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THE INVERTIBILITY FOR LINEAR COMBINATIONS OF BOUNDED LINEAR OPERATORS WITH CLOSED RANGE

GUOJUN HAI, CHEGEN BAO AND ALATANCANG CHEN

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Abstract. In this paper, it is given that the sufficient and necessary conditions for the invertibility of linear combinations of bounded linear operators with closed range. Furthermore, some related results are obtained.

1. Introduction

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces. We use $\mathscr{B}(\mathscr{H},\mathscr{K})$ to denote the set of all bounded linear operators from $\mathscr{H}$ into $\mathscr{K}$ and $\mathscr{B}(\mathscr{H})=\mathscr{B}(\mathscr{H},\mathscr{H})$. If $T\in\mathscr{B}(\mathscr{H},\mathscr{K})$, we use $\mathscr{R}(T)$, $\mathscr{N}(T)$ and T^* to denote the range space, the null space and the adjoint of T, respectively. For a closed subspace $\mathscr{M}\subset\mathscr{H}$, its orthogonal complement is denoted by $\mathscr{M}^{\perp}$.

Let $T\in\mathscr{B}(\mathscr{H},\mathscr{K})$. If there exists $S\in\mathscr{B}(\mathscr{K},\mathscr{H})$ such that $ST=I_{\mathscr{H}}$, then T is called left invertible; If there exists $S\in\mathscr{B}(\mathscr{K},\mathscr{H})$ such that $TS=I_{\mathscr{K}}$, then T is called right invertible. An operator T is invertible if T is both left and right invertible. It is obvious that T is left invertible if and only if T^* is right invertible. From [5, pp. 347-348] one can find that T is left invertible if and only if T is injective and $\mathscr{R}(T)$ is closed; T is right invertible if and only if T is surjective.

Recall [5, pp. 36-37] that an operator $P\in\mathscr{B}(\mathscr{H})$ is idempotent if $P^2=P$. It is evident that if P is idempotent, then both $\mathscr{R}(P)$ and $\mathscr{N}(P)$ are closed, and $\mathscr{H}=\mathscr{N}(P)\dotplus\mathscr{R}(P)$. Conversely, if $\mathscr{M}$ and $\mathscr{L}$ are closed subspace of $\mathscr{H}$, and $\mathscr{H}=\mathscr{M}\dotplus\mathscr{L}$, then there is an idempotent $P\in\mathscr{B}(\mathscr{H})$ such that $\mathscr{R}(P)=\mathscr{M}$ and $\mathscr{N}(P)=\mathscr{L}$. If $P\in\mathscr{B}(\mathscr{H})$ is idempotent and $\mathscr{N}(P)=\mathscr{R}(P)^{\perp}$, then P is called the orthogonal projection of $\mathscr{H}$ onto $\mathscr{R}(P)$. Clearly, $\mathscr{M}$ is a closed subspace of $\mathscr{H}$ if and only if there exists a unique orthogonal projection P of $\mathscr{H}$ onto $\mathscr{M}$. We will denote the orthogonal projection P of $\mathscr{H}$ onto $\mathscr{M}$ by $P_{\mathscr{M}}$. It is well known that $P_{\mathscr{M}}^*=P_{\mathscr{M}}$ and $I-P_{\mathscr{M}}=P_{\mathscr{M}^{\perp}}$. Define an operator $P'_{\mathscr{M}}\in\mathscr{B}(\mathscr{H},\mathscr{M})$ in the following way: $P'_{\mathscr{M}}x=P_{\mathscr{M}}x$ for any $x\in\mathscr{H}$.

For $T\in\mathscr{B}(\mathscr{H},\mathscr{K})$, if there exists an operator $S\in\mathscr{B}(\mathscr{K},\mathscr{H})$ such that

$$STS=S,\ \ TST=T,\ \ (TS)^*=TS,\ \ (ST)^*=ST,$$

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then S is called a Moore-Penrose inverse of T, and denoted by $T^{\dagger}$. It is well known that T has a Moore-Penrose inverse if and only if $\mathscr{R}(T)$ is closed. Furthermore, the Moore-Penrose inverse $T^{\dagger}$ of T is unique(if $T^{\dagger}$ exists). If T is a closed range operator, from the above equations we can see that $TT^{\dagger}=P_{\mathscr{R}(T)}$ and $T^{\dagger}T=I-P_{\mathscr{N}(T)}$.

For orthogonal projections P and Q on $\mathscr{H}$, D. Buckholtz [3] has shown that $P-Q$ is invertible if and only if $\mathscr{H}=\mathscr{R}(P)\dot{+}\mathscr{R}(Q)$. Motivated by this results, many authors have considered questions concerning the idempotents and orthogonal projections(see [2, 6, 7, 12, 13, 11, 17, 20, 21, 16, 19]). For idempotent matrices P and Q, J. Groβ and G. Trenkler [17] have considered the nonsingularity of $P-Q$, and obtained that if $P-Q$ is nonsingular, then so is $P+Q$, and then J. K. Baksalary and O. M. Baksalary in [2] have proved that the nonsingularity of $P+Q$ is equivalent to the nonsingularity of any linear combination $\alpha P+\beta Q$, where $\alpha,\beta\in\mathbb{C}$ and $\alpha+\beta\neq 0$. For idempotent operators P and Q on an infinite dimensional Hilbert space, H. Du, X. Yao and C. Deng [11] have extend the main results of [2] to infinite dimensional Hilbert space case, and proved that the invertibility of $\alpha P+\beta Q$ is independent of the choice of α and β, where $\alpha+\beta\neq 0$, $\alpha\beta\neq 0$. Furthermore, the Fredholmness and stability theorems of linear combinations of idempotents were considered in [13, 16, 21], respectively. It need to mention that the method used in the above papers rely strongly on the idempotency of idempotent or the positivity of the orthogonal projection.

Inspired by the results of [2, 11, 13, 21, 16], we are interested in the question that what we say about a linear combination of bounded linear operators? In this paper, we investigate the invertibility of linear combinations of bounded linear operators with closed range. As an application of our results the invertibility of linear combinations of EP operators is considered. In this paper a systematic use is made of operator matrix representations, and of generalized inverses of closed range operators.

2. Main result

Our main result of this paper is the following theorem.

THEOREM 2.1. *Let $A\in\mathscr{B}(\mathscr{H})$ and $B\in\mathscr{B}(\mathscr{H})$ be closed range operators, and let $\alpha,\beta\in\mathbb{C}\setminus\{0\}$. Assume that $\mathscr{M}=B^{\dagger}(\mathscr{R}(A)\cap\mathscr{R}(B))$ and $\mathscr{S}=(A^*)^{\dagger}(\mathscr{R}(A^*)\cap\mathscr{R}(B^*))$. Then $\alpha A+\beta B$ is invertible if and only if the following statements hold:*

(i) $\mathscr{N}(A)\cap\mathscr{N}(B)=\{0\}$, $\mathscr{R}(A)^{\perp}\cap\mathscr{R}(B)^{\perp}=\{0\}$;

(ii) *Both* $A^{\dagger}A(I-B^{\dagger}B)$ *and* $(I-AA^{\dagger})BB^{\dagger}$ *are closed range operators;*

(iii) $P'_{\mathscr{S}}(\alpha AB^{\dagger}B+\beta AA^{\dagger}B)|_{\mathscr{M}}:\mathscr{M}\longrightarrow\mathscr{S}$ *is invertible.*

For the proof of Theorem 2.1 we need a result, which is well known so its proof will be omitted.

LEMMA 2.2. *Let $\mathscr{H}_1$, $\mathscr{H}_2$, $\mathscr{K}_1$ and $\mathscr{K}_2$ be Hilbert spaces, and let $A\in\mathscr{B}(\mathscr{H}_1,\mathscr{K}_1)$, $B\in\mathscr{B}(\mathscr{H}_2,\mathscr{K}_2)$ and $C\in\mathscr{B}(\mathscr{H}_2,\mathscr{K}_1)$. Assume that*

$$M=\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}:\mathscr{H}_1\oplus\mathscr{H}_2\longrightarrow\mathscr{K}_1\oplus\mathscr{K}_2.$$

(i) M *is invertible if and only if* A *is left invertible,* B *is right invertible and* $C_1 = P'_{\mathscr{R}(A)^\perp} C|_{\mathscr{N}(B)} : \mathscr{N}(B) \longrightarrow \mathscr{R}(A)^\perp$ *is invertible.*

(ii) If $C = 0$*, then* M *is invertible if and only if* A *and* B *are invertible.*

(iii) If any two of operators A*,* B *and* M *are invertible, then the third is invertible.*

Proof of Theorem 2.1. Let A and B be closed range operators on $\mathscr{H}$, and let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

Claim 1. $\alpha A + \beta B$ is an invertible operator if and only if

$$\begin{bmatrix} \alpha A & \beta B \\ -I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix} : \mathscr{H} \oplus \mathscr{H} \longrightarrow \mathscr{H} \oplus \mathscr{H}$$

is an invertible operator.

This directly follows from

$$\begin{bmatrix} \alpha A + \beta B & 0 \\ 0 & I_{\mathscr{H}} \end{bmatrix} = \begin{bmatrix} I_{\mathscr{H}} & -\beta B \\ 0 & I_{\mathscr{H}} \end{bmatrix} \begin{bmatrix} \alpha A & \beta B \\ -I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix} \begin{bmatrix} I_{\mathscr{H}} & 0 \\ I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix}.$$

Claim 2. The operator matrix

$$\begin{bmatrix} \alpha A & \beta B \\ -I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix} : \mathscr{H} \oplus \mathscr{H} \longrightarrow \mathscr{H} \oplus \mathscr{H}$$

is invertible if and only if the operator

$$\begin{bmatrix} \alpha P'_{\mathscr{R}(A)} A P'_{\mathscr{N}(A)^\perp}|_{\mathscr{N}(B)} & P'_{\mathscr{R}(A)}(\beta B + \alpha A)|_{\mathscr{N}(B)^\perp} \\ 0 & \beta P'_{\mathscr{R}(A)^\perp} P'_{\mathscr{R}(B)} B|_{\mathscr{N}(B)^\perp} \end{bmatrix}$$

from $\mathscr{N}(B) \oplus \mathscr{N}(B)^\perp$ into $\mathscr{R}(A) \oplus \mathscr{R}(A)^\perp$ is invertible.

Indeed, since

$$\begin{bmatrix} \alpha A & \beta B \\ -I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix}$$

as an operator from $\mathscr{N}(A)^\perp \oplus \mathscr{N}(A) \oplus \mathscr{N}(B) \oplus \mathscr{N}(B)^\perp$ into $\mathscr{R}(A) \oplus \mathscr{R}(A)^\perp \oplus \mathscr{N}(A)^\perp \oplus \mathscr{N}(A)$ has the following operator matrix representation

$$\begin{bmatrix} \alpha A_1 & 0 & 0 & \beta B_1 \\ 0 & 0 & 0 & \beta B_2 \\ -I_{\mathscr{N}(A)^\perp} & 0 & C_1 & C_2 \\ 0 & -I_{\mathscr{N}(A)} & C_3 & C_4 \end{bmatrix},$$

there exist invertible operators

$$U = \begin{bmatrix} I_{\mathscr{R}(A)} & 0 & \alpha A_1 & 0 \\ 0 & I_{\mathscr{R}(A)^\perp} & 0 & 0 \\ 0 & 0 & I_{\mathscr{N}(A)^\perp} & 0 \\ 0 & 0 & 0 & I_{\mathscr{N}(A)} \end{bmatrix}$$

on $\mathscr{R}(A)\oplus\mathscr{R}(A)^{\perp}\oplus\mathscr{N}(A)^{\perp}\oplus\mathscr{N}(A)$ and

$$V=\begin{bmatrix} I_{\mathscr{N}(A)^{\perp}} & 0 & C_1 & C_2 \\ 0 & I_{\mathscr{N}(A)} & C_3 & C_4 \\ 0 & 0 & I_{\mathscr{N}(B)} & 0 \\ 0 & 0 & 0 & I_{\mathscr{N}(B)^{\perp}} \end{bmatrix}$$

on $\mathscr{N}(A)^{\perp}\oplus\mathscr{N}(A)\oplus\mathscr{N}(B)\oplus\mathscr{N}(B)^{\perp}$ such that

$$U\begin{bmatrix} \alpha A & \beta B \\ -I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix}V=\begin{bmatrix} 0 & 0 & \alpha A_1C_1 & \alpha A_1C_2+\beta B_1 \\ 0 & 0 & 0 & \beta B_2 \\ -I_{\mathscr{N}(A)^{\perp}} & 0 & 0 & 0 \\ 0 & -I_{\mathscr{N}(A)} & 0 & 0 \end{bmatrix}.$$

Thus

$$\begin{bmatrix} \alpha A & \beta B \\ -I_{\mathscr{H}} & I_{\mathscr{H}} \end{bmatrix}:\mathscr{H}\oplus\mathscr{H}\longrightarrow\mathscr{H}\oplus\mathscr{H}$$

is invertible if and only if

$$\begin{bmatrix} \alpha A_1C_1 & \beta B_1+\alpha A_1C_2 \\ 0 & \beta B_2 \end{bmatrix}:\mathscr{N}(B)\oplus\mathscr{N}(B)^{\perp}\longrightarrow\mathscr{R}(A)\oplus\mathscr{R}(A)^{\perp}$$

is invertible. Now, Claim 2 directly follows from

$$\begin{aligned} &A_1=P'_{\mathscr{R}(A)}A|_{\mathscr{N}(A)^{\perp}}, && B_1=P'_{\mathscr{R}(A)}B|_{\mathscr{N}(B)^{\perp}}, && B_2=P'_{\mathscr{R}(A)^{\perp}}B|_{\mathscr{N}(B)^{\perp}}, \\ &C_1=P'_{\mathscr{N}(A)^{\perp}}|_{\mathscr{N}(B)}, && C_2=P'_{\mathscr{N}(A)^{\perp}}|_{\mathscr{N}(B)^{\perp}}. \end{aligned}$$

Claim 3. The operator matrix

$$\begin{bmatrix} \alpha P'_{\mathscr{R}(A)}AP'_{\mathscr{N}(A)^{\perp}}|_{\mathscr{N}(B)} & P'_{\mathscr{R}(A)}(\alpha A+\beta B)|_{\mathscr{N}(B)^{\perp}} \\ 0 & \beta P'_{\mathscr{R}(A)^{\perp}}P'_{\mathscr{R}(B)}B|_{\mathscr{N}(B)^{\perp}} \end{bmatrix}$$

from $\mathscr{N}(B)\oplus\mathscr{N}(B)^{\perp}$ into $\mathscr{R}(A)\oplus\mathscr{R}(A)^{\perp}$ is invertible if and only if the statements (i), (ii) and (iii) in Theorem 2.1 hold true.

In fact, let

$$\begin{aligned} \widetilde{A}&=P'_{\mathscr{R}(A)}AP'_{\mathscr{N}(A)^{\perp}}|_{\mathscr{N}(B)}:\mathscr{N}(B)\longrightarrow\mathscr{R}(A), \\ \widetilde{B}&=P'_{\mathscr{R}(A)^{\perp}}P'_{\mathscr{R}(B)}B|_{\mathscr{N}(B)^{\perp}}:\mathscr{N}(B)^{\perp}\longrightarrow\mathscr{R}(A)^{\perp}, \\ \widetilde{C}&=P'_{\mathscr{R}(A)}(\alpha A+\beta B)|_{\mathscr{N}(B)^{\perp}}:\mathscr{N}(B)^{\perp}\longrightarrow\mathscr{R}(A). \end{aligned}$$

By Lemma 2.2 (i) one can see that

$$\begin{bmatrix} \alpha\widetilde{A} & \widetilde{C} \\ 0 & \beta\widetilde{B} \end{bmatrix}:\mathscr{N}(B)\oplus\mathscr{N}(B)^{\perp}\longrightarrow\mathscr{R}(A)\oplus\mathscr{R}(A)^{\perp}$$

is an invertible operator if and only if $\widetilde{A}$ is a left invertible operator, $\widetilde{B}$ is a right invertible operator and $\widetilde{C}_1 = P'_{\mathscr{R}(\widetilde{A})^\perp}\widetilde{C}|_{\mathscr{N}(\widetilde{B})} : \mathscr{N}(\widetilde{B}) \longrightarrow \mathscr{R}(\widetilde{A})^\perp$ is invertible. Since $P'_{\mathscr{R}(A)}A|_{\mathscr{N}(A)^\perp} : \mathscr{N}(A)^\perp \longrightarrow \mathscr{R}(A)$ is invertible, it follows that

$$\widetilde{A} = P'_{\mathscr{R}(A)}A|_{\mathscr{N}(A)^\perp}P'_{\mathscr{N}(A)^\perp}|_{\mathscr{N}(B)}$$

is left invertible if and only if $P'_{\mathscr{N}(A)^\perp}|_{\mathscr{N}(B)} : \mathscr{N}(B) \longrightarrow \mathscr{N}(A)^\perp$ is left invertible, which is equivalent to the fact that $\mathscr{N}(A)\cap\mathscr{N}(B) = \{0\}$ and $P_{\mathscr{N}(A)^\perp}P_{\mathscr{N}(B)}$ has closed range. In a similar way one can show that $\widetilde{B}$ is right invertible if and only if $\mathscr{R}(A)^\perp \cap \mathscr{R}(B)^\perp = \{0\}$ and $P_{\mathscr{R}(A)^\perp}P_{\mathscr{R}(B)}$ has closed range. On the other hand, note that

$$\begin{aligned}
\mathscr{N}(\widetilde{B}) &= \mathscr{N}(P_{\mathscr{R}(A)^\perp}P_{\mathscr{R}(B)}BP_{\mathscr{N}(B)^\perp}) \cap \mathscr{N}(B)^\perp \\
&= B^\dagger \mathscr{N}(P_{\mathscr{R}(A)^\perp}P_{\mathscr{R}(B)}) \\
&= B^\dagger(\mathscr{R}(A)\cap\mathscr{R}(B)), \\
\mathscr{R}(\widetilde{A})^\perp &= \mathscr{R}(P_{\mathscr{R}(A)}AP_{\mathscr{N}(A)^\perp}P_{\mathscr{N}(B)})^\perp \cap \mathscr{R}(A) \\
&= \mathscr{N}(P_{\mathscr{R}(B^*)^\perp}P_{\mathscr{R}(A^*)}A^*P_{\mathscr{N}(A^*)^\perp}) \cap \mathscr{N}(A^*)^\perp \\
&= (A^*)^\dagger(\mathscr{R}(A^*)\cap\mathscr{R}(B^*)),
\end{aligned}$$

and hence, by

$$\begin{aligned}
AA^\dagger &= P_{\mathscr{R}(A)}, \\
A^\dagger A &= I - P_{\mathscr{N}(A)}, \\
BB^\dagger &= P_{\mathscr{R}(B)}, \\
B^\dagger B &= I - P_{\mathscr{N}(B)},
\end{aligned}$$

we obtain Claim 3.

Now, Theorem 2.1 directly follows from Claims 1, 2 and 3. □

REMARK 1. Note that

$$\begin{aligned}
A^\dagger A(I - B^\dagger B) &= P_{\mathscr{N}(A)^\perp}P_{\mathscr{N}(B)}, \\
(I - AA^\dagger)BB^\dagger &= P_{\mathscr{R}(A)^\perp}P_{\mathscr{R}(B)},
\end{aligned}$$

and hence one can infer that $A^\dagger A(I - B^\dagger B)$ has closed range if and only if $A|_{\mathscr{N}(B)}$ has closed range; $(I - AA^\dagger)BB^\dagger$ has closed range if and only if $B^*|_{\mathscr{N}(A^*)}$ has closed range. On the other hand, by [8, 12] we can also see that the statement (ii) in Theorem 2.1 has some equivalent conditions.

3. Some related results

From Theorem 2.1 we obtain some related results.

PROPOSITION 3.1. *Let* $A \in \mathscr{B}(\mathscr{H})$ *and* $B \in \mathscr{B}(\mathscr{H})$, *and let* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *If* $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ *or* $\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}$, *then* $\alpha A + \beta B$ *is invertible if and only if*

(i) $\mathscr{R}(A)$ *and* $\mathscr{R}(B)$ *are closed;*

(ii) $P'_{\mathscr{N}(A)^{\perp}}|_{\mathscr{N}(B)} : \mathscr{N}(B) \longrightarrow \mathscr{N}(A)^{\perp}$ *and* $P'_{\mathscr{R}(A)^{\perp}}|_{\mathscr{R}(B)} : \mathscr{R}(B) \longrightarrow \mathscr{R}(A)^{\perp}$ *are invertible.*

Proof. The sufficiency directly follows from Lemma 2.2 (iii) and the proof of Theorem 2.1. Now suppose that $\alpha A + \beta B$ is invertible. If $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$, then by the invertibility of $\alpha A + \beta B$ one can infer that

$$\mathscr{H} = \mathscr{R}(\alpha A + \beta B) \subset \mathscr{R}(A) + \mathscr{R}(B),$$

which implies $\mathscr{R}(A) + \mathscr{R}(B) = \mathscr{H}$. This, together with $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$, shows that $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are closed by [15, Theorem 2.3]; If $\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}$, note that the invertibility of $\alpha A + \beta B$ implies that of $\overline{\alpha} A^* + \overline{\beta} B^*$, and hence a similar argument gives that $\mathscr{R}(A^*)$ and $\mathscr{R}(B^*)$ are closed. From the Banach closed range theorem it follows that $\mathscr{R}(A)$ and $\mathscr{R}(B)$ are closed. This proves (i). (ii) follows from Lemma 2.2 (ii) and the proof of Theorem 2.1. $\square$

As a consequence of Proposition 3.1 we obtain the following result.

COROLLARY 3.2. *Let* $A \in \mathscr{B}(\mathscr{H})$ *and* $B \in \mathscr{B}(\mathscr{H})$. *If* $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ *or* $\mathscr{R}(A^*) \cap \mathscr{R}(B^*) = \{0\}$, *then the invertibility of* $\alpha A + \beta B$ *is independent of the choice of* α *and* β, *where* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.

REMARK 2. It is worthy to be mentioned that

(i) For idempotent operators, the result in Corollary 3.2 was appeared in [11, p. 1455].

(ii) Corollary 3.2 can be seen as an extension of [1, Proposition 3.15]: if $A \in \mathscr{B}(\mathscr{H})$, $B \in \mathscr{B}(\mathscr{H})$ and $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$, then $A - B$ is invertible if and only if $A + B$ is invertible.

PROPOSITION 3.3. *Let* A *and* B *be closed range operators on* $\mathscr{H}$, *and let* $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. *If* $AB^{\dagger}B = AA^{\dagger}B = 0$, *then* $\alpha A + \beta B$ *is invertible if and only if the following four equations hold:*

$$\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}, \qquad \mathscr{N}(A)^{\perp} \cap \mathscr{N}(B)^{\perp} = \{0\},$$
$$\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}, \qquad \mathscr{R}(A)^{\perp} \cap \mathscr{R}(B)^{\perp} = \{0\}.$$

Proof. If $AB^{\dagger}B = AA^{\dagger}B = 0$, then $A^{\dagger}A(I - B^{\dagger}B) = A^{\dagger}A$ and $(I - AA^{\dagger})BB^{\dagger} = BB^{\dagger}$, and so the statement (ii) in Theorem 2.1 automatically holds true. Thus, $\alpha A + \beta B$ is invertible if and only if the statements (i) and (iii) in Theorem 2.1 hold true. It is not hard to find that the statement (iii) in Theorem 2.1 and Lemma 2.2 (iii) implies that $\mathscr{R}(A) \cap \mathscr{R}(B) = \{0\}$ and $\mathscr{N}(A)^{\perp} \cap \mathscr{N}(B)^{\perp} = \{0\}$. This proves Proposition 3.3. $\square$

One can see from Proposition 3.3 that

COROLLARY 3.4. *Let A and B be closed range operators on $\mathscr{H}$. If $AB^{\dagger}B = AA^{\dagger}B = 0$, then the invertibility of $\alpha A + \beta B$ is independent of the choice of α and β, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$.*

In the following, we consider the invertibility of linear combination of EP operators. Recall that $T \in \mathscr{B}(\mathscr{H})$ is called an EP operator if $\mathscr{R}(T)$ is closed and $TT^{\dagger} = T^{\dagger}T$. Clearly, $T \in \mathscr{B}(\mathscr{H})$ is an EP operator if and only if $\mathscr{R}(T) = \mathscr{R}(T^*)$. For some related results of EP operators, see [4, 9, 10, 14, 18, 22].

Applying Theorem 2.1 to EP operators, we get the following results.

PROPOSITION 3.5. *Let $A \in \mathscr{B}(\mathscr{H})$ and $B \in \mathscr{B}(\mathscr{H})$ be EP operators, and let $\alpha, \beta \in \mathbb{C} \setminus \{0\}$. Assume that $\mathscr{M} = B^{\dagger}(\mathscr{R}(A) \cap \mathscr{R}(B))$ and $\mathscr{S} = (A^*)^{\dagger}(\mathscr{R}(A) \cap \mathscr{R}(B))$. Then $\alpha A + \beta B$ is invertible if and only if the following statements hold true:*

(i) $\mathscr{N}(A) \cap \mathscr{N}(B) = \{0\}$;

(ii) $A^{\dagger}A(I - B^{\dagger}B)$ *has closed range;*

(iii) $P'_{\mathscr{S}}(\alpha AB^{\dagger}B + \beta AA^{\dagger}B)|_{\mathscr{M}} : \mathscr{M} \longrightarrow \mathscr{S}$ *is invertible.*

Proof. If $T \in \mathscr{B}(\mathscr{H})$ is an EP operator, then

$$\mathscr{R}(T) = \mathscr{R}(T^*),\ \mathscr{R}(T)^{\perp} = \mathscr{N}(T),\ TT^{\dagger} = T^{\dagger}T.$$

Now, Proposition 3.5 follows from Theorem 2.1. $\square$

In Particular, we have

COROLLARY 3.6. *Let $A \in \mathscr{B}(\mathscr{H})$ be an orthogonal projection, and let $B \in \mathscr{B}(\mathscr{H})$ be an EP operator. Assume that $C = P'_{\mathscr{R}(A)\cap\mathscr{R}(B)}(AB^{\dagger})|_{\mathscr{R}(A)\cap\mathscr{R}(B)} \in \mathscr{B}(\mathscr{R}(A) \cap \mathscr{R}(B))$, and $\rho(C)$ denotes the resolvent set of C. Then the invertibility of $\alpha A + \beta B$ is independent of the choice of α and β, where $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $-\frac{\beta}{\alpha} \in \rho(C)$.*

Proof. If A is an orthogonal projection, then the statement (iii) in Proposition 3.5 is equivalent to the fact that $C + \frac{\beta}{\alpha}I$ is an invertible operator on $\mathscr{R}(A) \cap \mathscr{R}(B)$, which implies $-\frac{\beta}{\alpha} \in \rho(C)$. This, together with Proposition 3.5, shows Corollary 3.6. $\square$

REMARK 3. In Corollary 3.6, if A and B are orthogonal projections, then $C = I_{\mathscr{R}(A)\cap\mathscr{R}(B)}$, and so $\rho(C) = \mathbb{C}\setminus\{1\}$, which implies that $-\frac{\beta}{\alpha} \in \rho(C)$ if and only if $\alpha+\beta \neq 0$. Thus Corollary 3.6 shows that the invertibility of $\alpha A+\beta B$ is independent of the choice of α and β, where $\alpha,\beta \in \mathbb{C}\setminus\{0\}$ and $\alpha+\beta \neq 0$. This coincide with [11, Theorem 1].

Finally, we give the following result, which has been obtained in [22, Theorem 2.1] when the space is finite dimensional.

PROPOSITION 3.7. *Let* $A \in \mathscr{B}(\mathscr{H})$ *and* $B \in \mathscr{B}(\mathscr{H})$ *be EP operators, and let* $\alpha,\beta \in \mathbb{C}\setminus\{0\}$. *If* $AB = 0$, *Then* $\alpha A+\beta B$ *is invertible if and only if* $\mathscr{R}(A)\cap\mathscr{R}(B) = \{0\}$ *and* $\mathscr{N}(A)\cap\mathscr{N}(B) = \{0\}$

Proof. If A and B are EP operators and $AB = 0$, it is not hard to find that $AA^\dagger = A^\dagger A$, $BB^\dagger = B^\dagger B$ and $BA = 0$. Now, Proposition 3.7 follows from Corollary 3.4. □

COROLLARY 3.8. *Let* $A \in \mathscr{B}(\mathscr{H})$ *and* $B \in \mathscr{B}(\mathscr{H})$ *be EP operators, and let* $AB = 0$. *Then the invertibility of* $\alpha A+\beta B$ *is independent of the choice of* α *and* β, *where* $\alpha,\beta \in \mathbb{C}\setminus\{0\}$.

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Guojun Hai
School of Mathematical Sciences
Inner Mongolia University
Hohhot, 010021 P. R. China
e-mail: 3695946@163.com

Chegen Bao
School of Mathematical Sciences
Inner Mongolia University
Hohhot, 010021 P. R. China
e-mail: 1196598781@qq.com

Alatancang Chen
Huhhot University for Nationalities
Hohhot, 010051 P. R. China
e-mail: altc1@imu.edu.cn