OPERATOR MONOTONICITY OF A 2–PARAMETER FAMILY OF FUNCTIONS AND $\exp\{f(x)\}$ RELATED TO THE STOLARSKY MEAN

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Abstract. We consider operator monotonicity of a 2-parameter family of functions including the representing function of the Stolarsky mean, which is constructed by integration of the function $[(1-\alpha)+\alpha x^p]^{\frac{1}{p}}$, representing the weighted power mean, of $\alpha \in [0,1]$. We also think about operator monotonicity of $\exp\{f(x)\}$ for a continuous function $f(x)$ defined on $(0,\infty)$.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$, and $\mathscr{B}(\mathscr{H})$ be the set of all bounded linear operators on $\mathscr{H}$. An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be positive if and only if $\langle Ax, x \rangle \geqslant 0$ for all $x \in \mathscr{H}$. We denote a positive operator A by $A \geqslant O$ and a set of all positive operators in $\mathscr{B}(\mathscr{H})$ by $\mathscr{B}(\mathscr{H})_+$. For self-adjoint operators $A, B \in \mathscr{B}(\mathscr{H})$, $A \leqslant B$ means $B - A$ is positive. $A > 0$ (A is strictly positive) means A is positive and invertible. We assume that a function is not a constant throughout this paper. A continuous function $f(x)$ defined on an interval I in $\mathbb{R}$ is called an operator monotone function, provided $A \leqslant B$ implies $f(A) \leqslant f(B)$ for every pair $A, B \in \mathscr{B}(\mathscr{H})$ whose spectra $\sigma(A)$ and $\sigma(B)$ lie in I. We call $f(x)$ a Pick function if $f(x)$ has an analytic continuation to the upper half-plane $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}$ and $f(z)$ maps from $\mathbb{C}^+$ into itself, where $\operatorname{Im} z$ means the imaginary part of z. It is well known that a Pick function is an operator monotone function and conversely an operator monotone function is a Pick function (Löwner's theorem, cf. [3]). In this article we consider operator monotonicity of some functions by using this fact.

A map $\mathfrak{M}(\cdot, \cdot) \colon \mathscr{B}(\mathscr{H})_+^2 \to \mathscr{B}(\mathscr{H})_+$ is called an *operator mean* [4] if the operator $\mathfrak{M}(A, B)$ satisfies the following four conditions for $A, B \in \mathscr{B}(\mathscr{H})_+$;

(1) $A \leqslant C$ and $B \leqslant D$ imply $\mathfrak{M}(A, B) \leqslant \mathfrak{M}(C, D)$,

(2) $C(\mathfrak{M}(A, B))C \leqslant \mathfrak{M}(CAC, CBC)$ for all self-adjoint $C \in \mathscr{B}(\mathscr{H})$,

(3) $A_n \searrow A$ and $B_n \searrow B$ imply $\mathfrak{M}(A_n, B_n) \searrow \mathfrak{M}(A, B)$,

(4) $\mathfrak{M}(I, I) = I$.

Next theorem is so important to study operator means;

THEOREM K-A. (Kubo-Ando [4]) *For any operator mean* $\mathfrak{M}(\cdot,\cdot)$, *there uniquely exists an operator monotone function* $f \geqslant 0$ *on* $[0,\infty)$ *with* $f(1) = 1$ *such that*

$$f(x)I = \mathfrak{M}(I,xI), \quad x \geqslant 0.$$

Then the following hold:

(1) *The map* $\mathfrak{M}(\cdot,\cdot) \mapsto f$ *is a one-to-one onto affine mapping from the set of all operator means to the set of all non-negative operator monotone functions on* $[0,\infty)$ *with* $f(1) = 1$. *Moreover,* $\mathfrak{M}(\cdot,\cdot) \mapsto f$ *preserves the order, i.e., for* $\mathfrak{M}(\cdot,\cdot) \mapsto f$, $\mathfrak{N}(\cdot,\cdot) \mapsto g$,

$$\mathfrak{M}(A,B) \leqslant \mathfrak{N}(A,B) \ (A,B \in \mathscr{B}(\mathscr{H})_+) \iff f(x) \leqslant g(x) \ (x \geqslant 0).$$

(2) *When* $A > 0$, $\mathfrak{M}(A,B) = A^{\frac{1}{2}} f(A^{\frac{-1}{2}} B A^{\frac{-1}{2}}) A^{\frac{1}{2}}$.

The function $f(x)$ is called the *representing function* of $\mathfrak{M}(\cdot,\cdot)$. From this theorem, it is enough to consider operator monotone functions when we study operator means.

The 1-parameter family of operator monotone functions $\{S_p(x)\}_{p \in [-2,2]}$ ([3]);

$$S_p(x) = \left(\frac{p(x-1)}{x^p - 1} \right)^{\frac{1}{1-p}}$$

is one of the most famous family of operator monotone functions on $(0,\infty)$. $S_p(x)$ is a representing function of the Stolarsky mean, and $-2 \leqslant p \leqslant$ is optimal for which $S_p(x)$ is operator monotone [6]. Namely, $S_p(x)$ is not operator monotone if $p \in (-\infty, -2) \cup (2, \infty)$. This family interpolates many famous operator monotone functions, for example,

$$S_1(x) := \lim_{p \to 1} S_p(x) = \exp\left(\frac{x \log x}{x-1} - 1 \right),$$

which is the representing function of the identric mean. The exponential function $\exp(x)$ is well known as a function which is not operator monotone, in contrast with its inverse function $\log x$ is so. But there exists a function $f(x)$ such that $\exp\{f(x)\}$ is an operator monotone function besides constant, like $S_1(x)$. In general, it is so difficult to check operator monotonicity of $\exp\{f(x)\}$ because $\exp\{f(x)\}$ is a composite function of the non-operator monotone function $\exp(x)$ with $f(x)$. In the following, we will obtain a characterization of such functions by using Löwner's theorem and Euler's formula. Thanks to this result, it has become easy to check operator monotonicity of $\exp\{f(x)\}$ by simple computation.

On the other hand, the 2-parameter family of operator monotone functions $\{F_{r,s}(x)\}_{r,s \in [-1,1]}$;

$$F_{r,s}(x) := \left(\frac{r(x^{r+s} - 1)}{(r+s)(x^r - 1)} \right)^{\frac{1}{s}}$$

is constructed in [7] by integration the function $[(1-\alpha) + \alpha x^p]^{\frac{1}{p}}$, which representing the weighted power mean, of the parameter $\alpha \in [0,1]$. This family interpolates

many well-known operator monotone functions except $\{S_p(x)\}_{p\in[-2,2]}$, for example, $F_{-1,-1}(x) = 2(x^{-1}+1)^{-1}$. Moreover, $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$ has monotonicity of r and s, namely, $-1 \leqslant r_1 \leqslant r_2 \leqslant 1$, $-1 \leqslant s_1 \leqslant s_2 \leqslant 1$ imply $F_{r_1,s_1}(x) \leqslant F_{r_2,s_2}(x)$. From this fact, we can easily get the following inequalities;

$$\frac{2x}{x+1} \leqslant \frac{x\log x}{x-1} \leqslant x^{\frac{1}{2}} \leqslant \frac{x-1}{\log x} \leqslant \exp\left(\frac{x\log x}{x-1}-1\right) \leqslant \frac{x+1}{2}. \qquad (\star)$$

If we put $r = 1$ and $s = p-1$, then $F_{r,s}(x)$ coincides with $S_p(x)$, and we obtain the fact that $S_p(x)$ is operator monotone for $0 \leqslant p \leqslant 2$. But we cannot prove operator monotonicity of $S_p(x)$ for $-2 \leqslant p < 0$ by the same way, because $s = p-1 \in [-1,1]$. So we think that the range of parameter of $\{F_{r,s}(x)\}_{r,s\in[-1,1]}$ such that $F_{r,s}(x)$ is operator monotone is not optimal.

In this article we treat this family as the following form

$$S_{p,\alpha}(x) := \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)}\right)^{\frac{1}{\alpha-p}}.$$

In [5], they have obtained an equivalent condition of (p, α) such that $S_{p,\alpha}(x)$ is operator monotone. However, their characterization have not given any concrete form of the range of (p, α), i.e., we have not known the complete form of the range of (p, α) such that $S_{p,\alpha}(x)$ is operator monotone, yet. On the other hand, we have obtained a part of the range of (p, α) such that $S_{p,\alpha}(x)$ is operator monotone in [7]. In this article, we shall extend the range of (p, α) from the results in [7].

In Section 2, we consider the range of parameter of $\{S_{p,\alpha}(x)\}$ in which the function is operator monotone, and try to extend it by using operator monotonicity of $S_p(x)$ and $F_{r,s}(x)$ for $p \in [-2,2]$ and $r,s \in [-1,1]$, respectively. In Section 3, we give a characterization of $f(x)$ such that $\exp\{f(x)\}$ is operator monotone, and by applying this result we get some examples of functions $f(x)$ such that $\exp\{f(x)\}$ is operator monotone.

2. Extension of the range of parameter (p, α) of $\{S_{p,\alpha}(x)\}$

In this section we try to extend the range of parameter (p, α) such that $S_{p,\alpha}(x)$ is operator monotone. So far, we have known that $S_{p,\alpha}(x)$ is operator monotone if

$$p - 1 \leqslant \alpha \leqslant p + 1, \quad p \in [-1,1]$$

from the operator monotonicity of $F_{r,s}(x)$ shown in [7]. In [5], they showed that the following function

$$h_{p,\alpha}(x) = \frac{\alpha(x^p - 1)}{p(x^\alpha - 1)}$$

is operator monotone if and only if $(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 \mid 0 < p - \alpha \leqslant 1, p \geqslant -1, \text{and } \alpha \leqslant 1\} \cup ([0,1] \times [-1,0]) \setminus \{(0,0)\}$. Also, if $(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 \mid 0 \leqslant p \leqslant 1, -1 \leqslant \alpha \leqslant 0 \text{ and } \alpha \leqslant p - 1\}$, then $\frac{1}{p-\alpha} \in [\frac{1}{2}, 1]$. From these results and Löwner-Heinz

inequality, we can find that $S_{p,\alpha}(x) = h_{p,\alpha}(x)^{\frac{1}{p-\alpha}}$ is operator monotone if $(p,\alpha) \in \{(p,\alpha) \in \mathbb{R}^2 | 0 \leqslant p \leqslant 1, \ -1 \leqslant \alpha \leqslant 0 \text{ and } \alpha \leqslant p-1\}$. Therefore, we obtain the fact that $S_{p,\alpha}(x)$ is operator monotone if

$$(p,\alpha) \in \{(p,\alpha) \in \mathbb{R}^2 \mid p-1 \leqslant \alpha \leqslant p+1, \ p \in [-1,1]\} \cup ([0,1] \times [-1,0]). \quad (*)$$

Firstly we think about a trivial part and a part to which the range of parameter cannot be extended.

Trivial part. If $\alpha = 1$, then $S_{p,1}(x)$ coincides with $S_p(x)$, so that $S_{p,\alpha}(x)$ is operator monotone if $\alpha = 1$, $-2 \leqslant p \leqslant 2$ by operator monotonicity of $S_p(x)$. Moreover, since $S_{p,-1}(x)$ coincides with $S_{-p}(x^{-1})^{-1}$ and $S_{-p}(x^{-1})^{-1}$ is operator monotone too, $S_{p,\alpha}(x)$ is operator monotone if $\alpha = -1$, $-2 \leqslant p \leqslant 2$. We can also find that $S_{p,\alpha}(x)$ is operator monotone if $p = 1$, $-2 \leqslant \alpha \leqslant 2$ and $p = -1$, $-2 \leqslant \alpha \leqslant 2$.

If $\alpha \to 0$, then

$$S_{p,0}(x) := \lim_{\alpha \to 0} S_{p,\alpha}(x) = \left(\frac{x^p - 1}{p \log x} \right)^{\frac{1}{p}} \quad (0 < x).$$

It is well known that the set of operator monotone functions is closed in the topology of pointwise convergence, and if $p = 2$, then

$$S_{2,0}(x) = \left(\frac{x+1}{2} \times \frac{x-1}{\log x} \right)^{\frac{1}{2}}.$$

It is easy to check that $S_{2,0}(x)$ is operator monotone. Also, if $f(x)$ is operator monotone, then $f(x^p)^{\frac{1}{p}}$ is operator monotone for all $p \in [-1,1]$. So we can find $S_{p,\alpha}(x)$ is operator monotone if $\alpha = 0$, $-2 \leqslant p \leqslant 2$.

On the other hand, there is a case where $S_{p,\alpha}(x)$ is operator monotone regardless of the value of p or α. If $\alpha = -p$, then

$$S_{p,-p}(x) = \left(\frac{p(x^{-p} - 1)}{(-p)(x^p - 1)} \right)^{\frac{1}{-2p}}$$

$$= \left(\frac{p(1 - x^p)}{(-p)x^p(x^p - 1)} \right)^{\frac{1}{-2p}}$$

$$= \left(\frac{1}{x^p} \right)^{\frac{1}{-2p}} = x^{\frac{1}{2}}.$$

Hence, we find that operator monotonicity of $S_{p,\alpha}(x)$ always holds if $\alpha = -p$.

A part to which the range of parameter cannot be extended.   If $\alpha = 2p$, then

$$S_{p,2p}(x) = \left(\frac{p(x^{2p} - 1)}{2p(x^p - 1)} \right)^{\frac{1}{2p-p}}$$

$$= \left(\frac{(x^p + 1)(x^p - 1)}{2(x^p - 1)} \right)^{\frac{1}{p}} = \left(\frac{x^p + 1}{2} \right)^{\frac{1}{p}}.$$

This function induces the power mean and is operator monotone if and only if $p \in [-1, 1]$ ([2]). So we can not extend the range of paramter $(*)$ such that $S_{p,\alpha}(x)$ is operator monotone when $\alpha = 2p$.

If $\alpha = p$, then

$$S_{p,p}(x) := \lim_{\alpha \to p} S_{p,\alpha}(x) = \exp\left\{ \frac{1}{p} \left(\frac{x^p \log x^p}{x^p - 1} - 1 \right) \right\}.$$

We will show that this function is operator monotone if $p \in [-1, 1]$, and is not if $\frac{5}{4} < |p|$ in Section 3. For example, when $p = 2$, $S_{p,p}(x)$ coincides with

$$\exp\left\{ \frac{1}{2} \left(\frac{x^2 \log x^2}{x^2 - 1} - 1 \right) \right\} = \exp\left\{ \frac{1}{2} \left(\frac{2x}{x+1} \frac{x \log x}{x - 1} - 1 \right) \right\},$$

and it is not operator monotone.

Extension from operator monotonicity of $\{S_p(x)\}_{p \in [-2,2]}$. From Löwner's theorem and operator monotonicity of the 1-parameter family $\{S_p(x)\}_{p \in [-2,2]}$, $z \in \mathbb{C}^+$ implies $S_p(z) \in \mathbb{C}^+$ for all $p \in [-2, 2]$, namely, the argument of $S_p(z)$ has the following property

$$0 < \arg\left(\frac{p(z-1)}{z^p - 1} \right)^{\frac{1}{1-p}} \left(= \frac{1}{1-p} \arg\left(\frac{p(z-1)}{z^p - 1} \right) \right) < \pi$$

$(z \in \mathbb{C}^+,\ -2 \leqslant p \leqslant 2)$. So we get

$$0 < \arg\left(\frac{p(z-1)}{z^p - 1} \right) < (1-p)\pi \quad (-2 \leqslant p < 1),$$

$$0 < \arg\left(\frac{z^p - 1}{p(z-1)} \right) < (p-1)\pi \quad (1 < p \leqslant 2),$$

respectively. By these inequalities we obtain

$$0 < \arg\left(\frac{p(z^\alpha - 1)}{\alpha(z^p - 1)} \right)^{\frac{1}{\alpha - p}}$$

$$= \frac{1}{\alpha - p} \left\{ \arg\left(\frac{p(z-1)}{z^p - 1} \right) + \arg\left(\frac{z^\alpha - 1}{\alpha(z - 1)} \right) \right\}$$

$$< \frac{1}{\alpha - p} \{ (\alpha - 1)\pi + (1-p)\pi \} = \pi$$

for the case $-2 \leqslant p < 1$, $1 < \alpha \leqslant 2$. On the other hand,

$$S_{-p}(x^{-1})^{-1} = \left(\frac{(-p)(x^{-1} - 1)}{(x^{-1})^{-p} - 1} \right)^{\frac{-1}{1-(-p)}}$$

$$= \left(\frac{(-p)(1-x)}{x(x^p - 1)} \right)^{\frac{-1}{1+p}}$$

$$= \left(\frac{x(x^p - 1)}{p(x - 1)} \right)^{\frac{1}{1+p}}$$

is operator monotone for $-2 \leqslant p \leqslant 2$ too. So we have

$$0 < \frac{1}{1+p} \arg\left(\frac{z(z^p-1)}{p(z-1)}\right) < \pi \quad (z \in \mathbb{C}^+, \ -2 \leqslant p \leqslant 2)$$

and get the following relation similarly for the case $-1 < p \leqslant 2$, $-2 \leqslant \alpha < -1$;

$$0 < \arg\left(\frac{p(z^\alpha-1)}{\alpha(z^p-1)}\right)^{\frac{1}{\alpha-p}}$$
$$= \frac{1}{p-\alpha}\left\{\arg\left(\frac{z(z^p-1)}{p(z-1)}\right) + \arg\left(\frac{\alpha(z-1)}{z(z^\alpha-1)}\right)\right\}$$
$$< \frac{1}{p-\alpha}\left\{(1+p)\pi - (1+\alpha)\pi\right\} = \pi.$$

Moreover, since $S_{p,\alpha}(x)$ is symmetric for p, α, we can extend the range of parameter $(*)$ symmetrically from the above results. Namely, we have

$$(-2 \leqslant p < 1, \ 1 < \alpha \leqslant 2) \longrightarrow (-2 \leqslant \alpha < 1, \ 1 < p \leqslant 2),$$

$$(-1 < p \leqslant 2, \ -2 \leqslant \alpha < -1) \longrightarrow (-1 < \alpha \leqslant 2, \ -2 \leqslant p < -1),$$

$$(p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 \mid 0 \leqslant p \leqslant 1, \ -1 \leqslant \alpha \leqslant 0 \text{ and } \alpha \leqslant p-1\}$$

$$\longrightarrow (p, \alpha) \in \{(p, \alpha) \in \mathbb{R}^2 \mid 0 \leqslant \alpha \leqslant 1, \ -1 \leqslant p \leqslant 0 \text{ and } p \leqslant \alpha-1\}.$$
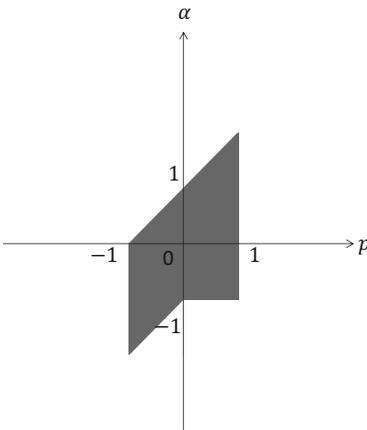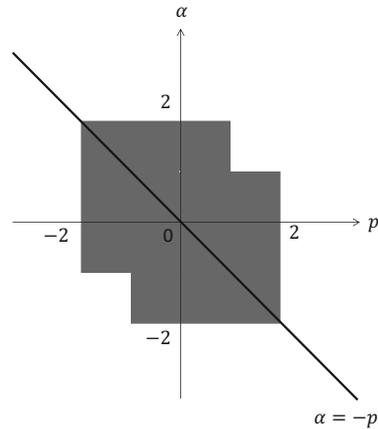

Figure 1: *Parameter range proved in* [5], [7]

Figure 2: *Extended parameter range of* $S_{p,\alpha}(x)$

From the above results, we regard $S_{p,\alpha}(x)$ as the representing function of the 2-parameter Stolarsky mean.

THEOREM 1. *Let*

$$S_{p,\alpha}(x) = \left(\frac{p(x^\alpha - 1)}{\alpha(x^p - 1)} \right)^{\frac{1}{\alpha - p}} \quad (x > 0).$$

Then $S_{p,\alpha}(x)$ is operator monotone if $(p,\alpha) \in \mathscr{A} \subset \mathbb{R}^2$, where

$$\mathscr{A} = \big([-2,1] \times [-1,2]\big) \cup \big([-1,2] \times [-2,1]\big) \cup \big\{(p,\alpha) \in \mathbb{R}^2 \mid \alpha = -p\big\}.$$

3. Operator monotonicity of $\exp\{f(x)\}$

3.1. Characterization

Here we give a characterization of a continuous function $f(x)$ on $(0,\infty)$ such that $\exp\{f(x)\}$ is an operator monotone function. It is clear that $f(x) = \log x$ satisfies this condition. The principal branch of $\mathrm{Log}\, z$ is defined as

$$\mathrm{Log}\, z := \log r + i\theta \ (z := re^{i\theta}, \quad 0 < \theta < 2\pi).$$

It is an analytic continuation of the real logarithmic function to $\mathbb{C}$. Moreover it is a Pick function, namely an operator monotone function, and satisfies $\mathrm{Im}\,\mathrm{Log}\, z = \theta$. In the following we think about the case $f(x)$ is not the logarithmic function:

THEOREM 2. *Let $f(x)$ be a continuous function on $(0,\infty)$. If $f(x)$ is not a constant or $\log(\alpha x)$ $(\alpha > 0)$, then the following are equivalent:*
(1) $\exp\{f(x)\}$ is an operator monotone function,
(2) $f(x)$ is an operator monotone function, and there exists an analytic continuation satisfying

$$0 < v(r,\theta) < \theta,$$

where

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \quad (0 < r, \ 0 < \theta < \pi).$$

REMARK 1. In the above theorem, functions $u(r,\theta)$ and $v(r,\theta)$ are real-valued continuous functions.

REMARK 2. In [1] Hansen proved a necessary and sufficient condition for $\exp\{F(\log x)\}$ to be an operator monotone function, that is, F admits an analytic continuation to $\mathbb{S} = \{z \in \mathbb{C} \mid 0 < \mathrm{Im}\, z < \pi\}$ and $F(z)$ maps from $\mathbb{S}$ into itself. A condition of Theorem 2 is more rigid than this statement.

Proof. $(1) \Longrightarrow (2)$. Since $\exp\{f(x)\}$ is operator monotone, $\log\{\exp\{f(x)\}\} = f(x)$ is operator monotone, too. Also $\exp\{f(x)\}$ is a Pick function, so there exists an analytic continuation to the upper half plane $\mathbb{C}^+$ and $z \in \mathbb{C}^+$ implies $\exp\{f(z)\} \in \mathbb{C}^+$. For $z = s + it \in \mathbb{C}^+$ $(s \in \mathbb{R}, \ 0 < t)$, let $f(z) = f(s+it) = p(s,t) + iq(s,t)$. Using Euler's formula, we obtain

$$\exp\{f(z)\} = \exp\{p(s,t)\}\big(\cos\{q(s,t)\} + i\sin\{q(s,t)\}\big).$$

So we have $\mathrm{Im}\exp\{f(z)\} = \exp\{p(s,t)\}\sin\{q(s,t)\}$, and hence $0 < \sin\{q(s,t)\}$. Also, $q(s,t)$ belongs to C^1, so $q(s,t)$ is continuous on its domain. From these facts, we can find that $2n\pi < q(s,t) < (2n+1)\pi$ holds for the unique $n \in \mathbb{N} \cup \{0\}$. Here by putting $z = re^{i\theta}$ $(0 < r,\ 0 < \theta < \pi)$, $f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$ again,

$$2n\pi < v(r,\theta) < (2n+1)\pi$$

holds for the unique $n \in \mathbb{N} \cup \{0\}$. On the other hand, from the operator monotonicity of $\exp\{f(x)\}$ and the assumption of Theorem 2, $x[\exp\{f(x)\}]^{-1}$ is a positive operator monotone function on $(0,\infty)$, too. So we get

$$
\begin{aligned}
z[\exp\{f(z)\}]^{-1} &= \exp\{\mathrm{Log}z - f(z)\} \\
&= \exp\{(\log r - u(r,\theta)) + i(\theta - v(r,\theta))\} \\
&= \exp\{\log r - u(r,\theta)\}\left(\cos\{\theta - v(r,\theta)\} + i\sin\{\theta - v(r,\theta)\}\right).
\end{aligned}
$$

From the above,

$$2m\pi < \theta - v(r,\theta) < (2m+1)\pi$$

holds for the unique $m \in \mathbb{N} \cup \{0\}$. Moreover, $0 < v(r,\theta)$ and $0 < \theta < \pi$ are required from the assumption, and hence

$$\theta - v(r,\theta) < \theta < \pi.$$

From these facts, $v(r,\theta)$ must satisfy

$$0 < \theta - v(r,\theta) < \pi, \tag{$**$}$$

so we get

$$0 < v(r,\theta) < \theta$$

by the left side inequality of $(**)$.

$(2) \Longrightarrow (1)$. Since $f(x)$ is a Pick function, and the set of all holomorphic functions is closed under composition, $\exp\{f(z)\}$ is a holomorphic function on the upper half plane $\mathbb{C}^+$. Let $z = re^{i\theta} \in \mathbb{C}^+$. From the assumption $0 < v(r,\theta) < \pi$,

$$0 < \sin\{v(r,\theta)\} \leqslant 1.$$

So we have

$$0 < \exp\{u(r,\theta)\}\sin\{v(r,\theta)\} = \mathrm{Im}\exp\{f(z)\}$$

and find

$$z \in \mathbb{C}^+ \Longrightarrow \exp\{f(z)\} \in \mathbb{C}^+. \quad \square$$

COROLLARY 1. *Let $f(x)$ be a continuous function on $(0,\infty)$, and assume $f(x)$ is not a constant or $\log(\alpha x)$ $(\alpha > 0)$. If $f(x)$ is not an operator monotone function or is an operator monotone function which does not satisfy*

$$v(r,\theta) < \pi,$$

then $\exp\{f(x)\}$ is not operator monotone, where

$$f(re^{i\theta}) = u(r,\theta) + iv(r,\theta) \quad (0 < r,\ 0 < \theta < \pi).$$

3.2. Applications

By Theorem 2, we can check numerically that $\exp\{f(x)\}$ is operator monotone or not if the imaginary part of $f(z)$ can be expressed concretely. Now we apply Theorem 2 and get some examples by "only" using simple computation.

EXAMPLE 1. (Harmonic mean)

$$H(x) = \frac{2x}{x+1}$$

is an operator monotone function on $[0,\infty)$, but $\exp\{H(x)\}$ is not operator monotone. Actually, by putting $z = re^{i\theta}$ $(0 < r, \ 0 < \theta < \pi)$, we have

$$H(z) = \frac{2(r^2 + r\cos\theta) + i(2r\sin\theta)}{r^2 + 1 + 2r\cos\theta}$$

and

$$v(r,\theta) := \operatorname{Im} H(z) = \frac{2r\sin\theta}{r^2 + 1 + 2r\cos\theta}.$$

When $r = 1$, $\theta = \frac{5}{6}\pi$, we get $v\left(1, \frac{5}{6}\pi\right) = 2 + \sqrt{3} > \frac{5}{6}$, hence we can find $\exp\{H(x)\}$ is not an operator monotone function by Theorem 2.

EXAMPLE 2. (Logarithmic mean)

$$L(x) = \frac{x-1}{\log x}$$

is an operator monotone function on $[0,\infty)$, but $\exp\{L(x)\}$ is not operator monotone. Actually, by putting $z = re^{i\theta}$ $(0 < r, \ 0 < \theta < \pi)$, we have

$$L(z) = \frac{\{(\log r)(r\cos\theta - 1) + r\theta\sin\theta\} + i\{(r\log r)\sin\theta - \theta(r\cos\theta - 1)\}}{(\log r)^2 + \theta^2}$$

and

$$v(r,\theta) := \operatorname{Im} L(z) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{(\log r)^2 + \theta^2}.$$

When $r = \exp\left\{\frac{\pi}{2}\right\}$, $\theta = \frac{\pi}{2}$, we get $v\left(\exp\left\{\frac{\pi}{2}\right\}, \frac{\pi}{2}\right) = \frac{\exp\{\frac{\pi}{2}\} + 1}{\pi} > \frac{\pi}{2}$ from

$$\pi^2 < 10 = 2 \times (4+1) < 2 \times \left(\exp\left\{\frac{\pi}{2}\right\} + 1\right).$$

So we find that $\exp\{L(x)\}$ is not an operator monotone function by Theorem 2.

EXAMPLE 3. (Dual of Logarithmic mean)

$$DL(x) = \frac{x \log x}{x - 1}$$

is an operator monotone function on $[0, \infty)$ and $\exp\{DL(x)\}$ is operator monotone, too. In the following we verify that $DL(x)$ satisfies the condition of Theorem 2:

By putting $z = re^{i\theta}$ $(0 < r,\ 0 < \theta < \pi)$, we have

$$DL(z) = \frac{r\left[\{(r - \cos\theta)\log r + \theta\sin\theta\} + i\{\theta(r - \cos\theta) - (\log r)\sin\theta\}\right]}{r^2 + 1 - 2r\cos\theta}$$

and

$$v(r, \theta) := \operatorname{Im} DL(z) = \frac{r}{r^2 + 1 - 2r\cos\theta}\{\theta(r - \cos\theta) - (\log r)\sin\theta\}.$$

In the following we show $0 < v(r, \theta) < \theta$.

(1) Proof of $v(r, \theta) < \theta$; $v(r, \theta) < \theta$ is equivalent to $r\{\theta\cos\theta - (\log r)\sin\theta\} < \theta$. By using the following inequalities

$$\theta\cos\theta \leqslant \sin\theta < \theta\ (0 < \theta < \pi), \quad r(1 - \log r) \leqslant 1 \quad (0 < r),$$

we obtain

$$r\{\theta\cos\theta - (\log r)\sin\theta\} \leqslant r\{\sin\theta - (\log r)\sin\theta\}$$
$$= r(1 - \log r)\sin\theta$$
$$\leqslant \sin\theta < \theta.$$

(2) Proof of $0 < v(r, \theta)$; $0 < v(r, \theta)$ is equivalent to $(\log r)\sin\theta < \theta(r - \cos\theta)$. When $1 \leqslant r$, since $\theta\cos\theta \leqslant \sin\theta < \theta\ (0 < \theta < \pi)$, $0 \leqslant \log r \leqslant r - 1$, we have

$$(\log r)\sin\theta < (r - 1)\theta < \theta(r - \cos\theta).$$

When $0 < r < 1$, since $\theta\cos\theta \leqslant \sin\theta < \theta\ (0 < \theta < \pi)$, $\log r < r - 1 < 0$, we have

$$(\log r)\sin\theta < (r - 1)\sin\theta$$
$$\leqslant (r - 1)\theta\cos\theta$$
$$= \theta(r\cos\theta - \cos\theta) < \theta(r - \cos\theta).$$

From (1) and (2), we have $0 < v(r, \theta) < \theta$.

EXAMPLE 4.

$$IL(x) := -L(x)^{-1} = -\frac{\log x}{x - 1}$$

is a negative operator monotone function on $(0, \infty)$ and $\exp\{IL(x)\}$ is operator monotone, too. In the following we verify that $IL(x)$ satisfies the condition of Theorem 2:

By putting $z = re^{i\theta}$ $(0 < r,\ 0 < \theta < \pi)$, we have

$$IL(z) = -\frac{\{(\log r)(r\cos\theta - 1) + r\theta\sin\theta\} + i\{\theta(r\cos\theta - 1) - (r\log r)\sin\theta\}}{r^2 + 1 - 2r\cos\theta}$$

and

$$v(r,\theta) := \operatorname{Im} IL(z) = \frac{(r\log r)\sin\theta - \theta(r\cos\theta - 1)}{r^2 + 1 - 2r\cos\theta}.$$

In the following we show $0 < v(r,\theta) < \theta$.

(1) Proof of $v(r,\theta) < \theta$; $v(r,\theta) < \theta$ is equivalent to $(\log r)\sin\theta + \theta\cos\theta < r\theta$. By using the following inequalities

$$\theta\cos\theta \leqslant \sin\theta < \theta\ (0 < \theta < \pi),\ \log r \leqslant r - 1\ (0 < r),$$

we obtain

$$\begin{aligned}
(\log r)\sin\theta + \theta\cos\theta &\leqslant (\log r)\sin\theta + \sin\theta \\
&= \sin\theta(\log r + 1) \\
&\leqslant r\sin\theta < r\theta.
\end{aligned}$$

(2) Proof of $0 < v(r,\theta)$; $0 < v(r,\theta)$ is equivalent to $r\{\theta\cos\theta - (\log r)\sin\theta\} < \theta$. Since $\theta\cos\theta \leqslant \sin\theta < \theta\ (0 < \theta < \pi)$, $r(1 - \log r) \leqslant 1\ (0 < r)$, we have

$$\begin{aligned}
r\{\theta\cos\theta - (\log r)\sin\theta\} &\leqslant r\{\sin\theta - (\log r)\sin\theta\} \\
&= \sin\theta\{r(1 - \log r)\} \\
&\leqslant \sin\theta < \theta.
\end{aligned}$$

From (1) and (2), we obtain $0 < v(r,\theta) < \theta$.

Results of Example 3 and Example 4 are extended as the following;

THEOREM 3. *Let*
$$DL_p(x) = \frac{x^p\log x}{x^p - 1}.$$
Then $\exp\{DL_p(x)\}$ *is an operator monotone function for all* $p \in [-1, 1] \setminus \{0\}$.

Proof. Firstly we show that $DL_p(x)$ satisfies the condition of Theorem 2 for the case $p \in (0, 1]$:

By putting $z = re^{i\theta}$ $(0 < r,\ 0 < \theta < \pi)$, we have

$$DL_p(z) = \frac{r^p\Big[\{(r^p - \cos(p\theta))\log r + \theta\sin(p\theta)\} + i\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\}\Big]}{r^{2p} + 1 - 2r^p\cos(p\theta)}$$

and

$$v(r,\theta) := \operatorname{Im} DL_p(z) = \frac{r^p}{r^{2p} + 1 - 2r^p\cos(p\theta)}\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\}.$$

In the following we show $0 < v(r, \theta) < \theta$.

(1) Proof of $v(r, \theta) < \theta$; $v(r, \theta) < \theta$ is equivalent to $r^p \theta \cos(p\theta) - (r^p \log r) \sin(p\theta) < \theta$.

$$
r^p \theta \cos(p\theta) - (r^p \log r) \sin(p\theta) \leqslant r^p \left(\frac{1}{p} \right) \sin(p\theta) - (r^p \log r) \sin(p\theta)
$$

$$
= \sin(p\theta) \left(\frac{1}{p} \right) (r^p - r^p \log r^p)
$$

$$
\leqslant \sin(p\theta) \left(\frac{1}{p} \right) < (p\theta) \left(\frac{1}{p} \right) = \theta.
$$

(2) Proof of $0 < v(r, \theta)$; $0 < v(r, \theta)$ equivalent to $(\log r) \sin(p\theta) < \theta(r^p - \cos(p\theta))$.
When $r = 1$, the inequality holds clearly.
When $1 < r$,

$$
(\log r) \sin(p\theta) = \left(\frac{1}{p} \right) (\log r^p) \sin(p\theta) < \left(\frac{1}{p} \right) (r^p - 1) \sin(p\theta)
$$

$$
\leqslant \left(\frac{1}{p} \right) (r^p - \cos(p\theta)) \sin(p\theta) \leqslant (r^p - \cos(p\theta))\theta.
$$

When $0 < r < 1$,

$$
(\log r) \sin(p\theta) = \left(\frac{1}{p} \right) (\log r^p) \sin(p\theta) < \left(\frac{1}{p} \right) (r^p - 1) \sin(p\theta)
$$

$$
\leqslant \left(\frac{1}{p} \right) (r^p - 1)(p\theta) \cos(p\theta) = \theta \left(r^p \cos(p\theta) - \cos(p\theta) \right)
$$

$$
\leqslant \theta(r^p - \cos(p\theta)).
$$

When $p \in [-1, 0)$,

$$
DL_p(z) = \frac{z^p \mathrm{Log} z}{z^p - 1} = \frac{z^{-p} z^p \mathrm{Log} z}{z^{-p}(z^p - 1)} = \frac{\mathrm{Log} z}{1 - z^{|p|}}
$$

and

$$
v'(r, \theta) := \mathrm{Im}\, DL_p(re^{i\theta}) = \frac{\theta(1 - r^{|p|} \cos(|p|\theta)) + (r^{|p|} \log r) \sin(|p|\theta)}{r^{2|p|} + 1 - 2r^{|p|} \cos(|p|\theta)}.
$$

In the following we show $0 < v'(r, \theta) < \theta$.

(3) Proof of $v'(r,\theta) < \theta$; $v'(r,\theta) < \theta$ is equivalent to $\theta\cos(|p|\theta) + (\log r)\sin(|p|\theta) < \theta r^{|p|}$.

$$\theta\cos(|p|\theta) + (\log r)\sin(|p|\theta) \leqslant \frac{\sin(|p|\theta)}{|p|} + (\log r)\sin(|p|\theta)$$

$$= \frac{\sin(|p|\theta)}{|p|}(1 + \log r^{|p|})$$

$$\leqslant \frac{\sin(|p|\theta)}{|p|}(1 + r^{|p|} - 1)$$

$$= \frac{\sin(|p|\theta)}{|p|}r^{|p|} < \theta r^{|p|}.$$

(4) Proof of $0 < v'(r,\theta)$; $0 < v'(r,\theta)$ is equivalent to $r^{|p|}(\theta\cos(|p|\theta) - (\log r)\sin(|p|\theta)) < \theta$.

When $r = 1$, the inequality holds clearly.

When $r \neq 1$,

$$r^{|p|}(\theta\cos(|p|\theta) - (\log r)\sin(|p|\theta)) \leqslant r^{|p|}\left(\frac{\sin(|p|\theta)}{|p|} - (\log r)\sin(|p|\theta)\right)$$

$$= r^{|p|}(1 - \log r^{|p|})\left(\frac{\sin(|p|\theta)}{|p|}\right)$$

$$< \frac{\sin(|p|\theta)}{|p|} < \theta. \quad \square$$

REMARK 3. $DL_p(x)$ doesn't satisfy the condition of Theorem 2 if $\frac{5}{4} < |p|$. If $p = 2$, for example, then

$$DL_2(x) = \frac{1}{2} \times \frac{2x}{x+1} \times \frac{x\log x}{x-1}.$$

From the inequality $(\star)$ in Section 1, we have

$$\frac{2x}{x+1} \times \frac{x\log x}{x-1} \leqslant x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x.$$

Since a positive operator monotone function on $(0,\infty)$ with $f(1) = 1$ must satisfy $x < f(x)$ if $0 < x < 1$,

$$\frac{2x}{x+1} \times \frac{x\log x}{x-1}$$

is not an operator monotone function. And hence, $DL_2(x)$ is not operator monotone. Next we will see more general case $\frac{5}{4} < |p|$. From a proof of Theorem 3, we have

$$v(r,\theta) := \operatorname{Im} DL_p(z) = \frac{r^p}{r^{2p} + 1 - 2r^p\cos(p\theta)}\{\theta(r^p - \cos(p\theta)) - (\log r)\sin(p\theta)\}.$$

By simple computation,

$$v(r, \theta) < \theta \Longleftrightarrow \big(l(p, r, \theta) =\big) r^p \cos(p\theta) - (r^p \log r) \frac{\sin(p\theta)}{\theta} < 1.$$

If $\theta = \frac{5\pi}{4p}$ and $r = e^{\frac{5\pi}{2p}} (= e^{2\theta})$, then

$$
\begin{aligned}
l(p, r, \theta) &= \frac{1}{\sqrt{2}} \left(-e^{2p\theta} + \frac{1}{\theta} \times e^{2p\theta} \times \log e^{2\theta} \right) \\
&= \frac{1}{\sqrt{2}} \left(-e^{2p\theta} + 2e^{2p\theta} \right) \\
&= \frac{1}{\sqrt{2}} \exp \left\{ \frac{5\pi}{2} \right\} > 1.
\end{aligned}
$$

Therefore $DL_p(x)$ is not operator monotone if $\frac{5}{4} < p$ from Theorem 2. We can also show the case $p < -\frac{5}{4}$ similarly.

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