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CORRIGENDUM: DERIVATIONS ON TERNARY RINGS OF OPERATORS

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Abstract. A minor error in the original paper leads to several revised proofs (sections 2 and 4 below) and two new questions (section 5). The main results in the original paper are unaffected.

1. Proposition 2.4

The proof of [3, Proposition 2.4 (iii)] contains a gap, as pointed out to the authors by Shaoze Pan. The authors have subsequently found a counter example (see section 2 below). Although part (iii) of [3, Proposition 2.4] is false, part (iv) remains an open question (See Question 1 in section 5). The main results in [3], namely, Theorems 2.2, 3.2 and 3.3, are unaffected.

The corrected statement of [3, Proposition 2.4] is the following, which has been proved in [3], and where (iii) and (iv) have been replaced by (iii') and (iv').

PROPOSITION 1.1. (Revision of Proposition 2.4 in [3]) *Let X be a TRO.*

(i) *Every TRO derivation is the strong operator limit of inner TRO derivations.*

(ii) *The triple derivations on X coincide with the TRO derivations.*

(iii') *Every inner triple derivation on X is an inner TRO derivation.*

(iv') *If all triple derivations of X are inner, then all TRO derivations of X are inner.*

In symbols, if X is any TRO, then with

- $\mathrm{Inn}_t(X)$ = the set of inner triple derivations of X
- $\mathrm{Inn}_{TRO}(X)$ = the set of inner TRO derivations of X
- $\mathscr{D}_{TRO}(X)$ = the set of all TRO derivations of X
- $\mathscr{D}_t(X)$ = the set of all triple derivations of X,

we have

$$\mathrm{Inn}_t(X) \subset \mathrm{Inn}_{TRO}(X) \subset \mathscr{D}_{TRO}(X) = \mathscr{D}_t(X).$$

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2. Theorem 3.3

The proof for [3, Theorem 3.3(iii)(3)] used [3, Proposition 2.4(iv)], but a direct proof can be given, following the outline in [1], so Theorem 3.3 of [3] is correct as stated. In the process, we discovered a counterexample to [3, Proposition 2.4(iii)]. In this section, we provide some details of that outline and the counterexample. We shall use the three Lemmas below.

The proofs of Lemmas 2.1 and 2.2 are simple modifications of the proofs of Lemma 3 and 4 and Proposition 3 of [1]. (Alternatively, the first statements in each of Lemmas 2.1 and 2.2 follow directly from Lemmas 3 and 4 in [1] and an obvious real version of Proposition 1.1(ii).) Lemma 2.3(i) is a deep result of Kaup, [2, Theorem 4.1], whereas Lemma 2.3(ii) is a simple modification of [1, Proposition 1].

LEMMA 2.1. *If X is a real Hilbert space, and $B(X,\mathbb{R})$ is identified with X with TRO product $xy^*z = \langle x|y\rangle z$, the TRO derivations of $B(X,\mathbb{R})$ correspond to the skew-symmetric operators on X, and the inner TRO derivations of $B(X,\mathbb{R})$ correspond to the skew-symmetric finite rank operators on X.*

LEMMA 2.2. *If D is a TRO derivation on a real Hilbert space X, and Y is another real Hilbert space, then the operator $\tilde{D}: B(X,Y) \to B(X,Y)$ given by $\tilde{D}(a) = a \circ D$ is a TRO derivation on $B(X,Y)$. Furthermore, if $\tilde{D}$ is an inner TRO derivation, then D has rank at most the Hilbert dimension of Y.*

LEMMA 2.3. *Let H and K be complex Hilbert spaces. Then*

(i) *There exist real Hilbert spaces X and Y such that $B(H,K)$ is the complexification of the real Banach space $B(X,Y)$.*

(ii) *If every TRO derivation of $B(H,K)$ is an inner TRO derivation, then every TRO derivation of $B(X,Y)$ is an inner TRO derivation.*

THEOREM 2.4. (Restatement of Theorem 3.3 in [3]) *Let $X = pM$ be a TRO, where M is a von Neumann algebra and p is a projection in M.*

(i) *If X is of type II^a_∞ or III^a, and has a separable predual, then every TRO derivation of X is an inner TRO derivation.*

(ii) *If M is of type III and countably decomposable, then every TRO derivation of $X = pM$ is an inner TRO derivation.*

(iii) *If $M = B(H)$ is a factor of type I, then:*

1. *If $\dim H < \infty$, then every TRO derivation of $X = pM$ is an inner TRO derivation.*
2. *If $\dim pH = \dim H$, then every TRO derivation of $X = pM$ is an inner TRO derivation.*
3. *If $\dim pH < \dim H = \infty$, then $X = pM$ admits outer TRO derivations.*

Proof. As noted, it is only necessary to prove (3) in (iii). We shall show that if H and K are Hilbert spaces with $\dim K < \dim H = \infty$, then $B(H,K)$ admits a TRO derivation which is not an inner TRO derivation. By Lemma 2.3, we may assume that H and K are real Hilbert spaces and $B(H,K)$ denotes the real linear bounded operators from H to K. Since H is infinite dimensional, there is a bounded skew-symmetric operator T with $T^2 = -I$. Thus by Lemma 2.1, T is a TRO derivation on H with rank equal to the dimension of H. If $\tilde{T}$ was an inner TRO derivation of $B(H,K)$, then by Lemma 2.2, the rank of T would be at most the dimension of K, a contradiction, since $\dim K < \dim H$. $\square$

Lemma 2.1 and Proposition 3 of [1] show that if X is an real Hilbert space, then

$$\mathrm{Inn}_t(X) = \mathrm{Inn}_{TRO}(X). \tag{2.1}$$

As shown in the following proposition, although (2.1) holds for finite dimensional complex Hilbert spaces (indeed, by pure algebra, for all finite dimensional complex TROs—see the proof of Proposition 3.3(iii)(1)) in [3]), it fails for complex TROs in general, and $\mathrm{Inn}_t(X)$ has codimension 1 in $\mathrm{Inn}_{TRO}(X)$ if $X = B(H,\mathbb{C}) \simeq H$, with H infinite dimensional.

PROPOSITION 2.5. *Let X be a complex Hilbert space.*

(i) *If X is finite dimensional, then (2.1) holds.*

(ii) *If X is infinite dimensional, then the inner TRO derivation $Dx = ix$ is not an inner triple derivation, and*

$$Inn_{TRO}(X) = Inn_t(X) \oplus \mathbb{C}D. \tag{2.2}$$

Proof. (i) Let $X = B(H,\mathbb{C}) \simeq H$, with H of finite dimension n, let δ be a TRO derivation of X, and identify X with $M_{1,n}(\mathbb{C})$. Set $a = \sum_{i=1}^n r_i^*\delta(r_i) \in M_n(\mathbb{C})$, where $r_i \in M_{1,n}(\mathbb{C})$ is the matrix with 1 in the $(1,i)$-position and zeros elsewhere. Then, since $r_i r_j^* = \delta_{ij}$, we have

$$\begin{aligned} a + a^* &= a1 + a^*1 = a(\sum_j r_j^* r_j) + (\sum_i r_i^* r_i)a^* = \sum_{i,j} r_i^*[\delta(r_i)r_j^* r_j + r_i\delta(r_j)^* r_j] \\ &= \sum_{i,j} r_i^*[\delta(r_i r_j^* r_j) - r_i r_j^*\delta(r_j)] = \sum_i r_i^*[\delta(r_i r_i^* r_i) - \delta(r_i)] = 0, \end{aligned}$$

and with $x = \sum_i x_i r_i$ $(x_i \in \mathbb{C})$,

$$xa = (\sum_i x_i r_i)(\sum_j r_j^*\delta(r_j)) = \sum_{i,j} x_i r_i r_j^*\delta(r_j) = \delta(x).$$

(ii) Let $X = B(H,\mathbb{C}) \simeq H$ be infinite dimensional. Suppose we had $a_k, b_k \in H$, $1 \leqslant k \leqslant n$ with

$$\begin{aligned} ix &= \sum_k \delta(a_k,b_k)x = \sum_k (\{a_k,b_k,x\} - \{b_k,a_k,x\}) \\ &= \sum_k (\langle a_k,b_k\rangle - \langle b_k,a_k\rangle)x + \sum_k (\langle x,b_k\rangle a_k - \langle x,a_k\rangle b_k). \end{aligned}$$

Taking $x \neq 0$ orthogonal to the a_k, b_k we conclude

$$\sum_k (\langle a_k, b_k\rangle - \langle b_k, a_k\rangle) = i \tag{2.3}$$

and

$$\sum_k (a_k \otimes b_k - b_k \otimes a_k)x = \sum_k (\langle x, b_k\rangle a_k - \langle x, a_k\rangle b_k) = 0,$$

so $\sum_k (a_k \otimes b_k)$ is self adjoint. Thus taking the trace in

$$2\sum_k (a_k \otimes b_k) = \sum_k (a_k \otimes b_k) + \sum_k (b_k \otimes a_k)$$

results in

$$\sum_k (\langle a_k, b_k\rangle + \langle b_k, a_k\rangle) = 2\sum_k (\langle a_k, b_k\rangle.$$

which when added to (2.3) yields a contradiction.

To prove (2.2), suppose that δ is an inner TRO derivation, say $\delta x = \alpha x + x\beta$ where $\alpha = -\alpha^* = \sum_i a_i b_i^* \in XX^*$, so $\alpha x = (\sum_i \langle a_i, b_i\rangle)x$ and $\beta = -\beta^* = \sum_j c_j^* d_j \in X^*X$, so $x\beta = \sum_j (d_j \otimes c_j^*)(x)$.

Thus $\delta = (\sum_i \langle a_i, b_i\rangle)\,\mathrm{Id}_X + \sum_j (c_j \otimes d_j^* - d_j \otimes c_j^*)$ and it follows that $\delta = i(\lambda - \mu)\,\mathrm{Id}_X + \sum_j \delta(c_j, d_j)$ where $i\lambda = \sum_i \langle a_i, b_i\rangle$ and $i\mu = \sum_j \langle c_j, d_j\rangle$ are purely imaginary, and $\delta(c,d)$ is the inner triple derivation $x \mapsto \{cdx\} - \{dcx\}$. $\square$

3. Proposition 3.7

The proof of [3, Proposition 3.7(ii)] used [3, Proposition 2.4(iv)] and therefore remains an open question (See Question 2 in section 5). The revised statement of [3, Proposition 3.7] is the following, where (ii) has been replaced by (ii'), and in the proof of (i), [3, Proposition 2.4(iv)] has been replaced by [3, Corollary 2.3].

PROPOSITION 3.1. (Revision of Proposition 3.7 in [3])

(i) *If a W*-TRO V acts on a separable Hilbert space and is of one of the types $I_{\infty,\infty}, II_{\infty,\infty}$ or III, then every triple derivation of V is an inner triple derivation and every TRO derivation of V is an inner TRO derivation.*

(ii') *Every triple derivation of every W^*-TRO of type $II_{1,\infty}$ is an inner triple derivation, if and only if every triple derivation of any W^*-TRO of type $II_{\infty,1}$ is an inner triple derivation.*

Proof. (i) is an immediate consequence of [3, Theorem 3.5(ii) and Theorem 3.5], and [3, Corollary 2.3]. (ii') is an immediate consequence of [3, Lemma 3.6]. $\square$

4. Lemma 3.8 and Proposition 3.9

Only the first statement in [3, Lemma 3.8] was proved, since the second statement followed from the unproved [3, Proposition 2.4(iv)]. Since the latter is not necessarily available we provide here a proof of the second statement, parallel to the proof of the first statement.

LEMMA 4.1. (Restatement of Lemma 3.8 of [3]) *Let E be a TRO and Ω a compact Hausdorff space.*

(i) *If every TRO derivation of $V := C(\Omega, E)$ is an inner TRO derivation, then the same holds for E.*

(ii) *If every triple derivation of $V := C(\Omega, E)$ is an inner triple derivation, then the same holds for E.*

Proof. The first statement was proved in [3].

Suppose every triple derivation of V is an inner triple derivation. If D is a triple derivation of E, then $\delta f(\omega) := D(f(\omega))$ is a triple derivation of V, as is easily checked. Then $\delta f = \sum_i \delta(a_i, b_i)f$ where $a_i, b_i \in V$. For $a \in E$, let $1 \otimes a \in V$ be the constant function equal to a. Then

$$\begin{aligned}
D(a) = D((1\otimes a)(\omega)) = \delta(1\otimes a)(\omega) &= \sum_i \delta(a_i, b_i)(1\otimes a)(\omega) \\
&= \sum_i \{a_i, b_i, 1\otimes a\}(\omega) - \{b_i, a_i, 1\otimes a\}(\omega) \\
&= \sum_i \{a_i(\omega), b_i(\omega), (1\otimes a)(\omega)\} - \{b_i(\omega), a_i(\omega), (1\otimes a)(\omega)\} \\
&= \sum_i \{a_i(\omega), b_i(\omega), a\} - \{b_i(\omega), a_i(\omega), a\} = \sum_i \delta(a_i(\omega), b_i(\omega))(a),
\end{aligned}$$

so that D is an inner triple derivation. $\square$

The proof of [3, Proposition 3.9], which involved only triple derivations, is easily adapted to prove a corresponding statement for TRO derivations.

PROPOSITION 4.2. (Extension of Proposition 3.9 of [3])
Let $V = \oplus_\alpha C(\Omega_\alpha, E_\alpha)$, where $E_\alpha = B(K_\alpha, H_\alpha)$. Then if every triple derivation of V is an inner triple derivation, or if every TRO derivation of V is an inner TRO derivation, then for every α, either $\dim E_\alpha < \infty$ or $\dim K_\alpha = \dim H_\alpha$.

5. Two more open questions

QUESTION 1. With reference to Proposition 1.1, does there exist a TRO admitting a triple derivation which is not an inner triple derivation, but which has the property that all TRO derivations are inner TRO derivations?

QUESTION 2. With reference to Proposition 3.1, if every TRO derivation of any W^*-TRO of type $II_{1,\infty}$ has only inner TRO derivations, then does every TRO derivation of any W^*-TRO of type $II_{\infty,1}$ have only inner TRO derivations? (See [3, Questions 2], where an equivalence was asserted, based on the unproven [3, Proposition 2.4(iv)].)

We note that by Proposition 3.1 above, a negative answer to Question 1 implies a positive answer to Question 2.

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