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ON THE LOCATION OF EIGENVALUES OF MATRIX POLYNOMIALS

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Abstract. A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix polynomial $P(z)$ if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. Note that each finite eigenvalue of $P(z)$ is a zero of the characteristic polynomial $\det(P(z))$. In this paper we establish some (upper and lower) bounds for eigenvalues of matrix polynomials based on the norm of their coefficient matrices and compare these bounds to those given by N. J. Higham and F. Tisseur [8], J. Maroulas and P. Psarrakos [12].

1. Introduction

Let $\mathbb{C}^{n\times n}$ be the set of all $n \times n$ matrices whose entries are in $\mathbb{C}$. For a *matrix polynomial* we mean the matrix-valued function of a complex variable of the form

$$P(z) = A_m z^m + \cdots + A_1 z + A_0, \tag{1}$$

where $A_i \in \mathbb{C}^{n\times n}$ for all $i = 0, \cdots, m$. If $A_m \neq 0$, $P(z)$ is called a matrix polynomial of *degree* m. When $A_m = I$, the identity matrix in $C^{n\times n}$, the matrix polynomial $P(z)$ is called a *monic*.

A number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of the matrix polynomial $P(z)$, if there exists a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. Then the vector x is called, as usual, an *eigenvector* of $P(z)$ associated to the eigenvalue λ. Note that each finite eigenvalue of $P(z)$ is a root of the *characteristic polynomial* $\det(P(z))$.

The *polynomial eigenvalue problem (PEP)* is to find an eigenvalue λ and a nonzero vector $x \in \mathbb{C}^n$ such that $P(\lambda)x = 0$. For $m = 1$, (PEP) is actually the *generalized eigenvalue problem (GEP)*

$$Ax = \lambda Bx,$$

and, in addition, if $B = I$, we have the standard eigenvalue problem

$$Ax = \lambda x.$$

For $m = 2$ we have the *quadratic eigenvalue problem (QEP)*.

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The theory of matrix polynomials was primarily devoted by two works, both of which are strongly motivated by the theory of *vibrating systems*: one by Frazer, Duncan, and Collar in 1938 [FDC], and the other by P. Lancaster in 1966 [10].

(QEPs), and more generally (PEPs), play an important role in applications to science and engineering. We refer to [19] for a survey on applications of (QEP). Moreover, we refer to the book of I. Gohberg, P. Lancaster and L. Rodman [6] for a theory of matrix polynomials and their applications.

There are algorithms to solve (QEPs), see the works of Hamarling, Munro and Tisseur [7, 2013] and Zeng and Su [20, 2014]. For (PEPs), there is some research on bounds of eigenvalues of matrix polynomials which were constructed in terms of the norms of coefficients of the given matrix polynomials. See, for example, the work of Higham and Tisseur [8, 2003], Maroulas and Psarrakos [12, 1997].

Computing eigenvalues of matrix polynomials (even computing eigenvalues of scalar matrices and finding roots of univariate polynomials) is a hard problem. There is an iterative method to compute these eigenvalues, see Simoncini and Perotti [16, 2006]. Moreover, when computing pseudospectra of matrix polynomials, which provide information about the global sensitivity of the eigenvalues, a particular region of the (possibly extended) complex plane must be identified that contains the eigenvalues of interest, and bounds clearly help to determine such region [18]. Therefore, it is useful to find the location of these eigenvalues.

Note that, if A_0 is singular then 0 is an eigenvalue of $P(z)$, and if A_m is singular then 0 is an eigenvalue of the matrix polynomial $z^m P(1/z)$. Therefore, to locate the eigenvalues of these matrix polynomials, *we always assume that A_0 and A_m are non-singular*.

The paper is organized as follows. In Section 2 we give bounds for matrix polynomials whose coefficients satisfy some special properties, in particular, we give a matrix version of Eneström-Kakeya's theorem. In Section 3 we establish matrix versions of some Cauchy's type theorems. In particular, we establish a matrix version of the theorem of Joyal, Labelle and Rahman (cf. [9], [13, Theorem 2.14]) and some of its corollaries. Moreover, we give also a matrix version of Datt and Govil's theorem [3, Theorem 1] and some other bounds. Finally, we give some numerical experiments in Section 4.

Notation. For a matrix $A \in \mathbb{C}^{n\times n}$, the notation $A \geqslant 0$ means *"A is positive semidefinite"*, i.e. for every vector $x \in \mathbb{C}^n$ we have $x^*Ax \geqslant 0$; $A > 0$ means *"A is positive definite"*, i.e. $x^*Ax > 0$ for every nonzero vector $x \in \mathbb{C}^n$. For two matrices $A, B \in \mathbb{C}^{n\times n}$, the notation $A \geqslant B$ means $A - B \geqslant 0$.

Throughout this paper, $\|\cdot\|$ denotes a subordinate matrix norm.

2. Eneström-Kakeya's theorem for matrix polynomials

In this section we give upper and lower bounds for eigenvalues of some special matrix polynomials. First of all we consider matrix polynomials with a dominant property.

THEOREM 2.1. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n\times n}$ satisfying the following dominant property:*

$$\|A_m\| > \|A_i\|, \forall\, i = 0, \cdots, m-1.$$

Then all eigenvalues λ of $P(z)$ locate in the open disk

$$|\lambda| < 1 + \|A_m\|\|A_m^{-1}\|.$$

In particular, for $n = 1$, we obtain the following corollary of Cauchy's theorem ([11, Theorem 27.2]; see also [2, Theorem 2.2]):

COROLLARY 2.1.1. *Let $p(z) = a_0 + a_1 z + \cdots + a_m z^m \in \mathbb{C}[z]$ such that $|a_m| > |a_i|$ for all $i = 0, \cdots, m-1$. Then all the roots of $p(z)$ locate in the open disk $|z| < 2$.*

The proof of this corollary uses the fact that when $n = 1$ we have $\|A_m\|\|A_m^{-1}\| = 1$.

Proof of Theorem 2.1. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ.
We have nothing to prove if $|\lambda| \leqslant 1$. Hence we may assume that $|\lambda| > 1$. Then we have

$$\begin{aligned}
\|P(\lambda)x\| &\geqslant |\lambda|^m \left[\|A_m x\| - \left\| \sum_{i=0}^{m-1} \frac{A_i x}{\lambda^{m-i}} \right\| \right] \geqslant |\lambda|^m \left[\|A_m^{-1}\|^{-1} - \sum_{i=0}^{m-1} \frac{\|A_i\|}{|\lambda|^{m-i}} \right] \\
&\geqslant |\lambda|^m \left[\|A_m^{-1}\|^{-1} - \sum_{i=0}^{m-1} \frac{\|A_m\|}{|\lambda|^{m-i}} \right] = |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \|A_m\|\|A_m^{-1}\| \sum_{i=1}^{m} \frac{1}{|\lambda|^i} \right] \\
&> |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \|A_m\|\|A_m^{-1}\| \sum_{i=1}^{\infty} \frac{1}{|\lambda|^i} \right] = |\lambda|^m \|A_m^{-1}\|^{-1} \left[1 - \frac{\|A_m\|\|A_m^{-1}\|}{|\lambda| - 1} \right] \\
&= \frac{|\lambda|^m \|A_m^{-1}\|^{-1}}{|\lambda| - 1} \left(|\lambda| - 1 - \|A_m\|\|A_m^{-1}\| \right).
\end{aligned}$$

Hence, if $|\lambda| \geqslant 1 + \|A_m\|\|A_m^{-1}\|$ we have $\|P(\lambda)x\| > 0$, a contradiction. It follows that $|\lambda| < 1 + \|A_m\|\|A_m^{-1}\|$, which completes the proof. □

The following theorem of Eneström and Kakeya is well-known.

THEOREM 2.2. ([15, Corollary 3]) *Let $p(z)$ be a polynomial in one variable given by*

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m,\ a_i \in \mathbb{R}, \forall i = 1, \cdots, m.$$

Suppose that

$$a_m \geqslant a_{m-1} \geqslant \cdots \geqslant a_0 \geqslant 0;\ a_m > 0.$$

If $z \in \mathbb{C}$ is a root of $p(z)$ then $\dfrac{a_0}{2a_m} \leqslant |z| \leqslant 1$.

A matrix version of Theorem 2.2 is given as follows.

THEOREM 2.3. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n\times n}$ satisfying*

$$A_m \geqslant A_{m-1} \geqslant \cdots \geqslant A_0 \geqslant 0;\ A_m > 0.$$

Then each eigenvalue λ of $P(z)$ satisfies

$$\frac{\lambda_{\min}(A_0)}{2\lambda_{\max}(A_m)} \leqslant |\lambda| \leqslant 1,$$

where $\lambda_{\min}(A_0)$ denotes the smallest eigenvalue of A_0 and $\lambda_{\max}(A_m)$ the largest eigenvalue of A_m.

Proof. A proof for the upper bound of $|\lambda|$ in this theorem was given by G. Dirr and H. K. Wimmer [4, Theorem 2.1] (see also in [17, Theorem 5.1]). Now we give a proof for the lower bound.

Firstly we observe that for a matrix $A \in \mathbb{C}^{n\times n}$, its smallest eigenvalue $\lambda_{min}(A)$ and its largest eigenvalue $\lambda_{max}(A)$ belong to the set

$$\{x^*Ax | x \in \mathbb{C}^n, \|x\| = 1\},$$

which is the *standard numerical range of A*. Hence for a unit vector $x \in \mathbb{C}^n$, we always have

$$\lambda_{min}(A) \leqslant x^*Ax \leqslant \lambda_{max}(A). \tag{2}$$

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$, and $u \in \mathbb{C}^n, \|u\| = 1$ an eigenvector of $P(z)$ associated to λ. Consider the polynomial

$$P_u(z) := u^*P(z)u = \sum_{i=0}^{m} (u^*A_i u) z^i.$$

Note that λ is a root of $P_u(z)$. Moreover, the hypothesis on the relation of A_i's implies that

$$u^*A_m u \geqslant u^*A_{m-1}u \geqslant \cdots \geqslant u^*A_0 u \geqslant 0, u^*A_m u > 0,$$

that is, the polynomial $P_u(z)$ satisfies the conditions given in Theorem 2.2. Applying this theorem for $P_u(z)$ we obtain

$$\frac{u^*A_0 u}{2u^*A_m u} \leqslant |\lambda|.$$

Then the required lower bound for $|\lambda|$ follows from (2). □

By applying Theorem 2.3 for the matrix polynomial $z^n P(\frac{1}{z})$ we obtain the following dual version of Theorem 2.3.

THEOREM 2.4. *Let $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ be a matrix polynomial whose coefficients $A_i \in \mathbb{C}^{n\times n}$ satisfying*

$$A_0 \geqslant A_1 \geqslant \cdots \geqslant A_m > 0.$$

Then each eigenvalue λ of $P(z)$ satisfied $|\lambda| \geqslant 1$.

We have also the following version of Eneström-Kakeya's theorem for polynomials.

THEOREM 2.5. (Eneström-Kakeya's theorem, Version 2, [1]) *Let* $p(z) = a_0 + a_1 z + \cdots + a_m z^m$ *be a polynomial whose coefficients* $a_i, i = 0, \cdots, m$ *are positive real numbers. Denote*

$$\alpha := \min_{0 \leqslant i \leqslant m-1} \left\{ \frac{a_i}{a_{i+1}} \right\}, \beta := \max_{0 \leqslant i \leqslant m-1} \left\{ \frac{a_i}{a_{i+1}} \right\}.$$

Then each root $z \in \mathbb{C}$ *of* $p(z)$ *satisfies the following inequalities*

$$\alpha \leqslant |z| \leqslant \beta.$$

Using the same method as given in the proof of Theorem 2.3, applying Theorem 2.5, we obtain the following bounds for eigenvalues of matrix polynomials whose coefficients are positive definite.

THEOREM 2.6. *Let* $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ *be a matrix polynomial whose coefficients* $A_i \in \mathbb{C}^{n \times n}$ *are positive definite. If* $\lambda \in \mathbb{C}$ *is an eigenvalue of* $P(z)$, *then*

$$\min_{i=0,\cdots,m-1} \left\{ \frac{\lambda_{min}(A_i)}{\lambda_{max}(A_{i+1})} \right\} \leqslant |\lambda| \leqslant \max_{i=0,\cdots,m-1} \left\{ \frac{\lambda_{max}(A_i)}{\lambda_{min}(A_{i+1})} \right\}.$$

3. Cauchy type theorems for matrix polynomials

In this section we establish some Cauchy type theorems for matrix polynomials of the form $P(z) = A_0 + A_1 z + \cdots + A_m z^m$ with A_m and A_0 non-singular. We should observe that the set of eigenvalues of $P(z)$ coincides to that of the monic matrix polynomial

$$A_m^{-1} P(z) = (A_m^{-1} A_0) + (A_m^{-1} A_1) z + \cdots + I z^m.$$

Therefore, because of the complexity in practice, *we concentrate to consider in this section the bounds for monic matrix polynomials.*

Firstly we state the Cauchy's theorem for monic matrix polynomials.

THEOREM 3.1. (Cauchy, [8, Lemma 3.1]) *Let* $P(z) = A_0 + A_1 z + \cdots + I z^m$ *be a monic matrix polynomial. Let* r *resp.* R *be the positive root of the polynomial*

$$h(z) = z^m + z^{m-1} \|A_{m-1}\| + \cdots + z \|A_1\| - \left\|A_0^{-1}\right\|^{-1}$$

resp.

$$g(z) = z^m - z^{m-1} \|A_{m-1}\| - \cdots - \|A_0\|.$$

Then each eigenvalue λ *of* $P(z)$ *satisfies*

$$r \leqslant |\lambda| \leqslant R.$$

Now we give some Cauchy type theorem for monic matrix polynomials.

THEOREM 3.2. *Let $P(z) = A_0 + A_1 z + \cdots + I z^m$. Denote*

$$M := \max_{i=0,\cdots,m-1} \|A_i\|.$$

Then all eigenvalues of $P(z)$ are contained in the closed disk

$$K(0, r_1) := \{z \in \mathbb{C} | \ |z| \leqslant r_1\},$$

where $r_1 := \max\{1, \delta\}$ and $\delta \neq 1$ is the positive root of the equation

$$z^{m+1} - (1+M)z^m + M = 0.$$

In particular, for $n = 1$ we obtain a Cauchy type theorem for polynomials [2, Theorem 3.2].

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ.
The conclusion is clear if $|\lambda| \leqslant 1$. Therefore we may assume that $|\lambda| > 1$. Then we have

$$\begin{aligned} \|P(\lambda)x\| &\geqslant \left[\|Ix\||\lambda|^m - \|\sum_{i=0}^{m-1} A_i x \lambda^i\| \right] \\ &\geqslant \left[|\lambda|^m - \sum_{i=0}^{m-1} \|A_i\| \|A_m^{-1}\| \lambda^i \right] \qquad (3) \\ &\geqslant \left[|\lambda|^m - M \sum_{i=0}^{m-1} \lambda^i \right] \qquad (4) \\ &= \left[|\lambda|^m - M \frac{|\lambda|^m - 1}{|\lambda| - 1} \right] = \frac{1}{|\lambda| - 1} \left(|\lambda|^{m+1} - (1+M)|\lambda|^m + M \right). \end{aligned}$$

In the lines above, from (3) to (4) we use the definition of M.
Note that the polynomial $f(z) := z^{m+1} - (1+M)z^m + M$ has exactly two positive real roots 1 and $\delta \neq 1$ by the Descartes' rule of signs, and $f(0) > 0$. It follows that

$$|f(z)| > 0 \text{ for all } z > \max\{\delta, 1\}.$$

Hence for $|\lambda| > r_1$ we have $\|P(\lambda)x\| > 0$, a contradiction. This completes the proof. □

COROLLARY 3.2.1. *Let $P(z) = A_0 + A_1 z + \cdots + I z^m$ be a monic matrix polynomial. Denote*

$$\widetilde{M} := \max_{i=0,\cdots,m} \|A_{m-i} - A_{m-i-1}\| \quad (A_m = I \text{ and } A_{-1} = 0).$$

Then all eigenvalues of $P(z)$ are contained in the closed disk $K(0, r_2)$, where $r_2 := \max\{1, \delta\}$ and $\delta \neq 1$ is the positive root of the equation

$$z^{m+2} - (1+\widetilde{M})z^{m+1} + \widetilde{M} = 0.$$

In particular, for $n=1$ we obtain [2, Theorem 3.3].

Proof. Consider the matrix polynomial

$$Q(z) := (1-z)P(z) = -Iz^{m+1} + \sum_{i=0}^{m}(A_{m-i} - A_{m-i-1})z^{m-i}.$$

Applying Theorem 3.2 for the polynomial $Q(z)$, observing that each eigenvalue of $P(z)$ is also an eigenvalue of $Q(z)$, we obtain the required result. □

THEOREM 3.3. *Let $P(z) = A_0 + A_1 z + \cdots + Iz^m$. Then all eigenvalues of $P(z)$ are contained in the open disk*

$$K^o(0, r_3) := \{z \in \mathbb{C} | \quad |z| < r_3\},$$

where $r_3 := 1 + M$ and M is defined as in Theorem 3.2.

In particular, for $n=1$ we obtain another Cauchy's theorem for polynomials [11, Theorem (27,2)].

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ.
As above, we may assume that $|\lambda| > 1$. Then we have

$$\begin{aligned}
\|P(\lambda)x\| &\geqslant |\lambda|^m \left[\|Ix\| - \| \sum_{i=0}^{m-1} \frac{A_i x}{\lambda^{m-i}} \| \right] \geqslant |\lambda|^m \left[1 - \sum_{i=0}^{m-1} \frac{\|A_i\|}{|\lambda|^{m-i}} \right] \\
&\geqslant |\lambda|^m \left[1 - M \sum_{i=1}^{m} \frac{1}{|\lambda|^i} \right] > |\lambda|^m \left[1 - M \sum_{i=1}^{\infty} \frac{1}{|\lambda|^i} \right] = |\lambda|^m \left[1 - \frac{M}{|\lambda| - 1} \right] \\
&= \frac{|\lambda|^m}{|\lambda| - 1} (|\lambda| - 1 - M).
\end{aligned}$$

Then, for $|\lambda| \geqslant 1 + M$ we have $\|P(\lambda)x\| > 0$, a contradiction. Thus $|\lambda| < 1 + M$. □

COROLLARY 3.3.1. *Let $P(z) = A_0 + A_1 z + \cdots + Iz^m$. Then all eigenvalues of $P(z)$ are contained in the open disk $K^o(0, r_4)$, where $r_4 := 1 + \widetilde{M}$ and $\widetilde{M}$ is defined as in Corollary 3.2.1.*

In particular, for $n=1$ we obtain [2, Theorem 3.4].

Proof. Consider the matrix polynomial

$$Q(z) := (1-z)P(z) = -Iz^{m+1} + \sum_{i=0}^{m}(A_{m-i} - A_{m-i-1})z^{m-i}.$$

Since each eigenvalue of $P(z)$ is also an eigenvalue of $Q(z)$, applying Theorem 3.3 for $Q(z)$ we have the conclusion. □

Next we give a matrix version of the theorem of Joyal, Labelle and Rahman, cf. [9], [13, Theorem 2.14].

THEOREM 3.4. *Let* $P(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + I z^m$ *be a monic matrix polynomial. Denote*

$$\alpha := \max_{i=0,\cdots,m-2} \|A_i\|.$$

Then each eigenvalue λ *of* $P(z)$ *is estimated by*

$$|\lambda| \leqslant \frac{1}{2}\left\{1 + \|A_{m-1}\| + \left[(1 - \|A_{m-1}\|)^2 + 4\alpha\right]^{\frac{1}{2}}\right\}.$$

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ.
By contradiction, assume

$$|\lambda| > \frac{1}{2}\left\{1 + \|A_{m-1}\| + \left[(1 - \|A_{m-1}\|)^2 + 4\alpha\right]^{\frac{1}{2}}\right\}.$$

It follows that

$$(|\lambda| - 1)(|\lambda| - \|A_{m-1}\|) - \alpha > 0. \tag{5}$$

Multiplying (5) by $|\lambda|^{m-1}$ and then dividing by $|\lambda| - 1$, we obtain

$$|\lambda|^m - \|A_{m-1}\|\,\lambda^{m-1} - \alpha\frac{|\lambda|^{m-1}}{|\lambda| - 1} > 0.$$

However,

$$\begin{aligned}\alpha\frac{|\lambda|^{m-1}}{|\lambda| - 1} > \alpha\frac{|\lambda|^{m-1} - 1}{|\lambda| - 1} &= \alpha(1 + |\lambda| + \cdots + |\lambda|^{m-2}) \\ &\geqslant \|(A_0 + A_1\lambda + \cdots + A_{m-2}\lambda^{m-2})x\|.\end{aligned}$$

On the other hand,

$$|\lambda|^m - \|A_{m-1}\|\,\lambda^{m-1} \leqslant \|(I \cdot \lambda^m + A_{m-1}\lambda^{m-1})x\|.$$

It follows that

$$\begin{aligned}0 &< |\lambda|^m - \|A_{m-1}\|\,\lambda^{m-1} - \alpha\frac{|\lambda|^{m-1}}{|\lambda| - 1} \\ &< \|(I \cdot \lambda^m + A_{m-1}\lambda^{m-1})x\| - \|(A_0 + A_1\lambda + \cdots + A_{m-2}\lambda^{m-2})x\| \\ &\leqslant \|(A_0 + A_1\lambda + \cdots + A_{m-2}\lambda^{m-2})x + (A_{m-1}\lambda^{m-1} + I \cdot \lambda^m)x\| = \|P(\lambda)x\|,\end{aligned}$$

a contradiction. Thus

$$\lambda \leqslant \frac{1}{2}\left\{1 + \|A_{m-1}\| + \left[(1 - \|A_{m-1}\|)^2 + 4\alpha\right]^{\frac{1}{2}}\right\}. \quad \square$$

By applying Theorem 3.4 for the monic matrix polynomial $z^m P(\frac{1}{z})$ we obtain the following lower bound for eigenvalues of $P(z)$.

COROLLARY 3.4.1. *Let* $P(z) = A_0 + A_1 z + \cdots + A_{m-1} z^{m-1} + Iz^m$. *Denote* $L_i := A_0^{-1} A_i$ *(* $i = 1, \ldots, m-1$*)*, $L_m = A_0^{-1}$, *and*

$$\beta := \max_{i=2,\cdots,m} \|L_i\|.$$

Then for each eigenvalue λ *of* $P(z)$ *we have*

$$|\lambda| \geqslant \frac{2}{1 + \|L_1\| + \left[(1 - \|L_1\|)^2 + 4\beta\right]^{\frac{1}{2}}}.$$

By applying Theorem 3.4 for the matrix polynomial $(1-z)P(z)$ we obtain

COROLLARY 3.4.2. *Let* $P(z) = A_0 + A_1 z + \cdots + A_{m-1} z^{m-1} + Iz^m$. *Denote*

$$\gamma := \max_{i=1,\cdots,m} \|A_{m-i} - A_{m-i-1}\| \qquad (A_{-1} = 0).$$

Then each eigenvalue λ *of* $P(z)$ *satisfies*

$$|\lambda| \leqslant \frac{1}{2}\left\{1 + \|I - A_{m-1}\| + \left[(1 - \|I - A_{m-1}\|)^2 + 4\gamma\right]^{\frac{1}{2}}\right\}.$$

Similarly, Corollary 3.4.2 yields the following lower bound.

COROLLARY 3.4.3. *Let* $P(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + Iz^m$. *Denote*

$$\gamma' := \max_{i=1,\cdots,m} \|L_i - L_{i+1}\| \qquad (L_{m+1} = 0).$$

Then each eigenvalue λ *of* $P(z)$ *satisfies*

$$|\lambda| \geqslant \frac{2}{1 + \|I - L_1\| + \left[(1 - \|I - L_1\|)^2 + 4\gamma'\right]^{\frac{1}{2}}}.$$

By applying Theorem 3.4 for the matrix polynomial $(Iz - A_{m-1})P(z)$ we obtain

COROLLARY 3.4.4. *Let* $P(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + Iz^m$. *Denote*

$$\delta := \max_{i=0,\cdots,m-1} \|A_{m-1} A_i - A_{i-1}\| \qquad (A_{-1} = 0).$$

Then each eigenvalue λ *of* $P(z)$ *satisfies*

$$|\lambda| \leqslant \frac{1}{2}(1 + \sqrt{1 + 4\delta}).$$

Corollary 3.4.4 yields the following lower bound.

COROLLARY 3.4.5. *Let* $P(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + I z^m$. *Denote*

$$\delta' := \max_{i=1,\cdots,m} \|L_1 L_i - L_{i+1}\| \qquad (L_{m+1} = 0).$$

Then each eigenvalue λ *of* $P(z)$ *is bounded below by*

$$|\lambda| \geqslant \frac{2}{1+\sqrt{1+4\delta'}}.$$

By applying Theorem 3.4 for the matrix polynomial $(I \cdot z + I - A_{m-1})P(z)$ we obtain

COROLLARY 3.4.6. *Let* $P(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + I z^m$. *Denote*

$$\varepsilon := \max_{i=0,\cdots,m-1} \|(I - A_{m-1})A_i + A_{i-1}\| \qquad (A_{-1} = 0).$$

Then each eigenvalue λ *of* $P(z)$ *is estimated by*

$$|\lambda| \leqslant 1 + \sqrt{\varepsilon}.$$

The following lower bound is obtained by applying Corollary 3.4.6 for the matrix polynomial $z^m P(\frac{1}{z})$.

COROLLARY 3.4.7. *Let* $P(z) = A_0 + A_1 z + \ldots + A_{m-1} z^{m-1} + I z^m$. *Denote*

$$\varepsilon' := \max_{i=1,\cdots,m} \|(I - L_1)L_i + L_{i+1}\| \qquad (L_{m+1} = 0).$$

Then each eigenvalue λ *of* $P(z)$ *bounded below by*

$$|\lambda| \geqslant \frac{1}{1+\sqrt{\varepsilon'}}.$$

Next we give the matrix version of the theorem of Datt and Govil [3, Theorem 1].

THEOREM 3.5. *Let* $P(z) = A_0 + A_1 z + \cdots + I z^m$ *be a monic matrix polynomial. Denote*

$$M = \max_{i=0,\cdots,m-1} \|A_i\|.$$

Then each eigenvalue λ *of* $P(z)$ *satisfies*

$$\frac{\|A_0^{-1}\|^{-1}}{2(1+M)^{m-1}(Mm+1)} \leqslant |\lambda| \leqslant 1 + \lambda_0 M,$$

where λ_0 *is a root of the equation* $x = 1 - \frac{1}{(Mx+1)^m}$ *in the interval* $(0,1)$.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ.

First we prove the upper bound for $|\lambda|$. We consider two cases:
The first case: $mM \leqslant 1$. In this case, if $|\lambda| > 1$, we have

$$\|P(\lambda)x\| \geqslant |\lambda|^m - mM|\lambda|^{m-1} \geqslant |\lambda|^m - |\lambda|^{m-1} > 0, \text{ a contradiction.}$$

It follows that $|\lambda| \leqslant 1 \leqslant 1+\lambda_0 M$ for all $\lambda_0 \in (0,1)$.
The second case: $mM > 1$. In this case the equation $x = 1 - \frac{1}{(Mx+1)^m}$ has a unique root $\lambda_0 \in (0,1)$ [3, Lemma 2]. Moreover, we have

$$\|P(\lambda)x\| \geqslant |\lambda|^m - M\sum_{j=0}^{m-1}|\lambda|^j = |\lambda|^m - M\frac{|\lambda|^m-1}{|\lambda|-1}.$$

If $|\lambda| > 1+M\lambda_0$, we can write $|\lambda| = 1+M\alpha$ with $\alpha > \lambda_0$. Then $\alpha > 1 - \frac{1}{(M\alpha+1)^m}$. It follows that

$$\|P(\lambda)x\| \geqslant (1+M\alpha)^m - \frac{(1+M\alpha)^m-1}{\alpha} > 0,$$

a contradiction. Thus $|\lambda| \leqslant 1+M\lambda_0$.

Now we prove the lower bound for $|\lambda|$. By contradiction, assume $|\lambda| < \frac{\|A_0^{-1}\|^{-1}}{2(1+M)^{m-1}(Mm+1)}$.
Let us consider the matrix polynomial $G(z) := (1-z)P(z)$.
We have

$$G(z) = A_0 + \sum_{i=1}^{m}(A_i - A_{i-1})z^i + Iz^m - A_{m-1}z^m - Iz^{m+1} =: A_0 + H(z).$$

Denote $R := 1+M$. Then for $|z| = R$, we have

$$\begin{aligned}\max_{|z|=R}\|H(z)x\| &\leqslant R^{m+1} + R^m + \|A_{m-1}\|R^m + \sum_{i=1}^{m-1}\|A_i - A_{i-1}\|R^i \\ &\leqslant R^m[R+1+M+2(m-1)M] = 2(1+M)^m(mM+1).\end{aligned}$$

It follows from the *maximal module principle* that for $|z| \leqslant R$ we have

$$\|H(z)x\| \leqslant 2(1+M)^m(mM+1).$$

Then for $|\lambda| < \frac{\|A_0^{-1}\|^{-1}}{2(1+M)^{m-1}(Mm+1)} < R$ we have

$$\begin{aligned}\|G(\lambda)x\| = \|A_0x + H(\lambda)x\| &\geqslant \left\|A_0^{-1}\right\|^{-1} - \|H(\lambda)x\| \\ &\geqslant \left\|A_0^{-1}\right\|^{-1} - \frac{|\lambda|}{1+M}\max_{|\lambda|\leqslant 1+M}\|H(\lambda)x\| \\ &\geqslant \left\|A_0^{-1}\right\|^{-1} - 2(1+M)^{m-1}(mM+1)|\lambda| > 0,\end{aligned}$$

a contradiction. Therefore

$$\frac{\left\|A_0^{-1}\right\|^{-1}}{2(1+M)^{m-1}(Mm+1)} \leqslant |\lambda|. \quad \square$$

If we do not wish to look for a root in the interval $(0,1)$ of the equation $x = 1 - \frac{1}{(Mx+1)^m}$, we use the following upper bound.

COROLLARY 3.5.1. *Let $P(z) = A_0 + A_1 z + \cdots + Iz^m$ be a monic matrix polynomial. Then each eigenvalue λ of $P(z)$ satisfies*

$$\frac{\left\|A_0^{-1}\right\|^{-1}}{2(1+M)^{m-1}(Mm+1)} \leqslant |\lambda| < 1 + \Big(1 - \frac{1}{(1+M)^m}\Big)M.$$

Proof. The proof follows from Theorem 3.5 and the fact that for a root λ_0 of the equation $x = 1 - \frac{1}{(Mx+1)^m}$ in the interval $(0,1)$, we have always $\lambda_0 < 1 - \dfrac{1}{(1+M)^m}$. $\square$

Next we give some other bounds for the magnitude of eigenvalues of monic matrix polynomials.

THEOREM 3.6. *Let $P(z) = A_0 + A_1 z + \cdots + Iz^m$ be a monic matrix polynomial. Denote*

$$M := \max_{i=0,\cdots,m-1} \|A_i\|, \quad M' := \max_{i=1,\cdots,m} \|A_i\|.$$

Then each eigenvalue λ of $P(z)$ satisfies

$$\frac{\left\|A_0^{-1}\right\|^{-1}}{\left\|A_0^{-1}\right\|^{-1} + M'} < |\lambda| < 1 + M.$$

In particular, for $n = 1$ we obtain [13, Theorem 2.2].

Proof. The upper bound is the one obtained in Theorem 3.3. Now we prove the lower bound.

Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ. If $|\lambda| \leqslant \frac{\left\|A_0^{-1}\right\|^{-1}}{\left\|A_0^{-1}\right\|^{-1} + M'}$, we have

$$\begin{aligned} \|P(\lambda)x\| &\geqslant \left\|A_0^{-1}\right\|^{-1} - \sum_{i=1}^{m} |\lambda|^i \|A_i\| \geqslant \left\|A_0^{-1}\right\|^{-1} - M' \sum_{i=1}^{m} |\lambda|^i > \left\|A_0^{-1}\right\|^{-1} - M' \frac{|\lambda|}{1-|\lambda|} \\ &= \frac{\left\|A_0^{-1}\right\|^{-1}(1-|\lambda|) - M'|\lambda|}{1-|\lambda|} \geqslant 0, \quad \text{a contradiction.} \end{aligned}$$

It follows that $|\lambda| > \frac{\left\|A_0^{-1}\right\|^{-1}}{\left\|A_0^{-1}\right\|^{-1} + M'}$. This completes the proof. $\square$

More generally, we have the following bounds.

THEOREM 3.7. *Let $P(z) = A_0 + A_1 z + \cdots + Iz^m$ be a monic matrix polynomial. Let $p,q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Denote*

$$M_p := \left(\sum_{i=0}^{m-1} \|A_i\|^p \right)^{\frac{1}{p}}, \; M_p' := \left(\sum_{i=1}^{m} \|A_i\|^p \right)^{\frac{1}{p}}.$$

Then each eigenvalue λ of $P(z)$ satisfies

$$\left[\frac{\|A_0^{-1}\|^{-q}}{(M_p')^q + \|A_0^{-1}\|^{-q}} \right]^{\frac{1}{q}} < |\lambda| < (1 + M_p^q)^{\frac{1}{q}}.$$

In particular, for $n = 1$ we obtain [13, Theorem 2.4]. Moreover, letting p tend to infinity (then q tends to 1), we obtain Theorem 3.6.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of $P(z)$ and $x \in \mathbb{C}^n$ a unit eigenvector of $P(z)$ associated to λ.
If $|\lambda| \geqslant (1 + M_p^q)^{\frac{1}{q}}$, we have

$$\|P(\lambda)x\| \geqslant |\lambda|^m - \sum_{i=0}^{m-1} \|A_i\| \, |\lambda|^i \tag{6}$$

$$\geqslant |\lambda|^m - \left(\sum_{i=0}^{m-1} \|A_i\|^p \right)^{\frac{1}{p}} \left(\sum_{i=0}^{m-1} |\lambda|^{iq} \right)^{\frac{1}{q}} \tag{7}$$

$$= |\lambda|^m \left[1 - \frac{M_p}{|\lambda|^m} \left(\sum_{i=0}^{m-1} |\lambda|^{iq} \right)^{\frac{1}{q}} \right] = |\lambda|^m \left[1 - M_p \left(\sum_{i=0}^{m-1} |\lambda|^{(i-m)q} \right)^{\frac{1}{q}} \right]$$

$$> |\lambda|^m \left[1 - M_p \left(\sum_{i=1}^{\infty} |\lambda|^{-iq} \right)^{\frac{1}{q}} \right] = |\lambda|^m \left[1 - M_p \frac{1}{(|\lambda|^q - 1)^{\frac{1}{q}}} \right] \geqslant 0,$$

a contradiction.

In the lines above, from (6) to (7) we use the well-known Hölder's inequality.
It follows that $|\lambda| < (1 + M_p^q)^{\frac{1}{q}}$.

Similarly we have $|\lambda| > \left[\frac{\|A_0^{-1}\|^{-q}}{(M_p')^q + \|A_0^{-1}\|^{-q}} \right]^{\frac{1}{q}}$. This completes the proof. □

4. Numerical experiments

We have already established several estimations for eigenvalues of matrix polynomials. It is in general not possible to compare the sharpness of these bounds. We can only compare them in some special cases by numerical examples. In order to get a good comparison throughout practical examples, we use random data in each example.

Moreover, we compare the sharpness of our bounds and those given by N. J. Higham and F. Tisseur [8], J. Maroulas and P. Psarrakos [12]. We compute and compare the bounds for two cases of the matrix coefficients: One with arbitrary random matrix coefficients, and the other one with symmetric matrix coefficients. The experiments were performed using the open source software OCTAVE (version 4.4.0).

EXAMPLE 4.1. Consider a 5×5 monic matrix polynomial $P(z)$ of degree $m = 9$ whose coefficient matrices are given by

$$A_i = 10^{i-3} rand(5),\ i = 0, \ldots, 8,$$

where $rand(5)$ denotes a 5×5 random matrix from the normal $(0,1)$ distribution.

The upper bounds obtained by Higham and Tisseur [8] are given in Table 5, while our new upper bounds are given in Table 6.

Lemmas	**Values**	**Comments**
2.3 (2.2)	3.5422×10^5	∞-norm based
2.3 (2.3)	2.4987×10^5	2-norm based
2.5 (2.13)	3.3493×10^5	∞-norm based
2.6 (2.14)	3.4651×10^5	Ostrowski, $\beta = 3/4$
2.11 (2.18)	2.4907×10^5	2-norm based
3.1	2.4827×10^5	Cauchy's theorem applied for P, 2-norm
3.1	2.4827×10^5	Cauchy's theorem applied for P_U, 2-norm
4.1	4.9654×10^5	2-norm based

Table 1: Higham and Tisseur's upper bounds

Theorems/Corollaries	**Values**	**Comments**
3.2, 3.2.1, 3.3, 3.3.1	2.4827×10^5	applied for P_U, 2-norm based
3.4, 3.4.2	2.4827×10^5	2-norm based
3.4.4, 3.4.6	2.4590×10^5	2-norm based
3.6	2.4827×10^5	applied for P_U, 2-norm

Table 2: New upper bounds

The upper bound given by Maroulas and Psarrakos equals to $1 + r_2$, with $r_2 = \max\{0.0059804, 0.065468, 0.84200, 0.87573, 25.012, 322.89, 322.74, 3.0513 \times 10^4, 2.7181 \times 10^5\} = 2.7181 \times 10^5$.

We can compute the maximal modulus of the eigenvalues of $P(z)$, which is exactly $\mathbf{2.4354 \times 10^5}$. Moreover, Corollary 3.4.4 and Corollary 3.4.6 give usually the best upper bounds.

The lower bounds obtained by Higham and Tisseur [8] are given in Table 7, while our new lower bounds are given in Table 8.

Lemmas	**Values**	**Comments**
2.2	9.1316×10^{-10}	2-norm
2.3 (2.1), 2.4 (2.5)	8.0663×10^{-10}	1-norm
2.3 (2.2), 2.4 (2.6)	6.0215×10^{-10}	∞-norm
2.3 (2.3), 2.4 (2.7)	1.0456×10^{-9}	2-norm
2.6	4.2286×10^{-8}	applied for $C_L(\alpha)$, $\beta = 1/4$

Table 3: Higham and Tisseur's lower bounds

Theorems	**Values**	**Comments**
3.4.1, 3.4.3	3.53×10^{-5}	2-norm based
3.4.5	0.71005×10^{-5}	2-norm based
3.4.7	0.71034×10^{-5}	2-norm based
3.5	9.4306×10^{-55}	2-norm based
3.6	2.4502×10^{-10}	2-norm based

Table 4: New lower bounds

The lower bound given by Maroulas and Psarrakos is $r_1 = 1.1436 \times 10^{-7}$.

We can compute the minimum modulus of the eigenvalues of $P(z)$, which is exactly **0.012037**. Hence the lower bounds obtained above are in general far away the expected one. However, compare together, Corollary 3.4.1 and Corollary 3.4.3 give usually the best lower bounds.

In the next example we compute and compare the obtained bounds for eigenvalues of monic matrix polynomials whose coefficients are symmetric random matrices.

EXAMPLE 4.2. Consider a 5×5 monic matrix polynomial $P(z)$ of degree $m = 9$ whose coefficient matrices are given by

$$A_i = (B_i + B_i^*)/2, \ i = 0, \ldots, 8,$$

where $B_0 = B_1 = rand(5)$, and $B_j = j * rand(5)$ for $j = 2, \ldots, 8$. Here $rand(5)$ denotes a 5×5 random matrix from the normal $(0,1)$ distribution.

The upper bounds obtained by Higham and Tisseur [8] are given in Table 5, while our new upper bounds are given in Table 6.

The upper bound given by Maroulas and Psarrakos equals to $1 + r_2$, with $r_2 = \max\{1.9326, 1.3831, 4.3089, 6.3952, 5.5836, 8.3366, 19.922, 7.6824, 6.9319\}$ $= 19.922$.

Lemmas	Values	Comments
2.3 (2.2)	99.050	∞-norm based
2.3 (2.3)	35.335	2-norm based
2.5 (2.13)	31.030	∞-norm based
2.11 (2.18)	28.504	2-norm based
3.1	20.62502	Cauchy's theorem applied for P, 2-norm
3.1	20.62502	Cauchy's theorem applied for P_U, 2-norm
4.1	39.542	2-norm based

Table 5: Higham and Tisseur's upper bounds

Theorems/Corollaries	Values	Comments
3.2	20.77125	2-norm based
3.2.1	19.77125	2-norm based
3.3	20.771	2-norm based
3.3.1	19.280	2-norm based
3.4	20.632	2-norm based
3.4.2	19.455	2-norm based
3.4.4	19.892	2-norm based
3.4.6	20.697	2-norm based
3.6	20.771	applied for P_U, 2-norm

Table 6: New upper bounds

We can compute the maximal modulus of the eigenvalues of $P(z)$, which is exactly $\mathbf{19.009}$. Moreover, Corollary 3.3.1 gives usually the best upper bounds.

The lower bounds obtained by Higham and Tisseur [8] are given in Table 7, while our new lower bounds are given in Table 8.

Lemmas	Values	Comments
2.2	0.0027435	2-norm
2.3 (2.1), 2.4 (2.5)	0.0084515	1-norm
2.3 (2.2), 2.4 (2.6)	0.0019335	∞-norm
2.3 (2.3), 2.4 (2.7)	0.0067592	2-norm
2.6	0.0093639	applied for $C_L(\alpha)$, $\beta = 1/4$

Table 7: Higham and Tisseur's lower bounds

The lower bound given by Maroulas and Psarrakos is $r_1 = 1.3606$.

Theorems	Values	Comments
3.4.1	0.073516	2-norm based
3.4.3	0.068055	2-norm based
3.4.5	0.058889	2-norm based
3.5	3.9543×10^{-15}	2-norm based
3.6	0.0024740	2-norm based

Table 8: New lower bounds

We can compute the minimum modulus of the eigenvalues of $P(z)$, which is exactly **0.15023**. Compare together, Corollary 3.4.1 and Corollary 3.4.3 give usually the best lower bounds.

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