condition that each of the seminorms

$$\|f\|_n = \max_{0 \leqslant j \leqslant n} \max_{0 \leqslant k \leqslant j+1} \sup_{z \in \mathbb{C}_-} |(\operatorname{Re} z)^k f^{(j)})(z)|$$

is finite, then we may define the Fourier–Borel and Stieltjes transforms of distributions in the dual space of $\mathscr{X}$, since $\mathscr{X}$ contains the exponentials $p \mapsto e^{pt}$ for $t > 0$ as well as the kernels $p \mapsto 1/(s - p)$ for $s \in \mathbb{C}_+$.

2. Stability and weighted Hankel operators

2.1. Stability

The following result may be seen as a natural generalization of the result of Montseny [11, Thm. 4.4], which applies to measures on $\mathbb{R}_+$.

PROPOSITION 2.1. *Let h be an impulse response given by the diffusive representation (2), where the associated measure μ satisfies (1). If in addition μ satisfies the condition*

$$\int_{\mathbb{C}_-} \frac{d|\mu|(p)}{|\operatorname{Re} p|} < \infty, \tag{3}$$

then the impulse response h lies in $L^1(0,\infty)$, thus defining a BIBO-stable system. For positive measures supported on $(-\infty,0)$ condition (3) is necessary and sufficient for BIBO stability.

Proof. We have

$$\int_0^\infty |h(t)|\,dt \leqslant \int_{t=0}^\infty \int_{p\in\mathbb{C}_-} |e^{tp}|\,d|\mu|(p)\,dt = \int_{p\in\mathbb{C}_-} \frac{d|\mu|(p)}{|\operatorname{Re} p|} < \infty,$$

by Fubini's theorem, and this implies the BIBO stability.

In the case that $\mu \geqslant 0$ and $\operatorname{supp}\mu \subset (-\infty,0)$, we have equality in the above, i.e.,

$$\int_0^\infty |h(t)|\,dt = \int_0^\infty h(t)\,dt = \int_{\mathbb{R}_-} \frac{d\mu(p)}{|p|}.$$

Hence if (3) fails to hold, the system is not BIBO stable (consider the constant input $u(t) = 1$). $\square$

2.2. Weighted Hankel operators

Achievable bounds in model reduction are linked to properties of the Hankel operator Γ, which we can define on $L^2(0,\infty)$ by

$$(\Gamma u)(t) = \int_0^\infty h(t+\tau)u(\tau)\,d\tau.$$

For finite-dimensional systems it is a finite-rank operator, and its rank is the McMillan degree of the system. If $h \in L^1$ the operator Γ is compact. So, defining its singular values as

$$\sigma_k(\Gamma) = \inf\{\|\Gamma - T\| : \operatorname{rank}(T) < k\},$$

we have $\sigma_k \to 0$. For effective H^∞ model reduction by balanced truncation or optimal Hankel-norm reduction we require Γ to be nuclear (see [5, 7]); that is, we require $\sum_{k=1}^\infty \sigma_k < \infty$. Indeed, the optimal H^∞ error E_k for a degree-k approximation is bounded by

$$\sigma_{k+1} \leqslant E_k \leqslant \sigma_{k+1} + \sigma_{k+2} + \dots.$$

For L^2 model reduction, the weighted Hankel operator Θ introduced by Glover [6], and defined by

$$(\Theta u)(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/4} h(t+\tau) \tau^{-1/4} u(\tau) \, d\tau$$

plays a significant role. It satisfies $\|\Theta\|_{HS} = \|h\|_{L^2}$, where HS denotes the Hilbert–Schmidt norm given by

$$\|\Theta\|_{HS}^2 = \sum_{k=1}^\infty \sigma_k^2.$$

Moreover rank(Θ) is the McMillan degree of the system (as for Hankel operators), meaning that L^2 errors for degree-k approximation are bounded below by

$$\left(\sigma_{k+1}^2 + \sigma_{k+2}^2 + \ldots\right)^{1/2}.$$

In order to study these and similar operators in the same framework, we define for measurable $w : (0,\infty) \to (0,\infty)$ the weighted Hankel operator $\Gamma_{h,w}$ on $L^2(0,\infty)$, by

$$(\Gamma_{h,w} u)(t) = \int_0^\infty w(t) h(t+\tau) w(\tau) u(\tau) \, d\tau, \tag{4}$$

which, if bounded, is self-adjoint whenever h is real-valued.

THEOREM 2.2. *Let w satisfy the condition $\psi_p \in L^2(0,\infty)$ for each $p \in \mathbb{C}_-$, where*

$$\psi_p(t) = w(t) e^{pt}.$$

If

$$\int_{\mathbb{C}_-} \|\psi_p\|_2^2 d|\mu|(p) < \infty, \tag{5}$$

then the weighted Hankel operator $\Gamma_{h,w}$ given by (2) and (4) is nuclear. In the case that $\mu \geqslant 0$ and μ is supported on $\mathbb{R}_-$, Condition (5) is necessary and sufficient for nuclearity.

Proof. Clearly by using the Hahn–Jordan decomposition of the real and imaginary parts of μ we may suppose without loss of generality that $\mu \geqslant 0$. We now adapt a proof of Howland [8] and define an operator T_0 by

$$T_0 u = \int_{\mathbb{C}_-} \langle u, \psi_p \rangle \psi_p \, d\mu(p) \qquad (u \in L^2(0,\infty)).$$

We then have that $T_0 = T$ and the nuclear norm of T is bounded by

$$\|T\| \leqslant \int_{\mathbb{C}_-} \|\psi_p\|_2^2 d\mu(p). \tag{6}$$

Finally, if $\mu \geqslant 0$ and μ is supported on $\mathbb{R}_-$, the elementary operators $u \mapsto \langle u, \psi_p \rangle \psi_p$ are all positive, and so equality holds in (6). $\quad\square$

The following corollary contains Howland's result on nuclearity of Hankel operators (the case $\alpha = 0$), as well as a result on the nuclearity of Glover's operators (the case $\alpha = -1/4$).

COROLLARY 2.3. *Suppose that $\alpha > -\frac{1}{2}$ and let $w(t) = t^\alpha$ for $t > 0$. Then the weighted Hankel operator $\Gamma_{h,w}$ is nuclear provided that*

$$\int_{\mathbb{C}_-} \frac{d|\mu|(p)}{|\operatorname{Re} p|^{2\alpha+1}} < \infty. \tag{7}$$

In the case that $\mu \geqslant 0$ is supported on $\mathbb{R}_-$, condition (7) is necessary and sufficient for nuclearity.

Proof. This follows directly from Theorem 2.2, noting that

$$\|\psi_p\|_2^2 = \int_0^\infty t^{2\alpha} e^{2(\operatorname{Re} p)t} \, dt = \int_0^\infty \left(\frac{u}{2x}\right)^{2\alpha} e^{-u} \frac{du}{2x},$$

where $x = -\operatorname{Re} p$ and $u = 2xt$. $\quad\square$

Note that nuclearity of the unweighted Hankel operator implies BIBO stability of the associated linear system [5], so that Corollary 2.3 directly implies Proposition 2.1.

REMARK 2.4. The example $d\mu(p) = (\sin p)\,dp$ for $p < 0$ leads to $h(t) = 1/(t^2 + 1)$ and a nuclear Hankel operator (as seen from [8, Thm 2.1]), showing that for signed measures condition (7) is not always necessary for nuclearity. There are further details and examples in [1, Chap. 3].

The Hilbert–Schmidt condition is rather easier to test, but we include the following specimen result for completeness.

PROPOSITION 2.5. *Suppose that $w(t) = t^\alpha$ with $\alpha > -1/2$. Then $\Gamma_{h,w}$ is Hilbert-Schmidt if and only if*

$$\int_0^\infty u^{4\alpha+1} |h(u)|^2 \, du < \infty.$$

If $\mu \geqslant 0$ is supported on $\mathbb{R}_-$, then this holds if and only if

$$\int_{\mathbb{R}_-} \int_{\mathbb{R}_-} \frac{d\mu(x)\,d\mu(y)}{|x+y|^{4\alpha+2}} < \infty.$$

Proof. It is well known (see e.g. [4, Chap. 2]) that an integral operator on a space $L^2(X)$, given by a measurable kernel $K(s,t)$, is Hilbert–Schmidt if and only $K \in L^2(X \times X)$. Since

$$\int_{t=0}^\infty \int_{\tau=0}^\infty w(t)^2 |h(t+\tau)|^2 w(\tau)^2 \, dt\, d\tau = \int_{u=0}^\infty \int_{\tau=0}^u (u-\tau)^{2\alpha} \tau^{2\alpha} |h(u)|^2 \, du\, d\tau$$

$$= \int_{u=0}^\infty \int_{\lambda=0}^1 u^{4\alpha} |h(u)|^2 (1-\lambda)^{2\alpha} \lambda^{2\alpha} u\, du\, d\lambda,$$

we have the first expression; then, using the formula (2) for h, we arrive at

$$C_1 \int_0^\infty u^{4\alpha+1} \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} e^{u(x+y)} \, d\mu(x) \, d\mu(y) \, du = C_2 \int_{\mathbb{R}_-} \int_{\mathbb{R}_-} \frac{d\mu(x) \, d\mu(y)}{|x+y|^{4\alpha+2}},$$

where C_1 and C_2 are constants depending only on α. $\quad\square$

A far more difficult question is the boundedness of $\Gamma_{h,w}$. For unweighted Hankel operators, much is known: for example, the reproducing kernel thesis holds [2], meaning that it is sufficient (and clearly also necessary) that $\sup_{s \in \mathbb{C}_-} \|\Gamma k_s\| / \|k_s\| < \infty$, where $k_s(t) = e^{st}$ (these act as reproducing kernels in $H^2(\mathbb{C}_+)$). However, it is not known whether this result generalises to weighted Hankel operators.

We shall take an approach based on results in [13] for unweighted Hankel operators, combined with very recent results from [10] on Carleson embeddings.

LEMMA 2.6. *Suppose that $\mu \geqslant 0$ is supported on $\mathbb{R}_-$, and that h is given by* (2). *Let w be a non-negative weight on $(0,\infty)$. Define $Z_\mu : L^2(0,\infty) \to L^2(\mathbb{C}_-, \mu)$ by*

$$Z_\mu f(s) = \int_0^\infty w(t) e^{st} f(t) \, dt.$$

Then $\Gamma_{h,w}$ is bounded if and only if Z_μ is bounded, and this holds if and only if the reversed Laplace transform $\mathscr{R}$ given by

$$(\mathscr{R}f)(s) = \int_0^\infty e^{st} f(t) \, dt$$

is a bounded operator from $L^2(0,\infty; dt/w(t)^2)$ into $L^2(\mathbb{C}_-, \mu)$.

Proof. If Z_μ is bounded, we have

$$\langle Z_\mu f, Z_\mu g \rangle = \int_{s \in \mathbb{R}_-} \int_0^\infty w(t) e^{st} f(t) \, dt \int_0^\infty w(\tau) e^{s\tau} \overline{g(\tau)} \, d\tau \, d\mu(s) = \langle \Gamma_{h,w} f, g \rangle,$$

so that $\Gamma_{h,w}$ is also bounded. Conversely, putting $f = g$, we see that the boundedness of $\Gamma_{h,w}$ implies the boundedness of Z_μ. The mapping $f \mapsto fw$ is an isometry between $L^2(0,\infty)$ and $L^2(0,\infty; dt/w(t)^2)$, and so the last assertion follows. $\quad\square$

The case $w(t) = 1$ is due to Widom, and is equivalent to the condition that μ is a Carleson measure in the space $H^2(\mathbb{C}_-)$, so that $\mu(-x,0) = O(x)$ as $x \to 0$ and $x \to \infty$ (see [13, 14]). We are able to extend this result to the class of power weights as follows.

THEOREM 2.7. *Let $w(t) = t^\alpha$ for $\alpha \in \mathbb{R}$, and let $\mu \geqslant 0$ be a measure supported on $\mathbb{R}_-$.*

(i) If $-1/2 < \alpha < 0$, then $\Gamma_{h,w}$ is bounded if and only there is a $\gamma > 0$ such that $\mu(-2x, -x) \leqslant \gamma x^{1+2\alpha}$ for all $x > 0$.

(ii) If $\alpha = 0$, then $\Gamma_{h,w}$ is bounded if and only if there is a $\gamma > 0$ such that if $\mu(-x, -0) \leqslant \gamma x$ for all $x > 0$.

(iii) If $\alpha > 0$, then $\Gamma_{h,w}$ is bounded if and only if there is a $\gamma > 0$ such that if $\mu(-x, -0) \leqslant \gamma x^{1+2\alpha}$ for all $x > 0$.

Proof. (i) By Lemma 2.6, boundedness of $\Gamma_{h,w}$ is equivalent to boundedness of the reversed Laplace transform $\mathscr{R}: L^2(0,\infty; dt/w(t)^2)$ into $L^2(\mathbb{C}_-, \mu)$. By [10, Thm. 3.11] this is equivalent to the condition $\mu(-2x, -x) \leqslant \gamma x^{1+2\alpha}$.

(ii) This is Widom's result [14]. See also [13, Thm. 2.5]. It can also be shown as in (iii) below.

(iii) Again, by Lemma 2.6, boundedness of $\Gamma_{h,w}$ is equivalent to boundedness of the reversed Laplace transform $\mathscr{R}: L^2(0,\infty; dt/w(t)^2)$ into $L^2(\mathbb{C}_-, \mu)$. We have $1/w(t)^2 = t^{-2\alpha}$, and since for $\beta > -1$ we have

$$\int_0^\infty e^{-2rt} r^\beta \, dr = \int_0^\infty e^{-u} \left(\frac{u}{2t}\right)^\beta \frac{du}{2t} = \frac{\Gamma(\beta+1)}{2^{\beta+1} t^{\beta+1}},$$

it follows from [9, Prop. 2.3] that with $2\alpha = \beta + 1$ the space $L^2(0,\infty; dt/w(t)^2)$ is isomorphic under the Laplace transform to a Zen space, which in this case is a weighted Bergman space with weight $|x|^{2\alpha-1} \, dx \, dy$. Then [9, Theorem 2.4] implies that $\mathscr{R}$ is bounded if and only if

$$\mu(-x, 0) \leqslant \gamma x \int_0^x r^{2\alpha-1} \, dr = \gamma' x^{1+2\alpha}$$

for constants $\gamma, \gamma' > 0$. $\square$

EXAMPLE 2.8. (i) Take $d\mu(x) = dx$, Lebesgue measure. Thus $h(t) = 1/t$. This μ satisfies the condition (ii) of Theorem 2.7, but not Condition (i) with $\alpha = -1/4$. Therefore the Hilbert–Hankel operator defined by

$$\Gamma u(t) = \int_0^\infty \frac{u(\tau)}{t+\tau} \, d\tau$$

is bounded on $L^2(0,\infty)$ (as is well-known), whereas the corresponding Glover operator defined by

$$\Theta u(t) = \int_0^\infty t^{-1/4} \frac{u(\tau)}{t+\tau} \tau^{-1/4} \, d\tau$$

is unbounded.

(ii) Next take $d\mu(x) = |x|^{-1/2} \, dx$ so that

$$h(t) = \int_0^\infty e^{-tx} \sqrt{x} \, dx = \sqrt{\pi} t^{-1/2}.$$

Now μ satisfies Condition (i) with $\alpha = -1/2$, but not Condition (ii). Therefore we conclude that the Hankel operator defined by

$$\Gamma u(t) = \int_0^\infty \frac{u(\tau)}{(t+\tau)^{1/2}} \, d\tau$$

is unbounded on $L^2(0,\infty)$ but the Glover operator defined by

$$\Theta u(t) = \int_0^\infty t^{-1/4} \frac{u(\tau)}{(t+\tau)^{1/2}} \tau^{-1/4} \, d\tau$$

is bounded. However Θ is not Hilbert–Schmidt, since $h \notin L^2(0, \infty)$, which incidentally provides an example asked for by K. Glover in conversation.

REMARK 2.9. Lemma 2.6 and Theorem 2.7 have partial extensions to sectorial measures supported on $\mathbb{C}_-$. The key observation is that we now have

$$\langle Z_\mu f, \overline{Z_\mu g} \rangle = \langle \Gamma_{h,w} f, \overline{g} \rangle,$$

so that boundedness of the Laplace–Carleson embedding is sufficient (although possibly not always necessary) for the boundedness of the integral operator. We leave the details to the interested reader.

REFERENCES

[1] A. BASHAR ABUSAKSAKA, *Properties of delay systems and diffusive systems*, Ph. D. thesis, University of Leeds (2015).
[2] F. F. BONSALL, *Boundedness of Hankel matrices*, J. London Math. Soc. (2) **29** (1984), no. 2, 289–300.
[3] C. CASENAVE AND G. MONTSENY, *Introduction to diffusive representation*, 4th IFAC syposium on system, structure and control, (2010), 15–17. https://hal.archives-ouvertes.fr/hal-00854381.
[4] E. B. DAVIES, *Linear operators and their spectra*, Cambridge Studies in Advanced Mathematics **106**, Cambridge University Press, Cambridge, 2007.
[5] K. GLOVER, R. F. CURTAIN AND J. R. PARTINGTON, *Realisation and approximation of linear infinite-dimensional systems with error bounds*, SIAM J. Control Optim. **26** (1988), no. 4, 863–898.
[6] K. GLOVER, J. LAM AND J. R. PARTINGTON, *Rational approximation of a class of infinite-dimensional systems: the L_2 case*, Progress in approximation theory, 405–440, Academic Press, Boston, MA, 1991.
[7] C. GUIVER AND M. R. OPMEER, *Model reduction by balanced truncation for systems with nuclear Hankel operators*, SIAM J. Control Optim. **52** (2014), no. 2, 1366–1401.
[8] J. S. HOWLAND, *Trace class Hankel operators*, Quart. J. Math. Oxford Ser. (2) **22** 1971 147–159.
[9] B. JACOB, J. R. PARTINGTON AND S. POTT, *On Laplace–Carleson embedding theorems*, J. Funct. Anal. **264** (2013), no. 3, 783–814.
[10] B. JACOB, J. R. PARTINGTON AND S. POTT, *Applications of Laplace–Carleson embeddings to admissibility and controllability*, SIAM J. Control Optim. **52** (2014), no. 2, 1299–1313.
[11] G. MONTSENY, *Diffusive representation of pseudo-differential time-operators*, ESAIM Proceedings **5** (1998), 159–175.
[12] G. MONTSENY, *Représentation diffusive*, Lavoisier Hermes-Science, Paris (2005).
[13] S. C. POWER, *Hankel operators on Hilbert space*, Research Notes in Mathematics **64**, Pitman (Advanced Publishing Program), Boston, Mass., London, 1982.
[14] H. WIDOM, *Hankel matrices*, Trans. Amer. Math. Soc. **121** (1966), 1–35.

Aolo Bashar Abusaksaka
School of Mathematics
University of Leeds
Leeds LS2 9JT, U.K.
e-mail: aolo_besher@yahoo.com

Jonathan R. Partington
School of Mathematics
University of Leeds
Leeds LS2 9JT, U.K.
e-mail: j.r.partington@leeds.ac.uk

Operators and Matrices
www.ele-math.com
oam@ele-math.com