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ON THE ESSENTIAL SPECTRA OF UNBOUNDED OPERATOR MATRICES WITH NON DIAGONAL DOMAIN AND AN APPLICATION

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Abstract. This paper is devoted to the investigation of the spectral stability of unbounded operator matrices with non diagonal domain in product of Banach spaces. Our results are aimed to characterize some essential spectra of this kind of operators in terms of the union of the essential spectra of the restriction of its diagonal operators entries. The abstract results are illustrated by an example of two-group transport equations with perfect periodic boundary conditions.

1. Introduction

Block operator matrices arise in various areas of mathematical physics such as ordinary differential equations [28], transport operator [11, 16, 22]. The spectral properties of block operator matrices are of vital importance as they govern for instance the solvability and stability of the underlying physical systems.

Spectral analysis is one of several techniques necessary for characterizing and investigating some essential spectra of block operator matrices with unbounded entries associated to the following operator:

$$\mathscr{A} := \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1}$$

defined on the product $E \times F$ of Banach spaces. These problems have attracted considerable attention and have been investigated by several authors involving the corresponding Schur-complement with maximal domain case (see [1, 13, 22, 29]). Later, many authors are interesting with the same problem where the domain consists of vectors satisfying one relation between their components expressed as: $\Gamma_X f = \Gamma_Y g$, for $\begin{pmatrix} f \\ g \end{pmatrix} \in (\mathscr{D}(A) \cap \mathscr{D}(C)) \times (\mathscr{D}(B) \cap \mathscr{D}(D))$ where Γ_X and Γ_Y are two linear operators (see [2, 16] for more details).

In [25], R. Nagel has paid attention to the research of the problem related to spectral properties of 2×2 operators matrices $\mathscr{A}$ with non diagonal domain $\mathscr{D}(\mathscr{A})$ defined by two relations between their components. Particularly, he presented some conditions

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on the entries of the operator matrix $\mathscr{A}$ in order to provide the expression of its resolvent.

In this paper, we deal with unbounded block operator matrix having the form (1) with domain containing two supplemented conditions relating the components entries with the continuous linear operators ϕ_i and ψ_i, $i = 1,2$, as:

$$\mathscr{D}(\mathscr{A}) := \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : \begin{array}{l} f \in \mathscr{D}(A_m),\ \phi_1(f) = \psi_2(g) \\ g \in \mathscr{D}(D_m),\ \phi_2(g) = \psi_1(f) \end{array} \right\},$$

and defined by

$$\mathscr{A}\begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} A_m & B \\ C & D_m \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix}, \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{D}(\mathscr{A}),$$

where A_m (resp. D_m) is closed densely defined linear operator with maximal domain $\mathscr{D}(A_m)$ (resp. $\mathscr{D}(D_m)$) in E (resp. F).

Under this conditions, new results and techniques are obtained to investigate some essential spectra of the operator matrix $\mathscr{A}$ in a fast manner of computations. More precisely, this study involves an elegant use of the notion of Fredholm-type properties of 2×2 operator matrices in order to characterize some essential spectra of $\mathscr{A}$ as the following form and independently of their Schur-complement but in terms of their diagonal components entries:

$$\sigma_{ek}(\mathscr{A}) := \sigma_{ek}(A_0) \cup \sigma_{ek}(D_0), \quad k = \{r, l, 4, 5, 6\},$$

where the operators A_0 and D_0 denote the restrictions of A_m and D_m to $\ker\phi_1$ and $\ker\phi_2$, respectively, (see Section 2 for the definition of the essential spectra $\sigma_{ek}(.)$). Our results provide an improvements of some earlier works ([1, 2, 16, 22, 25, 29]).

A typical example of a problem of one-dimensional problem of transport operator is given to show the efficiency and accuracy of this work on $X_p \times X_p$-space, where $X_p = L_p((-a,a)\times(-1,1), dxd\xi)$, $a > 0$ and $1 \leqslant p < \infty$ as follows:

$$\mathscr{A}_H = \begin{pmatrix} T_1 & K_{12} \\ K_{21} & T_2 \end{pmatrix}.$$

The operator $\mathscr{A}_H$ has a non diagonal domain expressed as follows:

$$\mathscr{D}(\mathscr{A}_H) = \left\{ \vartheta = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{D}(T_1)\times\mathscr{D}(T_2) :\ \vartheta^i = H\vartheta^o \right\},$$

where ϑ^o and ϑ^i represent the outgoing and the incoming fluxes related by the perfect periodic boundary operator H.

Each closed linear operator T_i, $i = 1,2$, is defined on its maximal domain

$$\mathscr{D}(T_i) := \left\{ f \in X_p \text{ such that } \xi\frac{\partial f}{\partial x} \in X_p \right\}$$

as:

$$\begin{cases} T_i : \mathscr{D}(T_i) \subset X_p \longrightarrow X_p \\ \qquad\qquad f \longmapsto T_i f, \quad (x,\xi) \longmapsto -\xi \dfrac{\partial f}{\partial x}(x,\xi) - \sigma_i(\xi) f(x,\xi) \end{cases}$$

and the bounded linear collision operator K_{ij}, for $(i,j) \in \{(1,2),(2,1)\}$ is defined on X_p by:

$$\begin{cases} K_{ij} : X_p \longrightarrow X_p \\ \qquad\qquad u \longmapsto K_{ij} u, \quad (x,\xi) \longmapsto \displaystyle\int_{-1}^{1} \kappa_{ij}(x,\xi,\xi') u(x,\xi')\, d\xi', \end{cases}$$

(see Section 4 for more details).

Next, we outline the content of the present study. In Section 2, we provide some basic notations and auxiliary results, which we apply in the proof of our main aim in Sections 3 and 4. New characterization of the some essential spectra of unbounded operator matrix with domain contains two supplemented conditions involving the perturbation theory of Fredholm operators is given in Section 3. Finally, Section 4 is devoted to illustrate the efficiency and accuracy ideas of this paper with a physical example of integro-differential operators with non maximal domain on Banach space.

2. Preliminaries results

In this section, we gather some auxiliary notations and definitions that we will need in the rest of the paper.

Let X and Y be two Banach spaces. We denote by $\mathscr{L}(X,Y)$ (resp. $\mathscr{C}(X,Y)$) the set of all bounded (resp. closed, densely defined) linear operators from X into Y. We denote by $\mathscr{K}(X,Y)$ the subspace of all compact operators of $\mathscr{L}(X,Y)$. For $A \in \mathscr{C}(X,Y)$, we write $\mathscr{D}(A) \subset X$ for the domain, $\ker(A) = \{x \in \mathscr{D}(A) : \quad Ax = 0\} \subset X$ for the null space and $\mathscr{R}(A) \subset Y$ for the range of A. The nullity, $\alpha(A)$, of A is defined as the dimension of $\ker(A)$ and the deficiency, $\beta(A)$, of A is defined as the codimension of $\mathscr{R}(A)$ in Y. The spectrum of A will be denoted by $\sigma(A)$. The resolvent set $\rho(A)$ of A is the complemented of $\sigma(A)$ in the complex plane.

An operator $A \in \mathscr{C}(X,Y)$ is semi-Fredholm if $\mathscr{R}(A)$ is closed and at least one of $\alpha(A)$ and $\beta(A)$ is finite. For such an operator, we define an index $i(A)$ by: $i(A) = \alpha(A) - \beta(A)$. Let $\Phi_+(X,Y)$ (resp. $\Phi_-(X,Y)$) denote the set of upper (resp. lower) semi-Fredholm operators, that is, the set of semi-Fredholm operators with $\alpha(A) < \infty$ (resp. $\beta(A) < \infty$). An operator A is Fredholm if it is both upper semi-Fredholm and lower semi-Fredholm. Let $\Phi(X,Y) = \Phi_+(X,Y) \cap \Phi_-(X,Y)$ denote the set of Fredholm operators from X into Y.

If $X = Y$, then $\mathscr{L}(X,Y)$, $\mathscr{C}(X,Y)$, $\mathscr{K}(X,Y)$, $\Phi(X,Y)$, $\Phi_+(X,Y)$ and $\Phi_-(X,Y)$ are replaced by $\mathscr{L}(X)$, $\mathscr{C}(X)$, $\mathscr{K}(X)$, $\Phi(X)$, $\Phi_+(X)$ and $\Phi_-(X)$ respectively.

A complex number λ is in Φ_{+A}, Φ_{-A} or Φ_A if $\lambda - A$ is in $\Phi_+(X)$, $\Phi_-(X)$ or $\Phi(X)$, respectively.

Let $A \in \mathscr{C}(X,Y)$. It follows from the closedness of A that $\mathscr{D}(A)$ endowed with the graph norm $\|.\|_A$ ($\|x\|_A = \|x\| + \|Ax\|$) is a Banach space denoted by X_A. Clearly, for $x \in \mathscr{D}(A)$, we have $\|Ax\| \leqslant \|x\|_A$, so $A \in \mathscr{L}(X_A, Y)$. Let B be a linear operator with $\mathscr{D}(A) \subseteq \mathscr{D}(B)$, then B is said to be A-defined. The restriction of B to $\mathscr{D}(A)$ will be denoted by $\hat{B}$. Moreover, if $\hat{B} \in \mathscr{L}(X_A, Y)$, we say that B is A-bounded.

In this work, we are interested with the theory of Fredholm inverse of unbounded operators. For this purpose, we start with the following definition considered as an extension of the definition of the Fredholm inverse for a bounded operator given by V. Müller in [24].

DEFINITION 2.1. Let X and Y be two Banach spaces. An operator $A \in \mathscr{C}(X,Y)$ is said to have a left (resp. a right) Fredholm inverse if there exists an operator $A_l \in \mathscr{L}(Y, X_A)$ (resp. $A_r \in \mathscr{L}(Y, X_A)$) such that $A_l\widehat{A} - I \in \mathscr{K}(X_A)$ (resp. $\widehat{A}A_r - I \in \mathscr{K}(Y)$). The operator A_l (resp. A_r) is called left (resp. right) Fredholm inverse of A.

We will denote by $\Phi_l(X,Y)$ (resp. $\Phi_r(X,Y)$) the set of operators which have left (resp. right) Fredholm inverse.

We denote the sets $\Phi_l^b(X,Y)$, $\Phi_r^b(X,Y)$, $\Phi^b(X,Y)$, $\Phi_+^b(X,Y)$ and $\Phi_-^b(X,Y)$ by $\Phi_l(X,Y) \cap \mathscr{L}(X,Y)$, $\Phi_r(X,Y) \cap \mathscr{L}(X,Y)$, $\Phi(X,Y) \cap \mathscr{L}(X,Y)$, $\Phi_+(X,Y) \cap \mathscr{L}(X,Y)$ and $\Phi_-(X,Y) \cap \mathscr{L}(X,Y)$, respectively, satisfying the following inclusions:

$$\Phi^b(X,Y) \subset \Phi_l^b(X,Y) \subset \Phi_+^b(X,Y)$$

and

$$\Phi^b(X,Y) \subset \Phi_r^b(X,Y) \subset \Phi_-^b(X,Y).$$

When dealing with closed, densely defined linear operator, A, on a Banach space, various notions of essential spectra involving the theory of Fredholm operators appear. In this work, we are concerned with some of them:

$$\begin{aligned}
\sigma_{er}(A) &:= \{\lambda \in \mathbb{C} : \ \lambda - A \notin \Phi_r(X)\},\\
\sigma_{el}(A) &:= \{\lambda \in \mathbb{C} : \ \lambda - A \notin \Phi_l(X)\},\\
\sigma_{e1}(A) &:= \{\lambda \in \mathbb{C} : \ \lambda - A \notin \Phi_+(X)\},\\
\sigma_{e2}(A) &:= \{\lambda \in \mathbb{C} : \ \lambda - A \notin \Phi_-(X)\},\\
\sigma_{e4}(A) &:= \{\lambda \in \mathbb{C} : \ \lambda - A \notin \Phi(X)\},\\
\sigma_{e5}(A) &:= \mathbb{C} \setminus \rho_5(A),\\
\sigma_{e6}(A) &:= \mathbb{C} \setminus \rho_6(A),
\end{aligned}$$

where $\rho_5(A) := \{\lambda \in \Phi_A : \ i(A-\lambda) = 0\}$ and $\rho_6(A)$ denotes the set of those $\lambda \in \rho_5(A)$ such that all scalars near λ are in $\rho(A)$.

Clearly, these sets can be ordered as:

$$\sigma_{e1}(A) \cap \sigma_{e2}(A) := \sigma_{e_3}(A) \subseteq \sigma_{e4}(A) \subseteq \sigma_{e5}(A) \subseteq \sigma_{e6}(A),$$

and

$$\sigma_{e1}(A) \subset \sigma_{el}(A) \subset \sigma_{e4}(A), \tag{2}$$

$$\sigma_{e2}(A) \subset \sigma_{er}(A) \subset \sigma_{e4}(A). \tag{3}$$

In this work, we are interested with several facts about perturbation theory of Fredholm operators. For this purpose, we recall the following definition.

DEFINITION 2.2. Let X and Y be two Banach spaces and let $F \in \mathscr{L}(X,Y)$.

(i) The operator F is called a Fredholm perturbation if $A+F \in \Phi(X,Y)$ whenever $A \in \Phi(X,Y)$.

(ii) The operator F is called an upper (resp. lower) semi-Fredholm perturbation if $A+F \in \Phi_+(X,Y)$ (resp. $A+F \in \Phi_-(X,Y)$) whenever $A \in \Phi_+(X,Y)$ (resp. $A \in \Phi_-(X,Y)$).

(iii) F is called a left (resp. right) Fredholm perturbation if $A+F \in \Phi_l(X,Y)$ (resp. $A+F \in \Phi_r(X,Y)$) whenever $A \in \Phi_l(X,Y)$ (resp. $A \in \Phi_r(X,Y)$).

We denote by $\mathscr{F}(X,Y)$ the set of Fredholm perturbation, by $\mathscr{F}_+(X,Y)$ (resp. $\mathscr{F}_-(X,Y)$) the set of upper semi-Fredholm (resp. lower semi-Fredholm) perturbation and by $\mathscr{F}_l(X,Y)$ (resp. $\mathscr{F}_r(X,Y)$) the set of left (resp. right) Fredholm perturbation.

If $X = Y$, we write $\mathscr{F}(X)$, $\mathscr{F}_+(X)$, $\mathscr{F}_-(X)$, $\mathscr{F}_l(X)$ and $\mathscr{F}_r(X)$ for $\mathscr{F}(X,X)$, $\mathscr{F}_+(X,X)$, $\mathscr{F}_-(X,X)$, $\mathscr{F}_l(X,X)$ and $\mathscr{F}_r(X,X)$, respectively.

REMARK 2.1. If in Definition 2.2 we replace $\Phi(X,Y)$, $\Phi_+(X,Y)$, $\Phi_-(X,Y)$, $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ by $\Phi^b(X,Y)$, $\Phi^b_+(X,Y)$, $\Phi^b_-(X,Y)$, $\Phi^b_l(X,Y)$ and $\Phi^b_r(X,Y)$, we obtain the sets $\mathscr{F}^b(X,Y)$, $\mathscr{F}^b_+(X,Y)$, $\mathscr{F}^b_-(X,Y)$, $\mathscr{F}^b_l(X,Y)$ and $\mathscr{F}^b_r(X,Y)$, respectively.

In [6], it is shown that $\mathscr{F}^b(X,Y)$, $\mathscr{F}^b_+(X,Y)$ and $\mathscr{F}^b_-(X,Y)$ are closed subset of $\mathscr{L}(X,Y)$ and if $X = Y$, then $\mathscr{F}^b(X) := \mathscr{F}^b(X,X)$, $\mathscr{F}^b_+(X) := \mathscr{F}^b_+(X,X)$ and $\mathscr{F}^b_-(X) := \mathscr{F}^b_-(X,X)$ are closed two-sided ideals of $\mathscr{L}(X)$.

In [15], it is shown that if $X = Y$, then $\mathscr{F}^b_l(X) := \mathscr{F}^b_l(X,X)$ $\mathscr{F}^b_r(X) := \mathscr{F}^b_r(X,X)$ are two-sided ideals of $\mathscr{L}(X)$, satisfying:

$$\mathscr{K}(X,Y) \subseteq \mathscr{W}(X,Y) \subseteq \mathscr{F}^b_+(X,Y) \subseteq \mathscr{F}^b_l(X,Y) \subseteq \mathscr{F}^b(X,Y) \tag{4}$$

and

$$\mathscr{K}(X,Y) \subseteq \mathscr{W}(X,Y) \subseteq \mathscr{F}^b_-(X,Y) \subseteq \mathscr{F}^b_r(X,Y) \subseteq \mathscr{F}^b(X,Y). \tag{5}$$

The following Theorem is fundamental for proofs of Section 3 and 4.

THEOREM 2.1. [15, Theorem 3.2] *Let X,Y and Z be Banach spaces. If the set $\Phi^b(Y,Z)$ is not empty, then*

$$F \in \mathscr{F}^b_l(X,Y) \text{ and } A \in \mathscr{L}(Y,Z), \text{ imply } AF \in \mathscr{F}^b_l(X,Z).$$

$$F \in \mathscr{F}^b_r(X,Y) \text{ and } A \in \mathscr{L}(Y,Z), \text{ imply } AF \in \mathscr{F}^b_r(X,Z).$$

Let us recall the following results on Fredholm perturbations theory of 2×2 block operator matrix established by M. Moalla et al. in [15].

THEOREM 2.2. [15, Theorem 3.1–3.2] *Let X_1 and X_2 be two Banach spaces and*

$$F := \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix},$$

where $F_{ij} \in \mathscr{L}(X_j,X_i)$, $i,j = 1,2$. *Then:*

(i) $F \in \mathscr{F}^b(X_1 \times X_2)$ *if and only if* $F_{ij} \in \mathscr{F}^b(X_j,X_i)$, $\forall i,j = 1,2$.

(ii) $F \in \mathscr{F}_l^b(X_1 \times X_2)$ *(resp.* $F \in \mathscr{F}_r^b(X_1 \times X_2)$*) if and only if* $F_{ij} \in \mathscr{F}_l^b(X_j,X_i)$, *(resp.* $F_{ij} \in \mathscr{F}_r^b(X_j,X_i)$*) for all* $i,j = 1,2$.

We recall stability results of essential spectra of unbounded operator subjected to Fredholm perturbation which is essential to provide the main purpose of this paper.

THEOREM 2.3. [3] *Let* X *be a Banach space,* T_1, T_2 *be two closed densely defined linear operators on* X.

(i) *If for some* $\lambda \in \rho(T_1) \cap \rho(T_2)$, *the operator* $(\lambda - T_1)^{-1} - (\lambda - T_2)^{-1} \in \mathscr{F}_r^b(X)$, *then*

$$\sigma_{er}(T_1) = \sigma_{er}(T_2).$$

(ii) *If for some* $\lambda \in \rho(T_1) \cap \rho(T_2)$, *the operator*

$$(\lambda - T_1)^{-1} - (\lambda - T_2)^{-1} \in \mathscr{F}_l^b(X),$$

then $\sigma_{el}(T_1) = \sigma_{el}(T_2)$.

When dealing with essential spectra of closed, densely defined linear operators on Banach spaces, one of the main problems consists of studying the invariance of the essential spectra of these operators subjected to various kind of perturbations. In this vein, we need to introduce the following definitions:

DEFINITION 2.3. Let X and Y be two Banach spaces. An operator $A \in \mathscr{L}(X,Y)$ is said to be weakly compact if $A(B)$ is relatively weakly compact in Y for every bounded $B \subset X$.

The family of weakly compact operators from X into Y is denoted by $\mathscr{W}(X,Y)$. If $X = Y$ the family of weakly compact operators on $X, \mathscr{W}(X) := \mathscr{W}(X,X)$ is a closed two-sided ideal of $\mathscr{L}(X)$ containing $\mathscr{K}(X)$ (see [7]).

DEFINITION 2.4. Let X and Y be two Banach spaces. An operator $A \in \mathscr{L}(X,Y)$ is said to be strictly singular if the restriction of A to any infinite-dimensional subspace of X is not an homeomorphism.

Let $S(X,Y)$ denote the set of strictly singular operators from X to Y. The concept of strictly singular operators was introduced in the pioneering paper by T. Kato [17]. In general, strictly singular operators are not compact (see [6, 17]). Note that, $S(X,Y)$ is a closed subspace of $\mathscr{L}(X,Y)$. If $X = Y$, $S(X) := S(X,X)$ is a closed two-sided ideal of $\mathscr{L}(X)$ containing $\mathscr{K}(X)$. If X is a separable Hilbert space, then $S(X) = \mathscr{K}(X)$. For basic properties of strictly singular operators, we refer readers to [6, 21, 30, 31].

REMARK 2.2. (i) Let X_p denotes the space $L_p(\Omega, d\mu)$ $(1 \leqslant p \leqslant \infty)$, where (Ω, Σ, μ) stands for a positive measure space.

According to Theorem 1 in [26], in a special case for $X_1 = L_1$-space (respectively $C(\Omega)$-spaces, with Ω is a compact Hausdorff space), we have

$$\mathscr{W}(X_1) = \mathscr{S}(X_1).$$

However, if $1 < p < \infty$, X_p is reflexive and then $\mathscr{L}(X_p) = \mathscr{W}(X_p)$. On the other hand, it follows from Theorem 5.2 in [7] that $\mathscr{K}(X_p) \subset_{\neq} \mathscr{S}(X_p) \subset_{\neq} \mathscr{W}(X_p)$ with $p \neq 2$. For $p = 2$ we have $\mathscr{K}(X_p) = \mathscr{S}(X_p) = \mathscr{W}(X_p)$.

Therefore, from [20, pp. 779], we get

$$\mathscr{F}_+(X_p) = \mathscr{F}_-(X_p) = \mathscr{F}(X_p) = \mathscr{S}(X_p) = \mathscr{C}\mathscr{S}(X_p), \quad p \geqslant 1.$$

(ii) Using the last equality and the Eqs. (4) and (5), we deduce that:

$$\mathscr{F}_r^b(X_p) = \mathscr{F}_l^b(X_p) = \mathscr{F}^b(X_p), \quad \forall p \geqslant 1.$$

Throughout this paper, we will consider the set of polynomially compact operators which denoted by $\mathscr{P}\mathscr{K}(X)$ and defined as:

$$\mathscr{P}\mathscr{K}(X) = \Big\{ A \in \mathscr{L}(X) \text{ such that there exists a nonzero complex polynomial}$$
$$P(z) = \sum_{k=0}^{n} a_k z^k \text{ satisfying } P(A) \in \mathscr{K}(X) \text{ and } P(1) \neq 0 \Big\}.$$

REMARK 2.3. (i) Note that in general, we have:

$$\mathscr{K}(X) \subset \mathscr{P}\mathscr{K}(X).$$

(ii) Let (Ω, μ) be a σ-finite measure space. Note that, if $X = L_1(\Omega, d\mu)$ (respectively $X = C(\Omega)$-spaces with Ω is a compact Hausdorff space), then we have:

$$\mathscr{W}(X) \subset \mathscr{P}\mathscr{K}(X).$$

Indeed, from [27] (respectively [9]), the product of two weakly compact linear operators in $L_1(\Omega)$ (respectively $C(\Omega)$, where Ω is a compact Hausdorff space) is compact. So, for $A \in \mathscr{W}(X)$ and taking $P(z) = z^2$, then $P(1) \neq 0$ and $P(A) \in \mathscr{K}(X)$, this implies that $A \in \mathscr{P}\mathscr{K}(X)$.

3. Essential spectra of operator matrix with non zero off diagonal entries

The main purpose of this section is to discuss the essential spectra of unbounded operator matrix $\mathscr{A}$ with non-diagonal domain in terms of the essential spectra of a diagonal operator matrix $\mathscr{A}_0$ associated to $\mathscr{A}$ which is easier to compute its essential spectra.

Let X, Y, E and F be Banach spaces. First, suppose that:

$(\mathscr{H}_1)$ A_m and D_m are closed, densely defined linear operators with domains $\mathscr{D}(A_m)$ in E and $\mathscr{D}(D_m)$ in F.

Consider the continuous linear operators ϕ_1, ϕ_2, ψ_1 and ψ_2 as in the following diagram:

$$\begin{tikzcd}
E \supset \mathscr{D}(A_m) \arrow[r, "\phi_1"] \arrow[dr, "\psi_1" near end] & X \\
F \supset \mathscr{D}(D_m) \arrow[r, "\phi_2"'] \arrow[ur, "\psi_2"' near end] & Y
\end{tikzcd}$$

where the Banach spaces X and Y (called "spaces of boundary conditions") endow $\mathscr{D}(A_m)$ and $\mathscr{D}(D_m)$ with the graph norm.

In addition, we shall assume that:

$(\mathscr{H}_2)$ ϕ_1 and ϕ_2 are surjective.

Let consider the unbounded operator matrix $\mathscr{A}$ in Banach space $E \times F$ corresponding to the matrix form

$$\mathscr{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for given bounded operator $B \in \mathscr{L}(\mathscr{D}(D_m), E)$ and $C \in \mathscr{L}(\mathscr{D}(A_m), F)$.

On the non diagonal domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{D}(A_m) \times \mathscr{D}(D_m) \text{ such that } \phi_1(f) = \psi_2(g) \text{ and } \phi_2(g) = \psi_1(f) \right\},$$

the operator $\mathscr{A}$ is defined by:

$$\mathscr{A}\begin{pmatrix} f \\ g \end{pmatrix} = \mathscr{A}_m \begin{pmatrix} f \\ g \end{pmatrix}, \quad \forall \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{D}(\mathscr{A}),$$

where the maximal operator $\mathscr{A}_m$ is expressed by:

$$\mathscr{A}_m := \begin{pmatrix} A_m & B \\ C & D_m \end{pmatrix}$$

on the maximal domain $\mathscr{D}(\mathscr{A}_m) := \mathscr{D}(A_m) \times \mathscr{D}(D_m)$.

As a first towards, the description of the essential spectra of unbounded operator matrix $\mathscr{A}$ with non diagonal domain will be investigated in terms of the essential spectra of the entries of an associated diagonal matrix operator $\mathscr{A}_0 := \begin{pmatrix} A_0 & 0 \\ 0 & D_0 \end{pmatrix}$ with diagonal domain in a good and an easy manner, where $A_0 := A_m|_{\ker \phi_1}$ and $D_0 := D_m|_{\ker \phi_2}$.

REMARK 3.1. From the definition of the operator A_0 (resp. D_0), based on the continuity assumptions on the ϕ_j, $j = 1, 2$, one can easily check that $\phi_1(\mathscr{D}(A_0)) = \{0\}$ (resp. $\phi_2(\mathscr{D}(D_0)) = \{0\}$) and thus the operator A_0 (resp. D_0) is closed, whence $\mathscr{D}(A_0)$ (resp. $\mathscr{D}(D_0)$) is a closed subspace of E (resp. F). Hence, the matrix operator $\mathscr{A}_0$ is closed.

Now, let us recall the following lemma explaining the relation between the matrix operators $\mathscr{A}$ and $\mathscr{A}_0$.

LEMMA 3.1. [8, Lemma 1.2]
(i) For $\lambda \in \rho(A_0)$ *(resp.* $\lambda \in \rho(D_0)$*) the following decomposition holds:*

$$\mathscr{D}(A_m) = \mathscr{D}(A_0) \oplus \ker(\lambda - A_m)$$

(resp. $\mathscr{D}(D_m) = \mathscr{D}(D_0) \oplus \ker(\lambda - D_m)$*).*
(ii) For $\lambda \in \rho(A_0)$ *and* $\lambda \in \rho(D_0)$*. Then,*

$$\phi_{1\lambda} := \phi_1|_{\ker(\lambda - A_m)}$$

and

$$\phi_{2\lambda} := \phi_2|_{\ker(\lambda - D_m)}$$

are continuous bijections from $\ker(\lambda - A_m)$ *onto* X *and from* $\ker(\lambda - D_m)$ *onto* Y*.*

As a direct consequence of the above Lemma, for $\lambda \in \rho(A_0) \cap \rho(D_0)$, the inverse of $\phi_{1\lambda}$ and $\phi_{2\lambda}$ will play an important role to define the bounded operators K_λ and L_λ as follows:

$$\begin{cases} K_\lambda : \mathscr{D}(D_m) \longrightarrow \mathscr{D}(A_m) \\ \qquad g \longmapsto K_\lambda(g) = \phi_{1\lambda}^{-1} \circ \psi_2(g) \end{cases} \quad \text{and} \quad \begin{cases} L_\lambda : \mathscr{D}(A_m) \longrightarrow \mathscr{D}(D_m) \\ \qquad f \longmapsto L_\lambda(f) = \phi_{2\lambda}^{-1} \circ \psi_1(f). \end{cases}$$

(see Lemma 2.5 in [25] for more explanation).

To introduce a fine decomposition of the unbounded operator matrix with non diagonal domain, we start to define for $\lambda \in \rho(A) \cap \rho(D)$, the bounded operators $\mathbb{G}(\lambda)$ and $\mathbb{F}(\lambda)$ as well:

$$\begin{cases} \mathbb{G}(\lambda) = -K_\lambda - (\lambda - A_0)^{-1} B \in \mathscr{L}(\mathscr{D}(D_m), \mathscr{D}(A_m)), \\ \\ \mathbb{F}(\lambda) = -L_\lambda - (\lambda - D_0)^{-1} C \in \mathscr{L}(\mathscr{D}(A_m), \mathscr{D}(D_m)). \end{cases}$$

The following factorization may be used to formulate the key tool for our investigations.

LEMMA 3.2. [25] *For* $\lambda \in \rho(A_0) \cap \rho(D_0)$*, consider the linear bounded operator matrix on* $\mathscr{D}(A_m) \times \mathscr{D}(D_m)$

$$\mathscr{Q}_\lambda = \begin{pmatrix} I & \mathbb{G}(\lambda) \\ \mathbb{F}(\lambda) & I \end{pmatrix}. \tag{6}$$

Then, we have

$$\lambda - \mathscr{A} := (\lambda - \mathscr{A}_0)\mathscr{Q}_\lambda, \quad on \ \mathscr{D}(\mathscr{A}).$$

The last decomposition may be used to describe the resolvent of block operator matrix $\mathscr{A}$. Therefore, our interest in the last part of this paper consists of showing what are the conditions that we impose on the operators entries of $\lambda - \mathscr{A}_0$ and $\mathscr{Q}_\lambda$ which make $\lambda - \mathscr{A}$ be invertible.

For this vein, let us introduced the invertibility of the matrix operator $\mathscr{Q}_\lambda$.

PROPOSITION 3.1. *Let $\lambda \in \rho(A_0) \cap \rho(D_0)$. Then, $\mathscr{Q}_\lambda$ is invertible (resp. injective) in $\mathscr{L}(\mathscr{D}(A_m) \times \mathscr{D}(D_m))$ if and only if $Id - \mathbb{G}(\lambda)\mathbb{F}(\lambda)$ (resp. $Id - \mathbb{F}(\lambda)\mathbb{G}(\lambda)$) is invertible (resp. injective) in $\mathscr{L}(\mathscr{D}(A_m))$ (resp. $\mathscr{L}(\mathscr{D}(D_m))$).*

Proof. We can easily derive the result from the Frobenuis-Schur factorization of the operator $\mathscr{Q}_\lambda$. $\square$

THEOREM 3.1. *Let $\lambda \in \rho(A_0) \cap \rho(D_0)$.*

(i) If $I - \mathbb{F}(\lambda)\mathbb{G}(\lambda) \in \mathscr{L}(\mathscr{D}(D_m))$ or $I - \mathbb{G}(\lambda)\mathbb{F}(\lambda) \in \mathscr{L}(\mathscr{D}(A_m))$ is invertible, then $\lambda - \mathscr{A}$ is invertible in $\mathscr{L}(E \times F)$.

(ii) If $\mathbb{F}(\lambda)\mathbb{G}(\lambda) \in P\mathscr{K}(\mathscr{D}(D_m))$ or $\mathbb{G}(\lambda)\mathbb{F}(\lambda) \in P\mathscr{K}(\mathscr{D}(A_m))$, then

$$\begin{aligned}\lambda - \mathscr{A} \text{ is invertible in } \mathscr{L}(E \times F) &\iff I - \mathbb{F}(\lambda)\mathbb{G}(\lambda) \in \mathscr{L}(\mathscr{D}(D_m)) \text{ is invertible} \\ &\iff I - \mathbb{G}(\lambda)\mathbb{F}(\lambda) \in \mathscr{L}(\mathscr{D}(A_m)) \text{ is invertible.}\end{aligned}$$

Proof. Let $\lambda \in \rho(A_0) \cap \rho(D_0)$.

(i) The result of this assertion was established in [25] and follows immediately from Lemma 3.2 and Proposition 3.1.

(ii) We prove the first equivalence, the second one can be done using the same argument.

The indirect implication follows from item (i).

Conversely, assume that $\lambda - \mathscr{A}$ is invertible in $\mathscr{L}(E \times F)$, that is $\lambda - \mathscr{A}$ is injective. Therefore, Lemma 3.2 in conjunction with Proposition 3.1 revel that $I - \mathbb{F}(\lambda)\mathbb{G}(\lambda)$ is injective. Following Theorem 2.2 in [12], under the assumption that $\mathbb{F}(\lambda)\mathbb{G}(\lambda) \in P\mathscr{K}(\mathscr{D}(D_m))$, amounts that $I - \mathbb{F}(\lambda)\mathbb{G}(\lambda))$ is invertible. $\square$

REMARK 3.2. (i) Theorem 3.1 shows that the results established in [25, Theorem 2.9] for compact operators remain valid in the context of polynomially compact operators. Therefore, it can be applied also for the case of weakly compact operators acting in $L_1(\Omega)$ or $C(\Omega)$ (see Remark 2.3-(ii)).

(ii) It follows from Theorem 3.1-(i), that if $\lambda \in \rho(A_0) \cap \rho(D_0)$ such that $1 \in \rho(\mathbb{F}(\lambda)\mathbb{G}(\lambda))$ or $1 \in \rho(\mathbb{G}(\lambda)\mathbb{F}(\lambda))$, then $\rho(\mathscr{A}) \neq \emptyset$, which ensures that $\mathscr{A}$ is closed operator matrix.

It is now our intention to memorize that the above result yields an easier formula for the computation of the essential spectra of unbounded operator matrix $\mathscr{A}$ with non-diagonal domain expressed as follows, for $\lambda \in \rho(A_0) \cap \rho(D_0)$ such that $1 \in \rho(\mathbb{F}(\lambda)\mathbb{G}(\lambda))$:

$$(\lambda - \mathscr{A})^{-1}$$
$$= \begin{pmatrix} \begin{array}{c}(\lambda - A_0)^{-1} + \\ (I - \mathbb{G}(\lambda)\mathbb{F}(\lambda))^{-1}\mathbb{G}(\lambda)\mathbb{F}(\lambda)(\lambda - A_0)^{-1}\end{array} & -(I - \mathbb{G}(\lambda)\mathbb{F}(\lambda))^{-1}\mathbb{G}(\lambda)(\lambda - D_0)^{-1} \\ \begin{array}{c}-\mathbb{F}(\lambda)(\lambda - A_0)^{-1} - \\ \mathbb{F}(\lambda)(I - \mathbb{G}(\lambda)\mathbb{F}(\lambda))^{-1}\mathbb{G}(\lambda)\mathbb{F}(\lambda)(\lambda - A_0)^{-1}\end{array} & \begin{array}{c}(\lambda - D_0)^{-1} + \\ \mathbb{F}(\lambda)(I - \mathbb{G}(\lambda)\mathbb{F}(\lambda))^{-1}\mathbb{G}(\lambda)(\lambda - D_0)^{-1}\end{array} \end{pmatrix}.$$

or

$$(\lambda-\mathscr{A})^{-1} \tag{7}$$
$$= \begin{pmatrix} \begin{array}{c}(\lambda-A_0)^{-1}\\ +\mathbb{G}(\lambda)(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)(\lambda-A_0)^{-1}\end{array} & \begin{array}{c}-\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\\ -\mathbb{G}(\lambda)(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\end{array} \\ -(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)(\lambda-A_0)^{-1} & \begin{array}{c}(\lambda-D_0)^{-1}\\ +(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\end{array}\end{pmatrix}.$$

Now, we are in the position to express the first main results of this section. We will denote by ${}^{C}\Omega$ the complement of a subset $\Omega\subset\mathbb{C}$.

THEOREM 3.2. *Let $\lambda\in\rho(A_0)\cap\rho(D_0)$ and $1\in\rho(\mathbb{F}(\lambda)\mathbb{G}(\lambda))$.*
Then, we have:
(i) If $\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\in\mathscr{F}_r^b(F,\mathscr{D}(A_m))$ and $\mathbb{F}(\lambda)(\lambda-A_0)^{-1}\in\mathscr{F}_r^b(E,\mathscr{D}(D_m))$, then

$$\sigma_{er}(\mathscr{A})=\sigma_{er}(A_0)\cup\sigma_{er}(D_0).$$

(ii) If $\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\in\mathscr{F}_l^b(F,\mathscr{D}(A_m))$ and $\mathbb{F}(\lambda)(\lambda-A_0)^{-1}\in\mathscr{F}_l^b(E,\mathscr{D}(D_m))$, then

$$\sigma_{el}(\mathscr{A})=\sigma_{el}(A_0)\cup\sigma_{el}(D_0).$$

(iii) If $\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\in\mathscr{F}^b(F,\mathscr{D}(A_m))$ and $\mathbb{F}(\lambda)(\lambda-A_0)^{-1}\in\mathscr{F}^b(E,\mathscr{D}(D_m))$, then

$$\sigma_{e_4}(\mathscr{A})=\sigma_{e_4}(A_0)\cup\sigma_{e_4}(D_0)$$

and

$$\sigma_{e_5}(\mathscr{A})\subseteq\sigma_{e_5}(A_0)\cup\sigma_{e_5}(D_0).$$

Moreover,
(iv) If ${}^{C}\sigma_{e4}(A_0)$ is connected, then

$$\sigma_{e_5}(\mathscr{A})=\sigma_{e_5}(A_0)\cup\sigma_{e_5}(D_0).$$

(v) If ${}^{C}\sigma_{e5}(\mathscr{A})$ and ${}^{C}\sigma_{e5}(D_0)$ are connected with $\rho(\mathscr{A})\neq\emptyset$, then

$$\sigma_{e6}(\mathscr{A})=\sigma_{e6}(A_0)\cup\sigma_{e6}(D_0).$$

Proof. In view of Theorem 2.2, to prove the Fredholmness perturbation of operator matrix $(\lambda-\mathscr{A})^{-1}-(\lambda-\mathscr{A}_0)^{-1}$ it suffices to show that all entries of this block operator matrix are Fredholm perturbations.

Consider $\lambda\in\rho(A_0)\cap\rho(D_0)$ such that $1\in\rho(\mathbb{F}(\lambda)\mathbb{G}(\lambda))$. It follows from Eq. (7) that:

$$(\lambda-\mathscr{A})^{-1}-(\lambda-\mathscr{A}_0)^{-1} \tag{8}$$
$$= \begin{pmatrix} \mathbb{G}(\lambda)(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)(\lambda-A_0)^{-1} & \begin{array}{c}-\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\\ -\mathbb{G}(\lambda)(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\\ \times\mathbb{F}(\lambda)\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\end{array} \\ -(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)(\lambda-A_0)^{-1} & (I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\end{pmatrix}.$$

(i) From the assumptions $\mathbb{F}(\lambda)(\lambda-A_0)^{-1}\in\mathscr{F}_r^b(E,\mathscr{D}(D_m))$, $\mathbb{G}(\lambda)(\lambda-D_0)^{-1}\in\mathscr{F}_r^b(F,\mathscr{D}(A_m))$ and Theorem 2.1, we deduce that the operators

$$\mathbb{G}(\lambda)(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)(\lambda-A_0)^{-1},$$

$$-(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)(\lambda-A_0)^{-1},$$

$$-\mathbb{G}(\lambda)(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)\mathbb{G}(\lambda)(\lambda-D_0)^{-1}$$

and

$$(I-\mathbb{F}(\lambda)\mathbb{G}(\lambda))^{-1}\mathbb{F}(\lambda)\mathbb{G}(\lambda)(\lambda-D_0)^{-1}$$

are right Fredholm perturbations operators.

According to Theorems 2.2 in [14] and 2.3, one gets $\sigma_{er}(\mathscr{A})=\sigma_{er}(\mathscr{A}_0)$. Therefore,

$$\sigma_{er}(\mathscr{A})=\sigma_{er}(A_0)\cup\sigma_{er}(D_0).$$

The use of Theorems 2.3 and 2.2 in [14] allows us to reach the result of assertion (ii) in a similar ways as in the item (i).

(iii) The results of this item are an immediate consequence of the items (i) and (ii).

(iv) According to item (iii), we will prove the opposite inclusion. For this purpose, let consider $\lambda\notin\sigma_{e5}(\mathscr{A})$, that is, $\lambda-\mathscr{A}$ is Fredholm operator if and only if $\lambda-A_0$ and $\lambda-D_0$ are Fredholm operators with

$$i(\lambda-\mathscr{A})=i(\lambda-A_0)+i(\lambda-D_0)=0.$$

Assume that ${}^C\sigma_{e4}(A_0)$ is connected. Hence, $\rho(A_0)$ is not empty. That is, for $\gamma\in\rho(A_0)$, we have $\gamma-A_0$ is Fredholm operator with null index. One has $\rho(A_0)\subset\rho_{e4}(A_0)$ and $i(\gamma-A_0)$ is constant on any component of Φ_{A_0} (see [6] for more details), we revels that $i(\lambda-A_0)=0$ for all $\lambda\in\rho_{e4}(A_0)$. Consequently, $i(\lambda-D_0)=0$.

(v) According to Lemma 2.1 in [14], we get

$$\sigma_{e6}(\mathscr{A})=\sigma_{e6}(A_0)\cup\sigma_{e6}(D_0).\quad\square$$

REMARK 3.3. In view of Eqs. (4) and (5), Theorem 3.2 remains true by considering compactness or weakly compactness assumptions.

We can translate the results of Theorem 3.2 in terms of Gustafson, Weidmann and Kato essential spectra as follows:

THEOREM 3.3. *Let* $\lambda\in\rho(A_0)\cap\rho(D_0)$ *and* $1\in\rho(\mathbb{G}(\lambda)\mathbb{F}(\lambda))$.
(i) *If* $(\lambda-\mathscr{A})^{-1}-(\lambda-\mathscr{A}_0)^{-1}\in\mathscr{F}_+^b(E\times F)$, *then*

$$\sigma_{e1}(\mathscr{A})=\sigma_{e1}(A_0)\cup\sigma_{e1}(D_0).$$

(ii) *If* $(\lambda-\mathscr{A})^{-1}-(\lambda-\mathscr{A}_0)^{-1}\in\mathscr{F}_-^b(E\times F)$, *then*

$$\sigma_{e2}(\mathscr{A})=\sigma_{e2}(A_0)\cup\sigma_{e2}(D_0).$$

(iii) $(\lambda-\mathscr{A})^{-1}-(\lambda-\mathscr{A}_0)^{-1}\in\mathscr{F}_+^b(E\times F)\cap\mathscr{F}_-^b(E\times F)$, *then*

$$\sigma_{e3}(\mathscr{A})=\sigma_{e3}(A_0)\cup[\sigma_{e1}(A_0)\cap\sigma_{e2}(D_0)]\cup[\sigma_{e2}(A_0)\cap\sigma_{e1}(D_0)]\cup\sigma_{e3}(D_0).$$

Proof. (i) For $\lambda\in\rho(A_0)\cap\rho(D_0)$ and $1\in\rho(\mathbb{G}(\lambda)\mathbb{F}(\lambda))$, we infer by Proposition 3.1 that $\lambda\in\rho(\mathscr{A})\cap\rho(\mathscr{A}_0)$. This property together with the fact that $(\lambda-\mathscr{A})^{-1}-(\lambda-\mathscr{A}_0)^{-1}\in\mathscr{F}_+^b(E\times F)$, leads from Theorem 3.2-(ii) in [19] to $\sigma_{e1}(\mathscr{A})=\sigma_{e1}(\mathscr{A}_0)$.
As $\mathscr{A}_0$ is a diagonal operator matrix, this shows that

$$\sigma_{e1}(\mathscr{A}_0)=\sigma_{e1}(A_0)\cup\sigma_{e1}(D_0).$$

So, we infer that $\sigma_{e1}(\mathscr{A})=\sigma_{e1}(A_0)\cup\sigma_{e1}(D_0)$.
(ii) A same reasoning allows us to reach the result of item (ii).
(iii) Assertion (iii) is a consequence of the items (i) and (ii).

REMARK 3.4. (i) It should be observe that Theorem 3.2 holds also true for maximal domain case, that is, suppose that the operators $\phi_k\equiv\psi_k\equiv 0$, $k=1,2$, then we obtain

$$\sigma_{ek}(\mathscr{A}):=\sigma_{ek}(A)\cup\sigma_{ek}(D),\ \ k=\{r,l,4,5,6\}.$$

So, in this case an amelioration of the results given in [1, 22, 29] can be obtained.

(ii) Note that for $\psi_2\equiv\psi_1\equiv 0$, we recover the case of unbounded operator matrix with non maximal domain and with one condition on its domain. Moreover, the analysis in our case is easier and better than those given in [2, 16] because, our procedure works deals with a new decomposition of this kind of operator matrix and provides under less assumptions an easier form of the essential spectra of $\mathscr{A}$ independently of the Schur complement, that is:

$$\sigma_{ek}(\mathscr{A}):=\sigma_{ek}(A_0)\cup\sigma_{ek}(D),\ \ k=\{r,l,4,5,6\}.$$

4. The motivating example

Once we obtain the well-possedness for the unbounded operator matrix with non diagonal domain, we can discuss a typical example motivating the abstract theoretical results. The main features of our approach already appear for integro-differential equation with a non diagonal domain.

4.1. Intergro-differential equation

In this subsection, we study the essential spectra of two-group transport operators with non maximal domain on $L_p\times L_p$, $1\leqslant p<\infty$.

We first make the functional setting of the problem. Let

$$X_p=L_p((-a,a)\times(-1,1),dxd\xi),\ \ a>0\ \text{ and }\ p\in[1,\infty),$$

equipped with the norm

$$\|f\|_p=\left(\int_{-a}^{a}\int_{-1}^{1}|f(x,\xi)|^p dxd\xi\right)^{\frac{1}{p}}.$$

We define the partial Sobolev space $\mathscr{W}_p$ by:

$$\mathscr{W}_p = \left\{ f \in X_p \ \text{ such that } \ \xi \frac{\partial f}{\partial x} \in X_p \right\}.$$

It is well known that any function $\varphi \in \mathscr{W}_p$ has traces on $\{-a\} \times (-1,0)$ and $\{a\} \times (0,1)$ which respectively belongs to the spaces X_p^o and X_p^i (see, for instance, [4]). They are denoted, respectively, by φ^o and φ^i, represent the outgoing and the incoming fluxes ("o" for outgoing and "i" for incoming) and given by:

$$\begin{cases} \varphi^i : \xi \in (0,1) & \longrightarrow \ \varphi(-a,\xi), \\ \varphi^i : \xi \in (-1,0) & \longrightarrow \ \varphi(a,\xi), \\ \varphi^o : \xi \in (-1,0) & \longrightarrow \ \varphi(-a,\xi), \\ \varphi^o : \xi \in (0,1) & \longrightarrow \ \varphi(a,\xi). \end{cases}$$

Now, we will consider on the product $X_p \times X_p$ of Banach spaces, the unbounded operator matrices

$$\mathscr{A}_H = \begin{pmatrix} T_1 & K_{12} \\ K_{21} & T_2 \end{pmatrix},$$

where the closed linear operator T_i, $i = 1,2$, is defined on its maximal domain $\mathscr{D}(T_i)$ as:

$$\begin{cases} T_i : \mathscr{D}(T_i) = \mathscr{W}_p \subset X_p \longrightarrow X_p \\ \qquad\qquad f \longmapsto T_i f, \quad (T_i f)(x,\xi) := -\xi \dfrac{\partial f}{\partial x}(x,\xi) - \sigma_i(\xi) f(x,\xi), \end{cases}$$

and the bounded linear collision operator K_{ij}, for $(i,j) \in \{(1,2),(2,1)\}$ is defined on X_p by:

$$\begin{cases} K_{ij} : X_p \longrightarrow X_p \\ \qquad u \longmapsto K_{ij} u, \quad (K_{ij} u)(x,\xi) := \displaystyle\int_{-1}^{1} \kappa_{ij}(x,\xi,\xi') u(x,\xi')\, d\xi'. \end{cases}$$

The operator $\mathscr{A}_H$ describes the transport of particles (neutrons, photons, molecules of gas, etc.) in a plane parallel domain with a width of $2a$ mean free paths. The function $f(.,.)$ (resp. $g(.,.)$) represents the number (or probability) density of gas particles having the position x and the direction cosine of propagation ξ. The variable ξ may be thought of as the cosine of the angle between the velocity of particles and the x-direction. The functions $\sigma_i(.) \in L^\infty(-1,1)$, $i = 1,2$ and $\kappa_{ij}(.,.,.)$, $(i,j) = \{(1,2),(2,1)\}$ (which is assumed to be measurable) are called, respectively, the collision frequency and the scattering kernel.

The operator $\mathscr{A}_H$ defined on its non diagonal domain $\mathscr{D}(\mathscr{A}_H)$ as:

$$\mathscr{D}(\mathscr{A}_H) := \left\{ \vartheta = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{W}_p \times \mathscr{W}_p \text{ such that } \vartheta^i = H \vartheta^o \right\},$$

where ϑ^o and ϑ^i represent the outgoing and the incoming fluxes related by the following perfect periodic boundaries operator H which is expressed by:

$$\begin{cases} H : X_p^o \times X_p^o \longrightarrow X_p^i \times X_p^i \\ \qquad \begin{pmatrix} f \\ g \end{pmatrix} \longmapsto H\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 & H_{12} \\ H_{21} & 0 \end{pmatrix}\begin{pmatrix} f \\ g \end{pmatrix} \end{cases}$$

where: H_{kj} is bounded operator and the boundary spaces X_p^o and X_p^i are given by:

$$X_p^o := L_p(\{-a\} \times (-1,0), |\xi|d\xi) \times L_p(\{a\} \times (0,1), |\xi|d\xi) := X_{1,p}^o \times X_{2,p}^o$$

and

$$X_p^i := L_p(\{-a\} \times (0,1), |\xi|d\xi) \times L_p(\{a\} \times (-1,0), |\xi|d\xi) := X_{1,p}^i \times X_{2,p}^i$$

(see [4] for more details).

Using the notations of Section 3, we write $\mathscr{A}_H$ as a 2×2 block operator matrices $\mathscr{A}_m := \begin{pmatrix} A_m & K_{12} \\ K_{21} & D_m \end{pmatrix}$ with diagonal entries A_m and D_m correspond to the maximal operators $A_m = T_1$ and $D_m = T_2$ on $\mathscr{D}(A_m) = \mathscr{D}(D_m) = \mathscr{W}_p \subset X_p = E = F$, the off-diagonal entries B and C correspond to the bounded collisions operators $B = K_{12}$ and $C = K_{21}$.

The corresponding transport problem presented with boundary conditions modeled by the relations $f^i = H_{12}g^o$ and $g^i = H_{21}f^o$ satisfying the following diagram:

$$\begin{tikzcd} X_p \supset \mathscr{D}(A_m) = \mathscr{W}_p \arrow[r, "\phi_1"] \arrow[dr, "\psi_2"'] & X_p^i \\ X_p \supset \mathscr{D}(D_m) = \mathscr{W}_p \arrow[r, "\phi_2"'] \arrow[ur, "\psi_1"] & X_p^i \end{tikzcd}$$

The above diagram takes the form of the previous section upon the following identification for the spaces and operators involved: $X = Y = X_p^i$, ϕ_k and ψ_k for $k = 1,2$ as the mapping:

$$\begin{cases} \phi_k : \mathscr{W}_p \longrightarrow X_p^i \\ \qquad \varphi \longmapsto \varphi^i \end{cases} \quad \begin{cases} \psi_2 : \mathscr{W}_p \longrightarrow X_p^i \\ \qquad g \longmapsto H_{12}g^o \end{cases} \text{ and } \begin{cases} \psi_1 : \mathscr{W}_p \longrightarrow X_p^i \\ \qquad f \longmapsto H_{21}f^o. \end{cases}$$

REMARK 4.1. (i) It should be noticed that, we deals with perfect periodic boundary conditions, i.e.,

$$\begin{cases} H_{12} : X_p^o \longrightarrow X_p^i \\ H_{12}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} 0 & I_{12} \\ I_{21} & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{cases}$$

where:

$$I_{12} : X_{2,p}^o \longrightarrow X_{1,p}^i, \quad u(a,\xi) \longrightarrow u(-a,\xi)$$

$$I_{21} : X^o_{1,p} \longrightarrow X^i_{2,p}, \quad u(-a,\xi) \longrightarrow u(a,\xi)$$

(ii) Obviously, H_{21} possesses the same structure as H_{12}, i.e.,

$$\begin{cases} H_{21} : X^o_p \longrightarrow X^i_p \\ H_{21}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{pmatrix} 0 & I_{12} \\ I_{21} & 0 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}. \end{cases}$$

Under these arguments, it suffices to verify the assumptions $(\mathscr{H}_1)$–$(\mathscr{H}_2)$ of the previous section. The first one is a consequence of the following remark.

REMARK 4.2. (i) It is well known from Remark 4.1 in [22], that the operators T_k, $k = 1,2$ are closed, densely defined linear operators with non empty resolvent set. Hence, the assumption $(\mathscr{H}_1)$ is satisfied.

(ii) Following Theorem 1 p. 252 in [5], the trace mapping ϕ_j, $j = 1,2$, are continuous and surjective, which ensure the validity of the hypothesis $(\mathscr{H}_2)$.

To compute the essential spectra of the operator matrix $\mathscr{A}_H$, we proceed in 5 steps:

Step 1. Consider the restrictions A_0 and D_0 of $A_m = T_1$ and $D_m = T_2$ to $\mathscr{D}(A_0) := \{f \in \mathscr{W}_p \text{ such that } f^i = 0\}$ and $\mathscr{D}(D_0) = \left\{g \in \mathscr{W}_p \text{ such that } g^i = 0\right\}$, respectively. Then, we obtain the operator matrix

$$\mathscr{A}_0 := \begin{pmatrix} A_m & 0 \\ 0 & D_m \end{pmatrix} := \begin{pmatrix} -\xi\frac{d}{dx} - \sigma_1(\xi) & 0 \\ 0 & -\xi\frac{d}{dx} - \sigma_2(\xi) \end{pmatrix}$$

on $X_p \times X_p$ with diagonal domain

$$\mathscr{D}(\mathscr{A}_0) = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \in \mathscr{W}_p \times \mathscr{W}_p \text{ such that } f^i = 0 \text{ and } g^i = 0 \right\}.$$

Step 2. The following Lemma is useful to formulate the proposed problem:

LEMMA 4.1. *Let $\lambda \in \rho(A_0) \cap \rho(D_0)$. The bounded operators K_λ and L_λ are expressed as:*

$$\begin{cases} K_\lambda : \mathscr{W}_p \longrightarrow \mathscr{W}_p \\ g \longmapsto (K_\lambda g)(x,\xi) = \chi_{(-1,0)}(\xi)(I_{21}g)(-a,\xi)e^{-\frac{(\lambda+\sigma_1(\xi))}{|\xi|}|a+x|} \\ \qquad +\chi_{(0,1)}(\xi)(I_{12}g)(a,\xi)e^{-\frac{(\lambda+\sigma_1(\xi))}{|\xi|}|a-x|}, \end{cases}$$

and

$$\begin{cases} L_\lambda : \mathscr{W}_p \longrightarrow \mathscr{W}_p \\ f \longmapsto (L_\lambda f)(x,\xi) = \chi_{(-1,0)}(\xi)(I_{21}f)(-a,\xi)e^{-\frac{(\lambda+\sigma_2(\xi))}{|\xi|}|a+x|} \\ \qquad +\chi_{(0,1)}(\xi)(I_{12}f)(a,\xi)e^{-\frac{(\lambda+\sigma_2(\xi))}{|\xi|}|a-x|}, \end{cases}$$

Proof. Let $\lambda \in \rho(A_0)\cap\rho(D_0)$. Note that the expression of K_λ and L_λ may be checked by steps:

$*$ We start firstly to revels the expression of $\ker(\lambda - A_m)$ and $\ker(\lambda - D_m)$. For this, we consider $\varphi \in \mathscr{D}(A_m)$ and $\gamma \in \mathscr{D}(D_m)$. A short computation revels that:

$$\varphi \in \ker(\lambda - A_m) \quad \text{means that} \quad \varphi(x,\xi) := \begin{cases} \varphi(-a,\xi)e^{-\frac{\lambda+\sigma_1(\xi)}{|\xi|}|a+x|}, & 0<\xi<1 \\ \varphi(a,\xi)e^{-\frac{\lambda+\sigma_1(\xi)}{|\xi|}|a-x|}, & -1<\xi<0 \end{cases}$$

$$\gamma \in \ker(\lambda - D_m) \quad \text{means that} \quad \gamma(x,\xi) := \begin{cases} \gamma(-a,\xi)e^{-\frac{\lambda+\sigma_2(\xi)}{|\xi|}|a+x|}, & 0<\xi<1 \\ \gamma(a,\xi)e^{-\frac{\lambda+\sigma_2(\xi)}{|\xi|}|a-x|}, & -1<\xi<0 \end{cases}$$

$*$ Secondly, taking into account the Remark 4.1, one finds that

$$\begin{cases} \varphi(-a,\xi) = I_{12}g(a,\xi), & \xi>0, \\ \varphi(a,\xi) = I_{21}g(-a,\xi), & \xi<0 \end{cases}$$

and

$$\begin{cases} \gamma(-a,\xi) = I_{12}f(a,\xi), & \xi>0, \\ \gamma(a,\xi) = I_{21}f(-a,\xi), & \xi<0 \end{cases}$$

satisfy $\phi_1(\varphi) = \psi_2(g)$ and $\varphi_2(\gamma) = \psi_1(f)$ for $f \in \mathscr{D}(A_m)$ and $g \in \mathscr{D}(D_m)$, which yields an explicit formula for K_λ and L_λ:

$$(K_\lambda g)(x,\xi) := \begin{cases} I_{12}g(a,\xi)e^{-\frac{\lambda+\sigma_1(\xi)}{|\xi|}|a+x|}, & 0<\xi<1, \\ I_{21}g(-a,\xi)e^{-\frac{\lambda+\sigma_1(\xi)}{|\xi|}|a-x|}, & -1<\xi<0 \end{cases}$$

$$(L_\lambda f)(x,\xi) := \begin{cases} I_{12}f(a,\xi)e^{-\frac{\lambda+\sigma_2(\xi)}{|\xi|}|a+x|}, & 0<\xi<1, \\ I_{21}f(-a,\xi)e^{-\frac{\lambda+\sigma_2(\xi)}{|\xi|}|a-x|}, & -1<\xi<0. \end{cases} \qquad \square$$

Step 3. To compute the essential spectra of $\mathscr{A}_H$, we shall prove the Fredholmness perturbations of the operators $\mathbb{G}(\lambda)$, $\mathbb{F}(\lambda)(\lambda - A_0)^{-1}$ and $\mathbb{G}(\lambda)(\lambda - D_0)^{-1}$. To do this, the following definition introduced by M. Kharroubi in [23] is required.

DEFINITION 4.1. [23] A collision operator K_{ij} in the form (4.1), is said to be regular if it satisfies the following conditions:

$$\begin{cases} \text{– the function } K_{ij}(.) \text{ is mesurable,} \\ \text{– there exists a compact subset } \mathscr{C} \subset \mathscr{L}(L_p((-1,1),d\xi)) \text{ such that:} \\ \quad K_{ij}(x) \in \mathscr{C} \text{ a.e. on } (-a,a), \\ \text{– } K_{ij}(x) \in \mathscr{K}(L_p((-1,1),d\xi)) \text{ a.e. on } (-a,a) \end{cases}$$

where $\mathscr{K}(L_p((-1,1),d\xi))$ is the set of compact operators on $L_p((-1,1),d\xi)$.

Taking into account the result of compactness and weakly compactness established by N. Moalla et al. in [22], we are in the position to state the following results:

LEMMA 4.2. *Let* $\lambda \in \rho(A_0)\cap\rho(D_0)$.

(i) If $H_{21} \in \mathscr{W}(X_1)$ *(resp.* $H_{12} \in \mathscr{W}(X_1)$*) and* $\dfrac{\kappa_{21}(x,\xi,\xi')}{|\xi'|}$ *(resp.* K_{12}*) defines a regular operator, then the operator* $\mathbb{F}(\lambda)(\lambda - A_0)^{-1}$ *(resp.* $\mathbb{G}(\lambda)$*) is weakly compact on* X_1.

(ii) If $H_{21} \in \mathscr{K}(X_p)$ *(resp.* $H_{12} \in \mathscr{K}(X_p)$*) and* K_{21} *(resp.* K_{12}*) defines a regular operator, then the operator* $\mathbb{F}(\lambda)(\lambda - A_0)^{-1}$ *(resp.* $\mathbb{G}(\lambda)$*) is compact on* X_p*, for* $p > 1$.

Proof. Let us written, for $\lambda \in \rho(A_0)\cap\rho(D_0)$, the operators $\mathbb{F}(\lambda)(\lambda - A_0)^{-1}$ and $\mathbb{G}(\lambda)$ as:

$$\mathbb{F}(\lambda)(\lambda - A_0)^{-1} := -L_\lambda(\lambda - A_0)^{-1} - (\lambda - D_0)^{-1}K_{21}(\lambda - A_0)^{-1}$$

and

$$\mathbb{G}(\lambda) := -K_\lambda - (\lambda - A_0)^{-1}K_{12}.$$

(i) The weak compactness assumption for the operator H_{21} (resp. H_{12}) revels that the operator L_λ (resp. K_λ) is weakly compacts on X_1.

Following Lemma 4.2 in [22] (resp. Lemma 3.1 in [10]), the operator $K_{21}(\lambda - A_0)^{-1}$ (resp. $(\lambda - A_0)^{-1}K_{12}$) is weakly compact on X_1.

Hence, the fact that the set $\mathscr{W}(X_1)$ is a closed two sided-ideal of $\mathscr{L}(X_1)$, allows us to conclude the desired results.

(ii) It is easy to check the results of this item, it is sufficient to use Theorem 2.2 in [18] and the compactness arguments of the operator H_{21} (resp. H_{12}) and the regularity of the collision operator K_{21} (resp. K_{12}). □

REMARK 4.3. For the remainder, we observe that if H_{12} is compact on X_p, $p > 1$ (resp. weakly compact on X_1), K_{12} defines a regular operator, then $\mathbb{F}(\lambda)\mathbb{G}(\lambda) \in \mathscr{K}(X_p)$ (resp. $\mathbb{F}(\lambda)\mathbb{G}(\lambda) \in \mathscr{W}(X_1)$). Hence, one has $[\mathbb{F}(\lambda)\mathbb{G}(\lambda)]^2 \in \mathscr{K}(X_p)$, $\forall p \geqslant 1$, we deduce that $\mathbb{F}(\lambda)\mathbb{G}(\lambda) \in P\mathscr{K}(X_p)$, $p \geqslant 1$.

Step 4. We claim that $1 \in \rho(\mathbb{F}(\lambda)\mathbb{G}(\lambda))$.

Indeed:

Let $\lambda \in \rho(A_0)\cap\rho(D_0)$ and $h \in \ker(I - \mathbb{F}(\lambda)\mathbb{G}(\lambda))$. Then, the following equation can be solved:

$$\begin{aligned}
&(I - \mathbb{F}(\lambda)\mathbb{G}(\lambda))h = 0\\
\Longleftrightarrow\ & L_\lambda\left[K_\lambda + (\lambda - A_0)^{-1}K_{12}\right]h + (\lambda - D_0)^{-1}K_{21}\left[K_\lambda + (\lambda - A_0)^{-1}K_{12}\right]h = h\\
\Longleftrightarrow\ & (\lambda - D_m)[L_\lambda\left[K_\lambda + (\lambda - A_0)^{-1}K_{12}\right]h]\\
&+(\lambda - D_m)(\lambda - D_0)^{-1}K_{21}\left[K_\lambda + (\lambda - A_0)^{-1}K_{12}\right]h = (\lambda - D_m)h,\\
&\ \text{since } L_\lambda \in \ker(\lambda - D_m)\\
\Longleftrightarrow\ & K_{21}(\lambda - A_0)^{-1}[(\lambda - A_0)K_\lambda + K_{12}]h = (\lambda - D_m)h
\end{aligned}$$

$$\iff K_{21}(\lambda - A_0)^{-1}K_{12}h = (\lambda - D_m)h, \text{ while } K_\lambda \in \ker(\lambda - A_m)$$
$$\iff [\lambda - D_m - K_{21}(\lambda - A_0)^{-1}K_{12}]h = 0.$$

Since we can find $\lambda \in \rho(A_0) \cap \rho(D_0) \cap \rho(D_0 + K_{21}(\lambda - A_0)^{-1}K_{12})$ (see [13]), then $\lambda - D_0 - K_{21}(\lambda - A_0)^{-1}K_{12}$ is invertible. So, one has $\lambda - D_m - K_{21}(\lambda - A_0)^{-1}K_{12}$ is injective, thus we conclude that $h = 0$. This argument yields the injectivity of the operator $I - \mathbb{F}(\lambda)\mathbb{G}(\lambda)$ and therefore, invertible by Remark 4.3. Following Theorem 3.1, allows us to deduce that the matrix operator $\mathscr{A}_H$ is invertible with bounded inverse.

It is now quite simple to characterize the essential spectra of $\mathscr{A}_H$.

Step 5. We compute easily the essential spectra of $\mathscr{A}_H$.

THEOREM 4.1. *Assume that $H_{21} \in \mathscr{S}(X_p)$ (resp. $H_{12} \in \mathscr{S}(X_p)$) and $\dfrac{\kappa_{21}(x,\xi,\xi')}{|\xi'|}$ (resp. K_{12}) defines a regular operator, then*

$$\sigma_{ei}(\mathscr{A}_H) = \sigma_{ei}(A_0) \cup \sigma_{ei}(D_0) = \{\lambda \in \mathbb{C} : \, Re\lambda \leqslant -\min(\lambda_1^*, \lambda_2^*)\}, \ \ \forall i \in \{r, l, 4, 5, 6\}.$$

Proof. Remark 4.3 in [22] revels that the essential spectra of the operators A_0 and D_0 (A_0 and D_0 are nothing else the streaming operators with vacuum boundary conditions) are expressed as:

$$\sigma_{ei}(A_0) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leqslant -\lambda_1^*\}, \ \ i \in \{4,5,6\}$$

and

$$\sigma_{ei}(D_0) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leqslant -\lambda_2^*\}, \ \ i \in \{4,5,6\}$$

where λ_k^* is defined for $k = 1, 2$ as:

$$\lambda_k^* := \liminf_{|\xi| \longrightarrow 0} \sigma_k(\xi), \ \ k = 1, 2.$$

Consequently, Eqs. (2) and (3) amounts that the right and left essential spectra of A_0 (resp. D_0) are just

$$\sigma_{er}(A_0) = \sigma_{el}(A_0) = \sigma_{ei}(A_0), \ \ (\text{resp. } \sigma_{er}(D_0) = \sigma_{el}(D_0) = \sigma_{ei}(D_0)), \ \ i \in \{4,5,6\}.$$

Therefore, according with Theorem 3.2, Remark 2.2 and Lemma 4.2, we have

$$\sigma_{ei}(\mathscr{A}_0) = \sigma_{ei}(A_0) \cup \sigma_{ei}(D_0) = \{\lambda \in \mathbb{C} \text{ such that } Re\lambda \leqslant -\min(\lambda_1^*, \lambda_2^*)\}$$

for $i \in \{r, l, 4, 5, 6\}$. $\square$

Conclusion. Sufficient conditions are reduced to the study of invertibility problem of unbounded operator matrices block 2×2. This study is applied to develop innovative ways leading to a rigorous study of spectral properties of matrix operator with non diagonal domain (see Theorems 3.2 and 3.3) in a fast manner of computation. Such a result exploit the resolvent expression involving an elegant use of the perturbation theory of Fredholm operators and improve under less hypotheses many earlier works.

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