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MAPS PRESERVING THE LOCAL SPECTRUM OF SOME MATRIX PRODUCTS

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Abstract. Let $\mathscr{M}_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices, and x_0 a nonzero vector in $\mathbb{C}^n$. For two fixed scalars μ and ν in $\mathbb{C}$ for which $(\mu,\nu)\neq(0,0)$, we characterize all maps φ on $\mathscr{M}_n(\mathbb{C})$ satisfying

$$\sigma_{\mu ST^*S+\nu T^*S}(x_0)=\sigma_{\mu\varphi(S)\varphi(T)^*\varphi(S)+\nu\varphi(T)^*\varphi(S)}(x_0),\ \ (S,\ T\in\mathscr{M}_n(\mathbb{C})).$$

This provides, in particular, a complete description of all maps on $\mathscr{M}_n(\mathbb{C})$ preserving the local spectrum of the skew double product "TS^*" or the skew triple product "TS^*T" of matrices. It also unifies and extends several known results on local spectrum preservers.

1. Introduction

Recently, linear and nonlinear local spectra preserver problems attracted the attention of a number of authors. Mainly, several authors described maps on matrices or operators that preserve local spectrum, local spectral radius and local inner spectral radius; see for instance [9, 11, 12, 13, 14, 15, 17, 21] and the references therein. In [11, 12], nonlinear surjective maps on Banach space operators preserving the local spectrum of the product or the triple product of operators have been investigated. In [1], the authors described all surjective maps on the algebra $\mathscr{B}(\mathscr{H})$ of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ that preserve the local spectrum of skew double or skew triple products of Hilbert space operators.

In what follows, let $\mathscr{M}_n(\mathbb{C})$ denote the algebra of all $n\times n$ complex matrices and let x_0 be a nonzero vector in $\mathbb{C}^n$. For any matrix $T\in\mathscr{M}_n(\mathbb{C})$, let $\sigma_T(x_0)$ denote the local spectrum of $T\in\mathscr{M}_n(\mathbb{C})$ at x_0, and T^* stands, as usual, for its adjoint. For two fixed scalars μ and ν in $\mathbb{C}$ with $(\mu,\nu)\neq(0,0)$ we describe all maps φ on $\mathscr{M}_n(\mathbb{C})$ satisfying

$$\sigma_{\mu ST^*S+\nu T^*S}(x_0)=\sigma_{\mu\varphi(S)\varphi(T)^*\varphi(S)+\nu\varphi(T)^*\varphi(S)}(x_0),\ \ (S,\ T\in\mathscr{M}_n(\mathbb{C})).$$

This provides, in particular, a complete description of all maps on $\mathscr{M}_n(\mathbb{C})$ preserving the local spectrum of the skew double product TS^* or the skew triple product TS^*T of matrices. This seems to be new and extends the main results of [1] to the finite dimensional setting without any restriction or additional condition on the map φ such as

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surjectivity. It also unifies and extends the main results of [5, 6] where maps preserving the local spectrum of the product and the triple product of matrices have been characterized. Furthermore, we characterize maps on $\mathscr{M}_n(\mathbb{C})$ preserving the skew Jordan product ST^*+T^*S of matrices. We thus provide a variant of the main result of [9].

2. Main result

Throughout this paper, let μ and ν be two scalars such that $(\mu,\nu)\neq(0,0)$, and define a map θ from $\mathscr{M}_n(\mathbb{C})\times\mathscr{M}_n(\mathbb{C})$ to $\mathscr{M}_n(\mathbb{C})$ by

$$\theta(S,T):=\mu STS+\nu TS,\ (S,\ T\in\mathscr{M}_n(\mathbb{C})).$$

Recall that the local resolvent set, $\rho_T(x)$, of a bounded linear operator T on a complex Banach space X at a point $x\in X$ is the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f:U\to X$ such that $(T-\lambda)f(\lambda)=x$, $(\lambda\in U)$. The local spectrum of T at x is defined by

$$\sigma_T(x):=\mathbb{C}\setminus\rho_T(x).$$

It is a (possibly empty) closed subset of $\sigma(T)$, the spectrum of T. Our references are the books [2] by P. Aiena and [26] by K. B. Laursen, M. M. Neumann which provide an excellent exposition as well as a rich bibliography of the local spectral theory. However the local spectra of matrices is well understood and can be found, for instance, in [14].

The study of linear and nonlinear local spectra preserver problems was initiated by A. Bourhim and T. J. Ransford in [15], and continued by several authors; see for instance the survey article [13] and the references therein. In [8, 11, 12], nonlinear surjective maps on Banach space operators preserving the local spectrum of the product, the triple product and Jordan product of operators have been investigated. In [5, 6, 9], nonlinear maps on $\mathscr{M}_n(\mathbb{C})$ preserving the local spectrum of the product, the triple product and Jordan product of matrices have been characterized. In [1], the current authors described all surjective maps φ on $\mathscr{B}(\mathscr{H})$, the algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$, that preserve the local spectrum of skew double and triple products of operators. The aim of this paper is to characterize the form of all maps (not supposed to be surjective or even linear) on $\mathscr{M}_n(\mathbb{C})$ that preserve the local spectrum of skew double and triple products of matrices.

THEOREM 2.1. *Let x_0 be a nonzero vector in $\mathbb{C}^n$. A map φ on $\mathscr{M}_n(\mathbb{C})$ satisfies*

$$\sigma_{\theta(S,T^*)}(x_0)=\sigma_{\theta(\varphi(S),\varphi(T)^*)}(x_0),\ \ (S,\ T\in\mathscr{M}_n(\mathbb{C}))\tag{1}$$

if and only if there are two unitary matrices U and V in $\mathscr{M}_n(\mathbb{C})$ and a scalar $\alpha\in\mathbb{C}$ such that $Ux_0=\alpha x_0$ and

$$\varphi(T)=\begin{cases}UTU^* & \text{if } \mu\neq0\\ VTU & \text{if } \mu=0\end{cases}\qquad (T\in\mathscr{M}_n(\mathbb{C})).\tag{2}$$

This theorem shows, in particular, that the main results of [1] remain valid without any additional assumption on the map φ such as the surjectivity. Its proof uses some arguments influenced by ideas from several papers including [9] but it requires new ingredients which will be established in sections 3 and 4. It is also somehow simpler than and different from the proofs of the main results of [5, 6] wherein the characterization of rank one nilpotent operators is employed. Although our aim is to specialize the main results of [1] from the context of Hilbert space operators to the case of complex square matrices. It is worth to mention that, with no extra efforts, a variant of Theorem 2.1 can be obtained characterizing all maps φ on $\mathscr{M}_n(\mathbb{C})$ satisfying

$$\sigma_{\theta(S,T)}(x_0) = \sigma_{\theta(\varphi(S),\varphi(T))}(x_0), \quad (S,\ T \in \mathscr{M}_n(\mathbb{C})).$$

This variant unifies and extends the main results of [5, 6] and shows that their proofs could be combined and simplified.

3. Preliminaries and auxiliary results

In this section, we collect some lemmas and introduce some concepts and notions needed for the proof of our main result. For a bounded linear operator T on a complex Banach space X and a point $x \in X$, the nonzero local spectrum introduced by A. Bourhim and J. Mashreghi in [11, 12] is defined by

$$\sigma_T^*(x) := \begin{cases} \{0\} & \text{if } \sigma_T(x) = \{0\}; \\ \sigma_T(x) \setminus \{0\} & \text{if } \sigma_T(x) \neq \{0\}. \end{cases}$$

For a matrix $T \in \mathscr{M}_n(\mathbb{C})$, let $\mathrm{Tr}(T)$, T^{tr} and T^* denote the trace, the transpose, and the adjoint of T, respectively. For two nonzero vectors $x, y \in \mathbb{C}^n$, let $x \otimes y$ be the matrix of rank at most one defined by $x \otimes y = xy^{tr}$, thus $(x \otimes y)(z) := \langle z, y \rangle x$, where $\langle z, y \rangle = z^{tr} y$, for all $z \in \mathbb{C}^n$.

The first lemma is an elementary result that describes the nonzero local spectrum of rank one matrices, and can be found in [15].

LEMMA 3.1. *For any vectors $x_0,\ x,\ y \in \mathbb{C}^n$, we have*

$$\sigma_{x\otimes y}^*(x_0) := \begin{cases} \{\langle y, x \rangle\} & \textit{if } \langle y, x_0 \rangle \neq 0; \\ \{0\} & \textit{if } \langle y, x_0 \rangle = 0. \end{cases}$$

The second lemma is a *local spectral identity principle* that provides necessary and sufficient conditions for two matrices to be the same. Before stating it, let us introduce a few more notations and recall some useful facts. Let GL_n be the set of all invertible matrices in $\mathscr{M}_n(\mathbb{C})$ and denote by I_n the identity matrix. Recall that Newburgh's theorem tells us that the spectrum function is continuous on $\mathscr{M}_n(\mathbb{C})$; see [4, Corollary 3.4.5]. While the local spectrum is not continuous on $\mathscr{M}_n(\mathbb{C})$ but it is lower semi-continuous on $\mathscr{M}_n(\mathbb{C})$; see [20, Corollary 2.3]. That is, if x_0 is a fixed vector in $\mathbb{C}^n$ and $(T_k)_{k\geqslant 1} \subset \mathscr{M}_n(\mathbb{C})$ is a converging sequence to a matrix $T \in \mathscr{M}_n(\mathbb{C})$, then $\sigma_T(x_0) \subset \liminf\limits_{k\to\infty} \sigma_{T_k}(x_0)$.

LEMMA 3.2. *Let $x_0 \in \mathbb{C}^n$ be a nonzero vector, and $A, B \in \mathscr{M}_n(\mathbb{C})$ be two matrices. Then $A = B$ if and only if $\sigma_{\theta(S,A^*)}(x_0) = \sigma_{\theta(S,B^*)}(x_0)$ for all $S \in GL_n$.*

Proof. Since '*only if*' part is trivial, we only have to prove the '*if*' part. So, assume that

$$\sigma_{\theta(S,A^*)}(x_0) = \sigma_{\theta(S,B^*)}(x_0),\ S \in GL_n, \tag{3}$$

and let us show that $A = B$.

In view of [9, Lemma 5.1], we may and shall assume that $\nu \neq 0$ and let us show that $\langle A^*x, y\rangle = \langle B^*x, y\rangle$ for all $x,\ y \in \mathbb{C}^n$. Fix two nonzero vectors x and y in $\mathbb{C}^n$, and set $S := x \otimes y$. Note that, since GL_n is dense in $\mathscr{M}_n(\mathbb{C})$, there is a sequence $(S_k)_k$ of invertible matrices that converges to S. Since

$$\theta(S, A^*) = [\mu\langle A^*x, y\rangle x + \nu A^*x] \otimes y,$$

and

$$\theta(S, B^*) = [\mu\langle B^*x, y\rangle x + \nu B^*x] \otimes y,$$

we have

$$\begin{aligned} \sigma(\theta(S,A^*)) &= \{0,\ \langle A^*x,y\rangle[\mu\langle x,y\rangle + \nu]\} \\ \sigma(\theta(S,B^*)) &= \{0,\ \langle B^*x,y\rangle[\mu\langle x,y\rangle + \nu]\}. \end{aligned} \tag{4}$$

Now, for any integer k, we have

$$\sigma_{\theta(S_k,A^*)}(x_0) = \sigma_{\theta(S_k,B^*)}(x_0) \subset \sigma(\theta(S_k,A^*)) \cap \sigma(\theta(S_k,B^*)).$$

This together with the continuity of the spectrum and the lower semi-continuity of the local spectrum imply that

$$\sigma_{\theta(S,A^*)}(x_0) \subset \sigma(\theta(S,A^*)) \cap \sigma(\theta(S,B^*))$$

and

$$\sigma_{\theta(S,B^*)}(x_0) \subset \sigma(\theta(S,A^*)) \cap \sigma(\theta(S,B^*)).$$

Therefore,

$$\sigma^*_{\theta(S,A^*)}(x_0) \cup \sigma^*_{\theta(S,B^*)}(x_0) \subset \sigma(\theta(S,A^*)) \cap \sigma(\theta(S,B^*)). \tag{5}$$

If $\langle x_0, y\rangle \neq 0$, then (4), (5) and Lemma 3.1 entail that

$$\langle A^*x,y\rangle[\mu\langle x,y\rangle + \nu] = \langle B^*x,y\rangle[\mu\langle x,y\rangle + \nu]. \tag{6}$$

If necessary, take a nonzero real scalar t such that $t\mu\langle x,y\rangle + \nu \neq 0$ and replace x by tx in (6) to get that $\langle A^*x,y\rangle = \langle B^*x,y\rangle$. If, however, $\langle x_0,y\rangle = 0$, take a nonzero vector z such that $\langle x_0,z\rangle \neq 0$ so that $\langle x_0,y+z\rangle \neq 0$. What has been proved previously shows that $\langle A^*x,z\rangle = \langle B^*x,z\rangle$ and $\langle A^*x,(y+z)\rangle = \langle B^*x,(y+z)\rangle$, and thus $\langle A^*x,y\rangle = \langle B^*x,y\rangle$ in this case too. This shows that $A = B$, and completes the proof. $\square$

As in [9, 14, 21], let us consider the set

$$\mathscr{S}_{n,x_0} = \{T \in \mathscr{M}_n(\mathbb{C}) : |\sigma(T)| = n \text{ and } T \text{ is cyclic with cyclic vector } x_0\},$$

where $|\mathscr{S}|$ denotes the cardinal of any subset set $\mathscr{S} \subset \mathbb{C}$. It is well known that $\mathscr{S}_{n,x_0}$ is an open dense subset of $\mathscr{M}_n(\mathbb{C})$ and

$$\sigma_T(x_0) = \sigma(T), \quad (T \in \mathscr{S}_{n,x_0}). \tag{7}$$

LEMMA 3.3. *If $\mathscr{O}$ is a nonempty open subset of $\mathscr{M}_n(\mathbb{C})$, then $\left\{\mu S^2 + \nu S : S \in \mathscr{O}\right\}$ is a spanning set of $\mathscr{M}_n(\mathbb{C})$.*

Proof. Pick a matrix $T \in \mathscr{M}_n(\mathbb{C})$ and assume that

$$\operatorname{Tr}\left([\mu S^2 + \nu S].T\right) = 0$$

for all $S \in \mathscr{O}$. To show that $\left\{\mu S^2 + \nu S : S \in \mathscr{O}\right\}$ is a spanning set of $\mathscr{M}_n(\mathbb{C})$, it suffices to prove that $T = 0$.

Given a matrix $S \in \mathscr{O}$, for every $R \in \mathscr{M}_n(\mathbb{C})$ and real scalar t small enough, we have $S + tR \in \mathscr{O}$ and

$$\begin{aligned} 0 &= \operatorname{Tr}\left([\mu(S+tR)^2 + \nu(S+tR)]T\right) \\ &= \operatorname{Tr}\left([\mu S^2 + \nu S].T\right) + \operatorname{Tr}([\mu(RS+SR) + \nu R]T)t + \mu \operatorname{Tr}\left(R^2 T\right)t^2. \end{aligned}$$

It then follows that $\operatorname{Tr}([\mu(RS+SR)+\nu R]T) = 0$ and $\mu \operatorname{Tr}(R^2T) = 0$ for all $R \in \mathscr{M}_n(\mathbb{C})$. If $\mu = 0$, then $\nu \neq 0$ and $\operatorname{Tr}(RT) = 0$ for all $R \in \mathscr{M}_n(\mathbb{C})$. This clearly implies that $T = 0$. If, however, $\mu \neq 0$ then $\operatorname{Tr}(R^2T) = 0$ for all $R \in \mathscr{M}_n(\mathbb{C})$. Since $\{R^2 : R \in \mathscr{M}_n(\mathbb{C})\}$ spans $\mathscr{M}_n(\mathbb{C})$, we conclude that $T = 0$ in this case too. The Lemma is therefore proved. $\square$

The following lemma plays a crucial role in the proof of our main result.

LEMMA 3.4. *For every $T_0 \in GL_n$, there is $S_0 \in GL_n$ and two open neighborhoods $\mathscr{V}_{S_0}$ and $\mathscr{V}_{\theta(S_0,T^*)}$ of S_0 and $\theta(S_0,T_0^*)$ such that the mapping $\theta(.,T_0^*) : S \mapsto \theta(S,T_0^*)$ is a diffeomorphism from $\mathscr{V}_{S_0}$ onto $\mathscr{V}_{\theta(S_0,T_0^*)}$.*

Proof. Firstly, observe that the partial S-derivative of $\theta(S,T^*)$ is given by

$$\partial_S\theta(S,T^*) \cdot H = H \cdot (\mu T^* S) + (\mu S + \nu I_n)T^* \cdot H, \ (H \in \mathscr{M}_n(\mathbb{C})),$$

and keep in mind that Sylvester's theorem [28, Theorem 2.4.4.1] tells us that $\partial_S\theta(S,T^*)$ is an isomorphism provided that

$$\sigma(\mu T^* S) \cap \sigma(-(\mu S + \nu I_n)T^*) = \emptyset. \tag{8}$$

Secondly, fix $T_0 \in GL_n$ and let us show that there is a matrix $S_0 \in GL_n$ such that (8) is satisfied for $T = T_0$ and $S = S_0$. If $\mu = 0$, then $\nu \neq 0$ and (8) is satisfied for any

$S_0 \in GL_n$. If $\mu \neq 0$, write $T_0^* = PRP^{-1}$, where $P \in GL_n$ and R is an upper triangular matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Let t is a positive real number, and D_t be the diagonal matrix with diagonal entries $t\overline{\mu\lambda_1}, \ldots, t\overline{\mu\lambda_n}$. Set $S_t := PD_tP^{-1}$ and note that $S_t \in GL_n$. Moreover, for t large enough, we have $t|\mu|^2(|\lambda_i|^2 + |\lambda_j|^2) + \nu\lambda_j \neq 0$ for all $(i,j) \in \{1, \ldots, n\}^2$ and then

$$\left\{\mu\lambda_i t\overline{\mu\lambda_i} \ : 1 \leqslant i \leqslant n\right\} \cap \left\{-(\mu t\overline{\mu\lambda_j} + \nu)\lambda_j \ : 1 \leqslant j \leqslant n\right\} = \emptyset.$$

This tells us that (8) is satisfied for T_0 and $S_0 := S_t$ for any t large enough. Therefore $\partial_S\theta(S_0, T_0{}^*)$ is an isomorphism; as desired.

Finally, the inverse function Theorem, applied to the function $S \mapsto \theta(S, T_0^*)$, tells us that there are two open neighborhoods $\mathscr{V}_{S_0}$ and $\mathscr{V}_{\theta(S_0,T_0^*)}$ of S_0 and $\theta(S_0, T_0^*)$ such the mapping $\theta(., T_0^*) : \mathscr{V}_{S_0} \to \mathscr{V}_{\theta(S_0,T_0^*)}$ is bijective and both $\theta(., T_0^*)$ and its inverse are continuously differentiable. This proves the lemma. $\square$

We close this section with the following lemma that tells us that a map φ on $\mathscr{M}_n(\mathbb{C})$ satisfying (1) is linear on GL_n.

LEMMA 3.5. *If a map φ on $\mathscr{M}_n(\mathbb{C})$ satisfies (1), then its restriction on GL_n is equal to a bijective linear mapping L.*

Proof. The proof breaks down into two steps.

Step 1. For every $T_0 \in GL_n$, there is an open neighborhood $\mathscr{V}_{T_0}$ of T_0 and an nonempty open set $\mathscr{O}_{T_0} \subseteq GL_n$ such that

$$\theta(S, T^*) \in \mathscr{S}_{n,x_0}, \ \text{ for all } \ (S,T) \in \mathscr{O}_{T_0} \times \mathscr{V}_{T_0}.$$

First, fix an invertible matrix $T_0 \in GL_n$ and let us show that

$$\Delta_{T_0,x_0} := \left\{S \in GL_n : \theta(S, T_0^*) \in \mathscr{S}_{n,x_0}\right\}$$

is a nonempty open set. Indeed, by Lemma 3.4, there is $S_0 \in GL_n$ and two open neighborhoods $\mathscr{V}_{S_0}$ and $\mathscr{V}_{\theta(S_0,T_0^*)}$ of S_0 and $\theta(S_0, T_0^*)$ for which $\theta(., T_0^*) : S \mapsto \theta(S, T_0^*)$ is a diffeomorphism from $\mathscr{V}_{S_0}$ onto $\mathscr{V}_{\theta(S_0,T_0^*)}$. Since $\mathscr{S}_{n,x_0}$ is an open dense set in $\mathscr{M}_n(\mathbb{C})$, we see that $\mathscr{S}_{n,x_0} \cap \mathscr{V}_{\theta(S_0,T_0^*)}$ is a nonempty open set. As $\theta(., T_0^*)$ is continuous on $\mathscr{M}_n(\mathbb{C})$, we have $\theta(., T_0^*)^{-1}\left(\mathscr{S}_{n,x_0} \cap \mathscr{V}_{\theta(S_0,T_0^*)}\right)$ is a nonempty open set too. Therefore, by the density of GL_n on $\mathscr{M}_n(\mathbb{C})$, we conclude that $GL_n \cap \theta(., T_0^*)^{-1}\left(\mathscr{S}_{n,x_0} \cap \mathscr{V}_{\theta(S_0,T_0^*)}\right)$ is a nonempty open set contained in Δ_{T_0,x_0}. Hence Δ_{T_0,x_0} is a nonempty open set.

Now, since the map $(S,T) \mapsto \theta(S, T^*)$ is continuous from $\mathscr{M}_n(\mathbb{C}) \times \mathscr{M}_n(\mathbb{C})$ to $\mathscr{M}_n(\mathbb{C})$, we obtain that

$$W := \left\{(S,T) \in GL_n \times GL_n \ : \ \theta(S, T^*) \in \mathscr{S}_{n,x_0}\right\}$$

is an open subset of $GL_n \times GL_n$ and $\Delta_{T_0,x_0} \times \{T_0\} \subseteq W$. Thus, there is an open neighborhood $\mathscr{V}_{T_0}$ of T_0 and an nonempty open set $\mathscr{O}_{T_0} \subseteq GL_n$ such that $\mathscr{O}_{T_0} \times \mathscr{V}_{T_0} \subseteq W$; as desired.

Step 2. We show that φ restricted on GL_n is equal to a bijective linear mapping L.

We first show that for every $T_0 \in GL_n$, there is an open neighborhood $\mathscr{V}_{T_0} \subseteq GL_n$ of T_0, such that φ is agree with an invertible linear mapping $L_{T_0} : \mathscr{M}_n(\mathbb{C}) \to \mathscr{M}_n(\mathbb{C})$ on $\mathscr{V}_{T_0}$. Indeed, choose $\mathscr{V}_{T_0}$ and $\mathscr{O}_{T_0} \subseteq GL_n$ as in Step 1.

From (1) and (7), we see that

$$\sigma(\theta(S,T^*)) = \sigma(\theta(\varphi(S),\varphi(T)^*)) \text{ and } \left|\sigma(\theta(S,T^*))\right| = n,$$

for all $(S,T) \in \mathscr{O}_{T_0} \times \mathscr{V}_{T_0}$. It then follows that

$$\mathrm{Tr}\big([\mu S^2 + \nu S]T^*\big) = \mathrm{Tr}\big([\mu\varphi(S)^2 + \nu\varphi(S)]\varphi(T)^*\big)$$

for all $(S,T) \in \mathscr{O}_{T_0} \times \mathscr{V}_{T_0}$. By Lemma 3.3, the set $\{\mu S^2 + \nu S : S \in \mathscr{O}_{T_0}\}$ spans $\mathscr{M}_n(\mathbb{C})$, and thus it contains a basis of $\mathscr{M}_n(\mathbb{C})$. Now, following the same argument as the one in the proof of Assertion 1 of [16, Theorem 2.1], one sees that the restriction of φ on $\mathscr{V}_{T_0}$ is equal to an invertible linear mapping L_{T_0}. Since GL_n is arcwise connected, using the arguments of Assertion 2 of the proof of [16, Theorem 2.1], we conclude that the map φ is equal to a bijective linear map L on GL_n, and the proof is complete. $\square$

4. Maps preserving the spectrum of a product of matrices

In this section, we give a characterization of maps φ on $\mathscr{M}_n(\mathbb{C})$ satisfying

$$\sigma(\theta(S,T^*)) = \sigma(\theta(\varphi(S),\varphi(T)^*)), \ (S,T \in \mathscr{M}_n(\mathbb{C})). \tag{9}$$

Such a characterization is new and will serve in the proof of the Theorem 2.1.

PROPOSITION 4.1. *A map φ on $\mathscr{M}_n(\mathbb{C})$ satisfies (9) if and only if there are two unitary matrices U and V in $\mathscr{M}_n(\mathbb{C})$ such that $V = U^*$ whenever $\mu \neq 0$ and either*

$$\varphi(T) = UTV, \ (T \in \mathscr{M}_n(\mathbb{C})), \tag{10}$$

or

$$\varphi(T) = UT^{tr}V, \ (T \in \mathscr{M}_n(\mathbb{C})). \tag{11}$$

Proof. For the "*if*" part, we need only to prove that the mapping $T \mapsto T^{tr}$ satisfies (9) since it is obvious that (9) holds provided that φ takes the form (10). Indeed, assume that $\varphi(T) = T^{tr}$ for all $T \in \mathscr{M}_n(\mathbb{C})$ and note that

$$\begin{aligned}
\sigma(\theta(S,T^*)) &= \sigma((\mu S + \nu)T^*S)\\
&= \sigma(T^*S(\mu S + \nu))\\
&= \sigma(T^*(\mu S + \nu)S)\\
&= \sigma(ST^*(\mu S + \nu))\\
&= \sigma((ST^*(\mu S + \nu))^{tr})\\
&= \sigma((\mu S + \nu)^{tr}(T^*)^{tr}S^{tr})\\
&= \sigma((\mu S^{tr} + \nu)(T^{tr})^*S^{tr})\\
&= \sigma(\theta(S^{tr},(T^{tr})^*))
\end{aligned}$$

for all $S,T \in \mathscr{M}_n(\mathbb{C})$; as claimed.

Now, assume that φ satisfies (9) and let us shows that φ has the desired forms. First, we note that along the same lines as the proof of Lemma 3.5, one shows that φ restricted on GL_n is equal to a bijective linear map $L: \mathscr{M}_n(\mathbb{C}) \to \mathscr{M}_n(\mathbb{C})$. Thus, the continuity of both the spectrum and the linear mapping L implies that

$$\sigma(\theta(S,T^*)) = \sigma(\theta(L(S),L(T)^*)) \tag{12}$$

for all $T,\ S \in \mathscr{M}_n(\mathbb{C})$. Take $S = tI_n$ in (12), where t is a nonzero scalar satisfying $\mu t + \nu \neq 0$, we obtain

$$\sigma((\mu t+\nu)tT^*) = \sigma((\mu L(tI_n)+\nu I_n)L(T)^*L(tI_n)),\ T \in \mathscr{M}_n(\mathbb{C}). \tag{13}$$

Take $T = I_n$ and plug in (13) to see that $L(tI_n)$ and $(\mu L(tI_n)+\nu I_n)$ are invertible matrices. This together with (13) show L preserves invertible matrices, and thus there are two invertible matrices M and N in $\mathscr{M}_n(\mathbb{C})$ such that L has one of the forms $T \mapsto MTN$ or $T \mapsto MT^{tr}N$; see [23].

To show that L takes either the form (10) or (11), we first show that M^*M and NN^* are scalar matrices. We may and shall assume that $L(T) = MTN$ for all $T \in \mathscr{M}_n(\mathbb{C})$ as the case when L takes the second form is dealt by similarity. Let $x,\ y,\ h,\ l \in \mathbb{C}^n$ be four vectors and note that, for $T^* := y \otimes x$ and $S := h \otimes l$, the identity (12) gives

$$\sigma\big(\langle h,x\rangle[(\mu\langle y,l\rangle h+\nu y)\otimes l]\big) = \sigma\big(\langle M^*Mh,x\rangle[(\mu\langle NN^*y,l\rangle NMh+\nu NN^*y)\otimes l]\big).$$

Hence,

$$\{0;\langle h,x\rangle\langle y,l\rangle[\mu\langle h,l\rangle+\nu]\} = \{0;\langle M^*Mh,x\rangle\langle NN^*y,l\rangle[\mu\langle NMh,l\rangle+\nu]\},$$

and

$$\langle h,x\rangle\langle y,l\rangle[\mu\langle h,l\rangle+\nu] = \langle M^*Mh,x\rangle\langle NN^*y,l\rangle[\mu\langle NMh,l\rangle+\nu]. \tag{14}$$

By the way of contradiction, suppose that M^*M is not a scalar matrix so that there exists a nonzero vector $h_1 \in \mathbb{C}^n$ such that M^*Mh_1 and h_1 are linearly independent. Thus, there is a nonzero vector $x_1 \in \mathbb{C}^n$ such that $\langle M^*Mh_1,x_1\rangle = 0$ and $\langle h_1,x_1\rangle = 1$. Then (14) applied to $y = h = h_1$, $x = x_1$ and $l = tx_1$ for an arbitrary $t \in \mathbb{C}$ entails that $\mu t+\nu = 0$ for all scalars t. This contradicts the fact that $(\mu,\nu) \neq (0,0)$ and shows that M^*M is a scalar matrix; as desired. Similarly, we show that NN^* is a scalar matrix too, and thus there are two positive scalars α and β such that $M^*M = \alpha I_n$ and $NN^* = \beta I_n$.

Second, we show that $\alpha\beta = 1$ so that $U := \frac{1}{\sqrt{\alpha}}M$ and $V := \frac{1}{\sqrt{\beta}}N$ are unitary matrices and

$$UTV = \frac{1}{\sqrt{\alpha\beta}}MTN = MTN = L(T),$$

for all $T \in \mathscr{M}_n(\mathbb{C})$. Indeed, if $\mu = 0$, then (14) trivially implies that $\alpha\beta = 1$. If, however, $\mu \neq 0$, we prove that NM is a scalar matrix. By the way of contradiction, assume that NM is not a scalar matrix so that there is a nonzero vector $h_1 \in \mathbb{C}^n$ such that NMh_1 and h_1 are linearly independent. Therefore, for every $t \in \mathbb{C}$, there is a nonzero

$l_1 \in \mathbb{C}^n$ such that $\langle NMh_1, l_1\rangle = -\frac{\nu}{\mu}$ and $\langle h_1, l_1\rangle = t$ and (14) applied to $y = h = h_1$ and $x = l = l_1$ gives $\mu t + \nu = 0$. This contradiction shows that $NM = \gamma I_n$ for some nonzero scalar $\gamma \in \mathbb{C}$ and (14) becomes

$$\langle h,x\rangle\langle y,l\rangle[\mu\langle h,l\rangle+\nu] = \alpha\beta\langle h,x\rangle\langle y,l\rangle[\mu\gamma\langle h,l\rangle+\nu]$$

for all $x,\ y,\ h,\ l \in \mathbb{C}^n$. In particular, when $x = h$ and $y = l$, we get

$$(1-\alpha\beta\gamma)\mu\langle h,l\rangle + (1-\alpha\beta)\nu = 0$$

for all $h, l \in \mathbb{C}^n$, and $\alpha\beta\gamma = 1$ and $(1-\alpha\beta)\nu = 0$. This implies that γ is positive too, and note that, since $NM = \gamma I_n$, $M^*M = \alpha I_n$ and $NN^* = \beta I_n$, we have

$$\gamma^2 I_n = NN^*M^*M = \alpha\beta I_n.$$

Hence, $\gamma^2 I_n = \alpha\beta I_n$, and $\gamma^3 = \alpha\beta\gamma = 1$. Clearly, $\gamma = 1$ and $\alpha\beta = 1$; as desired.

Finally, let us show that φ has one of the desired forms. Observe that the map $\varphi \circ L^{-1}$ satisfies (9) and thus replacing φ by $\varphi \circ L^{-1}$, we may and shall assume that $\varphi(T) = T$ for all $T \in GL_n$, and then prove that $\varphi(T) = T$ for all $T \in \mathscr{M}_n(\mathbb{C})$. Indeed, for every $S \in GL_n$ and $T \in \mathscr{M}_n(\mathbb{C})$, we have

$$\begin{aligned}
\sigma\left((\mu S^2+\nu S)T^*\right) &= \sigma\left(S(\mu S+\nu I_n)T^*\right) = \sigma\left((\mu S+\nu I_n)T^*S\right)\\
&= \sigma\left(\theta(S,T^*)\right) = \sigma\left(\theta(\varphi(S),\varphi(T)^*)\right)\\
&= \sigma\left(\theta(S,\varphi(T)^*)\right) = \sigma\left((\mu S+\nu I_n)\varphi(T)^*S\right)\\
&= \sigma\left((\mu S^2+\nu S)\varphi(T)^*\right).
\end{aligned}$$

By the continuity of the spectrum and the density of GL_n on $\mathscr{M}_n(\mathbb{C})$, we deduce that

$$\sigma\left((\mu S^2+\nu S)T^*\right) = \sigma\left((\mu S^2+\nu S)\varphi(T)^*\right)$$

for all S and $T \in \mathscr{M}_n(\mathbb{C})$. Now, observe that for every $S \in \mathscr{M}_n(\mathbb{C})$ of rank at most one, each of the matrices $(\mu S^2+\nu S)T^*$ and $(\mu S^2+\nu S)\varphi(T)^*$ is of rank at most one. Thus

$$\operatorname{Tr}\left((\mu S^2+\nu S)T^*\right) = \operatorname{Tr}\left((\mu S^2+\nu S)\varphi(T)^*\right) \tag{15}$$

for all $T \in \mathscr{M}_n(\mathbb{C})$ and all matrices $S \in \mathscr{M}_n(\mathbb{C})$ of rank at most one. Note that for every rank one matrix $S \in \mathscr{M}_n(\mathbb{C})$, we have $\mu S^2+\nu S = (\mu\operatorname{Tr}(S)+\nu)S$. This and (15) entail that

$$\operatorname{Tr}(ST^*) = \operatorname{Tr}(S\varphi(T)^*)$$

for all $T \in \mathscr{M}_n(\mathbb{C})$ and all matrices $S \in \mathscr{M}_n(\mathbb{C})$ of rank at most one such that $\mu\operatorname{Tr}(S)+\nu \neq 0$. Now, we can easily show that the set of all matrices $S \in \mathscr{M}_n(\mathbb{C})$ of rank at most one for which $\mu\operatorname{Tr}(S)+\nu \neq 0$ spans $\mathscr{M}_n(\mathbb{C})$. Hence, $\operatorname{Tr}(ST^*) = \operatorname{Tr}(S\varphi(T)^*)$ for all $S,\ T \in \mathscr{M}_n(\mathbb{C})$, and thus $\varphi(T) = T$ for all $T \in \mathscr{M}_n(\mathbb{C})$. This finishes the proof. $\square$

5. Proof of the main result

Checking the "*if*" part is straightforward. For the "*only if*" part, assume that φ satisfies (1) and let us show that φ has the desired form. By Lemma 3.5, φ is a continuous map and is equal to a bijective linear map L on GL_n.

First, we prove that there are two unitary matrices U and V and a nonzero scalar $\alpha \in \mathbb{C}$ such that $V = U^*$ if $\mu \neq 0$, $Vx_0 = \alpha x_0$ and $L(T) = UTV$. Indeed, the identities (1) and (7) together with the density of $\mathscr{S}_{n,x_0}$ in $\mathscr{M}_n(\mathbb{C})$ and the continuity of both the spectrum and the map L imply that

$$\sigma\left(\theta(S,T^*)\right) = \sigma\left(\theta(L(S),L(T)^*)\right), \; (S,\; T \in \mathscr{M}_n(\mathbb{C})). \tag{16}$$

If $\mu \neq 0$, Proposition 4.1 tells us that there is a unitary matrix $U \in \mathscr{M}_n(\mathbb{C})$ such that either $L(T) = UTU^*$ for all $T \in \mathscr{M}_n(\mathbb{C})$ or $L(T) = UT^{tr}U^*$ for all $T \in \mathscr{M}_n(\mathbb{C})$. Choose a scalar t such that $\mu t^2 + \nu t \neq 0$, in view of (1), we have

$$\begin{aligned}
\sigma_{(\mu t^2+\nu t)T^*}(x_0) &= \sigma_{\theta(tI_n,T^*)}(x_0) \\
&= \sigma_{\theta(\varphi(tI_n),\varphi(T)^*)}(x_0) \\
&= \sigma_{\theta(L(tI_n),L(T)^*)}(x_0) \\
&= \sigma_{(\mu t^2+\nu t)L(T)^*}(x_0)
\end{aligned}$$

for all $T \in GL_n$. Hence,

$$\sigma_{L(T)^*}(x_0) = \sigma_{T^*}(x_0)$$

for all $T \in GL_n$, and thus [9, Lemma 3.8 and Lemma 3.9] tell us that $Ux_0 = \alpha x_0$ for some nonzero scalar α and L takes only the form $T \mapsto UTU^*$ on $\mathscr{M}_n(\mathbb{C})$.

If $\mu = 0$, Proposition 4.1 yields two unitary matrices U and V in $\mathscr{M}_n(\mathbb{C})$ such that either $L(T) = UTV$ for all $T \in \mathscr{M}_n(\mathbb{C})$ or $L(T) = UT^{tr}V$ for all $T \in \mathscr{M}_n(\mathbb{C})$. To show that L cannot take the second form, suppose for the sake of contradiction that $L(T) = UT^{tr}V$ for all $T \in \mathscr{M}_n(\mathbb{C})$. It then follows from (1) that

$$\begin{aligned}
\sigma_{\nu S}(x_0) &= \sigma_{\theta(S,I_n^*)}(x_0) \\
&= \sigma_{\theta(\varphi(S),\varphi(I_n)^*)}(x_0) \\
&= \sigma_{\nu L(I_n)^*L(S)}(x_0) \\
&= \sigma_{\nu V^*S^{tr}V}(x_0)
\end{aligned}$$

for all $S \in GL_n$. This contradicts [9, Lemma 3.9] and shows that $L(T) = UTV$ for all $T \in \mathscr{M}_n(\mathbb{C})$. In view of (1), we obtain that

$$\begin{aligned}
\sigma_{\nu S}(x_0) &= \sigma_{\theta(S,I_n^*)}(x_0) \\
&= \sigma_{\nu L(I_n)^*L(S)}(x_0) \\
&= \sigma_{\nu V^*SV}(x_0),
\end{aligned}$$

for all $S \in GL_n$. Hence, by [9, Lemma 3.8], we have $Vx_0 = \alpha x_0$ for some nonzero scalar $\alpha \in \mathbb{C}$; as desired.

Now, we are in a position to show that φ has the asserted form. Keep in mind that we have shown that $\varphi(S) = L(S)$ for all $S \in GL_n$ and that there are two unitary matrices U and V and a nonzero scalar $\alpha \in \mathbb{C}$ such that $V = U^*$ if $\mu \neq 0$, $Vx_0 = \alpha x_0$ and $L(T) = UTV$. For every $S \in GL_n$ and $T \in \mathscr{M}_n(\mathbb{C})$, we have

$$\begin{aligned}
\sigma_{\theta(S,\varphi(T)^*)}(x_0) &= \sigma_{\mu S\varphi(T)^*S+\nu\varphi(T)^*S}(x_0) \\
&= \sigma_{\mu\varphi(U^*SV^*)\varphi(T)^*\varphi(U^*SV^*)+\nu\varphi(T)^*\varphi(U^*SV^*)}(x_0) \\
&= \sigma_{\mu(U^*SV^*)T^*(U^*SV^*)+\nu T^*(U^*SV^*)}(x_0) \\
&= \sigma_{\mu L(U^*SV^*)L(T)^*L(U^*SV^*)+\nu L(T)^*L(U^*SV^*)}(x_0) \\
&= \sigma_{\mu SL(T)^*S+\nu L(T)^*S}(x_0) \\
&= \sigma_{\theta(S,L(T)^*)}(x_0).
\end{aligned}$$

Thus, using Lemma 3.2, we conclude that $\varphi(T) = L(T)$ for all $T \in \mathscr{M}_n(\mathbb{C})$, and the main result is proved; as desired.

6. Concluding remarks and comments

Although our main result "Theorem 2.1" specializes the main results of [1] from the context of Hilbert space operators to the case of complex square matrices, we would like to mention that, with no extra efforts, the characterization of all maps φ on $\mathscr{M}_n(\mathbb{C})$ satisfying

$$\sigma_{\theta(S,T)}(x_0) = \sigma_{\theta(\varphi(S),\varphi(T))}(x_0), \quad (S,\ T \in \mathscr{M}_n(\mathbb{C})) \tag{17}$$

can be obtained.

THEOREM 6.1. *Let x_0 be a nonzero vector in $\mathbb{C}^n$. A map φ on $\mathscr{M}_n(\mathbb{C})$ satisfies (17) if and only if there is an invertible matrix A and a complex scalar λ such that $Ax_0 = x_0$ and*

$$\varphi(T) = \lambda ATA^{-1}, \ (T \in \mathscr{M}_n(\mathbb{C})), \tag{18}$$

with

$$\begin{cases} \lambda^3 = 1 & \text{if} \quad \nu = 0; \\ \lambda = \pm 1 & \text{if} \quad \mu = 0; \\ \lambda = 1 & \text{if} \quad \mu\nu \neq 0. \end{cases} \tag{19}$$

This theorem unifies and extends the main results of [5, 6] and shows that their proofs could be combined and simplified. Even we left its proof for the reader, we mention that some variants of the auxiliary results need to be established. In particular, maps φ on $\mathscr{M}_n(\mathbb{C})$ satisfying

$$\sigma(\theta(S,T)) = \sigma(\theta(\varphi(S),\varphi(T))), \ (S,T \in \mathscr{M}_n(\mathbb{C})) \tag{20}$$

should be characterized. The expectation is that such a map φ is either an automorphism or an antiautomorphism of $\mathscr{M}_n(\mathbb{C})$ multiplied by a scalar λ satisfying (19).

We close this section by observing that following the same lines of the proof of [9, Theorem 2.2], the complete characterization of nonlinear maps on $\mathscr{M}_n(\mathbb{C})$ preserving the local spectrum of skew Jordan product $ST^* + T^*S$ of matrices can be obtained.

THEOREM 6.2. *Let x_0 be a nonzero vector in $\mathbb{C}^n$. A map φ on $\mathscr{M}_n(\mathbb{C})$ satisfies*

$$\sigma_{\varphi(T)\varphi(S)^*+\varphi(S)^*\varphi(T)}(x_0) = \sigma_{TS^*+S^*T}(x_0), \quad (T,S \in \mathscr{M}_n(\mathbb{C})) \tag{21}$$

if and only if there is a unimodular scalar γ and a unitary matrix U such that $Ux_0 = x_0$ and $\varphi(T) = \gamma UTU^*$ for all $T \in \mathscr{M}_n(\mathbb{C})$.

Proof. Checking the "*if*" part is straightforward, and we therefore will only deal with the "*only if*" part. Assume that φ satisfies (21), and let us show that φ has the desired form. Following the same lines of the proof of [9, Theorem 2.2], one can shows that the restriction of φ on the open dense set $\Omega_n(\mathbb{C}) := \{A \in GL_n : \sigma(A) \cap \sigma(-A) = \emptyset\}$ is continuous and equals to a bijective linear map L. This together with (21), the continuity of both L and the spectrum and the density of $\Omega_n(\mathbb{C})$ in $\mathscr{M}_n(\mathbb{C})$ imply, just as in the proof of [9, Theorem 2.2], that

$$\sigma(TS^* + S^*T) = \sigma(L(T)L(S)^* + L(S)^*L(T)) \tag{22}$$

for all S and T in $\mathscr{M}_n(\mathbb{C})$. By [19, Theorem 4.1], there is a unitary matrix U and a scalar λ with $|\lambda| = 1$ such that either

$$L(T) = \lambda UTU^*, \ (T \in \mathscr{M}_n(\mathbb{C})),$$

or

$$L(T) = \lambda UT^{tr}U^*, \ (T \in \mathscr{M}_n(\mathbb{C})).$$

It then follows that

$$2\sigma_T(x_0) = \sigma_{\varphi(T)\varphi(I_n)^*+\varphi(I_n)^*\varphi(T)}(x_0) = 2\sigma_{\bar{\lambda}L(T)}(x_0)$$

for all $T \in \Omega_n(\mathbb{C})$. This together with [9, Lemma 3.8, Lemma 3.9] show that L takes only and the only form

$$L(T) = \lambda UTU^*, (T \in \mathscr{M}_n(\mathbb{C})),$$

and $Ux_0 = \alpha x_0$ for some nonzero scalar $\alpha \in \mathbb{C}$.

Finally, let us show that φ has the asserted form. For every $T \in \Omega_n(\mathbb{C})$ and $S \in \mathscr{M}_n(\mathbb{C})$, we have

$$\begin{aligned}
\sigma_{T\varphi(S)^*+\varphi(S)^*T}(x_0) &= \sigma_{\varphi(U^*TU)\varphi(S)^*+\varphi(S)^*\varphi(U^*TU)}(x_0) \\
&= \sigma_{U^*TUS^*+S^*U^*TU}(x_0) \\
&= \sigma_{L(U^*TU)L(S)^*+L(S)^*L(U^*TU)}(x_0) \\
&= \sigma_{TL(S)^*+L(S)^*T}(x_0).
\end{aligned}$$

By [9, Lemma 3.3], we get $\varphi(T) = L(T) = \lambda UTU^*$ for all $T \in \mathscr{M}_n(\mathbb{C})$, and the proof is therefore complete. □

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