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KOROVKIN TYPE APPROXIMATION THEOREMS IN WEIGHTED SPACES VIA POWER SERIES METHOD

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Abstract. In this paper we consider power series method which is also member of the class of all continuous summability methods. We study a Korovkin type approximation theorem for a sequence of positive linear operators acting from a weighted space C_{ρ_1} into a weighted space B_{ρ_2} with the use of the power series method which includes both Abel and Borel methods. We also consider the rates of convergence of these operators.

1. Introduction

In the development of the theory of approximation by positive linear operators, the Korovkin theory has big importance. The classical Korovkin type theorems provide conditions for whether a given sequence of positive linear operators converges to the identity operator in the space of continuous functions on a compact interval [1, 11]. Korovkin type theorems have also been studied in L_p spaces [3, 8, 14, 15]. This theory has been extended with the use of summability methods since they provide a nonconvergent sequence to converge [2, 6, 7, 10, 12, 13, 16]. This is the motivation behind Fejer's famous theorem showing Cesàro method being effective in making the Fourier series of continuous periodic function to converge [4]. In this paper we consider power series method which is also member of the class of all continuous summability methods. This paper has two main goals. The first one is to investigate a Korovkin type approximation theorem for a sequence of positive linear operators acting from a weighted space C_{ρ_1} into a weighted space B_{ρ_2} with the use of the power series method which includes Abel method and Borel method. The second is to focus on the rate of convergence of these operators. The main purpose of using summability theory has always been to make a nonconvergent sequence to converge.

First of all, we recall some basic definitions and notations used in the paper. The function ρ is called a weight function if it is continuous on $\mathbb{R}$ with non-increasing on $(-\infty,0)$, non-decreasing on $(0,+\infty)$

$$\lim_{|x|\to\infty} \rho(x) = \infty \text{ and } \rho(0) = 1.$$

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Then the space of real valued functions f defined on $\mathbb{R}$ and satisfying for all $x \in \mathbb{R}$, $|f(x)| \leqslant M_f \rho(x)$ is called weighted space and denoted by B_ρ, where M_f is a constant depending on the function f. The subspace C_ρ of B_ρ is given by

$$C_\rho := \{f \in B_\rho : f \text{ is continuous over } \mathbb{R}\}$$

The spaces B_ρ and C_ρ are Banach spaces with the norm

$$\|f\|_\rho := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{\rho(x)}.$$

Now let ρ_1 and ρ_2 be two weight functions. Assume also that the condition

$$\lim_{|x|\to\infty} \frac{\rho_1(x)}{\rho_2(x)} = 0 \tag{1}$$

holds. It is clear that $C_{\rho_1} \subset C_{\rho_2}$ and $B_{\rho_1} \subset B_{\rho_2}$. If $\{L_n\}$ is a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} then the operator norm is given by

$$\|L\|_{C_{\rho_1}\to B_{\rho_2}} := \sup_{\|f\|_{\rho_1}=1} \|Lf\|_{\rho_2} = \|L(\rho_1)\|_{\rho_2}.$$

Let (p_n) be real sequence with $p_0 > 0$ and $p_n \geqslant 0$ $(n \in \mathbb{N})$, and such that the corresponding power series $p(t) := \sum_{n=0}^{\infty} p_n t^n$ has radius of convergence R with $0 < R \leqslant \infty$. If, for all $t \in (0,R)$,

$$\lim_{t\to R^-} \frac{1}{p(t)} \sum_{n=0}^{\infty} x_n p_n t^n = L$$

then we say that $x = (x_n)$ is convergent in the sense of power series method. Note that the power series method is regular if and only if

$$\lim_{t\to R^-} \frac{p_n t^n}{p(t)} = 0, \quad \text{for each } n \in \mathbb{N} \tag{2}$$

holds [5]. Let $\{L_n\}$ be a sequence of positive linear operators from C_{ρ_1} into B_{ρ_2} such that for every $f \in C_{\rho_1}$

$$\sup_{0<t<R} \frac{1}{p(t)} \sum_{n=0}^{\infty} \|L_n(\rho_1)\|_{\rho_1} p_n t^n < \infty \tag{3}$$

holds. Throughout the paper, the operators fulfill conditions (2) and (3). Consider the operator V_t from C_{ρ_1} into B_{ρ_2} defined by

$$V_t\{(f(y);x)\} := \frac{1}{p(t)} \sum_{n=0}^{\infty} L_n(f(y);x) p_n t^n$$

for each $t \in (0,R)$. In this study we present a Korovkin type theorem in weighted spaces via power series method.

2. Korovkin type theorem in weighted spaces

In this section we give a Korovkin type approximation of a function f by means of a sequence of positive linear operators from a weighted space C_{ρ_1} into a weighted space B_{ρ_2} with the use of the power series method. Now we can give our main theorem.

THEOREM 1. *If*

$$\lim_{t\to R^-} \|V_t F_i - F_i\|_{\rho_1} = 0, \quad i = 0,1,2, \tag{4}$$

where $F_i(x) = \frac{x^i \rho_1(x)}{1+x^2}$ *then we have for any* $f \in C_{\rho_1}$

$$\lim_{t\to R^-} \|V_t f - f\|_{\rho_2} = 0.$$

REMARK 1. Notice that $V_t(F_i) = V_t(F_i) - F_i + F_i$ and $F_i \in B_{\rho_1}$ for $i = 0,1,2$, then by the hypothesis (4) one can get $V_t(F_i) - F_i \in B_{\rho_1}$ which gives $V_t(F_i) \in B_{\rho_1}$. Since $\rho_1 = F_0 + F_2$, we have $V_t(\rho_1) \in B_{\rho_1}$. Also by condition (3) we can write, for a given $f \in C_{\rho_1}$ that $\|V_t f\|_{\rho_1} \leqslant \sup_{0<t<R} \|V_t\|_{C_{\rho_1}\to B_{\rho_1}} \|f\|_{\rho_1} \leqslant M_1 \|f\|_{\rho_1}$. This implies $V_t(f) \in B_{\rho_1}$. Therefore we obtain $V_t(C_{\rho_1}) \subset B_{\rho_1}$. Using (1) and the fact that $\sup_{0<t<R} \|V_t\|_{C_{\rho_1}\to B_{\rho_2}} = \sup_{0<t<R} \|V_t(\rho_1)\|_{\rho_2} \leqslant M_2 < \infty$ we observe that $V_t\{(.);x\}$ is also a positive linear operator from C_{ρ_1} into B_{ρ_2}.

We need a lemma which is used in the proof of the main theorem.

LEMMA 1. Under the assumptions of Theorem 1, we have for any $s > 0$, for any $f \in C_{\rho_1}$

$$\lim_{t\to R^-} \|V_t f - f\|_{\rho_2,[-s,s]} := \lim_{t\to R^-} \sup_{|x|\leqslant s} \frac{|V_t(f;x) - f(x)|}{\rho_2(x)} = 0.$$

Proof. Let $f \in C_{\rho_1}$ and $s > 0$. For given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $y \in \mathbb{R}$ and $|x| \leqslant s$

$$|f(y) - f(x)| < \varepsilon + (y-x)^2 F_0(y) K_{\rho_1}(x,\delta) \tag{5}$$

where $K_{\rho_1}(x,\delta) := 4M_f \rho_1(x)(\frac{1+x^2}{\delta^2} + 1)$. Observe that

$$|V_t(f(y);x) - f(x)| \leqslant V_t(|f(y) - f(x)|;x) + |f(x)||V_t(1;x) - 1|.$$

By (5) we obtain

$$|V_t(f(y);x) - f(x)| \leqslant \varepsilon + K_{\rho_1}(x,\delta)|V_t((y-x)^2 F_0(y);x)| + (\varepsilon + |f(x)|)|V_t(1;x) - 1|.$$

Also observe that

$$\begin{aligned} |V_t((y-x)^2 F_0(y);x)| \leqslant{}& |V_t(F_2(y);x) - F_2(x)| + 2|x||V_t(F_1(y);x) - F_1(x)| \\ &+ x^2 |V_t(F_0(y);x) - F_0(x)| \end{aligned}$$

and

$$F_0(x)|V_t(1;x)-1| \leqslant |V_t(F_0(y);x)-F_0(x)|+V_t(|F_0(y)-F_0(x)|;x).$$

Then we get

$$\|V_t f-f\|_{\rho_2,[-s,s]} \leqslant M(\varepsilon+\sum_{i=0}^{2}\|V_tF_i-F_i\|_{\rho_1})$$

for all $t\in(0,R)$ and for some $M>0$. Taking limit as $t\to R^-$ we obtain the desired result. $\square$

Now we are ready to prove our main result.

Proof of Theorem 1. For given $\varepsilon>0$, pick an s_0 such that $\rho_1(x)\leqslant\varepsilon\rho_2(x)$ for all $|x|>s_0$. We may write for $f\in C_{\rho_1}$,

$$\begin{aligned}\|V_t f-f\|_{\rho_2} &\leqslant \sup_{|x|\leqslant s_0}\frac{|V_t(f;x)-f(x)|}{\rho_2}+\sup_{|x|>s_0}\frac{|V_t(f;x)-f(x)|}{\rho_2}\\ &\leqslant \|V_t f-f\|_{\rho_2,[-s_0,0]}+\varepsilon\|V_t f-f\|_{\rho_1}\\ &\leqslant \|V_t f-f\|_{\rho_2,[-s_0,s_0]}+\varepsilon(\|V_t(f)\|_{\rho_1}+\|f\|_{\rho_1})\end{aligned}$$

and using the lemma and remark, we complete the proof. $\square$

3. Rate of convergence

Throughout this section we assume that $\rho_1(x)=1+x^2$. Also we consider the following weighted modulus of continuity [9]

$$w_{\rho_1}(f;\delta)=\sup_{|x-y|\leqslant\delta}\frac{|f(y)-f(x)|}{\rho_1(x)+\rho_1(y)}$$

where δ is a constant and $f\in C_{\rho_1}$.

It is known that for all $f\in C_{\rho_1}$,

$$|f(y)-f(x)|\leqslant\{\rho_1(x)+\rho_1(y)\}w_{\rho_1}(f;\delta)\Big(2+\frac{|y-x|}{\delta}\Big)$$

which implies

$$|f(y)-f(x)|\leqslant 4\rho_1(x)\rho_1(y)w_{\rho_1}(f;\delta)\Big(1+\frac{(y-x)^2}{\delta^2}\Big).$$

Using the linearity and positivity of V_t for all $t \in (0,R)$, we have for any $\delta \geqslant 0$ and all $f \in C_{\rho_1}$

$$
\begin{aligned}
|V_t(f(y);x) - f(x)| &\leqslant V_t(|f(y)-f(x)|;x) + |f(x)||(V_t(F_0(y);x) - F_0(x))| \\
&\leqslant 4\rho_1(x) w_{\rho_1}(f;\delta) V_t\left(\rho_1(y) + \rho_1(y)\frac{(y-x)^2}{\delta^2}\right) \\
&\quad + |f(x)||V_t(F_0(y);x) - F_0(x)| \\
&\leqslant 4\rho_1(x) w_{\rho_1}(f;\delta)[|V_t(\rho_1(y);x) - \rho_1(x)| + \rho_1(x) \\
&\quad + \frac{1}{\delta^2} V_t(\rho_1(y)(y-x)^2;x)] + |f(x)||V_t(F_0(y);x) - F_0(x)|.
\end{aligned}
$$

Now we obtain that

$$
\begin{aligned}
\|V_t f - f\|_{\rho_2^2} \leqslant 4\|\rho_1\|_{\rho_2} w_{\rho_1}(f;\delta)\Big\{\|V_t(\rho_1) - \rho_1\|_{\rho_2} + \|\rho_1\|_{\rho_2} + \frac{1}{\delta^2}\|V_t(\rho_1\varphi_x)\|_{\rho_2}\Big\} \\
+ \|\rho_1\|_{\rho_2}\|f\|_{\rho_2}\|V_t(1) - 1\|_{\rho_1}
\end{aligned}
$$

provided that $V_t(\rho_1\varphi_x) \in B_{\rho_2}$. For example if we consider V_t from C_{ρ_2} into B_{ρ_2} and assume $\rho_1\varphi_x \in B_{\rho_2}$ then one can guarantee that $V_t(\rho_1\varphi_x) \in B_{\rho_2}$. In this case putting $\delta := \alpha(t) = \sqrt{\|V_t(\varphi_x)\|_{\rho_2}}$ and combining the above inequalities, we conclude,

$$
\begin{aligned}
\|V_t f - f\|_{\rho_2^2} &\leqslant 4\|\rho_1\|_{\rho_2} w_{\rho_1}(f;\delta)\{\|V_t(\rho_1) - \rho_1\|_{\rho_2} + \|\rho_1\|_{\rho_2} + 1\} \\
&\quad + \|\rho_1\|_{\rho_2}\|f\|_{\rho_2}\|V_t(1) - 1\|_{\rho_1}.
\end{aligned}
$$

REMARK 2. Let $\rho_1(x) = 1 + x^2$ and $\rho_2(x) = 1 + x^4$. In this case the test functions F_i become $F_i(x) = x^i$, $i = 0,1,2$. Consider the following classical Bernstein-Kantorovich operator $\{L_n\}$ which is defined by

$$
L_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)\,dt.
$$

Using the operators $\{L_n f\}$ define the sequence of positive linear operators $U := \{U_n\}$ as follows:

$$
U_n(f;x) = (1+s_n)L_n(f;x), \quad \text{for } f \in C_{\rho_1},
$$

where $s_n = 1$, n is square and 0 otherwise. Also let $R = 1$, $p(t) = \frac{1}{1-t}$ and for $n \geqslant 0$, $p_n = 1$. In this case the power series method coincides with Abel method. Note that $\{s_n\}$ is convergent to 0 in the sense of power series method. It is easy to see that

$$
U_n(F_0;x) = 1 + s_n
$$

$$
U_n(F_1;x) = (1+s_n)\left\{\frac{nx}{n+1} + \frac{1}{2(n+1)}\right\}
$$

$$U_n(F_2;x) = (1+s_n)\left\{\frac{n(n-1)x^2}{(n+1)^2} + \frac{2nx}{(n+1)^2} + \frac{1}{3(n+1)^2}\right\}.$$

Furthermore, since

$$\begin{aligned} U_n(\rho_1;x) &= 1 + s_n + (1+s_n)\left\{\frac{n(n-1)x^2}{(n+1)^2} + \frac{2nx}{(n+1)^2} + \frac{1}{3(n+1)^2}\right\} \\ &= 1 + s_n\left\{1 + \frac{n(n-1)x^2}{(n+1)^2} + \frac{2nx}{(n+1)^2} + \frac{1}{3(n+1)^2}\right\} \\ &< 6\rho_2(x), \end{aligned}$$

$\sup_{0<t<R} \|V_t\|_{C_{\rho_1} \to B_{\rho_2}} < H < \infty$ holds where

$$V_t\{(f(y);x)\} := \frac{1}{p(t)} \sum_{n=0}^{\infty} U_n(f(y);x) p_n t^n.$$

Since all conditions of our main theorem holds for all $f \in C_{\rho_1}$, $\lim_{t \to R^-} \|V_t f - f\|_{\rho_2} = 0$. However $\{s_n\}$ is not convergent to zero it is clear that the classical theorem does not hold.

It is noteworthy that

- in the case of $R = 1$, $p(t) = \dfrac{1}{1-t}$ and for $n \geqslant 0$, $p_n = 1$ the power series method coincides with Abel method which is a sequence-to-function transformation,
- in the case of $R = \infty$, $p(t) = e^t$ and for $n \geqslant 0$, $p_n = \dfrac{1}{n!}$ the power series method coincides with Borel method.

We can therefore give all of the theorems of this paper for Abel and Borel convergences.

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