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EIGENVALUE ASYMPTOTICS FOR ZAKHAROV–SHABAT SYSTEMS WITH LONG–RANGE POTENTIALS

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Abstract. We study the spectrum of Zakharov-Shabat (ZS) systems with long-range potentials that have infinitely many purely imaginary eigenvalues accumulating at the origin. We consider $N(s)$, the number of imaginary eigenvalues with imaginary part strictly larger than s. If the potential $q(t)$ is positive and falls off like $|t|^{-\gamma}$, $0<\gamma\leqslant 1$, and satisfies some additional technical conditions, we prove that $N(s)\sim\pi^{-1}\int_{\{t:q(t)>s\}}(q(t)^2-s^2)^{1/2}\,dt$. Therefore, we have a connection with the well known phase volume integral from quantum mechanics for the number of eigenvalues less than $-s^2$ for a Schrödinger operator with potential $-q(t)^2$. However, in contrast to Schrödinger operators, a major difficulty arises from the fact that the ZS system, since it is nonselfadjoint, may have eigenvalues that are not algebraically simple. We will pay special attention to this difficulty and prove a new result (Theorem 6.6) which says that nonsimple eigenvalues do not occur if s is sufficiently small.

1. Introduction

Consider the Zakharov-Shabat system

$$v'(t,\xi)=\begin{pmatrix}-i\xi & q(t)\\ -\overline{q}(t) & i\xi\end{pmatrix}v(t,\xi),\qquad t\in\mathbb{R}, \tag{1.1}$$

where the prime denotes differentiation with respect to $t\in\mathbb{R}$, ξ is a spectral parameter, and $q(t)$ is called the potential. The overbar denotes the complex conjugate. In general, q is a complex-valued function, but in this paper we will assume it to be real. The equation (1.1) can be written as an eigenvalue problem in the form

$$(H_0+Q)v=\xi v, \tag{1.2}$$

where

$$H_0=iJ\frac{d}{dt},\qquad J=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad v=\begin{pmatrix}v_1\\v_2\end{pmatrix}, \tag{1.3}$$

and (since q is real),

$$Q=\begin{pmatrix}0&-iq\\-iq&0\end{pmatrix}. \tag{1.4}$$

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The operator H_0 is selfadjoint on $L^2(\mathbb{R},\mathbb{C}^2)$ (on its maximal domain) but Q is skew-selfadjoint. This means that the eigenvalues ξ in (1.1) may very well be complex. We mention that eigenvalues are complex numbers ξ (with $\operatorname{Im}\xi > 0$) for which (1.1) (resp.(1.2)) has a solution in $v \in L^2(\mathbb{R},\mathbb{C}^2)$. We do not discuss the continuous spectrum in any detail but only mention that under the conditions of this paper the continuous spectrum coincides with the real axis ([2, Corollary 5.5, p. 174]). In our study of the eigenvalues, we may restrict ourselves to $\mathbb{C}^+$ (the open upper complex half plane), because, since q is real, if ξ is an eigenvalue, then $\overline{\xi}$, $-\xi$, $-\overline{\xi}$ are also eigenvalues. If ψ is the eigenfunction for the eigenvalue ξ of H_0+Q, then $JU\overline{\psi}$, $JU\psi$, and $\overline{\psi}$, are the eigenfunctions corresponding to $\overline{\xi}$, $-\xi$, $-\overline{\xi}$ respectively. Here

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.5}$$

Our main concern in this paper are potentials that are not integrable at infinity and decay like $|t|^{-\gamma}$ with $0 < \gamma \leqslant 1$ as $|t| \to \infty$ (or at least have such decay towards either $+\infty$ or $-\infty$). More precise conditions will be placed on $q(t)$ below. For such potentials it is typical that the spectrum may contain an infinite number of eigenvalues on the imaginary axis. For imaginary eigenvalues, we write

$$\xi = is, \qquad s > 0 \tag{1.6}$$

and let

$$N(s) = \#\{\text{eigenvalues of (1.1) with } \operatorname{Im}\xi > s\}. \tag{1.7}$$

The main goal of this paper is to discuss the asymptotic behavior of $N(s)$ as $s \to 0$. We were motivated by the desire to generalize previous results on the number of eigenvalues obtained in [12] to classes of potentials that are not in $L^1(\mathbb{R})$. Furthermore, on perusing the literature, we came across reference [18] where the existence of solitons for slowly decreasing potentials is discussed, *assuming* that there is an infinite set of complex eigenvalues including nonimaginary ones. Another reason for looking at nonintegrable potentials is that in applications to optical fibers the integral $\int q(t)^2\,dt$ is related to the total energy in a pulse [21, p. 41]. Unfortunately, under this condition the number of eigenvalues of (1.1) will typically be infinite. In this paper, we will show that under suitable conditions

$$N(s) \sim \frac{1}{\pi}\int_{\{t:q(t)>s\}} (q(t)^2 - s^2)^{1/2}\,dt, \qquad s \downarrow 0. \tag{1.8}$$

This reminds us of the well known counting formula from quantum mechanics (see, e.g., [20, Theorem XIII.82]) in terms of the phase volume for a Schrödinger operator with potential $-q(t)^2$ and eigenvalues below $-s^2$. It is natural to ask: Why $q(t)^2$? We do not have a simple answer but wish to point out the following connection. The system (1.1) can be converted to a direct sum of the two Schrödinger operators $(-d^2/dt^2 - q(t)^2 \pm iq'(t))\psi_\pm = -s^2\psi_\pm$ (cf. [16, Section 1.5]). If $q(t) \sim t^{-\gamma}$, then $q(t)^2 \sim t^{-2\gamma}$ and $q'(t) = O(t^{-1-\gamma})$. If $0 < \gamma < 1$, the quadratic term dominates so one could use this as a heuristic argument for why (1.8) might be true. Since the perturbation $\pm iq'(t)$ is not real, it is not immediately clear why there even have to exist infinitely many real eigenvalues. The case $\gamma = 1$ also remains inconclusive. We will use two main tools to obtain our results: The Prüfer transformation and a version of Dirichlet-Neumann decoupling (or bracketing), a technique that is familiar from the spectral theory of Schrödinger operators. The Prüfer transformation will be sufficient to give us the asymptotics of $N(s)$ when $\gamma = 1$. When $0 < \gamma < 1$, the Prüfer transformation runs into difficulties and so we use a decoupling method based on the Birman-Schwinger kernel, which turns out to be selfadjoint. In the ZS case, eigenvalues may have algebraic multiplicity greater than one. In fact, this happens when eigenvalues collide, for example, as a coupling constant varies (see [10], [11], [14]). This means that we will have to pay special attention to the multiplicity of eigenvalues.

The case $\gamma = 1$ deserves special mention because it lies on the borderline between integrable and nonintegrable potentials. To be specific, consider a family of potentials of the form $q_c(t) = c(1+|t|)^{-1}$ $(c > 0)$. The ZS system with such a potential always has infinitely many imginary eigenvalues and no nonimaginary complex eigenvalues (Theorems 8.3 and 6.2). The equivalent Schrödinger operator with potential $-q_c(t)^2 \pm iq_c'(t)$ therefore also has an infinite negative spectrum. On the other hand, if we ignore the term $\pm iq_c'(t)$ and just consider the Schrödinger operator with potential $-q_c(t)^2$, then it has exactly one eigenvalue if $c \leqslant 1/2$ but infinitely many eigenvalues when $c > 1/2$. The location of eigenvalues, if there are infinitely many, is discussed in [7] and the asymptotics of the number of eigenvalues is studied in [9]. Initially, we were uncertain if a similar situation might also arise in the ZS system. However, as it turns out, at least among the potentials considered in this paper, there is no case where the strength of the potential would determine whether the imaginary spectrum is finite or infinite.

The paper is organized as follows. In Section 2, we introduce the Prüfer transformation as a tool to obtain bounds and detailed asymptotic information on the solutions of (1.1) for ξ on the imaginary axis. It will be useful to have these results available when we turn to the study of the eigenvalue asymptotics. Since our potentials are not in in L^1, some of the results that were known from prior work, for example [12], have to be treated differently. In particular, we do not make use of Jost functions in this paper. Instead, we employ estimates for the solutions among which a topological method due to Ważewski [24] (see also [6, Chap. X]) has proved very useful. In Section 3, we prove bounds for the eigenfunctions for (1.1) for arbitrary $\xi \in \mathbb{C}^+$. In Section 4, we discuss the dependence of the Prüfer angle on s and in Section 5, we summarize some results on the dependence of the Prüfer angle on a coupling constant. These results are needed to discuss the behavior of eigenvalues as a coupling constant changes (part of this discussion takes place in Section 7). In Section 6, we address the question of multiple eigenvalues and prove that, under certain conditions, such eigenvalues do not occur at all (Theorem 6.2) or at least do not occur when s is sufficiently small. We felt that the proof of Theorem 6.6 is not suitable to be included in the main body of the paper in its entirety. So we decided to split the proof into two parts and to consign the second part to Appendix A. Section 7 introduces the Birman-Schwinger kernel as a device to estimate the number of eigenvalues above a given point on the imaginary axis. A somewhat surprising feature is that it lends itself nicely to another technique

– Dirichlet-Neumann decoupling (or bracketing). This concludes our preparations for Section 8 where we will prove (1.8) for power-law potentials. First we establish a result that says that if a potential is positive at least for large t and not in L^1, then there is always an infinite number of purely imaginary eigenvalues. Then we turn to a class of Coulomb-type potentials and determine the eigenvalue asymptotics by means of the Prüfer transformation alone. The main result of Section 8 is Theorem 8.7, which we prove by means of Dirichlet-Neumann decoupling. The proof of Theorem 8.7 rests on Lemma 8.6 whose proof is given in Appendix B. An extension of Theorem 8.7 to more general potentials is obtained in Corollary 8.8 by combining Dirichlet-Neumann decoupling with the Prüfer transformation.

2. The Prüfer angle

The main concern in this paper are the purely imaginary eigenvalues of (1.1) which, as indicated in (1.6), we write as $\xi = is$, with $s > 0$. The corresponding eigenfunctions may be taken to be real-valued. This allows us to write v in the form

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \rho\cos\theta \\ \rho\sin\theta \end{pmatrix}. \tag{2.1}$$

Then $\rho = \rho(t,s)$ and $\theta = \theta(t,s)$ satisfy the first-order differential equations

$$\theta' = -q(t) - s\sin(2\theta), \tag{2.2a}$$

$$\rho' = s\rho\cos(2\theta). \tag{2.2b}$$

The transformation from (1.1) to the polar coordinate representation (2.1) is referred to as a Prüfer transformation. In the context of the ZS system (1.1), the Prüfer transformation was used previously in [12] and [13], and more recently in [5],

There are two main hypotheses that play a role in this paper. The first one is:

HYPOTHESIS 1. Suppose that $q(t)$ is continuous, not identically zero, and $\lim_{|t|\to\infty} q(t) = 0$.

The second hypothesis will be introduced in Section 8. We consider Hypothesis 1 as a core requirement under which many of the theorems are true. However, in some instances conditions can be relaxed while in others they need to be made more stringent. We will comment on such modifications throughout the paper. In particular, it is possible to accomodate local integrable singularities in many places.

We first consider solutions of (2.2a) on an interval $(-\infty, t_0]$ which satisfy an initial condition at t_0. Following the approach in [12], we write (2.2a) in the form

$$\theta' + 2s\theta = -q(t) + sf(\theta), \qquad f(\theta) = 2\theta - \sin(2\theta), \tag{2.3}$$

and then convert (2.3) into the integral equation

$$\theta(t,s) = \theta(t_0,s)e^{-2s(t-t_0)} + e^{-2st}\int_t^{t_0} e^{2s\tau}q(\tau)\,d\tau - se^{-2st}\int_t^{t_0} e^{2s\tau}f(\theta(\tau,s))\,d\tau.$$

We remark that to some extent (2.3) falls under the class of differential equations considered in [3, p. 327], since $f(\theta) = o(\theta)$ as $\theta \to 0$. For example, it follows that if the intial value $\theta(t_0, s)$ is sufficiently small, then $\theta(t,s) \to 0$ as $t \to +\infty$ ([3, Theorem 3.1, p. 327]. However, for our purpose we need much more detailed information, especially for arbitrary initial conditions and as $t \to -\infty$. In the latter case, the problem becomes one of constructing a separatrix in solution space. This will be accomplished by taking into account the special form of the function $f(\theta)$.

Suppose that $\theta(t,s)$ is bounded on $(-\infty, t_0]$. Then the improper integrals in the next expression converge and we may write

$$\theta(t,s) = e^{-2st}\left[\theta(t_0,s)e^{2st_s} + \int_{-\infty}^{t_0} e^{2s\tau}q(\tau)\,d\tau - s\int_{-\infty}^{t_0} e^{2s\tau}f(\theta(\tau,s))\,d\tau\right] \tag{2.4a}$$

$$-e^{-2st}\int_{-\infty}^{t} e^{2s\tau}q(\tau)\,d\tau + se^{-2st}\int_{-\infty}^{t} e^{2s\tau}f(\theta(\tau,s))\,d\tau. \tag{2.4b}$$

Since the first term in (2.4b) vanishes as $t \to -\infty$ and the second term is bounded, the bracketed term in (2.4a) must vanish. Thus (2.4a,b) can be viewed as a fixed point problem for the operator T_α defined by

$$(T_\alpha\theta)(t,s) \stackrel{\text{def}}{=} -e^{-2st}\int_{-\infty}^{t} e^{2s\tau}q(\tau)\,d\tau + se^{-2st}\int_{-\infty}^{t} e^{2s\tau}f(\theta(\tau,s))\,d\tau. \tag{2.5}$$

So the equation to be solved is $T_\alpha\theta = \theta$, that is

$$\theta(t,s) = -e^{-2st}\int_{-\infty}^{t} e^{2s\tau}q(\tau)\,d\tau + se^{-2st}\int_{-\infty}^{t} e^{2s\tau}f(\theta(\tau,s))\,d\tau. \tag{2.6}$$

A suitable function space for this fixed point problem is $C(I_\alpha)$, the Banach space of the bounded continuous functions on $I_\alpha = (-\infty, \alpha]$ equipped with the sup norm denoted by $\|\ \|_{I_\alpha}$. Here α is any real number but appropriate choices will be made later. We will now construct a fixed point, called $\varphi_0(t,s)$, of (2.6), that converges to 0 as $t \to -\infty$ and which is also a solution of (2.2a). This then justifies the assumption made before (2.4a,b) that $\theta(t,s)$ be bounded. Using the positivity of $f(\theta)$ for $\theta \geqslant 0$ and its monotonicity for all θ, we obtain from (2.5) the estimate

$$\|(T_\alpha\theta)(\cdot,s)\|_{I_\alpha} \leqslant \frac{\|q\|_{I_\alpha}}{2s} + \frac{1}{2}f(\|\theta(\cdot,s)\|_{I_\alpha}). \tag{2.7}$$

For any $a \geqslant 0$, define the function $g_a(z)$ by

$$g_a(z) = a + \frac{f(z)}{2}, \qquad 0 \leqslant z \leqslant \frac{\pi}{2}. \tag{2.8}$$

So the right-hand side of (2.7) is equal to $g_a(\|\theta(\cdot,s)\|_{I_\alpha})$, with

$$a = a(\alpha,s) = \frac{\|q\|_{I_\alpha}}{2s}. \tag{2.9}$$

REMARKS.

1. When $q \in L^p(\mathbb{R})$, Hölder's inequality (with $p' = p/(p-1), p \geqslant 1$) gives us

$$e^{-2st}\int_{-\infty}^{t} e^{2s\tau}|q(\tau)|\,d\tau \leqslant (2sp')^{-1/p'}\left(\int_{-\infty}^{t}|q(\tau)|^p\,d\tau\right)^{1/p} \stackrel{\text{def}}{=} a_p,,$$

so that we can use (2.8), (2.9) with a replaced by a_p.

2. Instead of the constant a we could take

$$\widetilde{a} = \left\| e^{-2st}\int_{-\infty}^{t} e^{2s\tau} q(\tau)\,d\tau \right\|_{I_\alpha}.$$

Then $\widetilde{a} < a$ provided $\|q\|_{I_\alpha} \neq 0$ because

$$e^{-2s\alpha}\int_{-\infty}^{t} e^{2s\tau}(\|q\|_{I_\alpha} - q(\tau))\,d\tau > 0, \qquad t \leqslant \alpha,$$

since $q(t) \to 0$ as $t \to -\infty$.

We will make use of Remark 2 in the proof of Lemma 2.1 below.

When $0 \leqslant a \leqslant \frac{1}{2}$, let

$$z_1 = \frac{1}{2}\arcsin(2a), \qquad z_2 = \frac{\pi}{2} - \frac{1}{2}\arcsin(2a). \tag{2.10}$$

These are the fixed points of the function $g_a(z)$. They are distinct, with $z_1 < \frac{\pi}{4} < z_2$, if $0 \leqslant a < \frac{1}{2}$, and they coincide, $z_1 = z_2 = \frac{\pi}{4}$, if $a = \frac{1}{2}$. If $z_1 \neq z_2$, the fixed point z_1 is attractive and z_2 is repelling.

The next lemma is the key to the construction of θ and some of the later results. For $r > 0$, define the ball

$$B_\alpha(r) = \{\theta \in C(I_\alpha) : \|\theta\|_{I_\alpha} \leqslant r\}.$$

LEMMA 2.1. *Fix $s > 0$ and choose α such that the constant a in (2.9) satisfies $0 < a \leqslant \frac{1}{2}$. Then the mapping T_α leaves $B_\alpha(z_2)$ invariant and has a unique fixed point, φ_0, in $B_\alpha(z_2)$ so that $\varphi_0 \in B_\alpha(z_1)$. Moreover, for every $\theta_0 \in B_\alpha(z_2)$, the sequence $\{\theta_n\}$ defined by $\theta_n = T_\alpha\theta_{n-1}$ $(n = 1, 2, \dots)$ converges to φ_0.*

Proof. We suppress the argument s and sometimes also t in this proof. Let $\theta_0 \in B_\alpha(z_2)$. Since $a > 0$, $\|q\|_{I_\alpha} > 0$, and hence $\tilde{a} < a$, where $\tilde{a}$ is defined in Remark 2 above. Thus $0 < \tilde{a} < \frac{1}{2}$. Hence

$$\|\theta_1\|_{I_\alpha} = \|T_\alpha\theta_0\|_{I_\alpha} \leqslant g_{\tilde{a}}(\|\theta_0\|_{I_\alpha}) < g_a(z_2) = z_2.$$

Now define a sequence $\{\eta_n\}$, $n = 1, 2, \dots$, by $\eta_1 = \|\theta_1\|_{I_\alpha}$ and $\eta_n = g_a(\eta_{n-1})$. Since $\eta_1 < z_2$, $\eta_n \to z_1$ monotonically as $n \to \infty$ and, by (2.7), $\|\theta_n\|_{I_\alpha} \leqslant g(\eta_{n-1}) \leqslant \eta_n$. Fix $r_0 \in (z_1, \frac{\pi}{4})$. Then there is an N such that $\|\theta_n\|_{I_\alpha} < r_0$ for $n > N$. But T_α is a contraction on $B_\alpha(r_0)$ in view of the estimate

$$|f(\theta_1(t)) - f(\theta_2(t))| \leqslant 2(1 - \cos(2r))|\theta_1(t) - \theta_2(t)|.$$

Therefore

$$\|T_\alpha\theta_1 - T_\alpha\theta_2\|_{I_\alpha} \leqslant \frac{1}{2}\|f(\theta_1)-f(\theta_2)\|_{I_\alpha} \leqslant (1-\cos(2r))\|\theta_1-\theta_2\|_{I_\alpha}.$$

This establishes the existence and uniqueness of a fixed point, called φ_0, in $B_\alpha(r_0)$. Since r_0 may be arbitrarily close to z_1, $\varphi_0 \in B_\alpha(r_0)$. □

REMARKS.

1. In Lemma 2.1, the assumption $a>0$ is required because if $a=0$, then $q(t)\equiv 0$ on I_α, in which case $\theta(t,s)\equiv 0$ and $\theta(t,s)\equiv \pi/2$ are two fixed points. Remember that z is restricted to $[0,\pi/2]$.

2. The condition $a \leqslant \frac{1}{2}$ can be achieved by choosing α sufficiently small.

3. Any solution of (2.2a) satisfying $\limsup\limits_{t\to-\infty}|\theta(t,s)| < \frac{\pi}{2}$ must coincide with $\varphi_0(t,s)$ because $\theta(t,s)$ will be in some $B_\alpha(r)$ with $r<z_2<\frac{\pi}{2}$, provided α is sufficiently negative. Note that, by (2.10), $z_2\to\pi/2$ as $\alpha\to-\infty$ $(a\to 0)$.

We recall a comparison theorem ([23, Theorem IX and its Corollary, p. 96]) that will be useful in the sequel.

For a given differential equation $y'=f(t,y)$ where f satisfies a local Lipschitz condition in y and an absolutely continuous function h, let Ph denote its defect, that is, $Ph = h' - f(t,h)$.

LEMMA 2.2. *Under the above assumptions the following are true.*

(i) *Let ϕ,ψ be absolutely continuous in $J_+=[t_0,t_0+c]$ $(c>0)$ and suppose that $\phi(t_0)\leqslant\psi(t_0)$ and $P\phi\leqslant P\psi$ a.e. in J_+. Then $\phi<\psi$ in J_+ or there is a $t_1\in J_+$ such that $\phi=\psi$ in $[t_0,t_1]$, $\phi<\psi$ in $(t_1,t_0+c]$.*

(ii) *Let $J_-=[t_0-c,t_0]$ $(c>0)$ and suppose that $\phi(t_0)\leqslant\psi(t_0)$ and $P\phi\geqslant P\psi$ a.e. in J_-. Then $\phi<\psi$ in J_- or there is a $t_2\in J_-$ such that $\phi=\psi$ in $[t_2,t_0]$, $\phi<\psi$ in $[t_0-c,t_2]$.*

Define

$$q_+(t)=\max\{q(t),0\}, \qquad q_-(t)=\max\{-q(t),0\},$$

so that $q=q_+-q_-$.

LEMMA 2.3. *Suppose q satisfies Hypothesis 1. Let $0<\delta<\frac{\pi}{2}$ and fix $s>0$. Then every solution $\theta(t,s)$ of (2.2a) with the property that $\delta<\theta(t_n,s)<\pi-\delta$ for an infinite sequence $t_0>t_1>\dots$ tending to $-\infty$ must converge to $\frac{\pi}{2}$ as $t\to-\infty$.*

Proof. Set $\chi(t,s)=\theta(t,s)-\frac{\pi}{2}$. Then $\chi(t,s)$ satisfies the differential equation

$$\chi' = -q(t)+s\sin(2\chi). \tag{2.11}$$

Fix $\delta\in(0,\frac{\pi}{2})$ and let

$$\eta_\delta = \frac{2\sin(\delta)}{\pi-\delta}.$$

Then

$$|\sin(2\chi)| \geqslant \eta_\delta |\chi| \quad \text{if} \quad |\chi| \leqslant \frac{\pi}{2} - \frac{\delta}{2}. \tag{2.12}$$

In addition to (2.11) we consider the two initial value problems

$$\chi_\pm{}' = \mp q_\pm(t) + s\eta_\delta \chi_\pm, \qquad \chi_\pm(T,s) = \pm\frac{\pi}{2} \mp \delta, \tag{2.13}$$

on $(-\infty, T]$, where T will be chosen later. The solutions to (2.13) are

$$\chi_\pm(t,s) = \chi_\pm(T,s)\, e^{\eta_\delta s(t-T)} \pm e^{\eta_\delta st} \int_t^T e^{-\eta_\delta s\tau} q_\pm(\tau)\, d\tau, \qquad t \leqslant T. \tag{2.14}$$

Clearly, $\chi_+(t,s)$ is positive, $\chi_-(t,s)$ is negative, and both solutions approach 0 as $t \to -\infty$. Now we pick T so that

$$\sup_{t \leqslant T} \left(e^{\eta_\delta st} \int_t^T e^{-\eta_\delta s\tau} q_\pm(\tau)\, d\tau \right) < \frac{\delta}{2} \tag{2.15}$$

(for both q_+ and q_-). This can certainly be achieved, since the term in parentheses is bounded by $(\eta_\delta s)^{-1} \sup\limits_{t \leqslant T}[q_\pm(t)]$. In view of the initial conditions for $\chi_\pm(t,s)$ at $t = T$ in (2.13), it follows from (2.14), (2.15), and the triangle inequality, that

$$\begin{aligned} \chi_+(t,s) &< \frac{\pi}{2} - \frac{\delta}{2}, \qquad t \leqslant T, \\ \chi_-(t,s) &> -\frac{\pi}{2} + \frac{\delta}{2}, \qquad t \leqslant T. \end{aligned} \tag{2.16}$$

Thus both χ_+ and χ_- satisfy the inequalities in (2.12) for $t \leqslant T$. Now let Ph be the defect of a function h with respect to the differential equation (2.13). Then

$$P\chi = 0 = P\chi_+ + s(\sin(2\chi_+) - \eta_\delta\, \chi_+) + q_-(t) \geqslant P\chi_+.$$

Similarly, $P\chi \leqslant P\chi_-$. Hence, by Lemma 2.2(ii), if $\chi(t,s)$ is a solution of (2.11) with $|\chi(T,s)| \leqslant \frac{\pi}{2} - \delta$, then $\chi_-(t,s) \leqslant \chi(t,s) \leqslant \chi_+(t,s)$ for $t \leqslant T$ and $\chi_\pm(t,s) \to 0$ by (2.14). Hence $\chi(t,s) \to 0$. Finally, let $\theta(t,s)$ be a solution of (2.2a) satisfying the assumptions of the lemma. Then there is an n such that (2.15) holds with $T = t_n$ and $|\chi(t_n,s) = |\theta(t_n,s) - \frac{\pi}{2}| < \frac{\pi}{2} - \delta$. So the inequalities in (2.12) are satisfied at t_n and thus $\chi(t,s) \to 0$. Hence $\theta(t,s) \to \frac{\pi}{2}$, proving the lemma. □

REMARK. If $q \in L^p(\mathbb{R})$, $p \geqslant 1$, then (2.15) can be replaced by

$$(p'\eta_\delta s)^{-1/p'} \left(\int_{-\infty}^T |q_\pm(t)|^p\, dt \right)^{1/p} < \frac{\delta}{2}$$

and Lemma 2.3 still holds.

For every $\beta \in \mathbb{R}$, let $J_\beta = [\beta, \infty)$, and let the sup norm on $C(J_\beta)$ be denoted by $\|\ \|_{J_\beta}$.

THEOREM 2.4. *Suppose q satisfies Hypothesis 1. Then*

(i) *For each $k \in \mathbb{Z}$, there is a unique solution $\varphi_k(t,s)$ converging to $k\pi$ as $t \to -\infty$, and $\varphi_k(t,s) = \varphi_0(t,s) + k\pi$. The solution $\varphi_0(t,s)$ obeys*

$$-\frac{1}{2}\arcsin\left(\frac{\|q_+\|_{I_t}}{s}\right) \leqslant \varphi_0(t,s) \leqslant \frac{1}{2}\arcsin\left(\frac{\|q_-\|_{I_t}}{s}\right), \tag{2.17}$$

provided $\|q\|_{I_t} \leqslant s$. Both inequalities are strict unless $\|q_+\|_{I_t} = \|q_-\|_{I_t} = 0$.

(ii) *Suppose $\varphi(t,s)$ is a solution not identical to one of the solutions $\varphi_k(t,s)$. Then, as $t \to -\infty$, $\varphi(t,s)$ converges to $m\pi/2$ for some odd integer m; more precisely, if $\varphi_k(t_0,s) < \varphi(t_0,s) < \varphi_{k+1}(t_0,s)$ for some t_0, then $m = 2k+1$.*

(iii) *For each $k \in \mathbb{Z}$, there is a unique solution $\psi_k(t,s)$ converging to $(2k-1)\pi/2$ as $t \to +\infty$ and $\psi_k(t,s) = \psi_0(t,s) + k\pi$. The solution $\psi_0(t,s)$ obeys*

$$-\frac{1}{2}\arcsin\left(\frac{\|q_-\|_{J_t}}{s}\right) \leqslant \psi_0(t,s) + \frac{\pi}{2} \leqslant \frac{1}{2}\arcsin\left(\frac{\|q_+\|_{J_t}}{s}\right), \tag{2.18}$$

provided $\|q\|_{J_t} \leqslant s$. The inequalities are strict unless $\|q_+\|_{J_t} = \|q_-\|_{J_t} = 0$.

(iv) *Any solution not identical to one of the solutions $\psi_k(t,s)$ converges to $m\pi$ for some $m \in \mathbb{Z}$. If $\psi_k(t_0,s) < \psi(t_0,s) < \psi_{k+1}(t_0,s)$ for some t_0, then $m = k$.*

REMARKS.

1. Note that these solutions exist with the required asymptotic properties only when $s > 0$. At $s = 0$, the solution of (2.2a) is $\theta(t,0) = \theta(t_0,0) - \int_{t_0}^t q(\tau)\,d\tau$, which in general is not bounded at either $\pm\infty$, unless q is conditionally integrable there.

2. The solution $\psi_0(t,s)$ satisfies an integral equation similar to that for $\varphi_0(t,s)$ (see (2.5)). Let $\theta_0(t,s) = \frac{\pi}{2} + \psi_0(t,s)$. Then

$$\theta_0(t,s) = e^{2st}\int_t^\infty e^{-2s\tau}q(\tau)\,d\tau + e^{2st}\int_t^\infty e^{-2s\tau}f(\theta_0(\tau,s))\,d\tau. \tag{2.19}$$

This is also a fixed point problem. By reasoning as in the case of $\varphi_0(t,s)$, we obtain the bound

$$|\theta_0(t,s)| \leqslant \frac{1}{2}\arcsin\left(\frac{\|q\|_{J_t}}{s}\right), \qquad t \geqslant \omega(s), \tag{2.20}$$

in agreement with (2.18).

3. From Theorem 2.4, it follows that every solution of (2.2a) is bounded on the whole line and has limits as $t \pm \infty$.

Proof of Theorem 2.4. In (2.5), (2.6) replace q by q_+ (resp. q_-) and denote the corresponding operators and solutions by $T_\alpha^\pm$, θ^+ (resp. θ^-) and the iterates by $\theta_n(t,s)$, $\theta_n^\pm(t,s)$. Since $f(\theta)$ is monotone and maps positive (negative) functions to positive (negative) functions, we have that $\theta_n^+(t,s) \leqslant \theta_n(t,s) \leqslant \theta_n^-(t,s)$. Taking $n \to \infty$ and denoting the limits by $\varphi_0(t,s)$, $\varphi_0^\pm(t,s)$ gives $\varphi_0^+(t,s) \leqslant \varphi_0(t,s) \leqslant \varphi_0^-(t,s)$. The inequalities in (2.17) follow from Lemma 2.2 applied to $q_\pm(t)$, since the solutions $\varphi_0^\pm$ correspond to the fixed points of the mappings $T_\alpha^\pm$ associated with $q_\pm$.

That $\varphi_k(t,s) = \varphi_0(t,s) + k\pi$ is a consequence of the π-periodicity of the term $\sin(2\theta)$ in (2.2a). That the inequalities in (2.17) are strict follows from the fact that

$\tilde{a} < a$ (see Remark 2 below (2.9)); note that $\tilde{a} < a$ implies $z_1(\tilde{a}) < z_1(a)$. If $\|q_-\|_{I_t} = 0$ but $\|q_+\|_{I_t} \neq 0$ (or vice versa), then $\varphi_0(t,s) < 0$ (or $\varphi_0(t,s) > 0$), so both inequalities are strict.

In (ii) it suffices to consider the case $m = 1$. If

$$\varphi_0(t_0,s) < \varphi(t_0,s) < \varphi_1(t_0,s) = \varphi_0(t_0,s) + \pi,$$

then $\varphi(t,s)$ does not converge to 0 or π as $t \to -\infty$ by the uniqueness of $\varphi_0(t,s)$ and $\varphi_1(t,s)$. Hence there is a $\delta > 0$ and a sequence $\{t_n\}$ as required by Lemma 2.3. Thus $\varphi(t,s)$ must converge to $\pi/2$ as $t \to -\infty$.

The proofs of (iii) and (iv) are immediate if we define $\chi(t,s) = \theta(-t,s) - \frac{\pi}{2}$. Then

$$\chi' = q(-t) - s\sin(2\chi).$$

Set $y_0(t,s) = \widehat{\varphi}_0(-t,s) - \frac{\pi}{2}$, where $\widehat{\varphi}_0(t,s)$ is the solution constructed in Lemma 2.1, but with $q(t)$ replaced by $\widehat{q}(t) = -q(-t)$. Thus (2.18), together with the remaining assertion follow as in (i) and (ii). $\square$

We have included the next result because we believe it is nontrivial and gives the best possible answer for the asymptotics of $\varphi_0(t,s)$ as $t \to -\infty$ (or $\psi_0(t,s)$ as $t \to +\infty$) that we could think of under the given assumptions.

THEOREM 2.5. *Suppose that Hypothesis 1 holds and that $q(t) \geqslant 0$. Then*

$$\varphi_0(t,s) = -p(t,s)\,(1+o(1)), \qquad t \to -\infty, \tag{2.21}$$

where

$$p(t,s) = e^{-2st}\int_{-\infty}^{t} e^{2s\tau}q(\tau)\,d\tau.$$

Similarly,

$$\psi_0(t,s) = -\frac{\pi}{2} + \widetilde{p}(t,s)\,(1+o(1)), \qquad t \to +\infty, \tag{2.22}$$

where

$$\widetilde{p}(t,s) = e^{2st}\int_{t}^{\infty} e^{-2s\tau}q(\tau)\,d\tau.$$

Proof. First, if $q(t) \equiv 0$ on some interval $(-\infty, t_0)$, then (2.21) is obviously true. If the restriction of $q(t)$ to $(-\infty, 0]$ does not have compact support, then $p(t,s) > 0$ for all t. From (2.6) and by iteration, it is clear that

$$\varphi_0(t,s) < -p(t,s).$$

We only need this inequality for large negative t but it actually holds for all $t \in \mathbb{R}$ by a comparison argument, since $p(t,s)$ obeys the differential equation $p' = -2sp + q(t)$. It now suffices to show that for every $\varepsilon > 0$, we can find a t_ε such that

$$\varphi_0(t,s) \geqslant -(1+\varepsilon)\,p(t,s), \qquad t \leqslant t_\varepsilon.$$

This does not seem to follow easily from the integral equation unless one makes the stronger assumption that $q(t)$ is monotone. The method we use is due to Ważewski [24] and a lucid account can be found in [6, Chap. X]. The idea is to delimit a region Ω_ε in the (t,θ) plane of the form $\Omega_\varepsilon = \{(t,\theta) : t \leqslant t_\varepsilon; g_1(t) \leqslant \theta \leqslant g_2(t)\}$ so that the curves $g_1(t)$ and $g_2(t)$ consist of strict egress points for (2.2a). Then we can conclude that there exists a solution $\theta(t)$ such that $g_1(t) < \theta(t,s) < g_2(t)$ for $t < t_\varepsilon$. In our application, we will have $g_1(t) = -\arctan(\kappa p(t,s))$ and $g_2(t) = \delta$, where $\delta > 0$ may be chosen arbitrarily small. Let

$$u(t,\theta) = \sin\theta + \kappa p(t,s)\cos\theta,$$

where $\theta = \theta(t,s)$ is any solution of (2.2a). So $u(t,\theta) = 0$ when $\theta(t) = g_1(t)$. Computing the derivative of $u(t,\theta(t))$ when $u(t,\theta) = 0$ (so $\sin\theta = -\kappa p(t,s)\cos\theta$) we obtain

$$\frac{d}{dt}u(t,\theta(t)) = q(t)(\kappa - 1 - \kappa^2 p(t,s)^2)\cos\theta. \tag{2.23}$$

Now we pick any $\kappa > 1$ and notice that by choosing t sufficiently negative, we can make the term in parentheses positive. Thus we set $\kappa = 1+\varepsilon$ and define t_ε such that

$$\frac{\|q\|_{I_{t_\varepsilon}}}{2s} = \frac{\sqrt{\varepsilon}}{1+\varepsilon}. \tag{2.24}$$

Since $p(t,s) < \frac{\|q\|_{I_t}}{2s}$, t_ε has the property that for $t \leqslant t_\varepsilon$ the term in parentheses in (2.23) is positive. Since $u(t,\theta) > 0$ inside Ω_ε, a positive t-derivative of $u(t,\theta(t))$ on $\theta = g_1(t)$ means that for some $\varepsilon > 0$ and $t - \varepsilon < \tau < t$, $(\tau, \theta(\tau,s)) \notin \Omega_\varepsilon$. Thus $(t, g_1(t))$ is a strict egress point. The points on the line $\theta = g_2(t) = \delta$ are also strict egress points, since the right-hand side of (2.2a) is negative. By a variant of Theorem 2.1 (see also Theorem 3.1 and Corollary 3.1 in [6, Chap. X]), there exists a solution that is contained in Ω_ε for all $t < t_\varepsilon$ and which therefore must coincide with the solution $\varphi_0(t,s)$. The inequality $u(t,\theta) > 0$ translates into $\tan\theta \geqslant -(1+\varepsilon)p(t,s)$, hence

$$\varphi_0(t,s) \geqslant -(1+\varepsilon)p(t,s). \tag{2.25}$$

In view of (2.25), the first part of the theorem (eq.(2.21)) is proved. The proof of (2.22) is similar. □

3. Exponentially decaying solutions

An immediate consequence of Theorem 2.4 is the existence of a unique solution (up to constant multiples) of (1.1) for $\xi = is$ on the positive imaginary axis that decays exponentially as $t \to -\infty$. This becomes immediately clear if we integrate (2.2b) with $\theta = \varphi_0$ and an initial value $\rho(t_0,s)$ at $t = t_0$. We obtain

$$\rho(t,s) = \rho(t_0,s)e^{-s\int_t^{t_0}\cos(2\varphi_0(\tau,s))\,d\tau}. \tag{3.1}$$

The right-hand side goes to zero exponentially as $t \to -\infty$, since $\varphi_0(t,s) \to 0$. Similarly, there is a unique solution of (1.1) for ξ on the positive imaginary axis that decays as $t \to +\infty$. We only need to choose $\psi_0(t,s)$ instead of $\varphi_0(t,s)$. Then

$$\rho(t,s) = \rho(t_0,s)e^{s\int_{t_0}^{t}\cos(2\psi_0(\tau,s))\,d\tau}. \tag{3.2}$$

The right-hand side goes to zero since $\psi_0(t,s) \to -\frac{\pi}{2}$ as $t \to +\infty$.

By using the bounds (2.17) and (2.18), we can estimate the exponents in (3.1) and (3.2). Let $v^+(t,is)$ ($v^-(t,is)$) denote a solution that decays as $t \to +\infty$ ($t \to -\infty$).

THEOREM 3.1. *Under Hypothesis 1, the exponentially decaying solutions obey*
(i) $\|v^+(t,is)\| \leqslant \|v^+(t_0,is)\|\exp(-\int_{t_0}^{t}(s^2-\|q\|_{J_\tau}^2)^{1/2}\,d\tau$ *for* $t \geqslant t_0$ *where* t_0 *is such that* $\|q\|_{J_{t_0}} \leqslant s$,
(ii) $\|v^-(t,is)\| \leqslant \|v^-(t_0,is)\|\exp(-\int_{t}^{t_0}(s^2-\|q\|_{I_\tau}^2)^{1/2}\,d\tau$ *for* $t \leqslant t_0$ *where* t_0 *is such that* $\|q\|_{I_{t_0}} \leqslant s$.

The following bounds are also elementary (but nevertheless useful). They apply when $\xi \in \mathbb{C}^+$ to any solution $v(t,\xi)$ of (1.1).

THEOREM 3.2. *Suppose that Hypothesis 1 holds and let* $\xi = \alpha + i\beta$, *with* $\alpha \in \mathbb{R}$, $\beta > 0$. *Then, for any* $t_0 \in \mathbb{R}$,
(i) $\|v(t,\xi)\| \geqslant \|v(t_0,\xi)\|e^{-\beta(t-t_0)}$, $t \geqslant t_0$,
(ii) $\|v(t,\xi)\| \leqslant \|v(t_0,\xi)\|e^{\beta(t_0-t)}$, $t \leqslant t_0$.

Proof. If $A(\xi)$ is the matrix in (1.1), then $A(\xi)+A(\xi)^*$ has eigenvalues $\pm 2\beta$. From this the inequalities follow by integration (see [6, Chap. IV, Lemma 4.2]. Alternatively, we can use the identity $(|v_1|^2+|v_2|^2)' = 2\beta(|v_1|^2-|v_2|^2)$, estimate the right-hand side in an obvious way, and then integrate. □

We briefly discuss the construction of exponentially decaying solutions for arbitrary complex ξ with $\operatorname{Im}\xi = \beta > 0$. A solution having initial values $v_1(t_0,\xi)$ and $v_2(t_0,\xi)$ at $t = t_0$ satisfies the following system of integral equations (we temporarily drop the argument ξ from the notation)

$$\begin{aligned} v_1(t) &= e^{\beta(t-t_0)}v_1(t_0) - e^{\beta t}\int_t^{t_0} e^{-\beta\tau}q(\tau)v_2(\tau)\,d\tau, \\ v_2(t) &= v_2(t_0)e^{-\beta(t-t_0)} + e^{-\beta t}\int_t^{t_0} e^{\beta\tau}q(\tau)v_1(\tau)\,d\tau. \end{aligned} \tag{3.3}$$

Suppose that $v(t)$ is bounded on I_{t_0}. Then we may write the equation for v_2 as

$$v_2(t) = e^{-\beta t}\left(v_2(t_0)e^{\beta t_0} + \int_{-\infty}^{t_0} e^{\beta\tau}q(\tau)v_1(\tau)\,d\tau\right) - e^{-\beta t}\int_{-\infty}^{t} e^{\beta\tau}q(\tau)v_1(\tau)\,d\tau.$$

Since the second term on the right is bounded (and actually converges to zero as $t \to -\infty$), the term in parentheses must be zero. Therefore, v_2 satisfies

$$v_2(t) = -e^{-\beta t}\int_{-\infty}^{t} e^{\beta\tau}q(\tau)v_1(\tau)\,d\tau.$$

Choose $0 \leqslant \delta < \beta$ and set $w_1(t) = e^{-\delta t} v_1(t),\ w_2(t) = e^{-\delta t} v_2(t)$.

Then, for suitable t_0, the mapping $\mathscr{T}_{t_0} : C(I_{t_0})^2 \to C(I_{t_0})^2$ defined by

$$(\mathscr{T}_{t_0} w)(t) = \begin{pmatrix} (\mathscr{T}_{t_0} w)_1(t) \\ (\mathscr{T}_{t_0} w)_2(t) \end{pmatrix} = \begin{pmatrix} e^{(\beta-\delta)t} v_1(t_0) - e^{(\beta-\delta)t} \int_t^{t_0} e^{-(\beta-\delta)\tau} q(\tau) w_2(\tau) d\tau \\ -e^{-(\beta+\delta)t} \int_{-\infty}^t e^{(\beta+\delta)\tau} q(\tau) w_1(\tau) d\tau \end{pmatrix}$$

is a contraction. Here the norm is $\|(w_1, w_2)\|_{I_{t_0}} = \|w_1\|_{I_{t_0}} + \|w_2\|_{I_{t_0}}$. To see this, write

$$\mathscr{T}_{t_0} w^{(1)} - \mathscr{T}_{t_0} w^{(2)} = \begin{pmatrix} -e^{(\beta-\delta)t} \int_t^{t_0} e^{-(\beta-\delta)\tau} q(\tau) [w_2^{(1)}(\tau) - w_2^{(2)}(\tau)]\, d\tau \\ -e^{-(\beta+\delta)t} \int_{-\infty}^t e^{(\beta+\delta)\tau} q(\tau) [w_1^{(1)}(\tau) - w_1^{(2)}(\tau)]\, d\tau \end{pmatrix}.$$

Therefore

$$\begin{aligned} \|\mathscr{T}_{t_0} w^{(1)} - \mathscr{T}_{t_0} w^{(2)}\|_{I_{t_0}} &\leqslant \|q\|_{I_{t_0}} \left(\frac{\|w_2^{(1)} - w_2^{(2)}\|_{I_{t_0}}}{s - \beta} + \frac{\|w_1^{(1)} - w_1^{(2)}\|_{I_{t_0}}}{\beta + \delta} \right) \\ &\leqslant \frac{\|q\|_{I_{t_0}}}{\beta - \delta} \|w^{(1)} - w^{(2)}\|_{I_{t_0}}. \end{aligned}$$

This is a contraction provided $\dfrac{\|q\|_{I_{t_0}}}{\beta - \delta} < 1$, that is, if t_0 is negative and large in absolute value. The fixed point defines a unique solution of (3.3) (and thus (1.1)) which satisfies $\|v(t,\xi)\| = O(e^{\delta t})$ as $t \to -\infty$. This method also shows that $w(t,\xi)$ is analytic for $\xi \in \mathbb{C}^+$ because all the iterates are. Solutions decaying as $t \to -\infty$ are constructed in a similar way.

We provide the next theorem to show that the method of Waszewski [24] can also be applied here and yields a nontrivial bound for the decaying solutions.

THEOREM 3.3. *Suppose q satisfies Hypothesis 1. Let ξ be any complex number with $\operatorname{Im}\xi = \beta > 0$. Then (1.1) has solutions $v^{\pm}(t,\xi)$ that are unique up to constant multiples and satisfy the following inequalities:*

(i) *For every $\varepsilon \in (0,1)$, there is a t_ε such that for $t \leqslant t_\varepsilon$,*

$$|v_1^-(t,\xi)| \leqslant |v_1^-(t_\varepsilon,\xi)| e^{(1-\varepsilon)\beta(t-t_\varepsilon)}, \tag{3.4a}$$

$$|v_2^-(t,\xi)| \leqslant (1+\varepsilon) p(t,\beta) |v_1^-(t,\xi)|, \tag{3.4b}$$

where

$$p(t,\beta) = e^{-2\beta t} \int_{-\infty}^t e^{2\beta\tau} |q(\tau)|\, d\tau.$$

Furthermore, $\|q\|_{I_{t_\varepsilon}}/\beta \to 1$ as $\varepsilon \to 1$.

(ii) *For every $\varepsilon > 0$, there is a t_ε such that for $t \geqslant t_\varepsilon$,*

$$|v_1^+(t,\xi)| \leqslant (1+\varepsilon) \tilde{p}(t,\beta) |v_2^+(t,\xi)|, \tag{3.5a}$$

$$|v_2^+(t,\xi)| \leqslant |v_2^+(t_\varepsilon,\xi)| e^{-(1-\varepsilon)\beta(t-t_\varepsilon)}, \tag{3.5b}$$

where

$$\tilde{p}(t,\beta) = e^{2\beta t}\int_t^{\infty} e^{-2\beta\tau}|q(\tau)|\,d\tau.$$

Furthermore, $\|q\|_{J_{t_\varepsilon}}/\beta \to 1$ *as* $\varepsilon \to 1$.

Proof. Consider (3.4a,b). We suppress the superscript $(-)$ on v_1 and v_2 and the arguments t and ξ. We already have established the existence of a solution that goes to zero towards $-\infty$ by the fixed point method. So we need only consider the bounds in (3.4a) and (3.4b). In [6], the limit $t \to +\infty$ is considered. Instead of making the substitution $t \to -t$, we have written the proof so that it applies directly to our situation. For any $\kappa > 0$, define

$$u = |v_2|^2 - \kappa^2 p^2 |v_1|^2. \tag{3.6}$$

We need to show that $u' < 0$ when $u = 0$. This will imply that $u < 0$, which in turn leads to (3.4b). Note that $p' = -2\beta p + |q|$. Using this, together with (3.6) and (1.1), we obtain

$$\begin{aligned} u' &= -2\beta|v_2|^2 + 2\beta\kappa^2 p^2|v_1|^2 - 2\kappa^2 p|q||v_1|^2 - 2\kappa^2 p^2 q\,\mathrm{Re}(\overline{v}_1 v_2) - 2q\,\mathrm{Re}(\overline{v}_1 v_2) \\ &\leqslant -2\beta|v_2|^2 + 2\beta\kappa^2 p^2|v_1|^2 - 2\kappa^2 p|q||v_1|^2 + 2\kappa^2 p^2|q||v_1||v_2| + 2|q||v_1||v_2| \\ &= 2\kappa p|q|(1-\kappa+\kappa^2 p^2)|v_1|^2, \end{aligned} \tag{3.7}$$

where (3.6) when $u = 0$ was used in the last step. The right-hand side of (3.7) can be made negative for sufficiently large (negative) t by choosing $\kappa > 1$. The argument is now similar to that in the proof of Theorem 2.5. We again set $\kappa = 1 + \varepsilon$. Then (3.4b) follows from (3.6) since $u < 0$ in the region of interest. We choose t_ε so that

$$\frac{\|q\|_{I_{t_\varepsilon}}}{2\beta} = \frac{\sqrt{\varepsilon}}{\sqrt{2}\sqrt{1+\varepsilon}}. \tag{3.8}$$

This ensures that the term in parentheses in (3.7) is negative when $t \leqslant t_\varepsilon$. The reason why this is different from the choice in (2.24) is that we have to go one more step and estimate v_1. Using (3.8), we obtain

$$\kappa p|q| \leqslant \frac{\kappa\|q\|^2_{I_{t_\varepsilon}}}{2\beta} = \beta\varepsilon.$$

Therefore, for $t \leqslant t_\varepsilon$,

$$(|v_1|^2)' = 2\mathrm{Re}(-i\xi|v_1|^2 + q\overline{v}_1 v_2) \geqslant 2(\beta - \kappa p|q|)|v_1|^2 \geqslant 2\beta(1-\varepsilon)|v_1|^2.$$

Now an integration leads to (3.4b). As $\varepsilon \to 1$, the right-hand side of (3.8) goes to $1/2$. Part (i) is now proved. The proof of (3.5a,b) is similar. $\square$

4. The Prüfer angle as a function of s

We also need to know how the solutions $\varphi_k(t,s)$ and $\psi_k(t,s)$ behave as $s\to\infty$. More generally, we can ask this question about any solution $\theta(t,s)$ satisfying an arbitrary s-independent initial condition at some point t_0. This will be applied to ZS systems on a finite interval with suitable boundary conditions at the endpoints.

THEOREM 4.1. *Suppose q satisfies Hypothesis 1. Then the following are true.*

(i) *For every $k\in\mathbb{Z}$, $\varphi_k(t,s)\to k\pi$ as $s\to\infty$ uniformly on $(-\infty,\infty)$.*

(ii) *For every $k\in\mathbb{Z}$, $\psi_k(t,s)\to(2k-1)\pi/2$ as $s\to\infty$ uniformly on $(-\infty,\infty)$.*

(iii) *Given $t_0\in\mathbb{R}$, suppose $\theta(t_0,s)=\theta_0\in(0,\pi)\setminus\{\frac{\pi}{2}\}$ is independent of s. Let $T<t_0$. Then $\theta(t,s)\to\frac{\pi}{2}$ as $s\to\infty$ uniformly on $(-\infty,T]$. If $\theta_0=\frac{\pi}{2}$, then $\theta(t,s)\to\frac{\pi}{2}$ uniformly on $(-\infty,t_0]$.*

If $\theta_0\in(0,\frac{\pi}{2})$ and $T>t_0$, then $\theta(t,s)\to 0$ as $s\to\infty$ uniformly on $[T,\infty)$. If $\theta_0\in(\frac{\pi}{2},\pi)$ and $T>t_0$, then $\theta(t,s)\to\pi$ as $s\to\infty$ uniformly on $[T,\infty)$.

(iv) *Special cases:* (a) *Suppose $\theta_0=\frac{\pi}{2}$, $q(t_0)>0$ (resp. $q(t_0)<0$), or $q(t_0)=0$ and $q(t)>0$ (resp. $q(t)<0$) on some interval $(t_0,t_1]$. Let $T>t_0$. Then $\theta(t,s)$ converges to 0 (resp. π) uniformly on $[T,\infty)$.*

(b) *Suppose $\theta_0=0$. Then $\theta(t,s)$ converges to 0 uniformly on $[t_0,\infty)$. Suppose $\theta_0=0$ and $q(t_0)>0$ (resp. $q(t_0)<0$), or that $q(t_0)=0$ and $q(t)>0$ (resp. $q(t)<0$) on some interval $[t_1,t_0)$. Let $T<t_0$. Then $\theta(t,s)$ converges to $\frac{\pi}{2}$ (resp. $-\frac{\pi}{2}$) uniformly on $(-\infty,T]$.*

The above results do not cover every imaginable case. For example, potentials behaving like $q(t)=t\sin(1/t)$ near $t_0=0$ are not included.

Proof. (i) and (ii). It suffices to consider $\varphi_0(t,s)$. Assume that s is so large that $\|q\|_{\mathbb{R}}\leqslant s$ (here $\|q\|_{\mathbb{R}}=\sup_{t\in\mathbb{R}}|q(t)|$). Then the assertion follows from (2.17). The proof for $\psi_k(t,s)$ is similar.

Now consider an arbitrary solution $\theta(t,s)$ satisfying $\theta(t_0,s)=\theta_0\in(0,\pi)\setminus\{\frac{\pi}{2}\}$. For large s, by part (i), we have that $\varphi_0(t_0,s)<\theta_0<\varphi_1(t_0,s)$. Thus $\theta(t,s)$ converges to $\frac{\pi}{2}$ as $t\to-\infty$ by Theorem 2.4 (ii). As in the proof of Lemma 2.3, let $\chi(t,s)=\theta(t,s)-\frac{\pi}{2}\to 0$ and set $\delta=\min\{\theta_0,\pi-\theta_0\}$. Then, for large enough s, the inequalities in (2.15) and (2.16) are satisfied for this δ and for all $t\leqslant t_0$ (with $T=t_0$). Thus the upper and lower solutions corresponding to those in (2.14) can be constructed. Since the right-hand side of (2.14) tends to zero as $s\to\infty$, $\lim_{s\to\infty}\chi(t,s)=0$. Convergence is uniform on every interval $(-\infty,T]$, with $T<t_0$, but not uniform on $(-\infty,t_0]$. This can be seen from the first term on the right-hand side of (2.14). An exception occurs when $\theta(t_0,s)=\frac{\pi}{2}$. Then we can choose the initial values for $\chi_\pm(t_0,s)$ in (2.13) to be arbitrarily small. Choosing s sufficiently large thus makes the right-hand side of (2.14) as small as we please, uniformly in t for $t\leqslant t_0$. This proves the first assertion in (iii).

For the remaining assertions in (iii), similar upper/lower solution arguments can be found; the details are omitted. It remains to consider the special cases in (iv).

Consider the case $\theta_0=\pi/2$, $q(t_0)=0$, and $q(t)>0$ on $(t_0,t_1]$; the case with $q(t_0)>0$ is included in the proof. Since we may always take a smaller T, we may assume without loss that T is so close to t_0 that

$$t_0 < T < t_1 \qquad \text{and} \qquad \delta \stackrel{\text{def}}{=} \int_{t_0}^{T} q(\tau)\,d\tau < \frac{\pi}{2}.$$

Now consider $\theta(t,s)$ as long as it is between $\pi/2-\delta$ and $\pi/2$. Clearly $\pi/2 > \theta(t,0) \geqslant \pi/2-\delta$ for $t_0 < t \leqslant T$ and $\theta(T,0) = \pi/2-\delta$. Since

$$\sin(2\theta) > \frac{2\sin\delta}{\delta}\left(\frac{\pi}{2}-\theta\right), \qquad \theta \in \left(\frac{\pi}{2}-\frac{\delta}{2}, \frac{\pi}{2}\right),$$

a simple comparison argument shows that for every $s>0$ there is a unique $t = t_\delta(s)$ such that $\theta(t_\delta(s),s) = \pi/2-\delta$. Moreover, $t_\delta(s) \to t_0$ monotonically as $s \to \infty$. At $s=0$, $t_\delta(0) = T$. Now we pick $s = s_\delta$ so large that

$$\sup_{t \geqslant t_0} \left(e^{-\eta_\delta s_\delta t} \int_{t_0}^{t} e^{\eta_\delta s_\delta \tau} q_\pm(\tau)\,d\tau \right) < \frac{\delta}{2}.$$

On the interval $[t_\delta(s_\delta),\infty)$, we can now argue as in part (iii) by relying on upper/lower solutions constructed similarly to those in the proof of Lemma 2.3. For $s \geqslant s_\delta$ and $t \geqslant t_\delta(s_\delta)$, the upper/lower solutions will stay in the strip $|\theta| \leqslant \pi/2-\delta/2$ and converge to 0 as $s \to \infty$, uniformly on $[T,\infty)$ because $T > t_\delta(s_\delta)$. Hence $\theta(t,s)$ also converges to 0 uniformly on $[T,\infty)$. Part (iv)(a) is proved.

In similar fashion one proves the case when $\theta_0 = \frac{\pi}{2}$ at a point t_0 where $q(t_0) < 0$ (or $q(t_0) = 0$ and $q(t) < 0$ for t slightly larger than t_0).

If $\theta_0 = 0$ and $t \geqslant t_0$, then $\theta(t,s) \to 0$ as $s \to \infty$ uniformly in t, because this is the case for the solution $\varphi_0(t,s)$ for a potential q that is truncated and replaced by 0 on $(-\infty,t_0)$. This proves the first assertion of (iv), part (b).

If $\theta_0 = 0$, $q(t_0) = 0$ and $q(t) > 0$ on $[t_1,t_0)$, the solution will be positive on $[t_1,t_0)$ and shoot up towards $\pi/2$ when s is large because the term $-s\sin 2\theta$ will determine the slope for $t < t_1$. It is clear that the proof is similar to that of (iv), part (a), so we omit the details. $\square$

Let the derivative with respect to s be denoted by an overdot.

THEOREM 4.2. *The function $\varphi_0(s,t)$ is real analytic in s for every s and $\dot\varphi_0(t,s) \to 0$ as $t \to -\infty$.*

Proof. This follows from (2.6) and iteration. Every iterate $\theta_n(t,s)$ is analytic because the integrals appearing in them are absolutely convergent. It remains to prove the second part. The s-derivative of $\varphi_0(t,s)$ satisfies

$$\begin{aligned}\dot\varphi_0(t,s) = 2&\int_{-\infty}^{t} (t-\tau)e^{-2s(t-\tau)} q(\tau)\,d\tau \\ &+ \int_{-\infty}^{t} (1+2s(\tau-t))\,e^{-2s(t-\tau)} f(\varphi_0(\tau,s))\,d\tau \\ &+ s\int_{-\infty}^{t} e^{-2s(t-\tau)} f'(\varphi_0(\tau,s))\,\dot\varphi_0(\tau,s)\,d\tau.\end{aligned} \tag{4.1}$$

Fix $s > 0$. We can view (4.1) as a fixed point problem for $\dot{\varphi}_0(t,s)$. It follows that $\dot{\varphi}_0(t,s)$ is bounded as $t \to -\infty$. For t negative ($|t|$ large) the bounds in (2.17) hold and therefore $|\varphi_0(t,s)| = O(\|q\|_{I_t})$ as $t \to -\infty$. Then $f(\varphi_0(t,s)) = O(\|q\|_{I_t}^3)$ and $f'(\varphi_0(t,s)) = O(\|q\|_{I_t}^2)$. It follows that

$$|\dot{\varphi}_0(t,s)| \leqslant C_1(s)\|q\|_{I_t} + C_2(s)\|q\|_{I_t}^2 \|\dot{\varphi}_0(\tau,s)\|_{I_t}$$

with suitable constants $C_1(s)$ and $C_2(s)$. Upon taking the norm also on the left side we obtain, for t sufficiently negative,

$$\|\dot{\varphi}_0(t,s)\|_{I_t} \leqslant \frac{\widetilde{C}_1(s)\|q\|_{I_t}}{1 - C_3(s)\|q\|_{I_t}^2},$$

which goes to 0 as $t \to -\infty$. $\square$

We draw some consequences from Theorem 4.2. From (2.2a,b), by differentiating the first equation with respect to s and then integrating both equations over an interal $[t_0, t]$ with given s-dependent, differentiable, initial conditions $\rho(t_0,s)$ and $\theta(t_0,s)$, we obtain

$$\dot{\theta}(t,s) = \dot{\theta}(t_0,s) - \frac{1}{\rho(t,s)^2}\int_{t_0}^{t} \rho(\tau,s)^2 \sin(2\theta(\tau,s))\,d\tau.$$

When $\theta(t,s) = \varphi_0(t,s)$, Theorem 4.2 allows us to take $t_0 \to -\infty$, so

$$\dot{\varphi}_0(t,s) = -\frac{1}{\rho(t,s)^2}\int_{-\infty}^{t} \rho(\tau,s)^2 \sin(2\varphi_0(\tau,s))\,d\tau. \tag{4.2}$$

Analogously we obtain ($k \in \mathbb{Z}$)

$$\dot{\psi}_k(t,s) = \frac{1}{\rho(t,s)^2}\int_{t}^{\infty} \rho(\tau,s)^2 \sin(2\psi_k(\tau,s))\,d\tau. \tag{4.3}$$

(Note that the rhos in (4.2) and (4.3) are not the same.)

The functions

$$\chi_k(t,s) = \varphi_0(t,s) - \psi_k(t,s) \tag{4.4}$$

will play an important role in subsequent sections. If $\chi_k(t_0,s) = 0$ for some t_0 (s is fixed), we can conclude that $\varphi_0(t,s) = \psi_k(t,s)$ for all t. Then combining (4.2) and (4.3) gives

$$\dot{\chi}_k(t,s) = -\frac{1}{\rho(t,s)^2}\int_{-\infty}^{\infty} \rho(\tau,s)^2 \sin(2\varphi_0(\tau,s))\,d\tau.$$

The solution $v(t,is)$ corresponding to $\rho(t,s)$ and $\varphi_0(t,s)$ is then an eigenfunction and we can write, by using (2.1),

$$\dot{\chi}_k(t,s) = \frac{-1}{v_1(t,is)^2 + v_2(t,is)^2}\int_{-\infty}^{\infty} v_1(\tau,s)v_2(\tau,s)\,d\tau. \tag{4.5}$$

THEOREM 4.3.

(i) *Suppose that* $q(\tau) \geqslant 0$ *for* $\tau \leqslant t$, $\|q\|_{I_t} > 0$, *and* $\varphi_0(t,s) \geqslant -\frac{\pi}{2}$. *Then* $\dot{\varphi}_0(t,s) > 0$.

(ii) *If* $q(\tau) \geqslant 0$ *for* $\tau \geqslant t$, $\|q\|_{J_t} > 0$, *and* $\psi_0(t,s) \leqslant 0$, *then* $\dot{\psi}_0(t,s) < 0$.

(iii) *Suppose that* $q(t) \geqslant 0$ *for all* t. *Then for any fixed* $t \in \mathbb{R}$, *the equation* $\chi_0(t,s) = 0$ *has at most one root* (*in the variable* s) *and this root is independent of* t.

Proof. We know from (2.17) that $\varphi_0(\tau,s) \leqslant 0$ for $\tau \leqslant t$ and $\varphi_0(t,s) < 0$. If there exists a $\tau_0 < t$ such that $\varphi_0(\tau_0,s) = -\frac{\pi}{2}$, then $\varphi_0(\tau,s) \leqslant -\frac{\pi}{2}$ for $\tau \geqslant \tau_0$ by comparison with the differential equation $\psi' = -s\sin(2\psi)$ which has the constant solution $\psi(t,s) = -\frac{\pi}{2}$. This also shows that on $(\tau_0,t]$ either $\varphi_0(\tau,s) < -\frac{\pi}{2}$ or $\varphi_0(\tau,s) = -\frac{\pi}{2}$. The former contradicts the assumption, the latter is only possible if $q(\tau) = 0$ on $[\tau_0,t]$. Thus $\varphi_0(\tau,s) \geqslant -\frac{\pi}{2}$ for all $\tau \leqslant t$. It follows that the integral in (4.5) is strictly negative and the assertion is proved.

In the case of ψ_0, by (2.18), we have that $\psi_0(\tau,s) \geqslant -\frac{\pi}{2}$ for $\tau \geqslant t$ and $\psi_0(t,s) > -\frac{\pi}{2}$. The remaining steps are analogous to those in (i), using a comparison with the differential equation $\psi' = -s\sin(2\psi)$ and its constant solution $\psi(t,s) = 0$ to conclude that the integral in (4.5) is negative.

Turning to (iii), suppose that $s_1 > 0$ is a root of the equation $\chi_0(t,s) = 0$ at a fixed value t. It is clear that both $\varphi_0(t,s_1), \psi_0(t,s_1) \in [-\frac{\pi}{2},0]$, for otherwise $\chi_0(t,s)$ could not be zero. Hence the conditions of (i) and (ii) are fulfilled and we can conclude that $\dot{\chi}_0(t,s_1) = \dot{\varphi}_0(t,s_1) - \dot{\psi}_0(t,s_1) > 0$. This implies that there cannot be a second root. Of course, the root is independent of s by uniqueness of solutions. □

5. Dependence of the Prüfer angle on a coupling constant

In addition to the dependence on s, we need to have some basic formulas that allow us to determine the behavior of an eigenvalue as a coupling constant changes. We consider the family of potentials $\mu q(t)$ with $\mu \geqslant 0$ and are concerned with the dependence of an imaginary eigenvalue on μ, so $\xi(\mu) = is(\mu)$. Recall the function $\chi_k(t,s)$ from (4.4). We now write $\chi_k(t,s,\mu)$ and fix t. From $\chi_k(t,s(\mu),\mu) = 0$, we obtain

$$\frac{ds}{d\mu} = -\frac{\chi_{k,\mu}(t,s(\mu),\mu)}{\dot{\chi}_k(t,s(\mu),\mu)}. \tag{5.1}$$

(The subscript μ denotes the partial derivative with respect to μ.) We need to evaluate the numerator. From (2.2a) with potential $\mu q(t)$ in place of $q(t)$, we deduce that for any solution $\theta(t,s,\mu)$ having a μ-dependent initial value $\theta(t_0,s,\mu)$ at t_0, we have

$$\theta_\mu(t,s,\mu) = \theta_\mu(t_0,s,\mu)\frac{\rho(t_0,s,\mu)^2}{\rho(t,s,\mu)^2} - \frac{\int_{t_0}^{t} q(\tau)\rho(\tau,s,\mu)^2\,d\tau}{\rho(t,s,\mu)^2}. \tag{5.2}$$

Applying this to $\varphi_0(t,s,\mu)$ ($\psi_0(t,s,\mu)$) we get on letting $t_0 \to -\infty$ ($t_0 \to +\infty$),

$$
\begin{aligned}
\varphi_{0,\mu}(t,s,\mu) &= -\frac{\int_{-\infty}^{t} q(\tau)\rho(\tau,s,\mu)^2\,d\tau}{\rho(t,s,\mu)^2},\\
\psi_{k,\mu}(t,s,\mu) &= \frac{\int_{t}^{\infty} q(\tau)\rho(\tau,s,\mu)^2\,d\tau}{\rho(t,s,\mu)^2}
\end{aligned}
\tag{5.3}
$$

for every $k \in \mathbb{Z}$. Here we have used the fact that $\varphi_{0,\mu}(t,s,\mu) \to 0$ as $t \to -\infty$ and $\psi_{0,\mu}(t,s,\mu) \to 0$ as $t \to +\infty$. These facts are proved by employing an integral equation similar to that in the proof of Theorem 4.2; details are omitted. Putting (4.5), (5.2), (5.3), and (5.1) together yields

$$
\frac{ds}{d\mu} = -\frac{\int_{-\infty}^{\infty} q(t)(v_1(t,is,\mu)^2 + v_2(t,is,\mu)^2)\,dt}{2\int_{-\infty}^{\infty} v_1(t,is,\mu)v_2(t,is,\mu)\,dt}. \tag{5.4}
$$

The numerator could be simplified using

$$
\int_{-\infty}^{\infty} q(t)v_1(t,is,\mu)^2\,dt = \int_{-\infty}^{\infty} q(t)v_2(t,is,\mu)^2\,dt
$$

which follows from (1.1).

6. The eigenvalue equations and multiplicity

We have already seen that a purely imaginary eigenvalue occurs if the solution $\varphi_0(t,s)$ coincides with one of the branches $\psi_k(t,s)$. Equivalently, we could say that an imaginary eigenvalue occurs precisely when $L_{\varphi_0}(s) = \lim_{t\to+\infty}\varphi_0(t,s) = \frac{m\pi}{2}$ for an odd integer m (as in [12]). However, as a function of s, $L_{\varphi_0}(s)$ is a piecewise constant function and so not differentiable in s. In addition, the nonintegrability of q causes difficulties if we want to control the polar angle as $t \to \infty$. For these reasons, we prefer to work on a finite interval, namely

$$
\alpha(s) \leqslant t \leqslant \omega(s),
$$

where

$$
\alpha(s) = \sup\{t : |q(\tau)| \leqslant s \text{ if } \tau \leqslant t\}, \tag{6.1}
$$

$$
\omega(s) = \inf\{t : |q(\tau)| \leqslant s \text{ if } \tau \geqslant t\}. \tag{6.2}
$$

Outside the interval $[\alpha(s),\omega(s)]$ we have estimates that give us control over the Prüfer angle. Since we are interested in the number of imaginary eigenvalues with imaginary part greater than some $s_0 > 0$, we will match up the solution $\varphi_0(t,s)$ with one of the solutions $\psi_k(t,s)$ at the point $\omega(s_0)$. If such a match happens, $\xi = is$ is a purely imaginary eigenvalue. Thus the eigenvalue equations are

$$
\chi_k(\omega(s_0),s) = 0, \qquad s > s_0, \qquad k = 0,-1,-2,\dots. \tag{6.3}
$$

Note that if $s > s_0$, then $[\alpha(s), \omega(s)] \subset [\alpha(s_0), \omega(s_0)]$, so that if $t > \omega(s_0)$, then $\|q\|_{J_t} < s$ and if $t < \alpha(s_0)$, then $\|q\|_{I_t} < s$. Since in this paper the relevant potentials are nonnegative (and not identically zero), we know that $\varphi(\omega(s_0), s_0) < 0$. If there is a smallest (negative) integer k such that

$$0 > \psi_j(\omega(s_0), s_0) > \varphi_0(\omega(s_0), s_0) \qquad j = k, k+1, \dots, -1, 0, \tag{6.4}$$

then, as s increases from s_0 toward ∞, the values of $\varphi_0(\omega(s_0), s)$ must coincide with one of the values $\psi_j(\omega(s_0), s)$ at least once for each given j. Thus there are at least $|k| + 1$ purely imaginary eigenvalues. We can visualize what's going on by drawing graphs for the functions $\varphi_0(\omega(s_0), s)$ and $\psi_k(\omega(s_0), s)$ in the (s, θ) plane. Then eigenvalues correspond to the points of intersection, or possibly touchpoints (points of osculation) of the curve of $\varphi_0(\omega(s_0), s)$ with the "branches" $\psi_k(\omega(s_0), s)$ for $s > s_0$. We sometimes refer to the functions $\psi_k(t, s)$ as branches because taken together they comprise a multivalued function.

The question of whether or not a crossing of $\varphi_0(\omega(s_0), s)$ with one of the branches ψ_k is proper (strict) is very important for the later results in this paper. Clearly if, $\dot{\chi}_k(\omega(s_0), s) \neq 0$, then the crossing is proper. In light of (4.5), this happens if and only if $\int v_1 v_2 \, dt$ is nonzero. If this integral is negative, then $\dot{\chi}_k(\omega(s_0), s) > 0$, meaning the curve of $\varphi_0(\omega(s_0), s)$ crosses the branch $\psi_k(\omega(s_0), s)$ exactly once in the upward direction. Since we are interested in the eigenvalue asymptotics as $s \to 0$, the best way to ensure that the integral is nonzero is to show that $\int v_1 v_2 \, dt < 0$ for *all* s sufficiently small. If all crossings are proper, then the number of imaginary eigenvalues with $\operatorname{Im} \xi > s_0$ is equal to the number of branches ψ_k for which the inequalities in (6.4) hold.

An eigenvalue associated with a proper crossing has algebraic multiplicity 1. This follows from the the fact that the s-derivative of the Wronskian of linearly independent exponentially decaying solutions is nonzero (cf. (4.5) and (6.5) below). Nonproper crossings correspond to multiple eigenvalues of the ZS system and are typically associated with eigenvalue collisions that occur when a parameter built into the ZS system varies. Examples can be found in [10], [11] and a detailed study of eigenvalue collisions was made in [14].

We summarize the relevant formulas for the partial derivatives of the Wronskian with respect to the parameters s and μ. These results will be used in a discussion at the end of Section 7.

We denote by $v^+(t, \xi)$ ($v^-(t, \xi)$) any nontrivial solution of (1.1) that tends to 0 as $t \to +\infty$ ($t \to -\infty$).

Let

$$W(\xi) = W[v^-, v^+] = v_1^- v_2^+ - v_2^- v_1^+, \qquad \xi \in \mathbb{C}$$

denote the (constant) Wronskian of v^- and v^+.

THEOREM 6.1. *Suppose that $\xi \in \mathbb{C}^+$ is an eigenvalue for* (1.1). *Let C be the constant such that $v^+(t, \xi) = C v^-(t, \xi)$ for all $t \in \mathbb{R}$. Then*

$$\frac{dW(\xi)}{d\xi} = -2iC \int_{-\infty}^{\infty} v_1^-(t, \xi) v_2^-(t, \xi) \, dt. \tag{6.5}$$

Furthermore, if we consider the dependence of the Wronskian on a coupling constant μ*, the derivative of* $W(s,\mu)$ *with respect to* μ *is given by*

$$W_\mu(\xi,\mu) = C\int_{-\infty}^{\infty} q(t)\,(v_1^-(t,\xi,\mu)^2 + v_2^-(t,\xi,\mu)^2)\,dt. \tag{6.6}$$

Proof. Both (6.5) and (6.6) are well known (for (6.5), see [11, Appendix]). We omit a detailed proof and only mention that in case of (6.5) one starts with

$$W_\xi(\xi,\mu) = W[v_\xi^-, v^+] + W[v^-, v_\xi^+] = CW[v_\xi^-, v^-] + (1/C)W[v^+, v_\xi^+]$$

and then uses the fact that

$$(v_{1,\xi}{}^- v_2^- - v_{2,\xi}{}^- v_1^-)' = -2iv_1^- v_2^-.$$

Integration leads to the desired equality. In case of (6.6), one proceeds in similar fashion starting from

$$(v_{1,\mu}^- v_2^- - v_{2,\mu}^- v_1^-)' = q(t)((v_1^-)^2 + (v_2^-)^2). \qquad \square$$

For a special but physically relevant class of potentials, the so-called single-lobe potentials, it is known that nonimaginary and nonsimple eigenvalues cannot occur.

We say that a potential $q(t)$ is a single-lobe potential, if q is real-valued, of one sign, piecewise smooth, decaying to 0 at $\pm\infty$ and nondecreasing to the left of a point $t = t_0$ and nonincreasing to the right of $t = t_0$.

THEOREM 6.2. *If* q *is single-lobe, then all eigenvalues are purely imaginary and simple.*

In [11], the absence of nonimaginary eigenvalues for single-lobe potentials was proved. The tacit assumption that $q \in L^1(\mathbb{R})$ was also made because the focus was on potentials that only support a finite number of eigenvalues and properties of eigenfunctions pertinent to the L^1 case were used. It was also mentioned that the restriction of compact support can be removed by a perturbation argument based on an approximation of q by potentials of compact support. The algebraic simplicity of the imaginary eigenvalues was only proved under the assumption of compact support, however this could also be remedied by a perturbation argument. In [15, Theorem 9.4], an extension of Theorem 6.2 to certain multi-humped potentials was proved. We present here a variant of this theorem that is specifically tailored to the applications in this paper. It includes the single-lobe case. A more comprehensive review of these results may appear elsewhere.

THEOREM 6.3. *Suppose that* q *obeys Hypothesis 1, is positive, continuously differentiable, and that its local extrema are comprised of strict local maxima at* $a_1,\dots,a_N$ *and strict local minima at* $b_1,\dots,b_{N-1}$ *where*

$$-\infty < a_1 < b_1 < a_2 < \cdots < b_{N-1} < a_N < +\infty.$$

Suppose that

$$\sum_{k=1}^{N-1} \frac{1}{q(b_k)} < \sum_{k=1}^{N} \frac{1}{q(a_k)}.$$

Then all eigenvalues with sufficiently small imaginary parts are purely imaginary and simple.

The assumptions imply that $q'(t) \geqslant 0$ on $(-\infty, a_1]$ and $q'(t) \leqslant 0$ on $[a_N, \infty)$. The maxima/minima do not have to be strict. If the graph of q has a flat spot, there is some ambiguity in the choice of the a_k or b_k. One only has to make sure that q decreases between a_k and b_{k-1} and increases between b_k and a_{k+1}, but the increase or decrease does not have to be strict.

As preparation for the proof we recall a few facts from [11]. The crucial observation is that whenever $\xi = \alpha + i\beta$ is a nonimaginary eigenvalue (so $\alpha \neq 0$) and $v(t, is)$ is an eigenfunction, then

$$\int_{-\infty}^{\infty} (\overline{v}_1 v_2 + v_1 \overline{v}_2)\, dt = 0. \tag{6.7}$$

This either follows by manipulating the system (1.1) or, shorter, by noticing that $-\xi$ is an eigenvalue of $(H_0 + Q)^*$ with eigenfunction Uv where U is given in (1.5). Since $\alpha \neq 0$, v and Uv are orthogonal (as vector functions), (6.7) holds. Multiplying the first of (1.1) by $\overline{v}_1$ we obtain

$$\overline{v}_1 v_2 + v_1 \overline{v}_2 = \frac{1}{q(t)} [(\overline{v}_1 v_1)' - 2\beta |v_1|^2]. \tag{6.8}$$

Assuming t lies in a finite interval $[a, b]$, an integration by parts leads to

$$\int_a^b (\overline{v}_1 v_2 + v_1 \overline{v}_2)\, dt = \frac{|v_1(b)|^2}{q(b)} - \frac{|v_1(a)|^2}{q(a)} + \int_a^b \frac{|v_1|^2 q'(t)}{q(t)^2} dt - 2\beta \int_a^b \frac{|v_1|^2}{q(t)} dt. \tag{6.9}$$

Alternatively, using the second equation of (1.1), we obtain

$$\int_a^b (\overline{v}_1 v_2 + v_1 \overline{v}_2)\, dt = -\frac{|v_2(b)|^2}{q(b)} + \frac{|v_2(a)|^2}{q(a)} - \int_a^b \frac{|v_2|^2 q'(t)}{q(t)^2} dt - 2\beta \int_a^b \frac{|v_2|^2}{q(t)} dt. \tag{6.10}$$

Now we would like to take the limits $b \to +\infty$ in (6.9) and $a \to -\infty$ in (6.9) in view of the sign of $q'(t)$. In the case of power-law potentials, it is clear that these limits exist because the eigenfunctions decay exponentially and $1/q(t)$ only grows algebraically. But we can actually argue without making this assumption. Recall the bounds from (3.5a,b). We conclude that due to the monotone decay of $q(t)$,

$$\tilde{p}(t, \beta) \leqslant \frac{q(t)}{2\beta}.$$

So (with $v_1^+ = v_1$)

$$|v_1(t, \xi)| \leqslant c_\beta e^{-(1-\varepsilon)\beta t} q(t).$$

Now it is easy to see that the integrals in (6.9) converge at $+\infty$. Similarly, using (3.4a,b), we conclude that we can take $a \to -\infty$ in (6.10).

Proof of Theorem 6.3. Using (6.10) on $(-\infty, a_1]$ and $[b_k, a_{k+1}]$, and (6.9) on $[a_k, b_k]$ and $[a_N, \infty)$, we get

$$\begin{aligned}\int_{-\infty}^{\infty}(\overline{v}_1 v_2 + v_1\overline{v}_2)\,dt \leqslant &-\frac{|v_2(a_1,\xi)|^2}{q(a_1)} + \sum_{k=1}^{N-1}\left(\frac{|v_1(b_k,\xi)|^2}{q(b_k)} - \frac{|v_1(a_k,\xi)|^2}{q(a_k)}\right)\\ &+\sum_{k=1}^{N-1}\left(-\frac{|v_2(a_{k+1},\xi)|^2}{q(a_{k+1})} + \frac{|v_2(b_k,\xi)|^2}{q(b_k)}\right) - \frac{|v_1(a_N,\xi)|^2}{q(a_N)}\\ &= \sum_{k=1}^{N-1}\frac{\|v(b_k,\xi)\|^2}{q(b_k)} - \sum_{k=1}^{N}\frac{\|v(a_k,\xi)\|^2}{q(a_k)}.\end{aligned} \tag{6.11}$$

This is as in [15] but now we follow a different path. We use the bounds from Theorem 3.2 in the form

$$\|v(b_k,\xi)\| \leqslant \|v(a_1,\xi)\| e^{\beta(b_k-a_1)}, \qquad \|v(a_k,\xi)\| \geqslant \|v(a_1,\xi)\| e^{-\beta(a_k-a_1)}.$$

Inserting these in (6.11), we obtain

$$\text{r.h.s. of (6.11)} \leqslant \|v(a_1,\xi)\|^2\left(\sum_{k=1}^{N-1}\frac{e^{2\beta(b_k-a_1)}}{q(b_k)} - \sum_{k=1}^{N}\frac{e^{-2\beta(a_k-a_1)}}{q(a_k)}\right).$$

Taking $\beta \to 0$, we see that this expression is negative for small enough β precisely if

$$\sum_{k=1}^{N-1}\frac{1}{q(b_k)} < \sum_{k=1}^{N}\frac{1}{q(a_k)}. \qquad \square$$

The proof of the single-lobe theorem is immediate. Use (6.10) on $(-\infty, t_0]$ and (6.9) on $[t_0, \infty)$, where t_0 is a point where $q(t)$ attains its maximum.

LEMMA 6.4. *Suppose that q satisfies Hypothesis 1 and, more specifically, that $q(t) \sim q_0 t^\gamma$ as $t \to +\infty$ ($q_0 > 0$, $0 < \gamma \leqslant 1$). Let $v^+(t, is)$ be the decaying solution of (1.1). Then*

$$\int_{\omega(s)}^{\infty}\frac{v_1^+(t,is)^2}{q(t)}\,dt \geqslant c_\gamma s^{-2} \tag{6.12}$$

for some $c_\gamma > 0$.

Note that the power of s is independent of γ. The integral shows up as a negative contribution in (6.9) and (6.10) but multiplied by a factor s. We will take this factor into account at the end of the proof.

Proof. We normalize $v^+(t, is)$ such that $\|v^+(\omega(s), is)\| = 1$. The proof uses a lower bound for $|v_1^+(t, is)|$. The Prüfer angle associated with $v^+(t, is)$ is given by one of the branches $\psi_k(t,s)$ with $k=0,-1,-2,\dots$. From Theorem 2.4(ii) (or (2.19) and (2.20)), we see that

$$e^{2st}\int_t^\infty e^{-2s\tau}q(\tau)\,d\tau \leqslant \psi_k(t,s)-\frac{(2k-1)\pi}{2}\leqslant\frac{\pi}{4}\qquad t\geqslant\omega(s).$$

Therefore

$$\begin{aligned}|v_1^+(t,is)| = \rho(t,s)|\cos(\psi_k(t,s))| &\geqslant \frac{2\sqrt{2}}{\pi}\rho(t,s)\left(\psi_k(t,s)-\frac{(2k-1)\pi}{2}\right)\\ &\geqslant \frac{2\sqrt{2}}{\pi}e^{s\omega(s)}e^{st}\int_t^\infty e^{-2s\tau}q(\tau)\,d\tau.\end{aligned}$$

Here we have used the estimate $\sin x\geqslant 2\sqrt{2}\pi^{-1}x$ for $0\leqslant x\leqslant\frac{\pi}{4}$ and Theorem 3.2(i) in the form $\rho(t,s)\geqslant e^{-s(t-\omega(s))}$ (because of the normalization). Therefore, in view of the asymptotic form of $q(t)$, we can estimate (q_0 is absorbed in the constants)

$$\begin{aligned}\int_{\omega(s)}^\infty \frac{v_1^+(t,is)^2}{q(t)}\,dt &\geqslant c_1e^{2s\omega(s)}\int_{\omega(s)}^\infty e^{2st}t^\gamma\left(\int_t^\infty e^{-2s\tau}\tau^{-\gamma}\,d\tau\right)^2 dt\\ &= c_1(2s)^{-3+\gamma}e^{2s\omega(s)}\int_{2s\omega(s)}^\infty e^u u^\gamma\left(\int_u^\infty e^{-w}w^{-\gamma}\,dw\right)^2 du\\ &\geqslant c_2(2s)^{-3+\gamma}(2s\omega(s))^{-\gamma}\geqslant c_3s^{-2}.\end{aligned}$$

In the last step we have used that $\omega(s)\sim(q_0/s)^{1/\gamma}$ as $s\to 0$. $\square$

We remark that if we consider the integral in (6.9) involving $q'(t)/q(t)^2$, we can show that

$$\int_{\omega(s)}^\infty \frac{v_1^+(t,is)^2q'(t)}{q(t)^2}\,dt = o(s^{-2+\frac{1}{\gamma}}). \tag{6.13}$$

This integral is also negative (assuming $q(t)$ is decreasing) but the integral in (6.12) (even when multiplied by s) is dominant as $s\to 0$ if $0<\gamma\leqslant 1$. One can also show that (6.12) cannot in general be improved to $s^{-\beta}$ with $\beta>2$. We use the upper bound (3.5a) to do that.

The following transformation will also prove useful. We sometimes omit the argument t (and s) for brevity. Define

$$w=S^{-1}v,\qquad S=\begin{pmatrix}\dfrac{-s+i\sigma}{q} & \dfrac{-s-i\sigma}{q}\\ 1 & 1\end{pmatrix},\qquad \sigma=(q^2-s^2)^{1/2}, \tag{6.14}$$

with

$$S^{-1}=\begin{pmatrix}\dfrac{-iq}{2\sigma} & \dfrac{\sigma-is}{2\sigma}\\ \dfrac{iq}{2\sigma} & \dfrac{\sigma+is}{2\sigma}\end{pmatrix},\qquad \det S=\frac{2i\sigma}{q}. \tag{6.15}$$

Then the ZS system (1.1) is transformed into the new system

$$w' = \begin{pmatrix} -i\sigma + a_0 & -b_0 \\ -a_0 & i\sigma + b_0 \end{pmatrix} w \stackrel{\text{def}}{=} \mathscr{C} w, \tag{6.16}$$

where

$$a_0(t) = \frac{is(is+\sigma)q'(t)}{2q(t)\sigma^2}, \qquad b_0(t) = \frac{is(is-\sigma)q'(t)}{2q(t)\sigma^2}.$$

The eigenvalues of $\mathscr{C} + \mathscr{C}^*$ are

$$\lambda_1(t) = \frac{sq'(t)}{q(t)(s-q(t))}, \qquad \lambda_2(t) = \frac{sq'(t)}{q(t)(s+q(t))}.$$

Note that $\lambda_1(t)$ is in general not integrable at $t = \omega(s)$ or $t = \alpha(s)$ and that $\lambda_1(t)$ is positive and $\lambda_2(t)$ is negative in the typical situation when $t < \omega(s)$ and $s < q(t)$.

LEMMA 6.5. *Fix $s > 0$. Suppose that $w(t_0, s)$ is given and $w(t, s)$ is a solution of* (6.16) *for $t \leqslant t_0 < \omega(s)$. Suppose that $q'(\tau) \leqslant 0$ on $[t, t_0]$ and that $q(t_0) \geqslant s$. Then*

$$\|w(t,s)\|^2 \leqslant \frac{q(t)}{q(t)+s}\frac{q(t_0)+s}{q(t_0)}\|w(t_0,s)\|^2 \leqslant 2\|w(t_0,s)\|^2, \qquad t \leqslant t_0.$$

Proof. This follows by integrating the inequality

$$(\|w\|^2)' = w^*(\mathscr{C} + \mathscr{C}^*)w \geqslant \lambda_2(t)\|w\|^2$$

from t to t_0, which gives

$$\|w(t,s)\|^2 \leqslant \exp\left(-\int_t^{t_0} \lambda_2(\tau)\,d\tau\right)\|w(t_0,s)\|^2.$$

Computing the integral gives the first inequality. Since $q(t_0) \geqslant s$, the second inequality follows. $\square$

For the next theorem, which is new, we need more detailed assumptions. It was motivated by the idea that it should be true that eigenvalues with small enough imaginary part are simple, if the potential is essentially arbitrary on a finite interval and falls off monotonically outside it. The assumptions are:

Let $0 < \gamma_\pm < 1$, $q_{0,\pm} > 0$. Suppose that as $t \to \pm\infty$ the following asymptotics hold:

$$q(t) = q_{0,\pm}|t|^{-\gamma_\pm}[1 + \varepsilon_{0,\pm}(t)], \quad q'(t) = \mp q_{0,\pm}\gamma_\pm|t|^{-1-\gamma_\pm}[1 + \varepsilon_{1,\pm}(t)], \tag{6.17}$$

where

$$\varepsilon_{0,\pm}(t) = o(1), \qquad \varepsilon_{0,\pm}'(t) = o(|t|^{-1}), \qquad \varepsilon_{1,\pm}(t) = o(1). \tag{6.18}$$

$$q''(t) = O(|t|^{-2-\gamma_\pm}), \quad q'''(t) = O(|t|^{-3-\gamma_\pm}). \tag{6.19}$$

Since $\varepsilon_{1,\pm}(t) = \varepsilon_{0,\pm}(t) - \gamma_\pm^{-1} t \varepsilon_{0,\pm}'(t)$, the last relation in (6.18) follows from the first two.

Now there is a complication caused by the fact that we only need a subset of these properties depending on the value of γ. We therefore refine the assumptions to reflect this situation so that we can refer to them later.

($a_\pm$) $\frac{1}{2} < \gamma_\pm \leqslant 1$, together with (6.17) and (6.18) for $t \to \pm\infty$.

($b_\pm$) $0 < \gamma_\pm \leqslant \frac{1}{2}$, together with all three, (6.17)–(6.19), for $t \to \pm\infty$.

Hence we have four possibilities: (a_-, a_+), (a_-, b_+), (b_-, a_+), (b_-, b_+).

THEOREM 6.6. *Suppose q is continuously differentiable and satisfies one of the four conditions listed above. Then, if $v(t,is)$ is an eigenfunction, we have*

$$\int_{-\infty}^{\infty} v_1(t,is) v_2(t,is)\, dt < 0$$

if s is small enough. Hence, imaginary eigenvalues with small enough imaginary part are simple.

Proof. The assumptions imply that $q(t)$ is positive and strictly monotone outside some interval $[-T,T]$. It suffices to prove the theorem for the interval $[0,\infty)$ and to show that

$$\int_{0}^{\infty} v_1(t,is) v_2(t,is)\, dt < 0 \tag{6.20}$$

for sufficiently small s. We give the proof for $\frac{1}{2} < \gamma_+ \leqslant 1$. The proof for $0 < \gamma_+ \leqslant \frac{1}{2}$ is more involved and is given in Appendix A. So suppose that $\frac{1}{2} < \gamma_+ \leqslant 1$. The strategy is to use (6.12) on $[\omega(s), \infty)$ and to divide up the interval $[0, \omega(s)]$ into three intervals: $[0,t]$, $[T, t_\eta(s)]$, and $[t_\eta(s), \omega(s)]$. Here $t_\eta(s)$ is defined as

$$t_\eta(s) = \omega(s) - \frac{\eta}{s},$$

where $\eta > 0$ is arbitrary. However, when $\gamma = 1$, in which case $\omega(s) = O(s^{-1})$, we can simply replace $t_\eta(s)$ by T. On $[t_\eta(s), \omega(s)]$ and $[0,T]$, we will use Theorem 3.2 (ii) while to bridge the gap from T to $t_\eta(s)$ we will use Lemma 6.5. We now drop the super/subscript $+$ on v^+ and γ_+.

First step: $[t_\eta, \omega(s)]$. From Theorem 3.2(ii), we infer that $\|v(t,is)\| \leqslant e^{s(\omega(s)-t)}$ on this interval. Thus

$$\|v(t_\eta(s), is)\| \leqslant e^{\eta}. \tag{6.21}$$

Incidentally, this is the same bound as one gets from pertubation theory if the unperturbed problem is $H_0 + Q$ and the perturbation is s times the identity.

Second step: $[T, t_\eta(s)]$. We use the transformation S defined in (6.14). Since we need to go from $v(t_\eta(s), is)$ to $w(t_\eta(s), s)$, we need to estimate the matrix elements of S^{-1} in (6.15). We have, for some $c_1 > 0$,

$$q(t_\eta(s)) - s = s\left\{ \left(1 - \frac{\eta}{s\omega(s)}\right)^{-\gamma} \frac{1+\varepsilon_0(t_\eta(s))}{1+\varepsilon_0(\omega(s))} - 1 \right\} \geqslant c_1 \eta s^{\frac{1}{\gamma}},$$

since $\omega(s) \sim (q_0/s)^{\frac{1}{\gamma}}$, and

$$\frac{1+\varepsilon_0(t_\eta(s))}{1+\varepsilon_0(\omega(s))} = 1 + o\left(\frac{1}{s\omega(s)}\right).$$

The latter follows on using Taylor's theorem with remainder, which gives

$$\varepsilon_0(\omega(s)) - \varepsilon_0(t_\eta(s)) = \frac{c}{s}\,\varepsilon_0'(\widehat{\omega}(s)), \qquad t_\eta(s) < \widehat{\omega}(s) < \omega(s),$$

together with the second relation in (6.18). Since $q(t_\eta(s)) + s \geqslant 2s$, it follows that

$$q(t_\eta(s))^2 - s^2 \geqslant 2c_1\eta s^{\frac{1}{\gamma}+1}.$$

Also $q(t_\eta(s)) \leqslant c_2 s$ for small s, hence

$$\frac{q(t_\eta(s))}{\sigma(t_\eta(s))} \leqslant c_3\eta^{-1/2} s^{\frac{1}{2}-\frac{1}{2\gamma}}.$$

The right-hand side (maybe with a different constant in place of c_3) is an upper bound for the norm of S^{-1}. In conjunction with (6.21), this tells us that

$$\|w(t_\eta(s),s)\| \leqslant c_4\eta^{-1/2} e^\eta s^{\frac{1}{2}-\frac{1}{2\gamma}}.$$

Now, from Lemma 6.5, we get

$$\|w(t,s)\| \leqslant 2c_4\eta^{-1/2} e^\eta s^{\frac{1}{2}-\frac{1}{2\gamma}}, \qquad T \leqslant t \leqslant t_\eta(s).$$

Note that on $[T, t_\eta(s)]$, $q(t)$ is decreasing. At $t = T$, the transformation S is bounded. Hence

$$\|v(t,is)\| \leqslant c_5\eta^{-1/2} e^\eta s^{\frac{1}{2}-\frac{1}{2\gamma}}. \tag{6.22}$$

Third step: $[0,T]$. First, from Theorem 3.2(ii), we obtain (since $e^{s(T-t)} \leqslant e^{sT}$)

$$\|v(t,is)\| \leqslant c_5\eta^{-1/2} e^\eta s^{\frac{1}{2}-\frac{1}{2\gamma}} e^{sT}, \qquad 0 \leqslant t \leqslant T. \tag{6.23}$$

We now simply use the bound $|v_1 v_2| \leqslant (1/2)(v_1^2 + v_2^2)$ and immediately get that

$$\int_0^T |v_1||v_2|\,dt \leqslant c_6\eta^{-1} e^{2\eta} s^{1-\frac{1}{\gamma}} e^{sT}. \tag{6.24}$$

Now we return to the integral in (6.20) and take a look at (6.9). We write

$$\begin{aligned}\int_0^\infty v_1 v_2\,dt = &\int_0^T v_1 v_2\,dt - \frac{v_1(T)^2}{q(T)} + \int_T^\infty \frac{|v_1|^2 q'(t)}{q(t)^2}dt \\ &-2\beta\int_T^{\omega(s)} \frac{|v_1|^2}{q(t)}dt - 2s\int_{\omega(s)}^\infty \frac{|v_1|^2}{q(t)}dt.\end{aligned} \tag{6.25}$$

Of the five terms on the right, all but the first one are negative. The first one is $O(s^{1-\frac{1}{\gamma}})$ by (6.24), and the last one, by Lemma 6.5, is bounded above by $-c_\gamma s^{-1}$. Hence the right-hand side of (6.25) is negative if $1-\frac{1}{\gamma} > -1$, which is the case if $1/2 < \gamma \leqslant 1$. As mentioned above, the case $\gamma = 1$ is included by simply taking $t_\eta(s) = T$. This proves that the integral in (6.20) is negative. The integral from $-\infty$ to 0 can be handled in a similar way, completing the first part of the proof. The proof when $0 < \gamma \leqslant 1/2$ is given in Appendix A. □

We have carried the constant η along through to the end to see if the case $\gamma = 1/2$ could also be included by making a suitable choice for η. That does not seem to be obvious. That's why the case $\gamma = 1/2$ has also been relegated to Appendix A.

7. Dirichlet-Neumann decoupling

In this section, we introduce methods that will allow us to obtain the eigenvalue asymptotics when $0 < \gamma < 1$. The case $\gamma = 1$ is special, in fact simpler, and it will be handled with the Prüfer transformation alone. The decoupling method, as we will set it up for $0 < \gamma < 1$, forces us to exclude $\gamma = 1$. The tools we will be using include the Birman-Schwinger principle and an anolog of Dirichlet-Neumann decoupling (or Dirichlet-Neumann bracketing), the latter being well known in the context of Schrödinger operators. However, in contrast to the Schrödinger case, the operator whose eigenvalues we will estimate is the Birman-Schwinger kernel, not the operator $H_0 + Q$.

In this section we only consider nonnegative q. The first step is to factor the matrix Q from (1.4) as

$$Q = -Q_1 Q_2 \tag{7.1}$$

where

$$Q_1 = q^{1/2} I_2, \qquad Q_2 = i q^{1/2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{7.2}$$

With the help of the substitution $z = Q_2 v$, we can convert (1.2) to the eigenvalue problem

$$K(is)z = z, \tag{7.3}$$

where

$$K(is) = Q_2 (H_0 - is)^{-1} Q_1. \tag{7.4}$$

Conversely, from an eigenvector z, we can recover v by setting $v = (H_0 - is)^{-1} Q z$. The matrix kernel of $(H_0 - is)^{-1}$ is given by

$$R(is;t,\tau) = \begin{pmatrix} i\theta(\tau - t) e^{-s(\tau - t)} & 0 \\ 0 & i\theta(t-\tau) e^{-s(t-\tau)} \end{pmatrix}, \tag{7.5}$$

where θ denotes the Heaviside step function. Hence the kernel of $K(is)$ is given by

$$K(is;t,\tau) = \begin{pmatrix} 0 & -q(t)^{1/2} e^{-s(t-\tau)} \theta(t-\tau) q(\tau)^{1/2} \\ -q(t)^{1/2} e^{-s(\tau-t)} \theta(\tau - t) q(\tau)^{1/2} & 0 \end{pmatrix}. \tag{7.6}$$

and is called the Birman-Schwinger kernel. We see that $K(is;t,\tau)$ is a selfadjoint kernel. Since $JK(is)J=-K(is)$, the spectrum of $K(is)$ is symmetric about the origin. For $s>0$ the operator $K(is)$ is compact. For potentials satisfying Hypothesis 1, this follows by approximating q by a sequence q_n of potentials of compact support contained in $[-n,n]$. For potentials in L^p, the Birman-Schwinger kernel lies in a suitable trace ideal ([22], [4]). If $q\in L^1(\mathbb{R})$, then it is Hilbert-Schmidt. As mentioned above, we will employ Dirichlet-Neumann decoupling. This leads us to take a look at ZS systems on a finite interval with appropriate boundary conditions. Consider (1.1) on a finite interval $[a,b]$ under the boundary conditions:

Neumann b.c.: $v_1(a)=-v_2(a)$ and $v_1(b)=-v_2(b)$.

Dirichlet b.c.: $v_1(a)=v_2(a)$ and $v_1(b)=v_2(b)$.

Our terminology is in agreement with that introduced in [8]. We denote the corresponding ZS operators by $H^{\mathrm{N}}_{[a,b]}$, resp. $H^{\mathrm{D}}_{[a,b]}$.

The domain of these operators is the Sobolev space $\mathscr{H}^1([a,b],\mathbb{C}^2)$ whose elements satisfy the corresponding boundary conditions. The corresponding resolvents $(H^{\mathrm{N}}_{[a,b]}-\xi)^{-1}$, resp. $(H^{\mathrm{D}}_{[a,b]}-\xi)^{-1}$, will be denoted by $R^{\mathrm{N}}_{[a,b]}(\xi)$, resp. $R^{\mathrm{D}}_{[a,b]}(\xi)$, and their integral kernels by $R^{\mathrm{N}}_{[a,b]}(\xi;t,\tau)$, resp. $R^{\mathrm{D}}_{[a,b]}(\xi;t,\tau)$. For the associated Birman-Schwinger kernels, we write $K^{\mathrm{N,D}}_{[a,b]}(is)$, or $K^{\mathrm{N,D}}_{[a,b]}(is,q)$, if it is important to stress the dependence on q. If we pick a point $d\in\mathbb{R}$ and insert a boundary condition at d, then we write $\mathbb{R}\setminus\{d\}$ in place of $[a,b]$. The boundary conditions, say the Neumann type, are then understood to be of the form $v_1(d-,s)=-v_2(d-,s)$ and $v_1(d+,s)=-v_2(d+,s)$. So, the functions $v_1(t,is)$ and $v_2(t,is)$ need not be continuous across $t=d$. Generalizing this to n insertion points $t_1<\ldots<t_n$ inside an interval $[a,b]$, we use the notation $K^{\mathrm{N,D}}_{[a,b]\setminus\{t_1,t_2,\ldots,t_n\}}(is)$ to designate the Birman-Schwinger kernel associated with Neumann (resp. Dirichlet) boundary conditions at a,b and the points $t_1\ldots t_n$.

The crucial observation about the operators with boundary conditions is that certain resolvent differences are finite rank operators. We describe the scenarios that are relevant to us in detail and prefer an explicit approach over a more abstract one (using extension theory).

(a) One single Neumann b.c. at a point b. Then

$$R^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is;t,\tau)=\begin{pmatrix} ie^{-s(\tau-t)}\theta(\tau-t) & -ie^{-2sb}e^{s(t+\tau)} \\ 0 & ie^{-s(t-\tau)}\theta(t-\tau)\end{pmatrix}\qquad t,\tau\leqslant b,$$

$$R^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is;t,\tau)=\begin{pmatrix} ie^{-s(\tau-t)}\theta(\tau-t) & 0 \\ -ie^{2sb}e^{-s(t+\tau)} & ie^{-s(t-\tau)}\theta(t-\tau)\end{pmatrix}\qquad t,\tau\geqslant b,$$

$$R^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is;t,\tau)=0\qquad t<b,\tau>b\quad\text{and}\quad t>b,\tau<b.$$

Thus, by using (7.5), we obtain

$$R(is;t,\tau)-R^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is;t,\tau)=\chi(t,s)\psi(\tau,s)^T,\tag{7.7}$$

where

$$\chi(t,s)=\begin{pmatrix} ie^{st}\theta(b-t) \\ ie^{2sb}e^{-st}\theta(t-b)\end{pmatrix},$$

$$\psi(\tau,s)=\begin{pmatrix} e^{-s\tau}\theta(\tau-b) \\ e^{-2sb}e^{s\tau}\theta(b-\tau)\end{pmatrix}.$$

So this resolvent difference is rank one and selfadjoint. For the difference of the associated Birman-Schwinger kernels we get

$$K(is;t,\tau)-K^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is;t,\tau)=iq^{1/2}(t)U\chi(t,s)\psi^T(\tau,s)q^{1/2}(\tau),$$

where U is the matrix defined in (1.5). The eigenvalue of this difference is

$$i\int_{-\infty}^{\infty}\psi(t,s)^T U\chi(t,s)q(t)\,dt=-e^{2sb}\int_b^{\infty}e^{-2s\tau}q(\tau)\,d\tau-e^{-2sb}\int_{-\infty}^{b}e^{2s\tau}q(\tau)\,d\tau,$$

and is thus negative.

(b) Suppose we have a Neumann b.c. at b and we add, at some point $d<b$, an additional Neumann b.c.

Then a calculation shows that, for $t,\tau\leqslant b$ (only this range is relevant)

$$R^{\mathrm{N}}_{(-\infty,b]}(is;t,\tau)-R^{\mathrm{N}}_{(-\infty,b]\setminus\{d\}}(is;t,\tau)=\chi(t,s)\psi(\tau,s)^T \tag{7.8}$$

where

$$\chi(t,s)=\begin{pmatrix} ie^{st} \\ 0\end{pmatrix}\theta(d-t)+\frac{i}{e^{-2sd}-e^{-2sb}}\begin{pmatrix} -e^{-2bs}e^{st} \\ e^{-st}\end{pmatrix}\theta(t-d),$$

$$\psi(\tau,s)=\begin{pmatrix} e^{-s\tau}\theta(\tau-d) \\ (e^{-2sd}-e^{-2sb})e^{s\tau}\theta(d-\tau)-e^{-2sb}e^{s\tau}\theta(\tau-d)\end{pmatrix}.$$

Again, this is a selfadjoint kernel of rank one. The eigenvalue of the difference of the associated Birman-Schwinger kernels is

$$-(e^{-2ds}-e^{-2bs})\int_{-\infty}^{\mathrm{d}}e^{2st}q(t)\,dt-\frac{e^{-s(b-d)}}{\sinh[(b-d)s]}\int_d^b\cosh[2(b-t)s]q(t)\,dt,$$

and is thus negative (since $d<b$).

(c) This is case (a) but for a Dirichlet b.c. at b. We have

$$R(is;t,\tau)-R^{\mathrm{D}}_{\mathbb{R}\setminus\{b\}}(is;t,\tau)=\chi(t,s)\psi(\tau,s)^T, \tag{7.9}$$

where

$$\chi(t,s)=\begin{pmatrix} ie^{st}\theta(b-t) \\ -ie^{2sb}e^{-st}\theta(t-b)\end{pmatrix},$$

$$\psi(t,s)=\begin{pmatrix} e^{-st}\theta(t-b) \\ -e^{-2sb}e^{st}\theta(b-t)\end{pmatrix}.$$

The eigenvalue of the difference in the Birman-Schwinger kernels is

$$e^{2sb}\int_b^{\infty}q(t)e^{-2st}\,dt+e^{-2sb}\int_{-\infty}^{b}q(t)e^{2st}\,dt;$$

it is positive.

(d) We insert an additional Dirichlet b.c. at $d < b$. Then

$$R^{\mathrm{D}}_{(-\infty,b]}(is;t,\tau) - R^{\mathrm{D}}_{(-\infty,b]\setminus\{d\}}(is;t,\tau) = \psi(t,s)\phi^T(\tau,s), \tag{7.10}$$

where

$$\psi(t,s) = \begin{pmatrix} ie^{st} \\ 0 \end{pmatrix}\theta(d-t) - \frac{i}{e^{-2sd}-e^{-2sb}}\begin{pmatrix} e^{st}e^{-2bs} \\ e^{-st} \end{pmatrix}\theta(t-d),$$

$$\phi(\tau,s) = \begin{pmatrix} e^{-s\tau}\theta(\tau-d) \\ -(e^{-2sd}-e^{-2sb})e^{s\tau}\theta(d-\tau)+e^{-2sb}e^{s\tau}\theta(\tau-d) \end{pmatrix}.$$

Again we have a rank one difference. The eigenvalue of the associated Birman-Schwinger kernel is

$$(e^{-2ds}-e^{-2bs})\int_{-\infty}^{d} e^{2st}q(t)\,dt + \frac{e^{-s(b-d)}}{\sinh[(b-d)s]}\int_d^b \cosh[2(b-t)s]q(t)\,dt.$$

This eigenvalue is positive.

For a selfadjoint compact operator A and a positive number α, we let $N_\alpha[A]$ denote the number of eigenvalues of A larger than α. Relevant for us is the case $\alpha = 1$.

LEMMA 7.1. *For any $\alpha > 0$, we have*

$$N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is)] - 1 \leqslant N_\alpha[K(is)] \leqslant N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{b\}}(is)], \tag{7.11}$$

$$N_\alpha[K^{\mathrm{D}}_{\mathbb{R}\setminus\{b\}}(is)] \leqslant N_\alpha[K(is)] \leqslant N_\alpha[K^{\mathrm{D}}_{\mathbb{R}\setminus\{b\}}(is)] + 1, \tag{7.12}$$

$$N_\alpha[K^{\mathrm{N}}_{(-\infty,b]\setminus\{d\}}(is)] - 1 \leqslant N_\alpha[K^{\mathrm{N}}_{(-\infty,b]}(is)] \leqslant N_\alpha[K^{\mathrm{N}}_{(-\infty,b]\setminus\{d\}}(is)], \tag{7.13}$$

$$N_\alpha[K^{\mathrm{D}}_{(-\infty,b]\setminus\{d\}}(is)] \leqslant N_\alpha[K^{\mathrm{D}}_{(-\infty,b]}(is)] \leqslant N_\alpha[K^{\mathrm{D}}_{(-\infty,b]\setminus\{d\}}(is)] + 1. \tag{7.14}$$

Proof. The inequalities follow immediately from the fact that the resolvent differences in (7.7)–(7.10) are rank one and either positive or negative, together with the min-max principle (cf. [20, p. 274]). □

For this paper we need to generalize the lemma to a finite number of points where a boundary condition is inserted.

THEOREM 7.2. *Given a partition of the real line of the form $-\infty < t_1 < \cdots < t_n < \infty$, we have*

$$N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{t_1,\ldots,t_n\}}(is)] - n \leqslant N_\alpha[K(is)] \leqslant N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{t_1,\ldots,t_n\}}(is)], \tag{7.15}$$

$$N_\alpha[K^{\mathrm{D}}_{\mathbb{R}\setminus\{t_1,\ldots,t_n\}}(is)] \leqslant N_\alpha[K(is)] \leqslant N_\alpha[K^{\mathrm{D}}_{\mathbb{R}\setminus\{t_1,\ldots,t_n\}}(is)] + n. \tag{7.16}$$

Proof. We consider the first inequality in (7.15). The proof proceeds inductively by successively placing Neumann b.c. at t_n, t_{n-1}, etc., down to t_1. The base case with one boundary condition is established in Lemma 7.1. Suppose $t_k,\dots,t_n$ have been chosen and assume that

$$N_\alpha[K(is)] \geqslant N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{t_k,\dots,t_n\}}(is)] - (n-k+1)$$

holds. Then, by the direct sum decomposition induced by the boundary conditions, we have

$$N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{t_{k-1},\dots,t_n\}}(is)] = N_\alpha[K_{(-\infty,t_k]\setminus\{t_{k-1}\}^{\mathrm{N}}(is)}] + N_\alpha[K_{[t_k,\infty)\setminus\{t_{k+1},\dots,t_n\}]^{\mathrm{N}}(is)}].$$

Now using (7.13) we obtain

$$\begin{aligned}&\leqslant N_\alpha[K^{\mathrm{N}}_{(-\infty,t_k]}(is)] + 1 + N_\alpha[K^{\mathrm{N}}_{[t_k,\infty)\setminus\{t_{k+1},\dots,t_n\}}(is)]\\ &= N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{t_k,\dots,t_n\}}(is)] + 1 \leqslant N_\alpha[K(is)] + (n-k+2).\end{aligned}$$

Thus

$$N_\alpha[K(is)] \geqslant N_\alpha[K^{\mathrm{N}}_{\mathbb{R}\setminus\{t_{k-1},\dots,t_n\}}(is)] - (n-(k-1)+1),$$

proving the first inequality in (7.15). The other inequality in (7.15) and those in (7.16) are proved similarly. $\square$

Thus we can draw the conclusion that adding n Dirichlet boundary conditions decreases the number of eigenvalues of the Birman-Schwinger kernel above 1 by at most n. Adding n Neumann boundary conditions increases the number of eigenvalues above 1 by at most n.

Note that if we want to count eigenvalues with $\operatorname{Im}\xi > s$, we can ignore the potential for $t < \alpha(s)$ and $t > \omega(s)$. If we place, say, Neumann b.c. at $\alpha(s)$ and $\omega(s)$, then $K^N_{(-\infty,\alpha(s)]}(is)$ and $K^N_{[\omega(s),\infty)}(is)$ have no eigenvalues above 1, since the norms of $R^{\mathrm{N}}_{(-\infty,\alpha(s)]}(is)$ and $R^{\mathrm{N}}_{[\omega(s),\infty)}(is)$ are equal to s (the unperturbed differential operator is selfadjoint). In the next section we will insert Neumann or Dirichlet conditions at suitable points inside the interval $[\alpha(s),\omega(s)]$. To simplify matters further, we may just consider the interval $[0,\omega(s)]$.

THEOREM 7.3. *Suppose q satisfies Hypothesis 1 and is nonnegative. Fix $s_0 > 0$. Let k_0 ($k_0 \leqslant 0$) be the unique integer such that $\psi_{k_0}(\omega(s_0),s_0) \leqslant \varphi_0(\omega(s_0),s_0) < \psi_{k_0+1}(\omega(s_0),s_0)$. Then $N_1[K(is_0)] = |k_0|$.*

Proof. Since $q_- = 0$, we know that $\varphi_0(\omega(s_0),s_0) < 0$. Hence $k_0 \leqslant 0$. Now replace $q(t)$ by $\mu q(t)$ for $0 \leqslant \mu \leqslant 1$. Then the corresponding solution $\varphi_0(t,s;\mu)$ of (2.2a) approaches $\varphi_0(t,s;0) = 0$ as $\mu \to 0$, for any $s > 0$, $t \in \mathbb{R}$, and it does so monotonically by (5.3). Thus there are exactly $|k_0|$ distinct values $\mu_k \in (0,1)$ for which $\varphi_0(\omega(s_0),s_0;\mu_k) = \psi_k(\omega(s_0),s_0,\mu_k)$. These are precisely the coupling constants for which $\xi = is_0$ is an eigenvalue of (1.1) with potential $\mu_k q(t)$. Hence $\mu_k K(is_0)$ has eigenvalue 1 or, equivalently, $K(is_0)$ has eigenvalue $1/\mu_k > 1$. Thus $1/\mu_k$ is an eigenvalue of $K(is_0)$ that is counted in $N_1[K(is_0)]$, so $N_1[K(is_0)] = |k_0|$. $\square$

A similar result holds for Neumann and Dirichlet problems on a finite interval $[a,b]$. Let $\theta^{\mathrm{N}}(t,s)$ ($\theta^{\mathrm{D}}(t,s)$) denote the solution of (2.2a) corresponding to a Neumann b.c. (Dirichlet b.c.) at $t=a$ normalized such that $\theta^{\mathrm{N}}(a,s)=-\frac{\pi}{4}$ ($\theta^{\mathrm{D}}(a,s)=\frac{\pi}{4}$).

COROLLARY 7.4. *Suppose q is absolutely integrable on $[a,b]$. Fix $s_0>0$. Let m_0 $(m_0 \geqslant 1)$ denote the smallest integer such that $\theta^{\mathrm{N}}(b,s_0) < -(4m_0-3)\frac{\pi}{4}$. Then $N_1[K^{\mathrm{N}}_{[a,b]}(is_0)] = m_0$.*

Let n_0 $(n_0 \geqslant 1)$ denote the smallest integer such that $\theta^{\mathrm{D}}(b,s_0) < -(4n_0-1)\frac{\pi}{4}$. Then $N_1[K^{\mathrm{D}}_{[a,b]}(is_0)] = n_0$.

Proof. The argument follows that of the previous proof. Use (5.2) and the fact that as $\mu \to 0$, $\theta^{N}(b,s_0,\mu) \to -\operatorname{arccot}[\exp(2s_0(b-a))]$ and $\theta^{D}(b,s_0,\mu) \to \operatorname{arccot}[\exp(2s_0(b-a))]$. A look at the graph of these limits finishes the proof. □

In the next lemma, we consider two potentials q_1 and q_2. We indicate the dependence on q by an additional argument.

LEMMA 7.5. *Let $\alpha>0$. If $0 \leqslant q_1(t) \leqslant q_2(t)$ for all t, then*

$$N_\alpha[K(is,q_1)] \leqslant N_\alpha[K(is,q_2)]$$

and for any interval Δ (including semi-infinite intervals),

$$N_\alpha[K^{\mathrm{N,D}}_{\Delta}(is,q_1)] \leqslant N_\alpha[K^{\mathrm{N,D}}_{\Delta}(is,q_2)].$$

Proof. This follows immediately from Corollary 7.4 and the fact that the solution of (2.2a), either $\varphi_0(t,s)$ or $\theta^{\mathrm{N,D}}(t,s)$, obey

$$\varphi_0(t,s,q_2) \leqslant \varphi_0(t,s,q_1) \qquad \text{or} \qquad \theta^{\mathrm{N,D}}(t,s,q_2) \leqslant \theta^{\mathrm{N,D}}(t,s,q_1)$$

for any t in the appropriate interval and any $s>0$. □

With these results at hand we can now briefly describe how we will go about obtaining the asymptotics of $N(s)$ (see (1.7)). First, as already mentioned above, the Coulomb case is special because it can be handled by the Prüfer transformation alone. For potentials behaving like $|t|^{-\gamma}$ for large $|t|$, with $0<\gamma<1$, the Prüfer transformation does not seem to be the right tool to give us the leading asymptotics of $N(s)$. Thus we use the Birman-Schwinger kernel. Suppose for some s_0 the Birman-Schwinger kernel has a few eigenvalues above 1, say $\lambda_1(s_0) > \lambda_2(s_0) > \ldots > 1$, As s increases from s_0, these eigenvalues are given by analytic functions $\lambda_k(s)$. Their order is preserved, since as eigenvalue branches of a selfadjoint operator, $K(is)$, they cannot become degenerate, since the geometric multiplicity of an eigenvalue of (1.1) is always 1. We already know that the eigenvalues of the ZS system do not have to be algebraically simple. However, the eigenvalue of the ZS system with the largest imaginary part, that is, the eigenvalue associated with $\lambda_1(s)$, is always simple. This is a consequence of Theorem 4.3, which implies that $\dot{\chi}_0(t,s) > 0$ if $\xi = is$ is the eigenvalue in question, together with (4.5) and (6.5). For the other eigenvalue branches of $K(is)$, say $\lambda_2(s)$, it may happen that it touches the level line $\lambda = 1$ without crossing it, or that it crosses it but with zero slope. Then we have an eigenvalue of the ZS system with algebraic multiplicity greater than 1. We can actually make this connection more explicit as follows. Replace $q(t)$ by $\mu q(t)$ and let the eigenvalue of (1.1) be $is(\mu)$. Let $\lambda(s)$ be an eigenvalue branch of $K(is)$. Suppose that $\lambda(\hat{s}) = 1$, so $s(1) = \hat{s}$. Then $\mu\lambda(s(\mu)) = 1$ for the eigenvalue branch of $\mu K(is)$ emanating from $\hat{s}$ for μ near 1. On differentiating this relation with respect to μ and using $W_s = iW_\xi$, (6.5), and (5.4), we find that (at $\mu = 1$)

$$\dot{\lambda}(s) = -\frac{1}{s_\mu(1)} = \frac{W_s(\hat{s},1)}{W_\mu(\hat{s},1)}.$$

This shows that $\dot{\lambda}(s) = 0$ if and only if the s-derivative of the Wronskian is zero. More details on the connection between the multiplicity of an eigenvalue and the length of the associated Jordan chain can be found in [13, e.g., Theorem 4.10]. Since $\|K(is)\| \to 0$ as $s \to \infty$, which is true under the hypotheses in this paper (and also for $q \in L^p$, $p \geqslant 1$), the eigenvalue branches will converge to zero as $s \to \infty$, but they need not do so monotonically (except for $\lambda_1(s)$). If an eigenvalue branch undergoes multiple crossings at level $\lambda = 1$ for $s \geqslant s_0$, then it will count as 1 in $N_1[K(is_0)]$ but it produces several eigenvalues (as many as there are crossings) of the ZS system with imaginary part greater than s_0. Our strategy is to estimate the number of eigenvalues of the Birman-Schwinger kernel above 1. Then, if we also know that the eigenvalues of the ZS system are simple, at least for s sufficiently small, we can determine the eigenvalue asymptotics as given in (1.8).

We conclude this section with some remarks about a connection between the work in [5] and ours. Fix $s > 0$. Then, in [5], the behavior of $N_1[K(is,\mu q)]$ as $\mu \to \infty$ for $q \in L^1(\mathbb{R})$ is studied by means of the Prüfer transformation (in [5], μ is called $1/\gamma$ and s is called k). It seems to us that Dirichlet-Neumann decoupling would also be a viable approach to this problem, as in the Schrödinger case for the large coupling problem (see e.g., [17]).

8. Eigenvalue asymptotics for long-range potentials

We begin with two general results.

THEOREM 8.1. *Suppose q satisfies Hypothesis 1 and is nonnegative. Fix $s > 0$. Let N be the largest nonnegative integer such that*

$$\int_{\alpha(s)}^{\omega(s)} (q(t) - s)\,dt > \frac{(2N-1)\pi}{2}. \tag{8.1}$$

Then there are at least N purely imaginary eigenvalues with imaginary parts greater than s.

We emphasize that in (8.1) it is not required that $q(t) \geqslant s$ for all $t \in [\alpha(s), \omega(s)]$. If $q \in L^1(\mathbb{R})$ and $s = 0$, the result is known from [12].

Proof. From (2.2a), we conclude that

$$\varphi_0(\omega(s),s)-\varphi_0(\alpha(s),s)\leqslant -\int_{\alpha(s)}^{\omega(s)}(q(t)-s)\,dt.$$

By (2.17), $\varphi_0(\alpha(s),s)\in(-\frac{\pi}{4},0)$. Thus, by (8.1),

$$\varphi_0(\omega(s),s)<-\int_{\alpha(s)}^{\omega(s)}(q(t)-s)\,dt<-\frac{(2N-1)\pi}{2}.$$

Therefore the values of $\psi_k(\omega(s),s)$ for $k=0,-1,\dots,-(N-1)$ all lie above $\varphi_0(\omega(s),s)$. The assertion follows. □

THEOREM 8.2. *Suppose q is locally absolutely integrable on some interval $[-T,T]$, continuous and nonnegative outside $[-T,T]$, and converging to zero as $t\to\pm\infty$. Suppose that $q\notin L^1([T,\infty))$ (or $q\notin L^1((-\infty,-T])$. Then there exist infinitely many purely imaginary eigenvalues.*

If q is not of one sign, then the number of eigenvalues need not be infinite. In the special case when q is odd, there are no eigenvalues (see [12]).

Proof. Let $M\geqslant 0$ be a given integer. We will show that there exists an $s_0>0$ such that $N(s_0)>M$, where $N(s)$ is defined in (1.7). We consider the solution $\varphi_0(t,s)$. Since $q(t)$ is nonnegative on $(-\infty,-T)$, we know that for every $s>0$, $\varphi_0(-T,s)\leqslant 0$. Choose $t_N\geqslant T$ and s_0 such that

$$\int_{-T}^{t_N}(q(t)-s_0)\,dt>\frac{(2M-1)\pi}{2}$$

and $\omega(s_0)>t_N$. By the assumptions, this is certainly possible by first choosing t_N large enough and then choosing s_0 small enough. Then

$$\begin{aligned}\varphi_0(t_N,s_0)&\leqslant\varphi_0(-T,s_0)-\int_{-T}^{t_N}(q(\tau)-s_0)\,d\tau\\&\leqslant-\int_{-T}^{t_N}(q(\tau)-s_0)\,d\tau<-\frac{(2M-1)\pi}{2}.\end{aligned}$$

Then, since $q(t)\geqslant 0$ for $t\geqslant T$, $\varphi_0(t,s_0)<-(2N-1)\pi/2$ for all $t\geqslant t_N$. This means that $\varphi_0(t,s_0)<\psi_k(t,s_0)$ for $k=0,-1,\dots,-(N-1)$. In particular, this is the case at $t=\omega(s_0)$ (just to be consistent with our set-up of the eigenvalue equations in (6.3)). Hence there are at least M eigenvalues. Since M may be arbitrarily large, the theorem is proved. □

We are now able to determine the asymptotic behavior of $N(s)$ for a class of potentials with t^{-1} decay.

THEOREM 8.3. *Suppose that* $0 \leqslant q(t) \leqslant c|t|^{-1}$, *with* $c > 0$, *for* $|t| \geqslant T$, *and some* $T > 0$. *Let* $q(t)$ *be continuous and such that* $q \notin L^1(\mathbb{R})$. *Also, suppose that all imaginary eigenvalues with sufficiently small imaginary part are simple. Then*

$$N(s) \sim \frac{1}{\pi} \int_{\alpha(s)}^{\omega(s)} q(t)\,dt, \qquad s \to 0. \tag{8.2}$$

Proof. Either $\int_T^\infty q(t)\,dt$ or $\int_{-\infty}^{-T} q(t)\,dt$ does not exist as an improper integral. Thus $q(t)$ cannot have compact support and so either $\alpha(s) \to -\infty$ or $\omega(s) \to +\infty$ as $s \to 0$. From the upper bound on $q(t)$, it follows that $\omega(s) \leqslant c_1 s^{-1}$ and $\alpha(s) \geqslant -c_1 s^{-1}$. Hence

$$\int_{\alpha(s)}^{\omega(s)} (q(t) - s)\,dt = \int_{\alpha(s)}^{\omega(s)} q(t)\,dt + O(1).$$

A similar conclusion holds for the integral over $q(t) + s$. Thus, by (2.2a),

$$-\int_{\alpha(s)}^{\omega(s)} (q(t) + s)\,dt \leqslant \varphi_0(\omega(s), s) - \varphi_0(\alpha(s), s) \leqslant -\int_{\alpha(s)}^{\omega(s)} (q(t) - s)\,dt, \tag{8.3}$$

which implies that

$$\varphi_0(\omega(s), s) - \varphi_0(\alpha(s), s) \sim -\int_{\alpha(s)}^{\omega(s)} q(t)\,dt, \qquad s \to 0.$$

Since $\varphi_0(\alpha(s), s) \in (-\frac{\pi}{4}, 0]$, we have that

$$\varphi_0(\omega(s), s) \sim -\int_{\alpha(s)}^{\omega(s)} q(t)\,dt.$$

Thus, for any $\varepsilon \in (0,1)$ and s sufficiently small,

$$-(1+\varepsilon) \int_{\alpha(s)}^{\omega(s)} q(t)\,dt \leqslant \varphi_0(\omega(s), s) \leqslant -(1-\varepsilon) \int_{\alpha(s)}^{\omega(s)} q(t)\,dt.$$

Hence

$$N_-(s) \leqslant N(s) \leqslant N_+(s), \tag{8.4}$$

where

$$N_\pm(s) = \left\lfloor \frac{1}{2} + (1 \pm \varepsilon) \frac{1}{\pi} \int_{\alpha(s)}^{\omega(s)} q(t)\,dt \right\rfloor. \tag{8.5}$$

Now (8.2) follows from (8.4)–(8.5). □

The next corollary immediately follows from the theorem.

COROLLARY 8.4. *Suppose that* $q(t) \sim b_+ t^{-1}$ *as* $t \to +\infty$ *and* $q(t) \sim b_- |t|^{-1}$ *as* $t \to -\infty$, *with* $b_+ > 0$ *and* $b_- > 0$. *In addition, assume that all imaginary eigenvalues with small enough imaginary part are simple. Then*

$$N(s) \sim \frac{b_+ + b_-}{\pi} \ln(s^{-1}), \qquad s \to 0. \tag{8.6}$$

We note that the proof of Theorem 8.3 works because the lower and upper bounds in (8.3) have the same leading asymptotic behavior. For power-law potentials of the form $q(t) \sim t^{-\gamma}$ $(0 < \gamma < 1)$, this is not the case. Evaluating the lower and upper bounds in (8.3) gives

$$\int_{\alpha(s)}^{\omega(s)} (q(t) - s)\,dt = \frac{\gamma}{1-\gamma} s^{1-\frac{1}{\gamma}}(1 + o(1)),$$
$$\int_{\alpha(s)}^{\omega(s)} (q(t) + s)\,dt = \frac{2-\gamma}{1-\gamma} s^{1-\frac{1}{\gamma}}(1 + o(1)).$$

This gave us the hint that the true behavior might be somewhere between. In fact, as (1.8) shows, the integrand in the correct formula is the geometric mean of the integrands above.

We now come to the second major hypothesis in this paper.

HYPOTHESIS 2. Suppose that $q(t)$ has the following properties:
For some $\gamma \in (0,1)$, suitable constants $c_1,\ c_2,\ c_3 > 0$, and $T > 0$,
(i) $c_1|t|^{-\gamma} \leqslant q(t) \leqslant c_2|t|^{-\gamma}, \qquad |t| \geqslant T,$
(ii) $|q(t_1) - q(t_2)| \leqslant c_3[\min\{|t_1|,|t_2|\}]^{-\gamma-1}|t_1 - t_2| \qquad |t_1|,|t_2| \geqslant T.$

These conditions are essentially the same as those in [20, Theorem XIII.82]. except that here $q(t)$ is also required to be positive. The point of the restriction $|t| \geqslant T$ is that we want to give ourselves some freedom to choose the potential on the interval $[-T,T]$. Of course, in order to determine the asymptotics of $N(s)$, we also must be sure that the imaginary eigenvalues with small imaginary parts are simple and the best result we have in this regard is Theorem 6.6. Note that its assumptions are only marginally stronger than those of Hypothesis 2. The difference is that in Theorem 6.6 we require a somewhat more specific asymptotic behavior of q (cf. (6.17)–(6.19)). Under Hypothesis 2 alone, the potential would be allowed to have up and down swings (limited by the upper and lower bounds in (i)) even for large $|t|$. We think of Hypothesis 2 as a benchmark against which theorems like Theorem 6.6 can be measured, and it seems to us that there is still room for improvement in the results of Section 6. It is also clear that, as with Theorem 6.6, we can allow potentials with different γ on $t \geqslant T$ and $t \leqslant -T$, or potentials which are 0 on one side. Also, note that (ii) is a Lipschitz condition and implies that q is absolutely continuous. Hence q is differentiable a.e. and has a derivative of order $O(|t|^{-\gamma-1})$ as $|t| \to \infty$. In order to continue with the discussion of power-law potentials and to implement the decoupling method, we need to know the imaginary eigenvalues for the Dirichlet and Neumann problems on a finite interval $[0,d]$ for a constant potential q. The infinite discrete spectrum on the real axis does not concern us here. The eigenvalues $\xi_j = is_j$ in $\mathbb{C}^+$ are the same for both problems, except for a minor difference:

$$\text{Eigenvalues in } \mathbb{C}^+: \quad s_j = \left(q^2 - \frac{\pi^2 j^2}{d^2}\right)^{1/2},$$

where

$$j=1,2,\ldots,\left\lfloor\frac{qd}{\pi}\right\rfloor \qquad \text{Dirichlet b.c.,}$$
$$j=0,1,2,\ldots,\left\lfloor\frac{qd}{\pi}\right\rfloor \qquad \text{Neumann b.c..}$$

Since q is constant, eigenvalues of the Neumann or Dirichlet problem are simple, so $N_1[K^{\mathrm{N,D}}_{[0,d]}(is,q)]$ gives us the number of eigenvalues with imaginary part greater than s. Then

$$\frac{d}{\pi}(q^2-s^2)^{1/2}-1\leqslant N_1[K^{\mathrm{N,D}}_{[0,d]}(is,q)]\leqslant\frac{d}{\pi}(q^2-s^2)^{1/2}+1. \tag{8.7}$$

For proving the main result, Theorem 8.9, we have found it convenient to replace s by a suitable sequence $\{s_n\}$ that converges to zero. Specifically, the sequence will have the properties

$$s_n>s_{n+1}>0, \quad \lim_{n\to\infty}s_n=0, \quad \lim_{n\to\infty}\frac{s_n}{s_{n+1}}=1. \tag{8.8}$$

It will suffice to consider the eigenvalue problem on $[1,\infty)$ with Neumann or Dirichlet b.c. at 1. So we need to state the next lemma only for this interval. The reason for choosing the left endpoint at $t=1$ is that the proof of Lemma 8.6 (see Appendix B), is set up on the interval $[1,\infty)$ because this makes it notationally more convenient.

LEMMA 8.5. *Suppose q satisfies Hypothesis 2 and let s_n be a sequence as in* (8.8). *Let*

$$h(s)=\int_{\{t:t\geqslant 1,q(t)>s\}}(q(t)^2-s^2)^{1/2}\,dt. \tag{8.9}$$

Then $h(s_n)/h(s_{n+1})\to 1$.

Proof. We write

$$h(s_{n+1})-h(s_n)=A_1+A_2,$$

where

$$A_1=\int_{\{t:t\geqslant 1,q(t)>s_n\}}\left[(q(t)^2-s_{n+1}^2)^{1/2}-(q(t)^2-s_n^2)^{1/2}\right]dt, \tag{8.10}$$
$$A_2=\int_{\{t:t\geqslant 1,s_n\geqslant q(t)>s_{n+1}\}}(q(t)^2-s_{n+1}^2)^{1/2}\,dt. \tag{8.11}$$

Using the fact that the measure of the set $\{t:t\geqslant 1,q(t)>s_n\}$ is not greater than $\left(\frac{c_2}{s_n}\right)^{1/\gamma}$, together with the inequality

$$(q(t)^2-s_{n+1}^2)^{1/2}-(q(t)^2-s_n^2)^{1/2}\leqslant(s_n^2-s_{n+1}^2)^{1/2}, \qquad q(t)>s_n>s_{n+1},$$

we conclude from (8.10) that

$$|A_1| \leqslant c_2^{1/\gamma} s_n^{1-1/\gamma} \left(1 - \frac{s_{n+1}^2}{s_n^2}\right)^{1/2} = o(s_n^{1-1/\gamma}).$$

Considering A_2 in (8.11), we see that the set $\{t : t \geqslant 1, s_n \geqslant q(t) > s_{n+1}\}$ has measure bounded by

$$\left(\frac{c_2}{s_{n+1}}\right)^{1/\gamma} - \left(\frac{c_1}{s_n}\right)^{1/\gamma}.$$

Therefore, since the integrand is bounded by $(s_n^2 - s_{n+1}^2)^{1/2}$, we obtain

$$|A_2| \leqslant c_1^{1/\gamma} s_n^{1-1/\gamma} \left(1 - \frac{s_{n+1}^2}{s_n^2}\right)^{1/2} \left[\left(\frac{c_2 s_n}{c_1 s_{n+1}}\right)^{1/\gamma} - 1\right] = o(s_n^{1-1/\gamma}).$$

We conclude that

$$h(s_{n+1}) - h(s_n) = o(s_n^{1-1/\gamma}).$$

Since $h(s)$ is bounded below by $cs^{1-\frac{1}{\gamma}}$, the conclusion of the lemma follows. $\square$

In the next step we will estimate both sides of (8.9) and approximate them by integrals. We must be careful to keep the number of partition intervals under control, since each interval introduces an error of ± 1 in the eigenvalue count. So, we must make sure that the number of partition points is $o(s^{1-\frac{1}{\gamma}})$.

The sequence s_n we are going to use is defined as follows. We lie down a chain of knots given by

$$t_n = n^\delta \tag{8.12}$$

where δ is restricted by the following inequalities

$$\frac{1}{1-\gamma} < \delta < \frac{1}{1-2\gamma} \qquad \text{if } 0 < \gamma < \frac{1}{2}, \tag{8.13}$$

$$\delta > \frac{1}{1-\gamma} \qquad \text{if } \frac{1}{2} \leqslant \gamma < 1. \tag{8.14}$$

Then we define

$$s_n = c_1 t_n^{-\gamma} = c_1 n^{-\delta\gamma}. \tag{8.15}$$

Here c_1 is the constant in Hypothesis 2 (i), so $q(t_n) = s_n$.

Let

$$q^+(t) = \max q(t), \qquad t_k \leqslant t \leqslant t_{k+1},$$
$$q^-(t) = \min q(t), \qquad t_k \leqslant t \leqslant t_{k+1}.$$

Let $h^\pm(s)$ be given by (8.9) but with $q(t)$ replaced by $q^\pm(t)$.

LEMMA 8.6. *Suppose $q(t)$ satisfies Hypothesis 2. Let s_n be as in (8.15). Then*

$$\lim_{n\to\infty} h(s_n)/h^{\pm}(s_n) = 1.$$

Proof. The proof is given in Appendix B. □

REMARK. For the proof of Theorem 8.7 below it would suffice to only know that $h^+(s_n)/h^-(s_n) \to 1$. It turns out that the simplifications in the proof of Lemma 8.6 would be minimal, so we decided to prove the lemma as stated.

THEOREM 8.7. *Assume Hypothesis 2 and let q be positive and continuous. Then*

$$N_1[K(is,q)] \sim \pi^{-1}H(s), \qquad s \to 0, \tag{8.16}$$

where

$$H(s) = \int_{\{t:q(t)>s\}} (q(t)^2 - s^2)^{1/2}\,dt.$$

If, in addition, there is an $s_0 > 0$ such that all imaginary eigenvalues with $\operatorname{Im}\xi < s_0$ are simple, then

$$N(s) \sim \pi^{-1}H(s), \qquad s \to 0. \tag{8.17}$$

Proof. First we observe that, by (7.15),

$$N_1[K^{\mathrm{D}}_{\mathbb{R}\setminus\{-T,T\}}(is,q)] \leqslant N_1[K(is,q)] \leqslant N_1[K^{\mathrm{N}}_{\mathbb{R}\setminus\{-T,T\}}(is,q)]. \tag{8.18}$$

Clearly,

$$\begin{aligned} &N_1[K^{\mathrm{N}}_{\mathbb{R}\setminus\{-T,T\}}(is,q)] \\ &= N_1[K^{\mathrm{N}}_{(-\infty,-T]}(is,q)] + N_1[K^{\mathrm{N}}_{[-T,T]}(is,q)] + N_1[K^{\mathrm{N}}_{[T,\infty)}(is,q)]. \end{aligned} \tag{8.19}$$

There is a similar relation for the Dirichlet case. We now only discuss the Neumann b.c. and, when needed, state the results for Dirichlet b.c.

First, we note that the middle term on the right is bounded as $s \to 0$. To get an explicit bound, replace q by zero outside $[-T,T]$ and denote the truncated potential by $\widetilde{q}$. Then, by (7.15),

$$N_1[K^{\mathrm{N}}_{[-T,T]}(is,q)] = N_1[K^{\mathrm{N}}_{\mathbb{R}\setminus\{-T,T\}}(is,\widetilde{q})] \leqslant N_1[K(is,\widetilde{q})] + 2.$$

The number $N_1[K(is,\widetilde{q})]$ can be estimated in terms of the Hilbert-Schmidt norm of the Birman-Schwinger kernel, namely (cf. [13, Eq.(4.6)])

$$N_1[K(is,\widetilde{q})] \leqslant \|K(is,\widetilde{q})\|^2_{H.S.} \leqslant \left(\int_{-T}^{T} q(t)\,dt\right)^2.$$

Hence this term will not affect the asymptotics. We now focus on $N_1[K^{\mathrm{N}}_{[T,\infty)}(is,q)]$ but, for the analysis, replace it by $N_1[K^{\mathrm{N}}_{[1,\infty)}(is,q)]$. This can be done without loss and the reason for doing so was already mentioned in connection with Lemma 8.5. We now use the partition in terms of the points t_n and the sequence s_n defined in (8.12) and (8.15). Given s_n, the partition points are $1, 2^\delta, \ldots, \widetilde{t}_n$, where $\widetilde{t}_n = (n+k_n)^\delta = (s_n/c_2)^{-1/\gamma}$. Set $\Delta_j = [t_j, t_{j+1}]$, $j = 1, \ldots, n+k_n-1$. The interval $[t_{n+k_n}, \infty)$ does not contribute to the eigenvalue count because the potential on this interval is less than s. Note that $\omega(s_n) \leqslant \widetilde{t}_n$. By using (7.13) and successively inserting the points t_j ($j = 2, \ldots, n+k_n$) in $[1,\infty)$, together with Lemma 7.5, we have (with $s = s_n$)

$$N_1[K^{\mathrm{N}}_{[1,\infty)}(is_n, q)] \leqslant \sum_{j=1}^{n+k_n-1} N_1[K^{\mathrm{N}}_{\Delta_j}(is_n, q^+)]. \tag{8.20}$$

Note that there are $\lceil (s_n/c_2)^{-1/(\delta\gamma)} \rceil$ partition points. Since $\delta > (1-\gamma)^{-1}$ by (8.13), (8.14), this number is $o(s_n^{1-1/\gamma})$. Thus for Neumann b.c., the right-hand side of (8.20) is equal to

$$\sum_{j=1}^{n+k_n-1} N_1[K^{\mathrm{N}}_{\Delta_j}(is_n, q^+)] = \pi^{-1} h^+(s_n) + o(s_n^{-1/\gamma}). \tag{8.21}$$

The error term comes from the ± 1 summands in (8.7). Dividing (8.21) by $\pi^{-1}h(s_n)$, taking $n \to \infty$, and using (8.20) together with Lemma 8.6 leads to

$$\limsup_{n\to\infty} \left\{ \pi h(s_n)^{-1} N_1[K^{\mathrm{N}}_{[1,\infty)}(is_n, q)] \right\} \leqslant 1.$$

Now we need to show that this also holds if $s \to 0$ continuously and not through a special sequence. Given $s > 0$ (small enough) choose s_n such that $s_{n+1} \leqslant s < s_n$. Then

$$\begin{aligned}
\pi h(s)^{-1} N_1[K^{\mathrm{N}}_{[1,\infty)}(is, q)] &\leqslant \pi h(s_n)^{-1} N_1[K^{\mathrm{N}}_{[1,\infty)}(is, q)] \\
&\leqslant \pi h(s_n)^{-1} \sum_{j=1}^{n+k_n-1} N_1[K^{\mathrm{N}}_{\Delta_j}(is, q^+)] \\
&\leqslant \pi h(s_n)^{-1} \sum_{j=1}^{n+k_n-1} N_1[K^{\mathrm{N}}_{\Delta_j}(is_{n+1}, q^+)] \\
&\leqslant \pi h(s_n)^{-1} [\pi^{-1} h^+(s_{n+1}) + o(s_{n+1}^{-1/\gamma})].
\end{aligned}$$

Here we have used, in order, that $h(s)$ is decreasing, eq. (8.20), that for a constant potential with Neumann b.c. the number of imaginary eigenvalues with imaginary part greater than s increases as s decreases, and (8.21). Taking $n \to \infty$, we get, with the help of Lemmas 8.5 and 8.6,

$$\limsup_{s\to 0} \left\{ \pi h(s)^{-1} N_1[K^{\mathrm{N}}_{[1,\infty)}(is, q)] \right\} \leqslant 1. \tag{8.22}$$

Similarly, we argue that

$$\liminf_{s\to 0} \left\{ \pi h(s)^{-1} N_1[K^{\mathrm{D}}_{[1,\infty)}(is, q)] \right\} \geqslant 1. \tag{8.23}$$

We can do a similar analysis on the interval $(-\infty,0]$. Then, referring back to (8.19) and using (8.22), (8.23), we have shown that

$$\limsup_{s\to 0}\left\{\pi H(s)^{-1}N_1[K^{\mathrm{N}}_{\mathbb{R}\setminus\{-T,T\}}(is,q)]\right\}\leqslant 1. \tag{8.24}$$

and

$$\liminf_{s\to 0}\left\{\pi H(s)^{-1}N_1[K^{\mathrm{D}}_{\mathbb{R}\setminus\{-T,T\}}(is,q)]\right\}\geqslant 1. \tag{8.25}$$

Using (8.24) and (8.25) in (8.18), we obtain

$$\lim_{s\to 0}\left\{\pi H(s)^{-1}N_1[K(is,q)]\right\}=1.$$

Thus (8.16) is proved. The second assertion, (8.17), is a consequence of the additional assumptions. □

COROLLARY 8.8. *Suppose q satisfies Hypothesis 2 outside some interval $[-T,T]$. On $[-T,T]$, let $q\in L^1$. Then (8.16) holds true provided we restrict the integration to $|t|>T$. Also, (8.17) is true under the stated assumptions.*

Proof. We cannot use the Birman-Schwinger kernel, since q need not be positive on $[-T,T]$. So we must bridge the gap from $-T$ to T by using a Prüfer argument. We can also truncate the potential at $-T$ and replace it by zero on $[-T,\infty)$. Then, for the truncated problem, we can use the Birman-Schwinger kernel and make use of (8.16). We can then deduce that the Prüfer angle $\varphi_0(t,s)$ obeys

$$\varphi_0(-T,s)\sim -\int_{\{t:t<-T,q(t)>s\}}(q(t)^2-s^2)^{1/2}\,dt,\qquad s\to 0. \tag{8.26}$$

Similarly, for $\psi_k(t,s)$, we have

$$\psi_k(T,s)-\frac{(2k-1)\pi}{2}\sim\int_{\{t:t>T,q(t)>s\}}(q(t)^2-s^2)^{1/2}\,dt,\qquad s\to 0.$$

In fact any solution $\theta(t,s)$ has the property that (with $\theta(\infty,s)=\lim_{t\to\infty}\theta(t,s)$)

$$\theta(T,s)-\theta(\infty,s)\sim\int_{\{t:t>T,q(t)>s\}}(q(t)^2-s^2)^{1/2}\,dt,\qquad s\to 0. \tag{8.27}$$

Note that $q(t)>0$ for $t>T$, so that $\theta(T,s)$ is bigger than $\theta(\infty,s)$ for small s. In going from $-T$ to T the change in $\varphi_0(t,s)$, that is, $\varphi_0(T,s)-\varphi_0(-T,s)$, is bounded for small s as is easily seen from (2.2a) (actually it is bounded for all s by Lemma 4.3 in [5]). Combining this with (8.26) and (8.27), we conclude that (8.16) holds and that (8.17) follows as before. □

Appendix A: Proof of Theorem 6.6 ($0 < \gamma \leqslant 1/2$)

When t is near $\omega(s)$, we will transform (1.1) into a perturbed Airy equation. The motivation for this approach is the following. At the point $t = \omega(s)$, $q(t) = s$, and the matrix in (1.1) has the form

$$\begin{pmatrix} s & s \\ -s & -s \end{pmatrix}.$$

This matrix has eigenvalue 0 and is similar to a Jordan block. Thus, the phase curves are straight lines parallel to the eigenvector $(1,-1)^T$. Now, for $t < \omega(s)$, assuming $q(t)$ is larger than s, the eigenvalues of the matrix are $\pm i(q(t)^2 - s^2)^{1/2}$ and thus imaginary. This means oscillatory behavior. So we have the picture that, as t decreases from $\omega(s)$, the system has to transition out of the degenerate state at $t = \omega(s)$ into the oscillatory state. This will require a "short" adjustment period. It turns out that the oscillatory behavior really starts at $t = \omega(s) - 1/(2s)$ so that the transition interval is of order $1/s$. Hence this interval expands as $s \to 0$ but at a slower rate than $\omega(s)$, which grows like $s^{-1/\gamma}$. The analysis will show that the transition is of a type typically described by Airy functions whose behavior changes from exponential to oscillatory as the argument passes through zero from positive to negative values.

Proof. We start by converting the problem into a form that is suited for this analysis. For $0 \leqslant t \leqslant \omega(s)$, set

$$u = \frac{t}{\omega(s)}, \qquad 0 \leqslant u \leqslant 1, \tag{A.1}$$

and

$$f(u,s) = v(t, is). \tag{A.2}$$

$$f' = \lambda(s) \begin{pmatrix} 1 & s^{-1}q(u\,\omega(s)) \\ s^{-1}q(u\,\omega(s)) & -1 \end{pmatrix} f, \qquad \lambda(s) = s\,\omega(s).$$

Now set

$$z = 1 - u, \qquad g(z,s) = f(u,s), \qquad g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{A.3}$$

so that

$$g'(z,s) = -\lambda(s) \begin{pmatrix} 1 & s^{-1}q((1-z)\omega(s)) \\ -s^{-1}q((1-z)\omega(s)) & -1 \end{pmatrix} g.$$

(Here the prime denotes differentiation with respect to z.)

Define

$$h(z,s) = s^{-1}q((1-z)\omega(s)) \tag{A.4}$$

and make the substitution

$$\xi(z,s) = g_1(z,s)h(z,s)^{-1/2} \tag{A.5}$$

for the first component of $g(z,s)$. Then $\xi(z,s)$ obeys the second-order differential equation

$$\xi'' + \lambda(s)^2 p(z,s)\,\xi + g(z,s)\,\xi = 0, \tag{A.6}$$

where

$$p(z,s) = h(z,s)^2 - 1 - \frac{h'(z,s)}{\lambda(s)h(z,s)}, \tag{A.7}$$

$$g(z,s) = -\frac{3h'(z,s)^2}{4\,h(z,s)^2} + \frac{h''(z,s)}{2h(z,s)}. \tag{A.8}$$

In order to extract information about the behavior of $p(z,s)$ for small s we write it out explicitly using (6.17):

$$\begin{aligned} p(z,s) &= -1 + s^{-2}q((1-z)\omega(s))^2 + \frac{q'((1-z)\omega(s))}{sq((1-z)\omega(s))} \\ &= -1 + \frac{(1+\varepsilon_0((1-z)\omega(s)))^2}{(1-z)^{2\gamma}(1+\varepsilon_0(\omega(s)))^2} - \frac{\gamma}{\lambda(s)(1-z)}\frac{1+\varepsilon_1((1-z)\omega(s))}{1+\varepsilon_0((1-z)\omega(s))} \\ &= -1 + \frac{1}{(1-z)^{2\gamma}}(1+o(1)) - \frac{\gamma}{\lambda(s)(1-z)}(1+o(1)). \end{aligned} \tag{A.9}$$

Since $\lambda(s) \sim q_0^{1/\gamma} s^{1-1/\gamma} \to \infty$, we deduce that

$$\lim_{s\to 0} p(z,s) = -1 + \frac{1}{(1-z)^{2\gamma}} \overset{\text{def}}{=} p_0(z)$$

for every $z \in [0,1)$, uniformly in z on any compact subinterval of $[0,1)$. Similarly one shows that $p_z(z,s)$ converges uniformly on such intervals to the derivative of $p_0(z)$. More precisely,

$$p_z(z,s) = \frac{2\gamma}{(1-z)^{2\gamma+1}}(1+o(1)), \qquad s \to 0. \tag{A.10}$$

First, substituting $z=0$ in (A.9), gives

$$p(0,s) = -\frac{\gamma}{\lambda(s)}(1+o(1)), \tag{A.11}$$

which is negative. From the properties of $p(z,s)$, in particular (A.10) and (A.11), we deduce that $p(z,s)$ (in the variable z) has a unique root, called $p_1(s)$, located at

$$p_1(s) = \frac{1}{2\lambda(s)}(1+o(1)). \tag{A.12}$$

The point $z = p_1(s)$ is a turning point for (A.6). In the t variable it is approximately located at $\omega(s) - 1/(2s)$. It represents the point mentioned at the start of the proof where the behavior of the solutions changes from exponential to oscillatory. This will become clear shortly.

We introduce a new independent variable, called ζ, by

$$\zeta(z,s) = \left(\frac{3}{2}\int_{p_1(s)}^{z} \sqrt{p(\eta,s)}\,d\eta\right)^{2/3}, \qquad p_1(s) \leqslant z \leqslant 1. \tag{A.13}$$

$$\zeta(z,s) = -\left(\frac{3}{2}\int_z^{p_1(s)} \sqrt{-p(\eta,s)}\,d\eta\right)^{2/3}, \qquad 0 \leqslant z \leqslant p_1(s). \tag{A.14}$$

Let

$$\widehat{p}(z,s) = \frac{p(z,s)}{\zeta(z,s)}, \tag{A.15}$$

$$\xi(z,s) = \widehat{p}(z,s)^{-1/4}\,\Xi(\zeta,s),$$

and transform (A.6) into

$$\Xi_{\zeta\zeta}(\zeta,s) = \left[-\lambda(s)^2\zeta + \psi(\zeta,s)\right]\Xi(\zeta,s),$$

with

$$\psi(\zeta,s) = -\frac{\zeta\, g(z,s)}{p(z,s)} + \frac{5}{16\zeta^2} + \left\{4p(z,s)p_{zz}(z,s) - 5p_z(z,s)^2\right\}\frac{\zeta}{16p(z,s)^3},$$

where the relations

$$\zeta'(z,s) = \widehat{p}(z,s)^{1/2} \tag{A.16}$$

and (A.4)–(A.8) were used. The differential equation (A.6) has solutions of the form

$$\xi^{(1)}(z,s) = \widehat{p}(z,s)^{-1/4}\{\mathrm{Bi}(-\lambda^{2/3}\zeta) + \varepsilon^{(1)}(z,s)\} \tag{A.17}$$

$$\xi^{(2)}(z,s) = \widehat{p}(z,s)^{-1/4}\{\mathrm{Ai}(-\lambda^{2/3}\zeta) + \varepsilon^{(2)}(z,s)\} \tag{A.18}$$

where $\lambda = \lambda(s),\ \zeta = \zeta(z,s)$.

Accompanying (A.17), (A.18) is the error-control function [19, Ch.11, Sec.3, p. 399]

$$H(z,s) = \int_{p_1(s)}^{z}\left\{\frac{1}{|p(z,s)|^{1/4}}\frac{d^2}{dz^2}\left(\frac{1}{|p(z,s)|^{1/4}}\right) + \frac{g(z,s)}{|p(z,s)|^{1/2}} - \frac{5|p(z,s)|^{1/2}}{16|\zeta|^3}\right\}dz. \tag{A.19}$$

It enters into the error bounds through its total variation over $[p_1(s),z]$, resp., $[0,p_1(s)]$. Let $\mathscr{V}$ denote the variational operator, which, for $[p_1(s),z]$, is given by $\mathscr{V}_{[p_1(z),z]}(H) = \int_{p_1(s)}^{z}|H_\eta(\eta,s)|\,d\eta$. Then

$$\frac{|\varepsilon^{(1)}(z,s)|}{M(-\lambda^{2/3}\zeta)}, \frac{|\partial\varepsilon^{(1)}(z,s)/\partial z|}{\lambda^{2/3}\widehat{p}(z,s)^{1/2}N(-\lambda^{2/3}\zeta)}$$

$$\leqslant \frac{E(-\lambda^{2/3}\zeta)}{\kappa}\left[\exp\left\{\frac{\kappa\mathscr{V}_{[p_1(s),z]}(H(z,s))}{\lambda}\right\} - 1\right]. \tag{A.20}$$

For the definitions and properties of the functions M, N, E, and the constant κ we refer to [19, Chap. 11, Sec. 2, p. 394–397]. There is a similar error bound for $\varepsilon^{(2)}(z,s)$ associated with $\xi^{(2)}$. The only difference is that the factor $E(-\lambda^{2/3}\zeta)$ has to be replaced by $E(-\lambda^{2/3}\zeta)^{-1}$. One small difference between the solutions in (A.17), (A.18) and those in [19] is that the latter have error terms that vanish at different endpoints of an interval. In our case, we have both error terms vanishing at the same point ($z = p_1(s)$). Carrying out the analysis on p. 400 in [19], we have verified that the error bounds in (8) are also valid in our case.

From (A.19), we see that the second derivative of $p(z,s)$ with respect to z enters. This means we need up to three derivatives of $q(t)$, but it also turns out that we do not need the precise asymptotic forms of the second and third derivatives. That's the reason for the conditions in (6.17)–(6.19) as they are stated.

It now seems that we have to determine the coefficients c_1 and c_2 by matching up the function $v(t,is)$ with the general solution to (A.6),

$$\xi(z,s) = c_1\,\xi^{(1)}(z,s) + c_2\,\xi^{(2)}(z,s) \tag{A.21}$$

at $t = \omega(s)$. So we thought, until we realized that this is not necessary. It turns out that the coefficients are not needed in explicit form. They will enter into our analysis only through the norm $\|c\| = (c_1^2 + c_2^2)^{1/2}$.

To proceed with the analysis, we restrict z to an interval $[1-\beta, 1-\alpha]$ with $0 < \alpha < \beta < 1$. The corresponding t-interval is $[\alpha\omega(s), \beta\omega(s)]$ and thus expands as s decreases. As we will show now, it gives the dominant negative contribution to the integral in (6.20). We let

$$I(s;\alpha,\beta) = \int_{\alpha\omega(s)}^{\beta\omega(s)} \frac{|v_1(t,is)|^2}{q(t)}\,dt. \tag{A.22}$$

The factor s multiplying the integral will be taken into account later. We first transform the integral in (A.22) into an integral with respect to the z-variable using

$$v_1(t,is) = s^{-1/2} q(t)^{1/2} \xi(1 - t\omega(s)^{-1}, s). \tag{A.23}$$

This gives

$$I(s;\alpha,\beta) = \frac{\omega(s)}{s} \int_{1-\beta}^{1-\alpha} \xi(z,s)^2\,dz. \tag{A.24}$$

Now we insert (A.21) with $\xi^{(1)}(z,s)$, $\xi^{(2)}(z,s)$ as in (A.17) and (A.18) into (A.24). After a tedious calculation and estimating the error terms, we arrive at

$$I(s;\alpha,\beta) = 2^{-1}\pi^{-1}(\lambda^{-1/3}\omega(s)s^{-1}) \left(\int_{1-\alpha}^{1-\beta} p_0(z)^{-1/2}\,dz + o(1) \right) \|c\|^2, \tag{A.25}$$

where $\|c\|^2 = c_1^2 + c_2^2$. In the derivation of (A.25), we used the asymptotics of the Airy functions together with (A.13)–(A.15). The integrand $p_0(z)^{-1/2}$ arises through a combination of the factor $\widehat{p}(z,s)^{-1/4}$ and a factor $(\lambda^{3/2}\zeta(z,s))^{-1/4}$ from the asymptotics of the Airy functions, and the use of (A.15). By an application of the Riemann-Lebesgue lemma, one sees that the products of the sine and cosine terms from the asymptotics give a contribution that is of lower order than the leading term. This and the other lower order terms are subsumed under the $o(1)$ term in (A.25).

Now we construct the solution $v(t,is)$ at $t=\alpha\omega(s)$. We need to do this for both components. From (A.23), we have

$$\begin{aligned} v_1(\alpha\omega(s),is) &= s^{-1/2}q(\alpha\omega(s))^{1/2}\xi(1-\alpha,s) \\ &= s^{-1/2}q(\alpha\omega(s))^{1/2}\{c_1\xi^{(1)}(1-\alpha,s)+c_2\xi^{(2)}(1-\alpha,s)\}. \end{aligned} \tag{A.26}$$

The following relations will be used (as $s\to 0$)

$$\begin{aligned} q(\alpha\omega(s)) &\sim q_0\alpha^{-\gamma}\omega(s)^{-\gamma}, \\ s^{-1}q(\alpha\omega(s)) &\sim \alpha^{-\gamma} \\ p(1-\alpha,s) &\sim p_0(1-\alpha) = -1+\alpha^{-2\gamma}, \end{aligned}$$

which lead to

$$\xi^{(1)}(1-\alpha,s)=\pi^{-1/2}(-1+\alpha^{-2\gamma})^{-1/4}\lambda^{-1/6}\cos(\psi(s))+o(\lambda^{-1/6})$$

where

$$\psi(s)=\frac{\pi}{4}+\frac{2\lambda}{3}\zeta(1-\alpha,s)^{3/2}=\frac{\pi}{4}+\int_{p_1(s)}^{1-\alpha}\sqrt{p(\eta,s)}\,d\eta.$$

$\xi^{(2)}(1-\alpha,s)$ has the same form except that the cosine must be replaced by a sine. Therefore, by using (A.26), we obtain

$$v_1(\alpha\omega(s),is)=\pi^{-1/2}(1-\alpha^{2\gamma})^{-1/4}\lambda^{-1/6}\{c_1(\cos(\psi(s))+o(1))+c_2(\sin(\psi(s))+o(1))\}.$$

Thus

$$|v_1(\alpha\omega(s),is)|=O(\lambda^{-1/6})\,\|c\|. \tag{A.27}$$

The calculation for the second component uses $v_2=q(t)^{-1}v_1'-sq(t)^{-1}v_1$ and involves (A.23) and (A.13)–(A.16). The result is

$$\begin{aligned} v_2(\alpha\omega(s),is) = -\pi^{-1/2}(1-\alpha^{2\gamma})^{-1/4}\lambda^{-1/6}\Big\{&(c_1\alpha^{\gamma}+c_2(1-\alpha^{2\gamma})^{1/2})\,(\cos(\psi(s))+o(1)) \\ &+(c_2\alpha^{\gamma}-c_1(1-\alpha^{2\gamma})^{1/2})\,(\sin(\psi(s))+o(1))\Big\}. \end{aligned} \tag{A.28}$$

Therefore, combining (A.27) and (A.28) gives

$$\|v(\alpha\omega(s),is)\|=O(\lambda^{-1/6})\|c\|. \tag{A.29}$$

At the point $t_\alpha=\alpha\omega(s)$ the entries of the matrix S^{-1} in (6.15) are bounded in s. In fact,

$$\begin{aligned} \frac{q(t_\alpha(s))}{(q(t_\alpha(s))^2-s^2)^{1/2}} &\to (1-\alpha^{2\gamma})^{-1/2}, \\ \frac{s}{(q(t_\alpha(s))^2-s^2)^{1/2}} &\to \alpha^{\gamma}(1-\alpha^{2\gamma})^{-1/2}. \end{aligned}$$

Hence

$$S\to\begin{pmatrix} b & \overline{b} \\ 1 & 1\end{pmatrix},$$

where

$$b = -\alpha^{\gamma} + i(1-\alpha^{2\gamma})^{1/2},$$

$$S^{-1} \to i2^{-1}(1-\alpha^{2\gamma})^{-1/2}\begin{pmatrix} -1 & \overline{b} \\ 1 & -b \end{pmatrix}.$$

It is therefore clear that

$$\|w(\alpha\omega(s),s)\| = \|S^{-1}v(\alpha\omega(s),is)\| = O(\lambda^{-1/6})\|c\|.$$

Recall the meaning of T from the beginning of the proof of Theorem 6.6. Hence, by (A.29) and Lemma 6.5,

$$\|w(T,s)\| \leqslant \sqrt{2}\|w(\alpha\omega(s),s)\|,$$

and thus

$$\|w(t,s)\| = O(\lambda^{-1/6})\|c\|, \qquad T \leqslant t \leqslant \alpha\omega(s).$$

At T the transformation of $w(t,s)$ back to $v(t,is)$ is bounded, in fact

$$S \to \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \qquad s \to 0.$$

Hence $\|v(t,is)\| = O(\lambda^{-1/6})\|c\|$. As in the first part of the proof in the main body of the paper, we can use Theorem 3.2(ii). Consequently,

$$\int_0^T |v_1||v_2|dt = O(\lambda^{-1/3})\|c\|^2 \tag{A.30}$$

We already know from (A.25) that $sI(s;\alpha,\beta) = O(\lambda^{-1/3}\omega(s))$. In (6.25) this latter term is part of a negative term, so it overwhelms the contribution from (A.30). The theorem is proved. $\square$

Appendix B: Proof of Lemma 8.6

We prove Lemma 8.6 for $h^+(s)$ and comment on the changes needed for $h^-(s)$ at the end of the proof.

Proof. Since $\delta > 1$, the distance between consecutive knots t_n increases. It is straightforward to show that

$$\delta k^{\delta-1} \leqslant t_{k+1} - t_k \leqslant \delta 2^{\delta-1}k^{\delta-1},$$

$$t_{k+1} - t_k = \delta k^{\delta-1} + O(k^{\delta-2}), \qquad k \to \infty. \tag{B.1}$$

In view of Hypothesis 2(i) we know that $q(t) > s_n$ for $1 \leqslant t < t_n$. Thus the domain of integration in (8.9) for both $h(s)$ and $h^+(s)$ includes the interval $[1,t_n]$. Since the points t_n are the places where we will insert Neumann or Dirichlet conditions, we want n to be $o(s_n^{1-1/\gamma})$. Since $n = c_1^{1/(\delta\gamma)} s_n^{-1/(\delta\gamma)}$, this requires $\delta > (1-\gamma)^{-1}$, which, according to (8.13), (8.14), is satisfied for all $\gamma \in (0,1)$. In addition to t_n, we introduce the points

$$\widetilde{t}_n = c_2^{1/\gamma} s_n^{-1/\gamma} = \left(\frac{c_2}{c_1}\right)^{1/\gamma} t_n.$$

Then $q(t) < c_2 \widetilde{t}_n^{\,-\gamma} = s_n$ for $t > \widetilde{t}_n$ and thus

$$[1, t_n] \subset \{t : q(t) \geqslant s_n\} \subset [1, \widetilde{t}_n].$$

On $[t_n, \widetilde{t}_n]$, the difference $q(t) - s_n$ may change sign. So it is natural to split the proof into two parts: First we consider the interval $[1, t_n]$ and then the interval $[t_n, \widetilde{t}_n]$. The latter will be subdivided further depending on the sign of $q(t) - s_n$.

Step 1: Since the function $q^+(t)$ is constant on each interval $[t_k, t_{k+1}]$, we also use the notation q_k^+ for the value of $q^+(t)$ on $[t_k, t_{k+1}]$.

On the interval $[t_k, t_{k+1}]$, we have

$$((q_k^+)^2 - s_n^2)^{1/2} - (q(t)^2 - s_n^2)^{1/2} \leqslant \frac{(q_k^+)^2 - q(t)^2}{[(q_k^+)^2 - s_n^2]^{1/2} + (q(t)^2 - s_n^2)^{1/2}}$$

$$\leqslant \frac{(q_k^+)^2 - q(t)^2}{[(q_k^+)^2 - s_n^2]^{1/2}} \leqslant \frac{2c_2 c_3 t_k^{-2\gamma-1}(t_{k+1} - t_k)}{c_1 (t_k^{-2\gamma} - t_n^{-2\gamma})^{1/2}} \leqslant \frac{2c_2 c_3 t_k^{-\gamma-1} t_n^{\gamma}(t_{k+1} - t_k)}{c_1 (t_n^{2\gamma} - t_{n-1}^{2\gamma})^{1/2}}. \tag{B.2}$$

Here we have used (i) and (ii), and in particular the fact that $q_k^+ \geqslant q(t_k) \geqslant c_1 t_k^{-\gamma}$, to obtain the third inequality. In the last step, we have used that $t_k < t_n$ for $k = 1, 2, \ldots, n-1$. Now

$$\frac{t_n^{\gamma}}{(t_n^{2\gamma} - t_{n-1}^{2\gamma})^{1/2}} = \frac{\sqrt{n}}{\sqrt{2\gamma\delta}} + O(n^{-1/2}) \tag{B.3}$$

and from (B.1), we obtain

$$t_k^{-\gamma-1}(t_{k+1} - t_k) = \delta k^{-\delta\gamma-1} + O(k^{-\delta\gamma-2}). \tag{B.4}$$

Combining (B.2) through (B.4) and adding up the contributions for $n-1$ intervals, we get

$$\int_1^{t_n} \left[(q^+(t)^2 - s_n^2)^{1/2} - (q(t)^2 - s_n^2)^{1/2} \right] dt \leqslant c_4 \sqrt{n} \sum_{k=1}^{n-1} k^{-\delta\gamma+\delta-2}. \tag{B.5}$$

Note that an additional factor $k^{\delta-1}$ comes from the length of the interval $[t_k, t_{k+1}]$ (see (B.1)). By (8.13), (8.14), since $-\delta\gamma + \delta - 2 > -1$, the series in (B.5) is not summable if the summation were extended to infinity. This implies that

$$\text{r.h.s. of (B.5)} = O(n^{-\delta\gamma+\delta-\frac{1}{2}}) = O(s_n^{1-\frac{1}{\gamma}+\frac{1}{2\delta\gamma}}) = o(s_n^{1-\frac{1}{\gamma}}),$$

as desired.

Step 2: Consider the interval $[t_n, \widetilde{t_n}]$. Since the point $\widetilde{t_n}$ may not be equal to one of the t_k, we choose k_n to be the smallest nonnegative integer such that $\widetilde{t_n} \leqslant t_{n+k_n}$. Thus $t_{n+k_n-1} < \widetilde{t_n}$ and

$$\widetilde{t_n}^{\frac{1}{\delta}} - n \leqslant k_n < \widetilde{t_n}^{\frac{1}{\delta}} - n + 1.$$

Then the interval $(\widetilde{t_n}, t_{n+k_n}]$ (if nonempty) is a small overshoot on which $q(t) < s_n$. We keep this interval for notational convenience even though it makes no contribution to the integral in (8.9). Let

$$I_{n,k} = [t_{n+k}, t_{n+k+1}], \qquad k = 0, \dots, k_n - 1.$$

Then

$$[t_n, \widetilde{t_n}] \subset \bigcup_{k=0}^{k_n-1} [t_{n+k}, t_{n+k+1}].$$

Choose a number $\rho > 0$. It will be restricted further later. Then the index set $\mathscr{J} = \{0, \dots, k_n - 1\}$ contains two disjoint subsets B_1 and B_2 defined by

$$B_1 = \{k \in \mathscr{J} : 0 < ((q_{n+k}^+)^2 - s_n^2)^{1/2} < \rho\},$$

$$B_2 = \{k \in \mathscr{J} : ((q_{n+k}^+)^2 - s_n^2)^{1/2} \geqslant \rho\}.$$

For $k \notin B_1 \cup B_2$, we have that $q(t) \leqslant s_n$ on the entire interval $I_{n,k}$. To eliminate these intervals from further consideration we define

$$S_{+,n} = \{t \in [t_n, \widetilde{t_n}] : q(t) > s_n\}.$$

Thus the intervals $I_{n,k}$ relevant to us are those which have a nonempty intersection with $S_{+,n}$. The reason for choosing the sets B_1 and B_2 is that on B_1 the estimates are very simple and on B_2 the same estimates as in Step 1 can be used. For $k \in B_1$ and $t \in I_{n,k} \cap S_{+,n}$, we have that

$$(q_{n+k}^+)^2 - s_n^2)^{1/2} - (q(t)^2 - s_n^2)^{1/2} < \rho.$$

The estimate also holds (trivially) when $t \in I_{n,k} \setminus S_{+,n}$ and the term $(q(t)^2 - s_n^2)^{1/2}$ is absent, so we have

$$\int_{I_{n,k}} ((q_{n+k}^+)^2 - s_n^2)^{1/2}\, dt - \int_{I_{n,k} \cap S_{+,n}} (q(t)^2 - s_n^2)^{1/2}\, dt < \rho (t_{n+k+1} - t_{n+k}).$$

Therefore, adding up the contributions for $k \in B_1$, we obtain

$$\sum_{k \in B_1} \left(\int_{I_{n,k}} (q^+(t)^2 - s_n^2)^{1/2}\, dt - \int_{I_{n,k} \cap S_{+,n}} (q(t)^2 - s_n^2)^{1/2}\, dt \right)$$
$$< \rho(\tilde{t}_n - t_n) = \rho(c_2^{\frac{1}{\gamma}} - c_1^{\frac{1}{\gamma}}) s_n^{-\frac{1}{\gamma}}. \tag{B.6}$$

We want to make this $o(s_n^{1-\frac{1}{\gamma}})$ and therefore choose $\rho = s_n^{1+\sigma}$ for some $\sigma > 0$ yet to be determined. For $k \in B_2$ and $t \in I_{n,k} \cap S_{+,n}$, the estimates are similar to those in Step 1:

$$\begin{aligned}
((q_{n+k}^+)^2 - s_n^2)^{1/2} - (q(t)^2 - s_n^2)^{1/2} &\leqslant \frac{(q_{n+k}^+)^2 - q(t)^2}{((q_{n+k}^+)^2 - s_n^2)^{1/2}} \\
&\leqslant \rho^{-1}(q_{n+k}^+ - q(t))(q_{n+k}^+ + q(t)) \\
&\leqslant 2\rho^{-1} c_2 c_3 t_{n+k}^{-2\gamma-1}(t_{n+k+1} - t_{n+k}) \\
&\leqslant c_5 \rho^{-1}(n+k)^{-\delta(2\gamma+1)}(n+k)^{\delta-1} \\
&= c_5 \rho^{-1}(n+k)^{-2\delta\gamma-1}.
\end{aligned}$$

Thus for $k \in B_2$, we obtain

$$\int_{I_{n,k}} ((q_{n+k}^+)^2 - s_n^2)^{1/2}\, dt - \int_{I_{n,k}\cap S_{+,n}} (q(t)^2 - s_n^2)^{1/2}\, dt \leqslant c_6 \rho^{-1}(n+k)^{-2\delta\gamma+\delta-2}. \quad \text{(B.7)}$$

Here again, a factor $(n+k)^{\delta-1}$ has been included to account for the length of the integration interval. Again, the estimate also holds when $t \in I_{n,k} \setminus S_{+,n}$ and the second integral is absent. Under the assumptions in (8.13), (8.14), $-2\delta\gamma + \delta - 2 < -1$ for all $\gamma \in (0,1)$. Thus

$$\sum_{k\in B_2} (n+k)^{-2\delta\gamma+\delta-2} = O(n^{-2\delta\gamma+\delta-1}),$$

which, together with (B.4), gives

$$\begin{aligned}
\sum_{k\in B_2} \left(\int_{I_{n,k}} (q^+(t)^2 - s_n^2)^{1/2}\, dt - \int_{I_{n,k}\cap S_{+,n}} (q(t)^2 - s_n^2)^{1/2}\, dt \right)& \\
\leqslant c_7 \rho^{-1} n^{-2\delta\gamma+\delta-1} = O\left(s_n^{1-\frac{1}{\gamma}+\frac{1}{\delta\gamma}-\sigma} \right).& \quad \text{(B.8)}
\end{aligned}$$

This is $o(s_n^{1-\frac{1}{\gamma}})$ provided we choose

$$0 < \sigma < \frac{1}{\delta\gamma}. \quad \text{(B.9)}$$

Combining (B.5), (B.6), and (B.8) gives

$$h^+(s) - h(s) = o(s_n^{1-\frac{1}{\gamma}}).$$

This proves that $h(s_n)/h^+(s_n) \to 1$ as $s_n \to 0$.

The proof for $h^-(s)$ is similar and essentially contained in the above proof. We replace $q^+(t)$ by $q(t)$ and $q(t)$ by $q^-(t)$. Then, in the first step we consider the interval $[1, t_{n-1}]$ so that $q^-(t_{n-1}) > s_n$. On the interval $[t_{n-1}, \widetilde{t}_n]$ we consider, in place of B_1, the set where $0 < (q(t)^2 - s_n^2)^{1/2} < \rho$ and, in place of B_2, the set where $(q(t)^2 - s_n^2)^{1/2} \geqslant \rho$. The estimates are then similar to those above. $\square$

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