BERNSTEIN FUNCTIONS OF SEVERAL SEMIGROUP GENERATORS
ON BANACH SPACES UNDER BOUNDED PERTURBATIONS, II

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(*Communicated by V. V. Peller*)

Abstract. The paper deals with multidimensional Bochner-Phillips functional calculus. In the previous paper by the author bounded perturbations of Bernstein functions of several commuting semigroup generators on Banach spaces where considered, conditions for Lipschitzness and estimates for the norm of commutators of such functions where proved. Also in the one-dimensional case the Frechet differentiability of Bernstein functions of semigroup generators on Banach spaces where proved and a generalization of Livschits-Kreĭn trace formula derived. The aim of the present paper is to prove the Frechet differentiability of operator Bernstein functions and the Livschits-Kreĭn trace formula in the multidimensional setting.

1. Introduction

Yu. Daletskiĭ and S. G. Kreĭn pioneered the study of the problem of differentiability of functions of self-adjoint operators in [8]. Differential calculus for functions of several commuting Hermitian operators in Hilbert spaces was studied in [11]. For survey and bibliography of the theory that resulted see the article [1]. It should be stressed that all these work deal with Hilbert spaces only. The case of Banach spaces was considered in [33], [18], and [17]. In particular, the last paper by the author was devoted to Bernstein functions of several commuting semigroup generators on Banach spaces (they constitute the subject matter of the so called multidimensional Bochner-Phillips functional calculus). In [17] we gave inter alia conditions for their Lipschitzness and showed that such functions are $\mathscr{J}$ perturbations preserving where $\mathscr{J}$ is an arbitrary operator ideal, estimates for the norm of commutators were also obtained; in the one-dimensional case Frechet-differentiability and a trace formula were proved. The aim of the present paper is to prove the Frechet differentiability of operator Bernstein functions and the Livschits-Kreĭn trace formula in the multidimensional setting. (Apparently the multidimensional Livschits-Kreĭn trace formula established in the paper is also new for Hilbert space operators.)

So this work could be considered as a contribution to the multidimensional Bochner-Phillips functional calculus.

As was mentioned in [17], the one-dimensional Bochner-Phillips functional calculus is a substantial part of the theory of operator semigroups and finds important applications in the theory of random processes (see, e.g., [19], [20]). The foundations of multidimensional calculus were laid by the author in [21], [22], [23], [24], [25]. Below we recall some notions and facts from [6], [21], [22], and [23], which we need for formulating our results.

DEFINITION 1. [6] We say that a nonpositive function $\psi \in C^\infty((-\infty;0)^n)$ *belongs to the class* $\mathscr{T}_n$ (or is a nonpositive Bernstein function of n variables) if all of its first partial derivatives are absolutely monotone (a function in $C^\infty((-\infty;0)^n)$ is said to be absolutely monotone if it is nonnegative together with its partial derivatives of all orders).

Obviously, $\psi \in \mathscr{T}_n$ if and only if $-\psi(-s)$ is a nonnegative Bernstein function of n variables on $(0,\infty)^n$, and $\mathscr{T}_n$ is a cone under the pointwise addition of functions and multiplication by scalars. As is known [6] (see also [23], [26]), each function $\psi \in \mathscr{T}_n$ admits an integral representation of the form (here and in what follows, the dot denotes inner product in $\mathbb{R}^n$ and the expression $s \to -0$ means that $s_1 \to -0, \dots, s_n \to -0$)

$$\psi(s) = c_0 + c_1 \cdot s + \int_{\mathbb{R}^n_+ \setminus \{0\}} (e^{s \cdot v} - 1) d\mu(v) \quad (s \in (-\infty;0)^n), \qquad (*)$$

where $c_0 = \psi(-0) := \lim_{s \to -0} \psi(s)$, $c_1 = (c_1^j)_{j=1}^n \in \mathbb{R}^n_+$, $c_1^j = \lim_{s_j \downarrow -\infty} \psi(s)/s_j$, and μ is a positive measure on $\mathbb{R}^n_+ \setminus \{0\}$; μ are determined by ψ.

A lot of examples of Bernstein function of one variable one can found in [34] (see also [20], [26]).

Throughout the paper, $T_{A_1}, \dots, T_{A_n}$ denote pairwise commuting one-parameter C_0 semigroups (i.e., strongly continuous semigroups on $\mathbb{R}_+$) on a complex Banach space X with generators $A_1, \dots, A_n$ respectively satisfying the condition $\|T_{A_j}(t)\| \leqslant M_A$ ($t \geqslant 0$, $M_A = \text{const}$) (sometimes we write T_j instead of T_{A_j}). We denote the domain of A_j by $D(A_j)$ and set $A = (A_1, \dots, A_n)$. We put $R(t, A_i) := (tI - A_i)^{-1}$ for t in $\rho(A_i)$, the resolvent set for A_i. Hereafter, by the commutation of operators $A_1, \dots, A_n$ we mean the commutation of the corresponding semigroups. By $\text{Gen}(X)$ we denote the set of all generators of uniformly bounded C_0 semigroups on X and by $\text{Gen}(X)^n$, the set of all n-tuples $(A_1, \dots, A_n)$ where $A_j \in \text{Gen}(X)$. We put also $M := \max\{M_A, M_B\}$ for the pare A, B of n-tuples from $\text{Gen}(X)^n$. In the following $\mathscr{L}(X)$ denotes the algebra of linear bounded operators on X and I, the identity operator on X. An operator-valued function $T_A(u) := T_{A_1}(u_1) \dots T_{A_n}(u_n)$ ($u \in \mathbb{R}^n_+$) is an n-parameter C_0 semigroup; therefore, the linear manifold $D(A) := \cap_{j=1}^n D(A_j)$ is dense in X [10, Sec. 10.10].

DEFINITION 2. [23] The value of a function $\psi \in \mathscr{T}_n$ of the form $(*)$ at $A = (A_1, \dots, A_n)$ applied to $x \in D(A)$ is defined by

$$\psi(A)x = c_0 x + c_1 \cdot Ax + \int_{\mathbb{R}^n_+ \setminus \{0\}} (T_A(u) - I)x d\mu(u),$$

where $c_1 \cdot Ax := \sum_{j=1}^{n} c_1^j A_j x$.

Given $\psi \in \mathscr{T}_n$ and $t \geqslant 0$, the function $g_t(z) := e^{t\psi(z)}$ is absolutely monotone on $(-\infty; 0)^n$. It is also obvious that $g_t(z) \leqslant 1$. By virtue of the multidimensional version of the Bernstein-Widder Theorem (see, e.g., [6], [4]), there exists a unique bounded positive measure v_t on $\mathbb{R}_+^n$, such that, for $z \in (-\infty; 0)^n$, we have

$$g_t(z) = \int_{\mathbb{R}_+^n} e^{z \cdot u} dv_t(u).$$

DEFINITION 3. In the notation introduced above, we set

$$g_t(A)x = \int_{\mathbb{R}_+^n} T_A(u)x dv_t(u) \ (x \in X)$$

(the integral is understood in the sense of Bochner).

Obviously, $\|g_t(A)\| \leqslant M_A^n$. The map $g(A) : t \mapsto g_t(A)$ is a C_0 semigroup. In the one-dimensional case, it is called the *semigroup subordinate to T_A*. In [27] it was noticed that the closure of the operator $\psi(A)$ exists and is the generator of the C_0 semigroup $g(A)$ (cf. [23].) It suggests the following final version of the definition of the operator $\psi(A)$.

DEFINITION 4. [23] By the value of a function $\psi \in \mathscr{T}_n$ at an n-tuple $A = (A_1, \ldots, A_n)$ of commuting operators in $\mathrm{Gen}(X)$ we understand the generator of the semigroup $g(A)$, i.e., the closure of the operator defined in the Definition 2. This value is denoted by $\psi(A)$.

The functional calculus thus arising is called multidimensional Bochner-Phillips calculus, or $\mathscr{T}_n$-calculus.

In the sequel we assume for the sake of simplicity that $c_0 = c_1 = 0$ in the integral representation $(*)$ of the function $\psi \in \mathscr{T}_n$ (otherwise one should replace $\psi(s)$ by $\psi(s) - c_0 - c_1 \cdot s$).

The notation and constraints introduced above are used in what follows without additional explanations.

We shall use also the following results from [17].

THEOREM 1. [17, Theorem 1] *Let $\psi \in \mathscr{T}_n$. Then for every commuting families $A = (A_1, \ldots, A_n)$, and $B = (B_1, \ldots, B_n)$ from $\mathrm{Gen}(X)^n$ such that the operators $A_i - B_i$ are bounded, $D(A_i) = D(B_i)$ $(i = 1, \ldots, n)$ the operator $\psi(A) - \psi(B)$ is also bounded and*

$$\|\psi(A) - \psi(B)\| \leqslant -\frac{2e}{e-1} n M^n \psi\left(-\frac{M}{2n}\|A-B\|\right),$$

where $\|A - B\| := (\|A_1 - B_1\|, \ldots, \|A_n - B_n\|)$.

Below we shall assume that a (two sided) operator ideal $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ on X is *symmetrically normed* in the sense that $\|ASB\|_{\mathscr{J}} \leqslant \|A\|\|S\|_{\mathscr{J}}\|B\|$ for $A, B \in \mathscr{L}(X)$ and $S \in \mathscr{J}$. The following theorem shows that Bernstein functions are $\mathscr{J}$ perturbations preserving.

THEOREM 2. [17, Theorem 4] *Let* $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ *be an operator ideal on* X *and* $\psi \in \mathscr{T}_n$ *be such that* $\frac{\partial \psi}{\partial s_i}|_{s=-0} \neq \infty$ *for all* $i = 1, \ldots, n$. *For every commuting families* $A = (A_1, \ldots, A_n)$ *and* $B = (B_1, \ldots, B_n)$ *from* $\mathrm{Gen}(X)^n$ *such that* $A_i - B_i \in \mathscr{J}, D(A_i) = D(B_i)$ $(i = 1, \ldots, n)$ *the operator* $\psi(A) - \psi(B)$ *belongs to* $\mathscr{J}$, *too, and*

$$\|\psi(A) - \psi(B)\|_{\mathscr{J}} \leqslant M^{n+1} \sum_{i=1}^{n} \frac{\partial \psi}{\partial s_i}\bigg|_{s=-0} \|A_i - B_i\|_{\mathscr{J}}.$$

2. Differentiability

DEFINITION 5. (Cf. [17].) Let $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ be an operator ideal on X, $\psi \in \mathscr{T}_n$, A be the n-tuple of pairwise commuting operators from $\mathrm{Gen}(X)$. We call the bounded linear operator $\psi_A^{\nabla} : \mathscr{J}^n \to \mathscr{J}$ (transformator) the $\mathscr{J}$-*Frechet derivative* of the operator function ψ at the point A, if for every n-tuple $\Delta A \in \mathscr{J}^n$ such that $A_i + \Delta A_i \in \mathrm{Gen}(X)$ for all i and operators $A_i + \Delta A_i$ pairwise commute we have

$$\|\psi(A + \Delta A) - \psi(A) - \psi_A^{\nabla}(\Delta A)\|_{\mathscr{J}} = o(\|\Delta A\|_{\mathscr{J}}) \text{ as } \|\Delta A\|_{\mathscr{J}} := \sum_{i=1}^{n} \|\Delta A_i\|_{\mathscr{J}} \to 0.$$

Evidently, the Frechet derivative at the point A is unique.

Before we formulate our first result note that if $\partial \psi / \partial s_i|_{s=-0} \neq \infty$ the derivative $\partial \psi / \partial s_i$ of a function $\psi \in \mathscr{T}_n$ equals to

$$\int_{\mathbb{R}_+^n \setminus \{0\}} e^{s \cdot v} v_i d\mu(v) \quad (s \in (-\infty, 0]^n)$$

and the measure $v_i d\mu(v)$ is finite. So for every n-tuple A of pairwise commuting operators $A_i \in \mathrm{Gen}(X)$ the operator

$$\frac{\partial \psi(A)}{\partial s_i} := \int_{\mathbb{R}_+^n \setminus \{0\}} T_A(v) v_i d\mu(v)$$

exists and belongs to $\mathscr{L}(X)$.

THEOREM 3. *Let* $\psi \in \mathscr{T}_n$, *and* $\forall i \; \omega_i := \partial \psi / \partial s_i|_{s=-0} \neq \infty$. *Then for every* n-*tuple* A *of pairwise commuting operators* $A_i \in \mathrm{Gen}(X)$ *the* $\mathscr{L}(X)$-*Frechet derivative for the operator function* ψ *at the point* A *exists and*

$$\psi_A^{\nabla}(C) = \sum_{i=1}^{n} \frac{\partial \psi(A)}{\partial s_i} C_i \tag{1}$$

for every n-*tuple* $C \in \mathscr{L}(X)^n$.

Proof. The case $n = 1$ was considered in [17]. Now let $n \geqslant 2$. For the proof we need the following generalization of Theorem 5 from [26] on divided differences.

LEMMA 1. *Let the function* $\psi \in \mathscr{T}_n$ *has the integral representation* ($*$), *and* $\omega_i = \partial \psi / \partial s_i|_{s=-0} \neq \infty$ ($i = 1, \ldots, n$). *Then the function*

$$\varphi_i(s, s_{n+1}) := \begin{cases} \frac{\psi(s) - \psi(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n)}{s_i - s_{n+1}} - \omega_i, & \text{if } s_i \neq s_{n+1}, \\ \frac{\partial \psi(s)}{\partial s_i} - \omega_i, & \text{if } s_i = s_{n+1} \end{cases}$$

($s = (s_1, \ldots, s_n)$)) *belongs to* $\mathscr{T}_{n+1}$ *and has the integral representation*

$$\varphi_i(s, s_{n+1}) = \int\limits_{\mathbb{R}^{n+1}_+ \setminus \{0\}} (e^{s \cdot u + s_{n+1} u_{n+1}} - 1) d\mu_i(u),$$

where $d\mu_i(u)$ *is the image of the measure* $1/2 d\mu(v) dw$ *under the mapping* $u_j = v_j$ *if* $j \neq i, n+1$, $u_i = (v_i + w)/2$, $u_{n+1} = (v_i - w)/2$.

Proof. Assume that $s_i \neq s_{n+1}$. Then, putting $\hat{s}_i := (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$, we get from ($*$) that

$$\frac{\psi(s) - \psi(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n)}{s_i - s_{n+1}} - \omega_i = \int\limits_{\mathbb{R}^n_+ \setminus \{0\}} \frac{e^{s_i v_i} - e^{s_{n+1} v_i}}{s_i - s_{n+1}} e^{\hat{s}_i \cdot \hat{v}_i} d\mu(v) - \omega_i.$$

$$(2)$$

Since

$$\frac{e^{s_i v_i} - e^{s_{n+1} v_i}}{s_i - s_{n+1}} e^{\hat{s}_i \cdot \hat{v}_i} = \frac{1}{2} \int\limits_{-v_i}^{v_i} \left(e^{s_i \frac{v_i + w}{2} + s_{n+1} \frac{v_i - w}{2} + \hat{s}_i \cdot \hat{v}_i} - 1 \right) dw + v_i,$$

formula (2) implies that

$$\varphi_i(s, s_{n+1}) = \frac{1}{2} \int\limits_{\mathbb{R}^n_+ \setminus \{0\}} \int\limits_{-v_i}^{v_i} \left(e^{s_i \frac{v_i + w}{2} + s_{n+1} \frac{v_i - w}{2} + \hat{s}_i \cdot \hat{v}_i} - 1 \right) dw d\mu(v)$$

$$= \frac{1}{2} \int\limits_{\Omega_i} \left(e^{s_i \frac{v_i + w}{2} + s_{n+1} \frac{v_i - w}{2} + \hat{s}_i \cdot \hat{v}_i} - 1 \right) dw d\mu(v),$$

where $\Omega_i = \{(v, w) : v \in \mathbb{R}^n_+ \setminus \{0\}, w \in [-v_i, v_i]\}$.

Making the change of variables $u_j = v_j$ for $j \neq i, n+1$, $u_i = (v_i + w)/2$, $u_{n+1} = (v_i - w)/2$ in the last integral, we get

$$\varphi_i(s, s_{n+1}) = \int\limits_{\mathbb{R}^{n+1}_+ \setminus \{0\}} (e^{s \cdot u + s_{n+1} u_{n+1}} - 1) d\mu_i(u), \tag{3}$$

where $d\mu_i(u)$ is the image of the measure $1/2 d\mu(v) dw$ under the mapping $u_j = v_j$ if $j \neq i, n+1$, $u_i = (v_i + w)/2$, $u_{n+1} = (v_i - w)/2$.

The case $s_i = s_{n+1}$ follows from the first assertion of the lemma as $s_{n+1} \to s_i$. $\square$

Now we claim that for every commuting $A_1, \ldots, A_{n+1} \in \mathrm{Gen}(X)$, and for every i such that $A_i - A_{n+1} \in \mathscr{L}(X)$ the following equality holds for $x \in D(A)$

$$\varphi_i(A_1, \ldots, A_{n+1})(A_i - A_{n+1})x = \psi(A_1, \ldots, A_n)x - \psi(A_1, \ldots, A_{i-1}, A_{n+1}, A_{i+1}, \ldots, A_n)x$$

$$- \frac{\partial \psi}{\partial s_i}\Big|_{s=-0}(A_i - A_{n+1})x. \tag{4}$$

For the proof first note that in view of Definition 2 and formula (3) for $x \in D(A)$ we have

$$\varphi_i(A_1, \ldots, A_{n+1})(A_i - A_{n+1})x = \int\limits_{\mathbb{R}^{n+1}_+ \setminus \{0\}} (T_1(u_1) \ldots T_{n+1}(u_{n+1}) - I)(A_i - A_{n+1})x d\mu_i(u)$$

(for simplicity we write T_j instead of T_{A_j}). Let Ω_i be as in the proof of Lemma 1. If we put in the last integral $v_j = u_j (j \ne i, n+1)$, $v_i = u_i + u_{n+1}$, $w = u_i - u_{n+1}$, then (v, w) runs over Ω_i and we get

$$\varphi_i(A_1, \ldots, A_{n+1})(A_i - A_{n+1})x$$

$$= \frac{1}{2} \int\limits_{\mathbb{R}^n_+ \setminus \{0\}} d\mu(v) \int\limits_{-v_i}^{v_i} \left(T_i\left(\frac{v_i + w}{2}\right) T_{n+1}\left(\frac{v_i - w}{2}\right) \prod_{1 \le j \le n, j \ne i} T_j(u_j) - I \right)(A_i - A_{n+1})x dw$$

$$= \int\limits_{\mathbb{R}^n_+ \setminus \{0\}} \prod_{1 \le j \le n, j \ne i} T_j(u_j) \frac{1}{2} \int\limits_{-v_i}^{v_i} T_i\left(\frac{v_i + w}{2}\right) T_{n+1}\left(\frac{v_i - w}{2}\right)(A_i - A_{n+1})x dw d\mu(v)$$

$$- \frac{\partial \psi}{\partial s_i}\Big|_{s=-0}(A_i - A_{n+1})x. \tag{5}$$

Because of the following identity [17, p. 211] $(x \in D(A))$

$$\frac{1}{2} \int\limits_{-v_i}^{v_i} T_i\left(\frac{v_i + w}{2}\right) T_{n+1}\left(\frac{v_i - w}{2}\right)(A_i - A_{n+1})x dw = (T_i(u_i) - T_{n+1}(u_i))x,$$

formula (5) implies (4).

Now putting $A_{n+1} - A_i = \Delta A_i$ in the formula (3) $(\Delta A_i \in \mathscr{L}(X))$, we have for $x \in D(A)$

$$\psi(A_1, \ldots, A_{i-1}, A_i + \Delta A_i, A_{i+1}, \ldots, A_n)x - \psi(A_1, \ldots, A_n)x$$

$$= \varphi_i(A_1, \ldots, A_n, A_i + \Delta A_i)\Delta A_i x + \frac{\partial \psi}{\partial s_i}\Big|_{s=-0}\Delta A_i x. \tag{6}$$

By Theorem 1 the operator

$$\alpha_i(\Delta A_i) := \varphi_i(A_1, \ldots, A_n, A_i + \Delta A_i) - \varphi_i(A_1, \ldots, A_n, A_i)$$

is bounded and

$$\|\alpha_i(\Delta A_i)\| \leqslant -\frac{2e}{e-1}(n+1)M^{n+1}\varphi_i\left(-\frac{M}{2n+2}(0,\ldots,0,\|\Delta A_i\|)\right) \rightarrow$$

$$-\frac{2e}{e-1}(n+1)M^{n+1}\varphi_i(0,\ldots,0) = 0 \text{ as } \|\Delta A_i\| \rightarrow 0.$$

Thus the formula (6) entails the equality

$$\psi(A_1,\ldots,A_{i-1},A_i+\Delta A_i,A_{i+1},\ldots,A_n)x - \psi(A_1,\ldots,A_n)x$$

$$= \varphi_i(A_1,\ldots,A_n,A_i)\Delta A_i x + \left.\frac{\partial\psi}{\partial s_i}\right|_{s=-0}\Delta A_i x + \alpha_i(\Delta A_i)\Delta A_i x. \tag{7}$$

Let us show that

$$\varphi_i(A_1,\ldots,A_n,A_i)x = \frac{\partial\psi(A)}{\partial s_i}x - \left.\frac{\partial\psi}{\partial s_i}\right|_{s=-0}x. \tag{8}$$

To this end note that the Definition 2 implies in view of Lemma 1 that for $x \in D(A)$

$$\varphi_i(A_1,\ldots,A_n,A_i)x$$

$$= \int_{\mathbb{R}^{n+1}_+\setminus\{0\}} (T_1(u_1)\ldots T_{i-1}(u_{i-1})T_i(u_i+u_{n+1})T_{i+1}(u_{i+1})\ldots T_n(u_n)-I)xd\mu_i(u).$$

If we put here $v_j = u_j$ $(j \neq i, n+1)$, $v_i = u_i + u_{n+1}$, $w = u_i - u_{n+1}$, as in the proof of the formula (5), we get

$$\varphi_i(A_1,\ldots,A_n,A_i)x = \int_{\Omega_i}(T_1(v_1)\ldots T_n(v_n)-I)x\frac{1}{2}d\mu(v)dw$$

$$= \frac{1}{2}\int_{\mathbb{R}^n_+\setminus\{0\}}d\mu(v)\int_{-v_i}^{v_i}(T_1(v_1)\ldots T_n(v_n)-I)xdw$$

$$= \int_{\mathbb{R}^n_+\setminus\{0\}}T_1(v_1)\ldots T_n(v_n)v_id\mu(v) - \int_{\mathbb{R}^n_+\setminus\{0\}}v_id\mu(v) = \frac{\partial\psi(A)}{\partial s_i}x - \left.\frac{\partial\psi}{\partial s_i}\right|_{s=-0}x.$$

This completes the proof of formula (8). Since $D(A)$ is dense in X, formula (7) implies in view of (8) that $(i = 1,\ldots,n)$

$$\psi(A_1,\ldots,A_{i-1},A_i+\Delta A_i,A_{i+1},\ldots,A_n) - \psi(A_1,\ldots,A_n) = \frac{\partial\psi(A)}{\partial s_i}\Delta A_i + o(\|\Delta A_i\|).$$

It follows that

$$\psi(A+\Delta A) - \psi(A) = \psi(A_1+\Delta A_1,\ldots,A_n+\Delta A_n) - \psi(A_1,A_2+\Delta A_2,\ldots,A_n+\Delta A_n) + \ldots$$

$$+ \psi(A_1,\ldots,A_{n_1},A_n+\Delta A_n) - \psi(A_1,A_2,\ldots,A_n)$$

$$= \sum_{i=1}^n \frac{\partial\psi}{\partial s_i}(A_1,\ldots,A_i,A_{i+1}+\Delta A_{i+1},\ldots,A_n+\Delta A_n)\Delta A_i + o(\|\Delta A\|).$$

To complete the proof it suffices to show that

$$\frac{\partial \psi}{\partial s_i}(A_1, \ldots, A_i, A_{i+1} + \Delta A_{i+1}, \ldots, A_n + \Delta A_n) = \frac{\partial \psi}{\partial s_i}(A_1, \ldots, A_n) + \beta_i(\Delta A), \quad (9)$$

where $\|\beta_i(\Delta A)\| \to 0$ as $\|\Delta A\| \to 0$ $(i = 1, \ldots, n)$.

But

$$\frac{\partial \psi}{\partial s_i}(A_1, \ldots, A_i, A_{i+1} + \Delta A_{i+1}, \ldots, A_n + \Delta A_n)$$

$$= \int_{\mathbb{R}^n_+ \setminus \{0\}} T_{A_1}(v_1) \ldots T_{A_i}(v_i) T_{A_{i+1} + \Delta A_{i+1}}(v_{i+1}) \ldots T_{A_n + \Delta A_n}(v_n) v_i d\mu(v), \quad (10)$$

and it is known that $\|T_{A_j + \Delta A_j}(t) - T_{A_j}(t)\| \to 0$ as $\|\Delta A_j\| \to 0$ (see, e.g., [10, Theorem 13.5.8]). Since all semigroups T_{A_j} are bounded and measures $v_j d\mu(v)$ are finite, formula (9) follows from Lebesgue dominated convergence theorem [7, Ch. IV, Subsection 3.7, Corollary of Theorem 6]. This finishes the proof of Theorem 3. $\square$

Note that the condition $\forall i\ \partial \psi / \partial s_i|_{s=-0} \neq \infty$ is also necessary for the Frechet differentiability of the function ψ at every point A (take $A = (O, \ldots, O)$) but in the case of exponentially stable semigroups the following corollary holds.

COROLLARY 1. *Let $\psi \in \mathscr{T}_n$. Then for every n-tuple A of pairwise commuting operators from $\mathrm{Gen}(X)$ such that $\forall i\ \|T_{A_i}(t)\| \leqslant M e^{\omega t}$ with $\omega < 0$, the $\mathscr{L}(X)$-Frechet derivative for the operator function ψ at the point A exists and (1) holds.*

Proof. To use Theorem 3 we need the condition $\forall i\ \partial \psi / \partial s_i|_{s=-0} \neq \infty$. To bypass it we apply Theorem 3 to the function $\psi(s_1 + \omega, \ldots, s_n + \omega)$ from $\mathscr{T}_n$ and to the n-tuple $(A_1 - \omega I, \ldots, A_n - \omega I)$ from $\mathrm{Gen}(X)^n$. $\square$

THEOREM 4. *Let $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ be a symmetrically normed operator ideal on X, $\psi \in \mathscr{T}_n$, and $\partial \psi / \partial s_i|_{s=-0} \neq \infty$, $\partial^2 \psi / \partial s_i^2|_{s=-0} \neq \infty$ $(i = 1, \ldots, n)$. For every n-tuple A of pairwise commuting operators from $\mathrm{Gen}(X)$ the $\mathscr{J}$-Frechet derivative for the operator function ψ at the point A exists and (1) holds for every n-tuple $C \in \mathscr{J}^n$.*

Proof. As in the proof of previous theorem one can assume that $n \geqslant 2$. We proceed as in the proof of Theorem 3 with $\Delta A_i \in \mathscr{J}$, $\|\Delta A_i\|_{\mathscr{J}} \to 0$. Then Theorem 2 implies that the operator $\alpha_i(\Delta A_i)$ belongs to $\mathscr{J}$, too and for $\|\Delta A_i\|_{\mathscr{J}} \to 0$ we have

$$\|\alpha_i(\Delta A_i)\|_{\mathscr{J}} \leqslant M^{n+1} \frac{\partial \varphi_i}{\partial s_{n+1}}\bigg|_{s=-0} \|\Delta A_i\|_{\mathscr{J}} = M^{n+1} \frac{1}{2} \frac{\partial^2 \psi}{\partial s_i^2}\bigg|_{s=-0} \|\Delta A_i\|_{\mathscr{J}} \to 0. \quad (11)$$

Indeed,

$$\frac{\partial \varphi_i}{\partial s_{n+1}}\bigg|_{s=-0} := \lim_{s \to -0} \frac{\partial \varphi_i(s_1, \ldots, s_{n+1})}{\partial s_{n+1}}$$

$$= \lim_{s \to -0} \frac{\left(\psi(s) - \psi(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n)\right) - \frac{\partial \psi(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n)}{\partial s_i}(s_i - s_{n+1})}{(s_i - s_{n+1})^2}.$$

But, by the Taylor's formula (below ξ lies between $(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n)$ and s),

$$\psi(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n) - \psi(s) = \frac{\partial \psi(s)}{\partial s_i}(s_{n+1} - s_i) + \frac{1}{2}\frac{\partial^2 \psi(\xi)}{\partial s_i^2}(s_{n+1} - s_i)^2.$$

Hence,

$$\frac{\partial \varphi_i}{\partial s_{n+1}}\Bigg|_{s=-0} = \lim_{s \to -0} \frac{\left(\frac{\partial \psi(s)}{\partial s_i} - \frac{\partial \psi(s_1, \ldots, s_{i-1}, s_{n+1}, s_{i+1}, \ldots, s_n)}{\partial s_i}\right)(s_{n+1} - s_i) - \frac{1}{2}\frac{\partial^2 \psi(\xi)}{\partial s_i^2}(s_{n+1} - s_i)^2}{(s_{n+1} - s_i)^2}.$$

Applying the Taylor's formula to the first summand of the numerator in the right-hand side we deduce from the last equality that

$$\frac{\partial \varphi_i}{\partial s_{n+1}}\Bigg|_{s=-0} = \frac{1}{2}\frac{\partial^2 \psi}{\partial s_i^2}\Bigg|_{s=-0}$$

and the formula (11) follows.

Now consider

$$\beta_i(\Delta A) = \frac{\partial \psi}{\partial s_i}(A_1, \ldots, A_i, A_{i+1} + \Delta A_{i+1}, \ldots, A_n + \Delta A_n) - \frac{\partial \psi}{\partial s_i}(A_1, \ldots, A_n)$$

(see formula (9)). Since [10, Theorem 13.4.1]

$$T_{A_j + \Delta A_j}(t) = \sum_{m=0}^{\infty} S_m(t),$$

where

$$S_0 = T_{A_j}, \quad S_m(t) = \int_0^t T_{A_j}(t - \tau)\Delta A_j S_{m-1}(\tau)d\tau \quad (m \geqslant 1),$$

we have for $\|\Delta A_j\|_{\mathscr{J}} \to 0$

$$\|T_{A_j + \Delta A_j}(t) - T_{A_j}(t)\|_{\mathscr{J}} \leqslant \left(\sum_{m=1}^{\infty} \int_0^t \|T_{A_j}(t - \tau)\|\|S_{m-1}(\tau)\|d\tau\right)\|\Delta A_j\|_{\mathscr{J}} \to 0$$

(the series in the right-hand side converges, as the proof of Theorem 13.4.1 in [10] shows). So the formula (10) implies in view of Lebesgue dominated convergence theorem that $\|\beta_j(\Delta A)\|_{\mathscr{J}} \to 0$ as $\|\Delta A_j\|_{\mathscr{J}} \to 0$. The remaining part of the proof is the same as in Theorem 3. $\square$

In context of Theorem 4 there is an analog of Corollary 1 for exponentially stable semigroups, as well.

COROLLARY 2. *Let $(\mathscr{J}, \|\cdot\|_{\mathscr{J}})$ be a symmetrically normed operator ideal on X, $\psi \in \mathscr{T}_n$. Then for every n-tuple A of pairwise commuting operators from $\mathrm{Gen}(X)$ such that $\forall i\ \|T_{A_i}(t)\| \leqslant Me^{\omega t}$ with $\omega < 0$, the $\mathscr{J}$-Frechet derivative for the operator function ψ at the point A exists and (1) holds for every n-tuple $C \in \mathscr{J}^n$.*

The proof of this corollary is similar to the proof of corollary 1.

3. Trace formula

As is well known, the trace formula for a trace class perturbation of a self-adjoint operator was proved in a special case in [13] and in the general case in [12]. A survey of farther developments (in context of Hilbert spaces) and bibliography one can fined in [5], [31], see also recent papers [32], [2], [14], [15], and [16].

In this section we introduce a spectral shift function and prove a Livschits-Kreĭn trace formula for a trace class perturbations of generators of C_0-semigroups on Banach space with approximation property if this semigroups are holomorphic in the right half-plane and have a polynomial growth. Recall that if the Banach space X has the approximation property there is a continuous linear functional tr of norm 1 (*a trace*) on the operator ideal $(\mathfrak{S}_1, \|\cdot\|_{\mathfrak{S}_1})$ of nuclear operators on X (see, e. g., [9, p. 64]). In the following $\partial^\alpha \psi$ denotes a multi-index derivative of a function ψ,

$$\mathbb{C}_+^n := \{z \in \mathbb{C}^n : \mathrm{Re}(z_j) > 0,\ j = 1, \ldots, n\}.$$

THEOREM 5. *Let the Banach space X has the approximation property. Let A and B be n-tuples of generators of pairwise commuting bounded C_0-semigroups T_{A_j} and T_{B_j} respectively on X holomorphic in the half plane $\mathbb{C}_+$ and satisfying $\|T_{A_j}(\zeta)\|$, $\|T_{B_j}(\zeta)\| \leqslant M|\zeta|^{m_j}$ for some $m_j \in \mathbb{Z}_+$ ($\zeta \in \mathbb{C}_+, j = 1, \ldots, n$). If $\forall j\ A_j - B_j \in \mathfrak{S}_1$ there exists a unique distribution $\eta_{A,B}$ supported in $\mathbb{R}_+^n$ such that for every $\psi \in \mathscr{T}_n$ with $\partial^{2m+1}\psi\big|_{s=-0} \neq \infty$ ($m = (m_1, \ldots, m_n)$) we have*

$$\mathrm{tr}(\psi(A) - \psi(B)) = \int\limits_{\mathbb{R}_+^n \setminus \{0\}} \langle \eta_{A,B}(t), e^{-u \cdot t} \rangle d\mu(u),$$

where (as above) μ stands for the representing measure of ψ and $\langle \eta_{A,B}(t), e^{-u \cdot t} \rangle$ denotes the Laplace transform of $\eta_{A,B}$. In particular,

$$\mathrm{tr}(T_A(v) - T_B(v)) = \langle \eta_{A,B}(t), e^{-v \cdot t} \rangle\ (v \in \mathbb{R}_+^n \setminus \{0\}).$$

Proof. Consider the function

$$F(z) := T_A(z) - T_B(z) \quad (z \in \mathbb{C}_+^n).$$

It is easy to verify that

$$T_A(z) - T_B(z) = \left(\prod_{i=1}^{n-1} T_{A_i}(z_i)\right)(T_{A_n}(z_n) - T_{B_n}(z_n))$$

$$+ \left(\prod_{i=1}^{n-2} T_{A_i}(z_i) \right) (T_{A_{n-1}}(z_{n-1}) - T_{B_{n-1}}(z_{n-1})) T_{B_n}(z_n)$$

$$+ \ldots + T_{A_1}(z_1)(T_{A_2}(z_2) - T_{B_2}(z_2)) \prod_{i=3}^{n} T_{B_i}(z_i) + (T_{A_1}(z_1) - T_{B_1}(z_1)) \prod_{i=2}^{n} T_{B_i}(z_i). \quad (12)$$

Theorem 2 implies that $T_{A_i}(z_i) - T_{B_i}(z_i) \in \mathfrak{S}_1$. So, $F : \mathbb{C}_+^n \to \mathfrak{S}_1$ by formula (12).

But for all $x \in D(A), \mathrm{Re}(z_i) > 0$

$$(T_{A_i}(z_i) - T_{B_i}(z_i))x = \int\limits_{[0,z_i]} \frac{d}{ds}(T_{B_i}(z_i - s)T_{A_i}(s)x)ds$$

$$= \int\limits_{[0,z_i]} T_{B_i}(z_i - s)(A_i - B_i)T_{A_i}(s)xds. \quad (13)$$

Since for $s \in [0, z_i]$

$$\|T_{B_i}(z_i - s)(A_i - B_i)T_{A_i}(s)\|_{\mathfrak{S}_1} \leqslant M^2 (|z_i - s||s|)^{m_i} \|A_i - B_i\|_{\mathfrak{S}_1}$$
$$\leqslant M^2 (|z_i - s| + |s|)^{2m_i} \|A_i - B_i\|_{\mathfrak{S}_1}$$
$$= M^2 |z_i|^{2m_i} \|A_i - B_i\|_{\mathfrak{S}_1}, \quad (14)$$

both sides in (13) are bounded, formula (13) holds for all $x \in X$, and

$$\|T_{A_i}(z_i) - T_{B_i}(z_i)\|_{\mathfrak{S}_1} \leqslant M^2 |z_i|^{2m_i+1} \|A_i - B_i\|_{\mathfrak{S}_1}.$$

Now it follows from (12) that

$$\|F(z)\|_{\mathfrak{S}_1} \leqslant M^{n+1} \max_i \|A_i - B_i\|_{\mathfrak{S}_1} \prod_{i=1}^{n} |z_i|^{2m_i+1}.$$

Therefore if we put

$$f(z) := \mathrm{tr}F(z),$$

then

$$|f(z)| \leqslant \mathrm{const} \prod_{i=1}^{n} |z_i|^{2m_i+1}. \quad (15)$$

We clame that f is holomorphic in $\mathbb{C}_+^n$. In view of Hartogs Theorem it suffices to prove that f is separately holomorphic. To simplify the notation we shall show that f is holomorphic in z_1. Indeed, formula (12) yields that

$$F(z) = (T_{A_1}(z_1) - T_{B_1}(z_1))S_1(z_2, \ldots, z_n) + T_{A_1}(z_1)S_2(z_2, \ldots, z_n)$$

for some operators $S_1(z_2, \ldots, z_n) \in \mathscr{L}(X)$ and $S_2(z_2, \ldots, z_n) \in \mathfrak{S}_1$.

Then for every $z_1 \in \mathbb{C}_+$ and sufficiently small Δz_1 we have

$$F(z_1 + \Delta z_1, z_2, \ldots, z_n) - F(z)$$
$$= ((T_{A_1}(z_1 + \Delta z_1) - T_{B_1}(z_1 + \Delta z_1)) - (T_{A_1}(z_1) - T_{B_1}(z_1)))S_1(z_2, \ldots, z_n)$$
$$+ (T_{A_1}(z_1 + \Delta z_1) - T_{A_1}(z_1))S_2(z_2, \ldots, z_n). \quad (16)$$

Formula (13) implies the equality

$$
(T_{A_1}(z_1 + \Delta z_1) - T_{B_1}(z_1 + \Delta z_1)) - (T_{A_1}(z_1) - T_{B_1}(z_1))
$$

$$
= \int\limits_{[0,z_1+\Delta z_1]} T_{B_1}(z + \Delta z_1 - s)(A_1 - B_1)T_{A_1}(s)ds - \int\limits_{[0,z_1]} T_{B_1}(z_1 - s)(A_1 - B_1)T_{A_1}(s)ds
$$

$$
= \int\limits_{[0,z_1]} T_{B_1}(z_1 + \Delta z_1 - s)(A_1 - B_1)T_{A_1}(s)ds - \int\limits_{[0,z_1]} T_{B_1}(z_1 - s)(A_1 - B_1)T_{A_1}(s)ds
$$

$$
+ \int\limits_{[z_1,z_1+\Delta z_1]} T_{B_1}(z_1 + \Delta z_1 - s)(A_1 - B_1)T_{A_1}(s)ds
$$

$$
= \left(T_{B_1}\left(\frac{z_1}{2} + \Delta z_1\right) - T_{B_1}\left(\frac{z_1}{2}\right)\right) \int\limits_{[0,z_1]} T_{B_1}\left(\frac{z_1}{2} - s\right)(A_1 - B_1)T_{A_1}(s)ds
$$

$$
+ \int\limits_{[z_1,z_1+\Delta z_1]} T_{B_1}(z_1 + \Delta z_1 - s)(A_1 - B_1)T_{A_1}(s)ds.
$$

Taking into account formula (14) we have

$$
\|T_{A_1}(z_1 + \Delta z_1) - T_{B_1}(z_1 + \Delta z_1)) - (T_{A_1}(z_1) - T_{B_1}(z_1)\|_{\mathfrak{S}_1}
$$

$$
\leqslant \left\|T_{B_1}\left(\frac{z_1}{2} + \Delta z_1\right) - T_{B_1}\left(\frac{z_1}{2}\right)\right\| M^2 \left|\frac{z_1}{2}\right|^{2m_1} \|A_1 - B_1\|_{\mathfrak{S}_1}|z_1|
$$

$$
+ M^2|z_1 + \Delta z_1|^{2m_1}\|A_1 - B_1\|_{\mathfrak{S}_1}|\Delta z_1| \to 0 \quad (\Delta z_1 \to 0).
$$

Now, formula (16) shows that the map $F : \mathbb{C}_+^n \to \mathfrak{S}_1$, and consequently the function f, are continuous in z_1 (above we used the fact that holomorphic semigroups T_{A_1} and T_{B_1} are norm continuous on $\mathbb{C}_+$).

Moreover, since $z_1 \mapsto F(z_1, z_2, \ldots, z_n)$ is analytic in the half plane $\mathbb{C}_+$ with respect to the operator norm, we have for every closed path C located at this half plane that

$$
\oint\limits_C f(z_1, z_2, \ldots, z_n)dz_1 = \mathrm{tr} \oint\limits_C F(z_1, z_2, \ldots, z_n)dz_1 = 0.
$$

So by the Morera's Theorem the function $z_1 \mapsto f(z_1, z_2, \ldots, z_n)$ is analytic in the right half plane, as well. Now, since f is analytic in $\mathbb{C}_+^n$ and satisfies (15), there is a unique distribution $\eta_{A,B}$ supported in $\mathbb{R}_+^n$ such that $f(z) = \langle \eta_{A,B}(t), e^{-z \cdot t} \rangle$, the Laplace transform of $\eta_{A,B}$ (see, e.g., [28, Theorem 8.13.3]).

Since, by our hypothesis,

$$
\partial^{2m+1}\psi\big|_{s=-0} = \int\limits_{\mathbb{R}_+^n \setminus \{0\}} \prod_{i=1}^n u_i^{2m_i+1} d\mu(u) \neq \infty
$$

and (15) holds, Definition 2 and Theorem 2 imply that

$$\mathrm{tr}(\psi(A) - \psi(B)) = \int_{\mathbb{R}^n_+ \setminus \{0\}} \mathrm{tr}(T_A(u) - T_B(u)) d\mu(u)$$

$$= \int_{\mathbb{R}^n_+ \setminus \{0\}} f(u) d\mu(u) = \int_{\mathbb{R}^n_+ \setminus \{0\}} \langle \eta_{A,B}(t), e^{-u \cdot t} \rangle d\mu(u).$$

Putting $\mu = \delta_v$, the Dirac measure centered at v, we get the last conclusion of the theorem. $\square$

COROLLARY 3. *Under the conditions of theorem 5*

$$\eta_{A,B} = L^{-1} \mathrm{tr}(T_A - T_B),$$

where L^{-1} denotes the inverse of the n-dimensional Laplace transform in a distributional setting.

COROLLARY 4. *Under the conditions of theorem 5 for $\lambda \in \mathbb{C}^n_+$ let $R(\lambda, A) :=$ $\prod_{i=1}^n R(\lambda_i, A_i)$. Then $R(\lambda, A) - R(\lambda, B) \in \mathfrak{S}_1$ and*

$$\mathrm{tr}(R(\lambda, A) - R(\lambda, B)) = \int_{\mathbb{R}^n_+} \langle \eta_{A,B}(t), e^{-u \cdot t} \rangle e^{-u \cdot \lambda} du,$$

the iterated Laplace transformation of $\eta_{A,B}$ (the Stieltjes transform).

Proof. First note that $\mathbb{C}_+ \subseteq \rho(A_i)$. Since

$$R(\lambda_i, A_i) = \int_{\mathbb{R}_+} T_{A_i}(s) e^{-\lambda_i s} ds \ (\lambda_i \in \mathbb{C}_+; i = 1, \ldots, n),$$

we have

$$R(\lambda, A) - R(\lambda, B) = \int_{\mathbb{R}^n_+} (T_A(u) - T_B(u)) e^{-\lambda \cdot u} du. \tag{17}$$

Theorem 2 with $\psi(s) = e^{s \cdot u} - 1$, $\mathscr{J} = \mathfrak{S}_1$ implies that $T_A(u) - T_B(u) \in \mathfrak{S}_1$ and

$$\|T_A(u) - T_B(u)\|_{\mathfrak{S}_1} \leqslant M^{n+1} \sum_{i=1}^n u_i \|A_i - B_i\|_{\mathfrak{S}_1}. \tag{18}$$

It follows that

$$\int_{\mathbb{R}^n_+} \|T_A(u) - T_B(u)\|_{\mathfrak{S}_1} e^{-\lambda \cdot u} du < \infty.$$

Therefore in view of (17) we get $R(\lambda, A) - R(\lambda, B) \in \mathfrak{S}_1$ and

$$\mathrm{tr}(R(\lambda, A) - R(\lambda, B)) = \int_{\mathbb{R}^n_+} \mathrm{tr}(T_A(u) - T_B(u)) e^{-\lambda \cdot u} du = \int_{\mathbb{R}^n_+} \langle \eta_{A,B}(t), e^{-u \cdot t} \rangle e^{-u \cdot \lambda} du$$

by theorem 5. $\square$

REMARK 1. For the n-dimensional Stieltjes transform of distributions see [28, Ch.10], especially Theorem 10.8.1 therein, and [30].

REMARK 2. It was shown in [32] that Livschits-Kreǐn trace formula holds for arbitrary pairs of not necessarily bounded self-adjoint operators with trace class difference if and only if the corresponding function is operator Lipschitz. In [16], [15] Livschits-Kreǐn trace formulae (for operator Lipschitz functions) was extended to the case of pairs of maximal dissipative (m-dissipative) operators and pairs of contractions on Hilbert space. Since by [17, Corollary 2] (see also [34, Corollary 13.9]) every negative Bernstein function ψ in one variable such that $\psi'(-0) \neq \infty$ is operator Lipschitz in the class of generators of contractive C_0-semigroups (and, more generally, in any class of generators of uniformly bounded C_0-semigroups with common upper bound M), theorem 5 (in the case $n = 1$) and [17, Theorem 8] are consistent with results for Hilbert space operators mentioned above.

It should me mentioned also that when $n = 1$ and T_A and T_B are Hilbert space contractive semigroups the result of previous corollary is closely related to [14, Theorem 3.14].

COROLLARY 5. *Under the conditions of theorem 5 let bounded one-parameter C_0-semigroups $g_t(A) = T_{\psi(A)}(t)$ and $g_t(B) = T_{\psi(B)}(t)$ satisfy $\|T_{\psi(A)}(\zeta)\|$, $\|T_{\psi(B)}(\zeta)\| \leqslant M|\zeta|^k$ for some $k \in \mathbb{Z}_+$ ($\zeta \in \mathbb{C}_+$). Then*

$$\eta_{\psi(A),\psi(B)} = L_s^{-1} \int_{\mathbb{R}_+^n} \langle \eta_{A,B}(t), e^{-u \cdot t} \rangle dv_s(u), \qquad (19)$$

where L_s^{-1} denotes the inverse of the one-dimensional Laplace transform in a distributional setting with respect to s.

Proof. First note that bounded C_0-semigroups $T_{\psi(A)}$ and $T_{\psi(B)}$ are holomorphic in the half plane $\mathbb{C}_+$ by [3, Theorem 7.2]. Since $\psi(A) - \psi(B) \in \mathfrak{S}_1$, we have by corollary 3 and theorem 5 that

$$\eta_{\psi(A),\psi(B)} = L_s^{-1} \mathrm{tr}(g_s(A) - g_s(B)) = L_s^{-1} \mathrm{tr} \int_{\mathbb{R}_+^n} (T_A(u) - T_B(u)) dv_s(u)$$

$$= L_s^{-1} \int_{\mathbb{R}_+^n} \mathrm{tr}(T_A(u) - T_B(u)) dv_s(u) = L_s^{-1} \int_{\mathbb{R}_+^n} \langle \eta_{A,B}(t), e^{-u \cdot t} \rangle dv_s(u)$$

(the first integral converges in the sense of Bochner in the $\mathfrak{S}_1$-norm in view of (18)). □

REMARK 3. Since $g_s(z) := e^{s\psi(z)} = \int_{\mathbb{R}_+^n} e^{u \cdot z} dv_s(u)$, formula (19) formally can be written as

$$\eta_{\psi(A),\psi(B)} = L_s^{-1} \langle \eta_{A,B}(t), e^{s\psi(-t)} \rangle.$$

COROLLARY 6. (Cf. [17, Theorem 8].) *Let the Banach space X has the approximation property. Let A and B be generators of bounded C_0-semigroups T_A and T_B respectively on X holomorphic in the half plane $\mathbb{C}_+$ and satisfying $\|T_A(\zeta)\|$, $\|T_B(\zeta)\| \leqslant M|\zeta|^m$ $(m \in \mathbb{Z}_+, \zeta \in \mathbb{C}_+)$. If $A - B \in \mathfrak{S}_1$ there exists a unique distribution $\xi_{A,B}$ supported in $\mathbb{R}_+$ such that for every $\psi \in \mathscr{T}_1$ with $\psi^{(2m+1)}\big|_{s=-0} \neq \infty$ we have*

$$\mathrm{tr}(\psi(A) - \psi(B)) = \int\limits_{(0,+\infty)} \langle \xi_{A,B}(t), e^{-ut} \rangle u d\mu(u).$$

In particular,

$$\mathrm{tr}(T_A(v) - T_B(v)) = \langle \xi_{A,B}(t), e^{-vt} \rangle v.$$

Proof. Put $n = 1$ in theorem 5 and take for $\xi_{A,B}$ the antiderivative of $\eta_{A,B}$ supported in $\mathbb{R}_+$. □

To formulate our next corollaries we need some preparations. First note that the function $\psi_\lambda(s) := \log\lambda - \log(\lambda - s)$ $(\lambda > 0)$ belongs to $\mathscr{T}_1$ [20, Example 3]. So, for $A \in \mathrm{Gen}(X), \lambda > 0$ we can put

$$\log(\lambda I - A) := (\log\lambda)I - \psi_\lambda(A).$$

Note also that for $A, B \in \mathrm{Gen}(X), \lambda > 0$ such that $A - B$ is nuclear the operator

$$\log(\lambda I - B) - \log(\lambda I - A) = \psi_\lambda(A) - \psi_\lambda(B)$$

is nuclear by theorem 2.

DEFINITION 6. (Cf. [5, formula (3.25)].) Let the Banach space X has the approximation property. For $A, B \in \mathrm{Gen}(X), \lambda > 0$ such that $A - B$ is nuclear define the *perturbation determinant of the pair* (A, B) as follows

$$\Delta_{B/A}(\lambda) = \exp\mathrm{tr}(\log(\lambda I - B) - \log(\lambda I - A)).$$

REMARK 4. Since $\exp\mathrm{tr}S = \det\exp S$ for nuclear S, one can define perturbation determinant of a pair (A, B) by the formula

$$\Delta_{B/A}(\lambda) = \det\exp(\log(\lambda I - B) - \log(\lambda I - A)).$$

If, in addition, A and B commute, we have for $\lambda > 0$

$$\Delta_{B/A}(\lambda) = \det((\lambda I - B)(\lambda I - A)^{-1}) = \det(I + (A - B)(\lambda I - A)^{-1}).$$

Indeed, in this case using Dyson-Phillips series (see, e.g., [10, (13.2.4)]) it is easy to prove that

$$\exp(\psi_\lambda(B))\exp(\psi_\lambda(A) - \psi_\lambda(B)) = \exp(\psi_\lambda(A))$$

where $\exp(G)$ denotes $T_G(1)$ for a generator G of a C_0-semigroup T_G. Now, putting $g_t(z) = e^{t\psi_\lambda(z)} = \lambda^t(\lambda - z)^{-t}$ in definition 3, we get for $t = 1$ from the above equality, that

$$\exp(\psi_\lambda(A) - \psi_\lambda(B)) = (\lambda I - B)(\lambda I - A)^{-1}.$$

Another approach to the definition of perturbation determinant of a pair of closed operators on Hilbert space one can fined in [35, Section 8.1] and [14].

We shall use also the following notion of the Stieltjes transform of distributions [29]. Let $\alpha \leqslant 1$ be fixed real number. The space S_α of test functions consists of all complex-valued functions $\varphi \in C^\infty(0,\infty)$ such that

$$p_k(\varphi) := \sup_{t \in (0,\infty)} (1+t)^\alpha \left| \left(t \frac{d}{dt} \right)^k \varphi(t) \right| < \infty \ (k \in \mathbb{Z}_+).$$

The topology in S_α in determined by the family of seminorms $(p_k)_{k \in \mathbb{Z}_+}$. For every linear functional f from the dual space S'_α its Stieltjes transform F is defined by the rule

$$F(z) := \left\langle f(t), \frac{1}{t+z} \right\rangle, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

The function F is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$.

COROLLARY 7. (Cf. [12], and also [5, formula (3.1)]). *Let X, A, and B be as in the above corollary. Then*

$$\log \Delta_{B/A}(\lambda) = \left\langle \xi_{A,B}(t), \frac{1}{t+\lambda} \right\rangle \quad (\lambda > 0), \tag{20}$$

the Stieltjes transformation of $\xi_{A,B}$. So one can compute $\xi_{A,B}$ via inversion theorems for the Stieltjes transform (see, e.g., [28], [29]). In particular,

$$\xi_{A,B}(t) = \lim_{k \to \infty} \frac{(-t)^{k-1}}{k!(k-2)!} \frac{d^{2k-1}}{dt^{2k-1}} (t^k \log \Delta_{B/A}(t)) \tag{21}$$

(the limit is taken in the sense of distributions).

Proof. First note that by the proof of theorem 5 and corollary 6 the Laplace transformation of $\xi_{A,B}$ exists. So, by definition, $\xi_{A,B}(t) = e^{p_0 t} f(t)$ for some $p_0 > 0$ and tempered distribution f supported in $\mathbb{R}_+$. Since $e^{-p_0 t} S_0$ embeds in the Schwartz space $\mathscr{S}(\mathbb{R}_+)$, it follows that $\xi_{A,B} \in S'_0$.

On the other hand, $\psi_\lambda^{(2m+1)}\big|_{s=-0} \neq \infty$, and

$$\psi_\lambda(s) = \int\limits_0^\infty (e^{su} - 1) u^{-1} e^{-\lambda u} du.$$

Thus, by corollary 6

$$\log \Delta_{B/A}(\lambda) = \mathrm{tr}(\psi_\lambda(B) - \psi_\lambda(A)) = \int\limits_0^\infty \langle \xi_{A,B}(t), e^{-ut} \rangle e^{-\lambda u} du$$

$$= \left\langle \xi_{A,B}(t), \int\limits_0^\infty e^{-u(t+\lambda)} du \right\rangle = \left\langle \xi_{A,B}(t), \frac{1}{t+\lambda} \right\rangle \ (\lambda > 0).$$

Formula (21) follows from the real inversion theorem for the Stieltjes transform [29]. $\quad\square$

COROLLARY 8. (Cf. [12], and also [5, formula 3.7].) *Let X, A, and B be as in the corollary 6. The perturbation determinant of the pair (A, B) has analytic continuation to $\mathbb{C} \setminus (-\infty, 0]$ and*

$$\xi_{A,B}(t) = \frac{1}{2\pi i} \lim_{y \downarrow 0} \log \frac{\Delta_{B/A}(-t - iy)}{\Delta_{B/A}(-t + iy)} \quad (t > 0); \tag{22}$$

in particular, if $\xi_{A,B}$ is real-valued, formula (22) takes the form

$$\xi_{A,B}(s) = \frac{1}{\pi} \lim_{y \downarrow 0} \operatorname{Im}(\log(\Delta_{B/A}(-s - iy))) \quad (s > 0) \tag{23}$$

(limits in (22) and (23) are taken in the sense of distributions).

Proof. Since the right-hand side of the formula (20) is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$, define

$$\log \Delta_{B/A}(z) := \left\langle \xi_{A,B}(t), \frac{1}{t+z} \right\rangle, \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{24}$$

Now we can apply the complex inversion theorem for the Stieltjes transform [29] and get (22). Since for the real-valued $\xi_{A,B}$ formula (24) implies

$$\operatorname{Re}(\log(\Delta_{B/A}(-s + iy))) = \operatorname{Re}(\log(\Delta_{B/A}(-s - iy))),$$
$$\operatorname{Im}(\log(\Delta_{B/A}(-s + iy))) = -\operatorname{Im}(\log(\Delta_{B/A}(-s - iy))),$$

formula (23) follows from (22). $\square$

A substantial part of properties of perturbation determinant of pairs of operators on Hilbert space (see, e.g., [35, Section 8.1]) is valid for $\Delta_{B/A}$. For example, the following formula holds.

COROLLARY 9. (Cf. [35, Section 8.1, formula (4)].) *Let X, A, and B be as in the corollary 6. Then*

$$\frac{\Delta'_{B/A}(z)}{\Delta_{B/A}(z)} = \operatorname{tr}(R(z, B) - R(z, A)), \ z \in \rho(A) \cap \rho(B).$$

Proof. Differentiating (24) we get for $z \in \rho(A) \cap \rho(B)$ in view of corollary 4

$$\frac{\Delta'_{B/A}(z)}{\Delta_{B/A}(z)} = \left\langle \xi_{A,B}(t), \frac{d}{dz} \frac{1}{t+z} \right\rangle = -\left\langle \xi'_{A,B}(t), \frac{1}{t+z} \right\rangle$$
$$= -\left\langle \eta_{A,B}(t), \frac{1}{t+z} \right\rangle = \operatorname{tr}(R(z, B) - R(z, A)). \quad \square$$

REMARK 5. Formula (20) implies that $\lim_{\lambda \to +\infty} \Delta_{B/A}(\lambda) = 1$. It follows also from the definition 6 and corollary 8 that $\Delta_{B/A}(z)\Delta_{C/B}(z) = \Delta_{C/A}(z)$ for $z \in \mathbb{C} \setminus (-\infty, 0]$, and operators $A, B, C \in \operatorname{Gen}(X)$ such that the pairs (A, B) and (B, C) satisfy all the conditions of corollary 6.

COROLLARY 10. *If, in addition to the conditions mentioned in the corollary 6,* $\xi_{A,B}$ *is a measure, then*

$$\operatorname{tr}(\psi(A) - \psi(B)) = \int_{\mathbb{R}_+} \psi'(-t)d\xi_{A,B}(t).$$

It follows from the corollary 6 and Tonelli's Theorem.

REFERENCES

[1] A. B. ALEKSANDROV AND V. V. PELLER, *Operator Lipschitz functions*, Russian Mathematical Surveys **71** (2016), no. 4, 605–702.

[2] A. B. ALEKSANDROV AND V. V. PELLER, *Krein's trace formula for unitary operators and operator Lipschitz functions*, Funct. Anal and Appl. **50** (2016), no. 3, 167–175.

[3] C. BERG, K. BOYADZHIEV AND R. DELAUBENFELS, *Generation of generators of holomorphic semigroups*, J. Austral. Math. Soc. (Series A) **55** (1993), 246–269.

[4] CH. BERG, J. P. R. CHRISTENSEN, P. RESSEL, *Harmonic analysis on semigroups*, Grad. Texts in Math., vol.100, Springer-Verlag, New York-Berlin, 1984.

[5] M. S. BIRMAN AND D. R. YAFAEV, *The spectral shift function. The papers of M. G. Kreĭn and their further development*, Algebra i Analiz **4** (1992), 1–44 (Russian). English transl.: St. Petersburg Math. J. **4** (1993), 833 – 870.

[6] S. BOCHNER, *Harmonic analysis and the theory of probabylity*, University of California Press, Berkeley and Los Angeles, 1955.

[7] N. BOURBAKI, *Elements de mathematique. Livre VI. Integration. 2nd ed., Ch. 1 – 9*, Hermann, Paris, 1965–1969.

[8] YU. L. DALETSKII AND S. G. KREĬN, *Integration and differentiation of functions of Hermitian operators and application to the theory of perturbations* (Russian), Trudy Sem. Functsion. Anal., Voronezh. Gos. Univ. 1 (1956), 81–105.

[9] A. DEFANT AND K. FLORET, *Tensor norms and operator ideals*, North-Holland, Amsterdam, 1993.

[10] E. HILLE AND R. PHILLIPS, *Functional Analysis and Semigroups*, Amer. Math. Soc., Providence, R.I., 1957.

[11] E. KISSIN, D. POTAPOV, V. SHULMAN, AND F. SUKOCHEV, *Operator smoothness in Schatten norms for functions of several variables: Lipschitz conditions, differentiability and unbounded derivations*, Proc. London Math. Soc. **105**, (2012), no. 4, 661–702.

[12] M. G. KREĬN, *On a trace formula in perturbation theory*, Mat. Sbornik **33** (1953), 597–626 (Russian).

[13] I. M. LIFSHITZ, *On a problem in perturbation theory connected with quantum statistics*, Uspekhi Mat. Nauk **7** (1952), 171–180 (Russian).

[14] M. MALAMUD, H. NEIDHART, *Trace formulas for additive and non-additive perturbations*, Adv. in Math. **274** (2015), 736–832.

[15] M. MALAMUD, H. NEIDHART, V. PELLER, *Analytic operator Lipschitz functions in the disc and a trace formula for functions of contractions*, Functional Analysis and its Applications, **51** (2017), no. 3, 33–55, Preprint, arXiv:1705.07225 v1 [math. FA].

[16] M. MALAMUD, H. NEIDHARDT, V. PELLER, *A trace formula for functions of contractions and analytic operator Lipschitz functions*, Comptes Rendus Acad. Sci. Paris, Ser. I, **355** (2017), 806–811.

[17] A. R. MIROTIN, *Bernstein functions of several semigroup generators on Banach spaces under bounded perturbations*, Operators and Matrices **11** (2017), 199–217.

[18] A. R. MIROTIN, *On some functional calculus of closed operators on Banach space. III. Some topics in perturbation theory*, Izvestiya VUZ. Matematika **12** (2017), 24–34 (Russian); English transl.: Russian Math. **12**, to appear.

[19] A. R. MIROTIN, *On the $\mathcal{T}$-calculus of generators for C_0-semigroups*, Sib. Matem. Zh., **39** (1998), no. 3, 571–582; English transl.: Sib. Math. J. **39** (1998), no. 3, 493–503.

[20] A. R. MIROTIN, *Criteria for Analyticity of Subordinate Semigroups*, Semigroup Forum **78** (2009), no. 2, 262–275.

[21] A. R. MIROTIN, *Functions from the Schoenberg class $\mathcal{T}$ on the cone of dissipative elements of a Banach algebra*, Mat. Zametki **61** (1997), no. 4, 630–633; English transl.: Math. Notes **61** (1997), no. 3–4, 524–527.

[22] A. R. MIROTIN, *Functions from the Schoenberg class $\mathcal{T}$ act in the cone of dissipative elements of a Banach algebra, II*, Mat. Zametki **64** (1998), no. 3, 423–430; English transl.: Math. Notes **64** (1998), no. 3–4, 364–370.

[23] A. R. MIROTIN, *Multidimensional $\mathcal{T}$-calculus for generators of C_0 semigroups*, Algebra i Analiz **11** (1999), no. 2, 142–170; English transl.: St. Petersburg Math. J. **11** (1999), no. 2, 315–335.

[24] A. R. MIROTIN, *On some properties of the multidimensional Bochner-Phillips functional calculus*, Sib. Mat. Zhurnal, **52** (2011), no. 6, 1300 – 1312; English transl.: Siberian Mathematical Journal **52** (2011), no. 6, 1032–1041.

[25] A. R. MIROTIN, *On joint spectra of families of unbounded operators*, Izvestiya RAN: Ser. Mat. **79** (2015), no. 6, 145–170; English transl.: Izvestiya: Mathematics **79** (2015), no. 6, 1235–1259.

[26] A. R. MIROTIN, *Properties of Bernstein functions of several complex variables*, Mat. Zametki, **93** (2013), no. 2, 257–265; English transl.: Math. Notes **93** (2013), no. 2.

[27] A. R. MIROTIN, *On multidimensional Bochner-Phillips functional calculus*, Probl. Fiz. Mat. Tekh. **1** (2009), (2013)1, 63–66 (Russian).

[28] O. P. MISRA, J. L. LAVOINE, *Transform analysis of generalized functions*, North Holland, Amsterdam, 1986.

[29] J. N. PANDEX, *On the Stieltjes transform of generalized functions*, Proc. Camb. Phil. Soc. **71** (1972), (2013)1, 85–96.

[30] R. S. PATHAK, *A distributional generalized Stieltjes transformation*, Proc. Edinburgh Math. Soc. **20** (1976), no. 1, 15–22.

[31] V. V. PELLER, *The behavior of functions of operators under perturbations. A glimpse at Hilbert space operators*, 287–324, Oper. Theory Adv. Appl. **207**, Birkhauser Verlag, Basel, 2010. Preprint, `http://arxiv.org/abs/0904.1761` [math. FA].

[32] V. V. PELLER, *The Lifshitz-Krein trace formula and operator Lipschitz functions*, Proc. Amer. Math. Soc. **144** (2016), 5207–5215, doi: `http://dx.doi.org/10.1090/proc/13140`.

[33] J. ROZENDAAL, F. SUKOCHEV, AND A. TOMSKOVA, *Operator Lipschitz functions on Banach spaces*, Studia Mathematica **232** (2016), no. 1, 57–92.

[34] R. SHILLING, R. SONG, Z. VONDRACEK, *Bernstein functions. Theory and applications*, de Greyter, Berlin-New York, 2010.

[35] D. R. YAFAEV, *Mathematical scattering theory, volume 105 of Translations of Mathematical Monographs*, American Mathematical Society, Providence, RI, 1992.

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