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THE MATRIX TODA EQUATIONS FOR COEFFICIENTS OF A MATRIX THREE–TERM RECURRENCE RELATION

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Abstract. For $q\times q$ positive measures of the form $e^{-xt}\sigma(dx)$ on $[0,\infty)$ with respect to x and $t\geqslant 0$, we derive the matrix Toda equations for the three-term recurrence relation coefficients of the corresponding orthogonal matrix polynomials. Additionally, relations for the matrix version of the Volterra lattice and associated orthogonal polynomials are attained.

1. Introduction

The transformed scalar Toda lattice

$$\dot{\alpha}_n=\lambda_{n+1}-\lambda_n,\quad \dot{\lambda}_{n+1}=\lambda_{n+1}(\alpha_{n+1}-\alpha_n),\quad n=1,2,\ldots, \tag{1.1}$$

where $\alpha_n=\alpha_n(t)$ and $\lambda_n=\lambda_n(t)$ are the coefficients of the three-term recurrence relation

$$p_n(z,t)=(z-\alpha_n(t))p_{n-1}(z,t)-\lambda_n(t)p_{n-2}(z,t),\quad n=1,2,\ldots$$

with $p_0:=1$ and $p_{-1}:=0$, was considered in [30], [3], [35], [36], [34], [40], [38], [41], [4] and references therein. In these works, the complete integrability, the relations to orthogonal polynomials, inverse problems and the physical application of the Toda lattice are discussed. Here and in the sequel, the overdot denotes the derivative with respect to t.

The operator and matrix version of the Toda lattice was considered by Berezanskii and Gekhtman in [6] under the assumption that the Lax equation is satisfied. In [33], a perturbation of a measure similar to that considered in the present work but defined on the full line is studied. In [1], non-Abelian $2D$ Toda hierarchies via orthogonal matrix polynomials are studied. In [10], the bidimensional Toda lattice related to matrix coefficients of three-term relations is discussed. The Toda system for a certain Laguerre-type perturbation with the help of orthogonal matrix polynomials is treated in [11].

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In this work, we derive the *matrix* version of the Toda equations (1.1) for the $q\times q$ matrix coefficients $A_{r,j}(t)$, $B_{r,j}(t)$ of the relation

$$P_{r,j+1}(z,t) =(zI_q - A_{r,j}(t))P_{r,j}(z,t) - B^*_{r,j-1}(t)P_{r,j-1}(z,t), \quad j \geqslant 1, \tag{1.2}$$

$$P_{r,0}(z,t) =I_q, \quad P_{r,1}(z,t) = zI_q - A_{r,0}(t) \tag{1.3}$$

for $r=1,2$ and $t\in[0,+\infty)$. The $q\times q$ matrices $A_{r,j}(t)$, $B_{r,j}(t)$, $P_{r,j}(z,t)$ are constructed via the sequence of moments $(s_j(t))_{j=0}^{\infty}$ with

$$s_j(t) := \int\limits_{[0,\infty)} x^j e^{-tx}\sigma(dx) \tag{1.4}$$

and $t\in[0,+\infty)$, which are well-defined for $j\geqslant 0$. Here I_q denotes the $q\times q$ identity matrix, and A^* stands for the conjugate transpose of A. We assume that σ is a matrix-valued positive measure on $[0,\infty)$. Furthermore, in the sequel we assume that the Hankel block matrices

$$H_{1,j}(t) := \begin{pmatrix} s_0(t) & s_1(t) & \dots & s_j(t) \\ s_1(t) & s_2(t) & \dots & s_{j+1}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_j(t) & s_{j+1}(t) & \dots & s_{2j}(t) \end{pmatrix}, H_{2,j}(t) := \begin{pmatrix} s_1(t) & s_2(t) & \dots & s_{j+1}(t) \\ s_2(t) & s_3(t) & \dots & s_{j+1}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+1}(t) & s_{j+1}(t) & \dots & s_{2j+1}(t) \end{pmatrix} \tag{1.5}$$

are both positive definite for $j\geqslant 0$ and $t\in[0,+\infty)$. An accurate definition of the matrices $A_{r,j}(t)$, $B_{r,j}(t)$, and $P_{r,j}(x,t)$ will be given in Definitions 2.3 and 2.2, respectively. A proof of the three-term relation (1.2) is given without using the orthogonality properties of the matrix polynomials $P_{r,j}$ (as in [33, Proposition 3] and [19, Proposition 3.7]). We employ certain identities instead. See Section 3.

Note that $s_0(t)$ is the Laplace transform of the measure $\sigma(x)$; such a transform was studied in [7, Chapter II] by Christian Berg, Jens Christensen and Paul Ressel. See also [21], [39] and references therein.

Additionally, the matrix version of the Volterra lattice is proved in Proposition 5.1. The scalar version of the Volterra lattice is given by the relation [35, Equation (1.5)]

$$\dot{\lambda}_n = \lambda_n(\lambda_{n+1} - \lambda_{n-1}), \quad n = 2,3,\dots. \tag{1.6}$$

Moreover, we prove some relations that involve the derivative of the associated matrix polynomials of order k, where k is a nonnegative integer. See Section 6. The scalar version of these polynomials is defined by the recurrence relation [36, Equation (1.3)]:

$$p_n^{(k)}(z,t) = (z-\alpha_{n+k}(t))p_{n-1}^{(k)}(z,t) - \lambda_{n+k}(t)p_{n-2}^{(k)}(z,t), \quad n=1,2,\dots,$$

with $p_0^{(k)} := 1$ and $p_{-1}^{(k)} := 0$.

It is well-known that if the the matrices $H_{1,j}(t)$ and $H_{2,j}(t)$ are nonnegative definite matrices for $j\geqslant 0$ there is a solution to the Stieltjes matrix moment problem:

Given a sequence $(s_j(t))_{j\geqslant 0}$ of $q\times q$ matrices, find the set $\mathscr{M}_t$ of positive measures $e^{-tx}\sigma(dx)$ for x belonging to $[0,+\infty)$ and $t\in[0,+\infty)$ such that (1.4) holds for $j\geqslant 0$. Let $e^{-tx}\sigma(dx)\in\mathscr{M}_t$. The function

$$s(z,t) := \int\limits_{[0,+\infty)} \frac{e^{-xt}\sigma(dx)}{x-z}, \quad z\in\mathbb{C}\setminus[0,+\infty)$$

is called the Stieltjes transform of $e^{-tx}\sigma(dx)$. The asymptotic relation between the Stieltjes transform $s(z,t)$ and the moments $s_j(t)$ near the point $z=+\infty$ reads

$$s(z,t) = -\frac{s_0(t)}{z} - \frac{s_1(t)}{z^2} - \ldots - \frac{s_j(t)}{z^{j+1}} - \ldots. \tag{1.7}$$

From (1.4) for $[0,+\infty)$, we get the following obvious equality:

$$\dot{s}_j(t) = -s_{j+1}(t). \tag{1.8}$$

On other hand, by employing (1.7) and (1.8) we attain the following relation:

$$\dot{s}(z,t) = -s_0(t) - zs(z,t).$$

Further properties of the Stieltjes transform $s(z,t)$ will be studied elsewhere.

Throughout the work, we use some results and notations from [13]; in particular, we repeatedly employ the Schur complements $\widehat{H}_{r,j}$ (2.5), (2.6), as well as certain block partitions of the Hankel block matrices $H_{r,j}$ and their inverses. The main results of the present work are principally based on explicit algebraic and differential identities between the mentioned parts. See Section 3. Similar identities, in the frame of the Potapov approach [37], appeared in a number of works on matrix interpolation problems, which include matrix moment problems on the real axis and orthogonal matrix polynomials as well as orthogonal matrix functions. The Potapov approach consists of reducing a general interpolation problem (which includes the moment problem) into certain matrix inequalities for analytic functions that are solved with the help of the construction of the resolvent matrix, also known as the Nevanlinna matrix. See [31], [27], [20], [28], [15] and references therein.

The present work differs from previous works concerning the matrix Toda equations [6], [33], [1] as follows: Firstly, the perturbed measure σ is defined on $[0,+\infty)$ instead of on all real axis. Secondly, we decisively make use of the Schur complements $\widehat{H}_{r,j}$ to handle the coefficients $A_{r,j}$ and $B_{r,j}$, and we do not employ their integral representations. Thirdly, we do not use the orthogonality properties of polynomials $P_{r,j}$.

Future work can be devoted to the study of Lax pairs corresponding to the matrix Toda sequence; see Definition 4.1. Our motivation is to follow the papers [14], [17] and [16] in order to attain applications of the polynomials $P_{r,j}(x,t)$ to the control theory of systems described by differential equations and to the problem of the stability of polynomials with interval coefficients.

2. Preliminaries and notations

Throughout this paper, let q and p be positive integers. We will use $\mathbb{C}$, $\mathbb{N}_0$ and $\mathbb{N}$ to denote the set of all complex numbers, the set of all nonnegative integers and the set of all positive integers, respectively. The notation $\mathbb{C}^{q\times q}$ stands for the set of all complex $q\times q$ matrices. For the null matrix that belongs to $\mathbb{C}^{p\times q}$, we will write $0_{p\times q}$. We denote by 0_q the null matrix in $\mathbb{C}^{q\times q}$, respectively. In cases where the sizes of the null and the identity matrix are clear, we will omit the indices.

Let

$$y_{[j,k]}(t) := \text{column}\,(s_j(t), s_{j+1}(t), \ldots, s_k(t)), \quad 0 \leqslant j \leqslant k$$

and $y_{[j,k]}(t) = 0_q$, if $j > k$,

$$u_{1,0}(t) := 0_q,\, u_{1,j}(t) := \begin{pmatrix} 0_q \\ -y_{[0,j-1]}(t) \end{pmatrix}, \quad u_{2,j}(t) := -y_{[0,j]}(t), \quad j \geqslant 1. \tag{2.1}$$

Consider

$$Y_{1,j}(t) := y_{[j,2j-1]}(t), \quad Y_{2,j}(t) := y_{[j+1,2j]}(t), \qquad j \geqslant 1. \tag{2.2}$$

Let $R_j : \mathbb{C} \to \mathbb{C}^{(j+1)q\times(j+1)q}$ be given by

$$R_j(z) := (I_{(j+1)q} - zT_j)^{-1}, \qquad j \geqslant 0, \tag{2.3}$$

with

$$T_0 := 0_q, \;\; T_j := \begin{pmatrix} 0_{q\times jq} & 0_q \\ I_{jq} & 0_{jq\times q} \end{pmatrix}, \qquad j \geqslant 1.$$

Observe that for each $j \in \mathbb{N}_0$ the matrix–valued function R_j can be represented via

$$R_j(z) = \begin{pmatrix} I_q & 0_q & 0_q & \ldots & 0_q & 0_q \\ zI_q & I_q & 0_q & \ldots & 0_q & 0_q \\ z^2 I_q & zI_q & I_q & \ldots & 0_q & 0_q \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ z^j I_q & z^{j-1} I_q & z^{j-2} I_q & \ldots & zI_q & I_q \end{pmatrix}.$$

Furthermore, let

$$v_0 := I_q, \quad v_j := \begin{pmatrix} I_q \\ 0_{jq\times q} \end{pmatrix} = \begin{pmatrix} v_{j-1} \\ 0_q \end{pmatrix} \tag{2.4}$$

for positive integers j.

DEFINITION 2.1. A sequence of matrix moments $(s_j(t))_{j=0}^{\infty}$ is called a Stieltjes positive definite sequence if the Hankel block matrices $H_{1,j}(t)$ and $H_{2,j}(t)$ defined as in (1.5) are positive definite for $j \geqslant 0$ and $t \in [0,+\infty)$.

In the sequel, we consider only Stieltjes positive definite sequences.

For $t \in [0,+\infty)$, let $\widehat{H}_{1,j}$ (resp. $\widehat{H}_{2,j}$) denote the Schur complement of the block $H_{1,j-1}$ in $H_{1,j}$ (resp. of the block $H_{2,j-1}$ in $H_{2,j}$):

$$\widehat{H}_{1,0}(t) := s_0(t), \;\; \widehat{H}_{1,j}(t) := s_{2j}(t) - Y_{1,j}^*(t)H_{1,j-1}^{-1}(t)Y_{1,j}(t), \;\; j \geqslant 1, \tag{2.5}$$

$$\widehat{H}_{2,0}(t) := s_1(t), \;\; \widehat{H}_{2,j}(t) := s_{2j+1}(t) - Y_{2,j}^*(t)H_{2,j-1}^{-1}(t)Y_{2,j}(t), \;\; j \geqslant 1. \tag{2.6}$$

These matrices are positive definite matrices, as well as the matrices $H_{1,j}$ (resp. $H_{2,j}$). In the scalar case, the matrices $\widehat{H}_{1,j}$ and $\widehat{H}_{1,j}$ have the form

$$\widehat{H}_{1,j} = \frac{|H_{1,j}|}{|H_{1,j-1}|}, \qquad \widehat{H}_{2,j} = \frac{|H_{2,j}|}{|H_{2,j-1}|}, \tag{2.7}$$

where $|\widehat{H}_{r,j}|$ denotes the determinant of $\widehat{H}_{r,j}$. Equality (2.7) is readily proved by calculating the determinant of the Schur complement $\widehat{H}_{r,j}$. See [9, Proposition 8.2.3].

Usually we will omit the dependence of t in s_j, $H_{r,j}$, $Y_{r,j}$ and $u_{r,j}$ for $r = 1,2$. The matrix polynomials of the following definition were considered in [13, Definition 4.1] for $t = 0$.

DEFINITION 2.2. For $t \in [0,+\infty)$, let $(s_j(t))_{j=0}^{\infty}$ be an infinite Stieltjes positive definite sequence as in (1.4), and denote

$$P_{1,0}(z,t) := I_q, \;\; Q_{1,0}(z,t) := 0_q, \;\; P_{2,0}(z,t) := I_q, \;\; Q_{2,0}(z,t) := s_0. \tag{2.8}$$

For $j \geqslant 1$, let

$$P_{1,j}(z,t) := (-Y_{1,j}^* H_{1,j-1}^{-1}, I_q)R_j(z)v_j, \tag{2.9}$$

$$P_{2,j}(z,t) := (-Y_{2,j}^* H_{2,j-1}^{-1}, I_q)R_j(z)v_j, \tag{2.10}$$

$$Q_{1,j}(z,t) := -(-Y_{1,j}^* H_{1,j-1}^{-1}, I_q)R_j(z)u_{1,j} \tag{2.11}$$

and

$$Q_{2,j}(z,t) := -(-Y_{2,j}^* H_{2,j-1}^{-1}, I_q)R_j(z)u_{2,j}. \tag{2.12}$$

Note that the matrix polynomials $P_{r,j}$ and $Q_{r,j}$ satisfy the following orthogonality and integral properties.

REMARK 2.1. For $t \in [0,+\infty)$, let $(s_j(t))_{j\geqslant 0}$ be a Stieltjes positive definite sequence related to a positive measure $e^{-xt}\sigma(dx)$ as in (1.4). Let $P_{r,j}$ and $Q_{r,j}$ be as in Definition 2.2. Thus, the following equalities hold:

$$\int_{[0,+\infty)} e^{-tx}P_{1,j}(t,x)\sigma(dx)P_{1,k}^*(t,x) = \begin{cases} \widehat{H}_{1,j}(t), & \text{if } j = k, \\ 0_q, & \text{if } j \neq k, \end{cases}$$

$$\int_{[0,+\infty)} xe^{-tx}P_{2,j}(t,x)\sigma(dx)P_{2,k}^*(t,x) = \begin{cases} \widehat{H}_{2,j}(t), & \text{if } j = k, \\ 0_q, & \text{if } j \neq k, \end{cases}$$

for $t \in [0,+\infty)$ and $j,k \in \mathbb{N}_0$. Moreover,

$$Q_{1,j}(t,x) = \int\limits_{[0,+\infty)} \frac{1}{x-\tau}(P_{1,j}(t,x) - P_{1,j}(t,\tau))e^{-t\tau}\sigma(d\tau),$$

$$Q_{2,j}(t,x) = \int\limits_{[0,+\infty)} \frac{1}{x-\tau}(xP_{2,j}(t,x) - \tau P_{2,j}(t,\tau))e^{-t\tau}\tau\sigma(d\tau),$$

for $t \in [0,+\infty)$ and $j \in \mathbb{N}_0$.

The proof of these equalities can be verified by direct calculations as in [13, Remark D.6] and [13, Remark E.4] where the case $t = 0$ was proven.

DEFINITION 2.3. Let $\widehat{H}_{1,j}$ and $\widehat{H}_{2,j}$ be as in (2.5) and (2.6), respectively. For $t \in [0,+\infty)$, define

$$A_{1,0}(t) := \widehat{H}_{2,0}(t)\widehat{H}_{1,0}^{-1}(t), \tag{2.13}$$

$$A_{1,j}(t) := \widehat{H}_{2,j}(t)\widehat{H}_{1,j}^{-1}(t) + \widehat{H}_{1,j}(t)\widehat{H}_{2,j-1}^{-1}(t), \ j \geqslant 1, \tag{2.14}$$

$$A_{2,j}(t) := \widehat{H}_{1,j+1}(t)\widehat{H}_{2,j}^{-1}(t) + \widehat{H}_{2,j}(t)\widehat{H}_{1,j}^{-1}(t), \ j \geqslant 0. \tag{2.15}$$

For $r = 1,2$, $j \geqslant 0$ and $t \in [0,+\infty)$, denote

$$B_{r,j}(t) := \widehat{H}_{r,j}^{-1}(t)\widehat{H}_{r,j+1}(t). \tag{2.16}$$

In [13, Theorem 9.3(b)], the matrices $A_{r,j}$ and $B_{r,j}$ are obtained via the so-called Dyukarev-Stieltjes parameters [13, Definition 2.3].

In the scalar case, the matrices $A_{r,j}$ and $B_{r,j}$ have the form $A_{1,0} = \frac{s_1}{s_0}$, $A_{2,0} = \frac{s_2}{s_1}$:

$$A_{1,1} = \frac{|H_{2,1}|H_{1,0}}{H_{2,0}|H_{1,1}|} + \frac{|H_{1,1}|}{H_{1,0}H_{2,0}},$$

$$A_{1,j} = \frac{|H_{2,j}||H_{1,j-1}|}{|H_{2,j-1}||H_{1,j}|} + \frac{|H_{1,j}||H_{2,j-2}|}{|H_{1,j-1}||H_{2,j-1}|},$$

$$A_{2,j} = \frac{|H_{1,j+1}||H_{2,j-1}|}{|H_{1,j}||H_{2,j}|} + \frac{|H_{2,j}||H_{1,j-1}|}{|H_{2,j-1}||H_{1,j}|},$$

for $j \geqslant 1$. Furthermore, $B_{1,0} = \frac{|H_{1,1}|}{s_0^2}$, $B_{2,0} = \frac{|H_{2,1}|}{s_1^2}$ and $B_{r,j} = \frac{|H_{r,j-1}||H_{r,j+1}|}{|H_{r,j}|^2}$, for $j \geqslant 1$.

In Subsection 5.1, we prove the following proposition using identities of Section 3 without employing the orthogonality condition.

PROPOSITION 2.1. *Let $P_{r,j}$ for $r = 1,2$ be as in Definition 2.2, and let $A_{r,j}$, $B_{r,j}$ be as in Definition 2.3. The polynomials $P_{r,j}$, $r = 1,2$ satisfy the recurrence relation (1.2) and (1.3).*

Observe that relation (1.2) was proved in [19] with the help of the orthogonality condition. See also [29].

3. Algebraic and differential identities

In this section, we introduce relevant algebraic and differential identities, which will allow proving the main results of the present work. For each positive integer n, let

$$L_{1,n} := \left(\delta_{j,k+1} I_q\right)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}} \quad \text{and} \quad L_{2,n} := \left(\delta_{j,k} I_q\right)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}}, \tag{3.1}$$

where $\delta_{j,k}$ is the Kronecker symbol: $\delta_{j,k} := 1$ if $j = k$ and $\delta_{j,k} := 0$ if $j \neq k$.

For $j \geqslant 1$, let

$$\Sigma_{r,j} := \begin{pmatrix} -H_{r,j-1}^{-1} Y_{r,j} \\ I_q \end{pmatrix}, \quad r = 1,2, \tag{3.2}$$

$$Y_{0,j} := y_{[1,j]}, \quad Y_{3,j} := y_{[j+2,2j+1]}, \tag{3.3}$$

and

$$H_{3,j} := \begin{pmatrix} s_2 & s_3 & \dots & s_{j+2} \\ s_3 & s_4 & \dots & s_{j+3} \\ \vdots & \vdots & \vdots & \vdots \\ s_{j+2} & s_{j+1} & \dots & s_{2j+2} \end{pmatrix}, \quad j \geqslant 0. \tag{3.4}$$

PROPOSITION 3.1. *For $r = 1,2$, let $H_{r,j}$, $Y_{0,j}$, $Y_{3,j}$, v_j, $L_{1,j}$, $Y_{r,j}$, $\widehat{H}_{r,j}$ and $\Sigma_{r,j}$ be as in (1.5), (3.3), (2.4), (3.1), (2.2), (2.5), (2.6), (3.2) and (3.4), respectively. The following identities then hold:*

$$H_{2,j}^{-1} Y_{0,j+1} = v_j, \tag{3.5}$$

$$H_{1,j}^{-1} H_{2,j} = (L_{1,j}, H_{1,j}^{-1} Y_{1,j+1}), \tag{3.6}$$

$$H_{2,j}^{-1} H_{3,j} = (L_{1,j}, H_{2,j}^{-1} Y_{2,j+1}), \tag{3.7}$$

$$Y_{1,j+1}^* H_{1,j}^{-1} H_{2,j} - Y_{2,j+1}^* = -(0_q, \dots, 0_q, \widehat{H}_{1,j+1}), \tag{3.8}$$

$$-Y_{1,j}^* H_{1,j-1}^{-1} Y_{2,j} + s_{2j+1} = (0_q, \dots, 0_q, \widehat{H}_{1,j}) H_{2,j-1}^{-1} Y_{2,j} + \widehat{H}_{2,j}, \tag{3.9}$$

$$(0_q, -Y_{2,j}^* H_{2,j-1}^{-1}) H_{1,j} + Y_{1,j+1}^* = (0_q, \dots, 0_q, \widehat{H}_{2,j}), \tag{3.10}$$

$$(0_q, \dots, 0_q, \widehat{H}_{1,j}) H_{1,j-1}^{-1} = \widehat{H}_{1,j} \widehat{H}_{1,j-1}^{-1} (-Y_{1,j-1}^* H_{1,j-2}^{-1}, I_q), \tag{3.11}$$

$$(0_q, \dots, 0_q, \widehat{H}_{2,j}) H_{2,j-1}^{-1} = \widehat{H}_{2,j} \widehat{H}_{2,j-1}^{-1} (-Y_{2,j-1}^* H_{2,j-2}^{-1}, I_q), \tag{3.12}$$

$$\Sigma_{1,j}^* - \Sigma_{2,j-1}^* L_{1,j}^* + \widehat{H}_{2,j-1} \widehat{H}_{1,j-1}^{-1} \Sigma_{1,j-1}^* L_{2,j}^* = 0_{q \times jq}, \tag{3.13}$$

$$\Sigma_{2,j}^* - \Sigma_{1,j}^* + \widehat{H}_{1,j} \widehat{H}_{2,j-1}^{-1} \Sigma_{2,j-1}^* L_{2,j}^* = 0_{q \times jq}, \tag{3.14}$$

$$\Sigma_{1,j}^* H_{2,j} \Sigma_{1,j} = \widehat{H}_{1,j} \widehat{H}_{2,j-1}^{-1} \widehat{H}_{1,j} + \widehat{H}_{2,j}, \tag{3.15}$$

$$\begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1} Y_{2,j} \\ I_q \end{pmatrix}^* H_{1,j+1} \begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1} Y_{2,j} \\ I_q \end{pmatrix} = \Sigma_{2,j}^* H_{3,j} \Sigma_{2,j}, \tag{3.16}$$

$$\Sigma_{2,j}^* H_{3,j} \Sigma_{2,j} = \widehat{H}_{1,j+1} + \widehat{H}_{2,j} \widehat{H}_{1,j}^{-1} \widehat{H}_{2,j}. \tag{3.17}$$

Proof. Equalities (3.5) through (3.7) are readily verified. To prove (3.8), we use (3.6) and (2.5). We have

$$\begin{aligned}Y^*_{1,j+1}H^{-1}_{1,j}H_{2,j}-Y^*_{2,j+1}&=Y^*_{1,j+1}(L_{1,j},H^{-1}_{1,j}Y_{1,j+1})-Y^*_{2,j+1}\\&=(Y^*_{1,j+1}L_{1,j},Y^*_{1,j+1}H^{-1}_{1,j}Y_{1,j+1})-Y^*_{2,j+1}\\&=(0_q,\dots,0_q,Y^*_{1,j+1}H^{-1}_{1,j}Y_{1,j+1})-Y^*_{2,j+1}=-(0_q,\dots,0_q,\widehat{H}_{1,j+1}).\end{aligned}$$

Equality (3.9) is verified by employing (2.6) and (3.8). To verify (3.10), we use the equality

$$H_{1,j}=\begin{pmatrix} s_0 & Y^*_{0,j}\\ Y_{0,j} & H_{3,j-1}\end{pmatrix}, \tag{3.18}$$

as well as (3.5), (3.7) and (2.6). We have

$$\begin{aligned}(0_q,-Y^*_{2,j}H^{-1}_{2,j-1})H_{1,j}+Y^*_{1,j+1}=&(-Y^*_{2,j}H^{-1}_{2,j-1}Y_{0,j},\,-Y^*_{2,j}H^{-1}_{2,j}H_{3,j-1}+Y^*_{1,j+1}\\=&(-Y^*_{2,j}v_{j-1},-Y^*_{2,j}L_{1,j},-Y^*_{2,j}H^{-1}_{2,j-1}Y_{2,j})+Y^*_{1,j+1}\\=&(0_q,\dots,0_q,\widehat{H}_{2,j}).\end{aligned}$$

To prove (3.11) and (3.12), one employs the equality

$$H^{-1}_{r,j}=\begin{pmatrix} H^{-1}_{r,j-1} & 0_{jq\times q}\\ 0_{q\times jq} & 0_{q\times q}\end{pmatrix}+\begin{pmatrix} -H^{-1}_{r,j-1}Y_{r,j}\\ I_q\end{pmatrix}\widehat{H}^{-1}_{r,j}(-Y^*_{r,j}H^{-1}_{r,j-1},I_q). \tag{3.19}$$

This equality is valid for $r=1,2$ because $H_{1,j}$ and $H_{2,j}$ are positive definite matrices. To prove (3.13), we use (3.2), (3.1), (3.18), (3.10) and (3.11). We have

$$\begin{aligned}&\Sigma^*_{1,j}-\Sigma^*_{2,j-1}L^*_{1,j}+\widehat{H}_{2,j-1}\widehat{H}^{-1}_{1,j-1}\Sigma^*_{1,j-1}L^*_{2,j}\\&=(-Y^*_{1,j}H^{-1}_{1,j-1},I_q)-(0_q,-Y^*_{2,j-1}H^{-1}_{2,j-2},I_q)+\widehat{H}_{2,j-1}\widehat{H}^{-1}_{1,j-1}(-Y^*_{1,j-1}H^{-1}_{1,j-2},I_q,0_q)\\&=\left(\left(-Y^*_{1,j}+(0,-Y^*_{2,j-1}H^{-1}_{2,j-1})\begin{pmatrix} s_0 & Y^*_{0,j-1}\\ Y_{0,j-1} & H_{3,j-2}\end{pmatrix}\right)H^{-1}_{1,j-1},0_q\right)\\&\quad+\widehat{H}_{2,j-1}\widehat{H}^{-1}_{1,j-1}(-Y^*_{1,j-1}H^{-1}_{1,j-2},I_q,0_q)\\&=((-Y^*_{1,j}-(-Y_{2,j-1}v_{j-2},-Y^*_{2,j-1}L_{1,j-2},-Y^*_{2,j-1}H_{2,j-2}Y_{2,j-1}))H^{-1}_{1,j-1},0_q)\\&\quad+\widehat{H}_{2,j-1}\widehat{H}^{-1}_{1,j-1}(-Y^*_{1,j-1}H^{-1}_{1,j-2},I_q,0_q)\\&=-((0_q,\dots,0_q,\widehat{H}_{2,j-1})H^{-1}_{1,j-1},0_q)+\widehat{H}_{2,j-1}\widehat{H}^{-1}_{1,j-1}(-Y^*_{1,j-1}H^{-1}_{1,j-2},I_q,0_q)=0_{q\times jq}.\end{aligned}$$

Now we prove (3.14). We use (3.2), (3.1), (3.8) and (3.19) for $r=2$:

$$\begin{aligned}&\Sigma^*_{2,j}-\Sigma^*_{1,j}+\widehat{H}_{1,j}\widehat{H}^{-1}_{2,j-1}\Sigma^*_{2,j-1}L^*_{2,j}\\&=(-Y^*_{2,j}H^{-1}_{2,j-1}+Y^*_{1,j}H^{-1}_{1,j-1},0_q)+\widehat{H}_{1,j}\widehat{H}^{-1}_{2,j-1}(-Y^*_{2,j-1}H^{-1}_{2,j-2},I_q,0_q)\\&=-((0_q,\dots,0_q,\widehat{H}_{1,j})H^{-1}_{2,j-1},0_q)+\widehat{H}_{1,j}\widehat{H}^{-1}_{2,j-1}(-Y^*_{2,j-1}H^{-1}_{2,j-2},I_q,0_q)=0_{q\times jq}.\end{aligned}$$

We prove (3.15). By using (3.2) for $r=1$ and equality $H_{2,j}=\begin{pmatrix} H_{2,j-1} & Y_{2,j} \\ Y_{2,j}^* & s_{2j+1}\end{pmatrix}$, we have

$$\begin{aligned}\Sigma_{1,j}^* H_{2,j}\Sigma_{1,j} =&(-Y_{1,j}^* H_{1,j-1}^{-1}, I_q)\begin{pmatrix} H_{2,j-1} & Y_{2,j} \\ Y_{2,j}^* & s_{2j+1}\end{pmatrix}\begin{pmatrix} -H_{1,j-1}^{-1}Y_{1,j} \\ I_q\end{pmatrix}\\ =&(Y_{1,j}^* H_{1,j-1}^{-1} H_{2,j-1}-Y_{2,j}^*)H_{1,j-1}^{-1}Y_{1,j}-Y_{1,j}^*H_{1,j-1}^{-1}Y_{2,j}+s_{2j+1} \\ =&-(0_q,\dots,0_q,\widehat{H}_{1,j})(H_{1,j-1}^{-1}Y_{1,j}-H_{2,j-1}^{-1}Y_{2,j})+\widehat{H}_{2,j}\\ =&\widehat{H}_{1,j}\widehat{H}_{2,j-1}^{-1}\widehat{H}_{1,j}+\widehat{H}_{2,j}.\end{aligned} \tag{3.20}$$

In this chain of equalities, in the third equality we have added and subtracted the matrix $Y_{2,j}^* H_{2,j-1}^{-1} Y_{2,j}$. Furthermore, we used (3.8) and (2.6). Equality (3.16) follows by using the equality

$$H_{1,j+1}=\begin{pmatrix} H_{1,j} & Y_{1,j+1} \\ Y_{1,j+1}^* & s_{2j+2}\end{pmatrix}, \tag{3.21}$$

and we have

$$\begin{aligned}&\begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1}Y_{2,j} \\ I_q\end{pmatrix}^* H_{1,j+1}\begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1}Y_{2,j} \\ I_q\end{pmatrix}\\ =&\left((0_q,-Y_{2,j}^*H_{2,j-1}^{-1})H_{1,j}+Y_{1,j+1}^*\right)\begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1}Y_{2,j}\end{pmatrix}+(0_q,-Y_{2,j}^*H_{2,j-1}^{-1})Y_{1,j+1}+s_{2j+2}\\ =&Y_{2,j}^*H_{2,j-1}^{-1}H_{3,j-1}H_{2,j-1}^{-1}Y_{2,j}-Y_{3,j}^*H_{2,j-1}^{-1}Y_{2,j}-Y_{2,j}^*H_{2,j-1}^{-1}Y_{3,j}+s_{2j+2}\\ =&\begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I_q\end{pmatrix}^* H_{3,j}\begin{pmatrix} -H_{2,j-1}^{-1}Y_{2,j} \\ I_q\end{pmatrix}=\Sigma_{2,j}^*H_{3,j}\Sigma_{2,j}.\end{aligned} \tag{3.22}$$

The second equality follows from (3.18).

Now we prove (3.17). By employing (3.15) and equality (3.21), we have

$$\begin{aligned}&\Sigma_{2,j}^*H_{3,j}\Sigma_{2,j}\\ =&\left((0_q,-Y_{2,j}^*H_{2,j-1}^{-1})H_{1,j}+Y_{1,j+1}^*\right)\begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1}Y_{2,j}\end{pmatrix}+(0_q,-Y_{2,j}^*H_{2,j-1}^{-1})Y_{1,j+1}+s_{2j+2}\\ =&\left((0_q,-Y_{2,j}^*H_{2,j-1}^{-1})H_{1,j}+Y_{1,j+1}^*\right)H_{1,j}^{-1}\left(H_{1,j}\begin{pmatrix} 0_q \\ -H_{2,j-1}^{-1}Y_{2,j}\end{pmatrix}+Y_{1,j+1}\right)+\widehat{H}_{1,j+1}\\ =&(0_q,\dots,0_q,\widehat{H}_{2,j})H_{1,j}^{-1}\begin{pmatrix} 0_q \\ \cdots \\ 0_q \\ \widehat{H}_{2,j}\end{pmatrix}+\widehat{H}_{1,j+1}=\widehat{H}_{1,j+1}+\widehat{H}_{2,j}\widehat{H}_{1,j}^{-1}\widehat{H}_{2,j}.\end{aligned}$$

In this chain of equalities, in the second equality we have added and subtracted the matrix $Y_{1,j+1}^*H_{1,j}^{-1}Y_{1,j+1}$. Furthermore, we used (2.5). In the third equality, we employed (3.10). Finally, the last equality follows from (3.19) for $r=1$. □

REMARK 3.1. In [13], for $r=1,2$ the coefficient $A_{r,j}$ is defined as

$$A_{r,j} = M_{r,j}\widehat{H}_{r,j}^{-1}, \tag{3.23}$$

where $M_{r,j} := \Sigma_{r,j}^* H_{r+1,j}\Sigma_{r,j}$. By using (3.15) and (3.17), the equivalence between (3.23) and (2.14) (resp. (2.15)) is evident.

The following remark alludes to the equivalence between the representation of coefficient $A_{1,j}$ given in (3.23) and in [33, Proposition 3].

REMARK 3.2. The coefficient $A_{1,j}$ satisfies the following equality:

$$A_{1,j} = Y_{1,j+1}^* H_{1,j}^{-1}\lambda_j - Y_{1,j}^* H_{1,j-1}^{-1}\lambda_{j-1}$$

for $j \geqslant 1$, where λ_j is a $(j+1)q\times q$ matrix equal to $\operatorname{column}(0_q,\ldots,0_q,I_q)$.

Proof. By using the following equalities $Y_{1,j+1}^* - Y_{1,j}^* H_{1,j-1}^{-1} L_{1,j}^* H_{1,j} = \Sigma_{1,j}^* H_{2,j}$, $L_{1,j}^*\lambda_j = \lambda_{j-1}$ and $H_{1,j}^{-1}\lambda_j = \Sigma_{1,j}\widehat{H}_{1,j}^{-1}$, we have

$$Y_{1,j+1}^* H_{1,j}^{-1}\lambda_j - Y_{1,j}^* H_{1,j-1}^{-1}\lambda_{j-1} = \Sigma_{1,j}^* H_{2,j}\Sigma_{1,j}\widehat{H}_{1,j}^{-1}. \quad \square$$

In the following lemma, we calculate the derivative of the Schur complement $\widehat{H}_{r,j}$. We will employ (1.8), the obvious equalities

$$\dot{H}_{1,j} = -H_{2,j}, \qquad \dot{H}_{2,j} = -H_{3,j} \tag{3.24}$$

and the equality

$$\dot{A}^{-1} = -A^{-1}\dot{A}A^{-1} \tag{3.25}$$

that is valid for every $q\times q$ invertible and differentiable matrix $A = A(t)$ for $t\in(0,\infty)$.

LEMMA 3.1. *Let $\widehat{H}_{r,j}$ for $r=1,2$ be as in (2.5) and (2.6). For $t\in(0,+\infty)$, the following equalities are valid:*

$$\dot{\widehat{H}}_{1,j} = -\widehat{H}_{2,j} - \widehat{H}_{1,j}\widehat{H}_{2,j-1}^{-1}\widehat{H}_{1,j}, \tag{3.26}$$

$$\dot{\widehat{H}}_{2,j} = -\widehat{H}_{1,j+1} - \widehat{H}_{2,j}\widehat{H}_{1,j}^{-1}\widehat{H}_{2,j}. \tag{3.27}$$

Proof. To prove (3.26), we use (1.8), the first equality of (3.24), the equality $\dot{Y}_{1,j} = -Y_{2,j}$ and (3.25). We have

$$\begin{aligned}\dot{\widehat{H}}_{1,j} =& \dot{s}_{2j} - \dot{Y}_{1,j}^* H_{1,j-1}^{-1} Y_{1,j} - Y_{1,j}^*(H_{1,j-1}^{-1})^{\cdot} Y_{1,j} - Y_{1,j}^* H_{1,j-1}^{-1}\dot{Y}_{1,j}\\ =& -(Y_{1,j}^* H_{1,j-1}^{-1} H_{2,j-1} - Y_{2,j}^*) H_{1,j-1}^{-1} Y_{1,j} + Y_{1,j}^* H_{1,j-1}^{-1} Y_{2,j} - s_{2j+1} = \Sigma_{1,j}^* H_{2,j}\Sigma_{1,j}\\ =& -\widehat{H}_{1,j}\widehat{H}_{2,j}^{-1}\widehat{H}_{1,j} - \widehat{H}_{2,j}.\end{aligned}$$

In this chain of equalities, in the third equality one uses (3.20) and (3.15).

Now we prove (3.27). By employing (1.8), the second equality of (3.24) and (3.25), we get

$$\begin{aligned}\dot{\widehat{H}}_{2,j} =&\dot{s}_{2j+1} - \dot{Y}^*_{2,j}H^{-1}_{2,j-1}Y_{2,j} - Y^*_{2,j}(H^{-1}_{2,j-1})^{\cdot}Y_{2,j} - Y^*_{2,j}H^{-1}_{2,j-1}\dot{Y}_{2,j}\\ =&-s_{2j+2} + Y^*_{3,j}H^{-1}_{2,j-1}Y_{1,j} - Y^*_{2,j}H^{-1}_{2,j-1}H_{3,j-1}H^{-1}_{2,j-1}Y_{2,j} + Y^*_{2,j}H^{-1}_{2,j-1}Y_{3,j}\\ =&-\Sigma^*_{2,j}H_{3,j}\Sigma_{2,j} = -\widehat{H}_{1,j+1} - \widehat{H}_{2,j}\widehat{H}^{-1}_{1,j}\widehat{H}_{2,j}.\end{aligned}$$

In this chain of equalities, the third equality follows from (3.22). $\square$

Finally, we calculate the derivative of $\Sigma_{r,j}$ with respect to t.

REMARK 3.3. For $r = 1,2$ and $t \in (0,+\infty)$, the following identity is valid:

$$\dot{\Sigma}_{r,j} = \widehat{H}_{r,j}\widehat{H}^{-1}_{r,j-1}(\Sigma_{r,j-1}, 0_q), \quad j \geqslant 1. \tag{3.28}$$

Proof. We use (3.2), the identity $\dot{Y}_{r,j} = -Y_{r+1,j}$, (3.25) and (3.24). We then have

$$\begin{aligned}(-Y^*_{r,j}H^{-1}_{r,j-1}, I_q)^{\cdot} =&((Y^*_{r+1,j} - Y^*_{r,j}H^{-1}_{r,j-1}H_{r+1,j-1})H^{-1}_{r,j-1}, 0_q)\\ =&\widehat{H}_{r,j}\widehat{H}^{-1}_{r,j-1}(-Y^*_{r,j}H^{-1}_{r,j-1}, I_q, 0_q). \quad \square\end{aligned}$$

Now we are ready to state and prove the main theorem of the present work.

4. Matrix Toda equation

In this section, we consider the matrix generalization of the transformed Toda lattice (1.1). In [10, Theorem 4 and Theorem 5], a similar generalization is performed for the coefficients of the three-term recurrence relation of bivariate orthogonal polynomials. These coefficients are matrices of different dimensions, and the measure that guarantees the orthogonality is a scalar measure. In [6, Example 2.2], a specific 2×2 Toda equation is considered.

The following remark readily follows from (2.14), (2.15), (3.26) and (3.27).

REMARK 4.1. Let $A_{r,j}$ and $\widehat{H}_{r,j}$ be as in (2.13) through (2.15). The following equality is then valid:

$$A_{r,j} = -\dot{\widehat{H}}_{r,j}\widehat{H}^{-1}_{r,j}, \quad r = 1,2, \quad j \geqslant 0. \tag{4.1}$$

THEOREM 4.1. *Let $A_{r,j}$ and $B_{r,j}$ be as in Definition 2.3. For $r = 1,2$, the following identities are valid:*

$$\dot{A}_{r,0} = -B^*_{r,0}, \tag{4.2}$$

$$\dot{A}_{r,j} = B^*_{r,j-1} - B^*_{r,j}, \tag{4.3}$$

$$\dot{B}^*_{r,j} = B^*_{r,j}A_{r,j} - A_{r,j+1}B^*_{r,j}, \tag{4.4}$$

for $j \geqslant 1$ and $t \in (0,+\infty)$.

Proof. We prove (4.2) for $r=1$. We have

$$\begin{aligned}\dot{A}_{1,0} =&(s_1 s_0^{-1})^{\cdot} = -(s_2 - s_1 s_0^{-1} s_1) s_0^{-1} = -\widehat{H}_{1,1}\widehat{H}_{1,0}^{-1} = -B_{1,0}^{*},\\ \dot{A}_{2,0} =&(s_2 s_0^{-1})^{\cdot} = -(s_3 - s_2 s_1^{-1} s_2) s_1^{-1} = -\widehat{H}_{2,1}\widehat{H}_{2,0}^{-1} = -B_{2,0}^{*}.\end{aligned}$$

Now we prove (4.3) for $r=1$. By employing (2.14), (3.26) and (2.16) for $r=2$, we get

$$\begin{aligned}\dot{A}_{1,j} =&\dot{\widehat{H}}_{2,j}\widehat{H}_{1,j}^{-1} - \widehat{H}_{2,j}\widehat{H}_{1,j}^{-1}\dot{\widehat{H}}_{1,j}\widehat{H}_{1,j}^{-1} + \dot{\widehat{H}}_{1,j}\widehat{H}_{2,j-1}^{-1} - \widehat{H}_{1,j}\widehat{H}_{2,j-1}^{-1}\dot{\widehat{H}}_{2,j-1}\widehat{H}_{2,j-1}^{-1}\\ =&\widehat{H}_{1,j}\widehat{H}_{1,j-1}^{-1} - \widehat{H}_{1,j+1}\widehat{H}_{1,j}^{-1} = B_{1,j-1}^{*} - B_{1,j}^{*}.\end{aligned}$$

In a similar manner for $r=2$, one can prove (4.3).

Next we prove (4.4). By using (2.16) and (4.1), we get

$$\begin{aligned}\dot{B}_{r,j}^{*} =& \dot{\widehat{H}}_{r,j+1}\widehat{H}_{r,j}^{-1} - \widehat{H}_{r,j+1}\widehat{H}_{r,j}^{-1}\dot{\widehat{H}}_{r,j}\widehat{H}_{r,j}^{-1} = -A_{r,j+1}\widehat{H}_{r,j+1}\widehat{H}_{r,j}^{-1} + \widehat{H}_{r,j+1}\widehat{H}_{r,j}^{-1}A_{r,j}\\ =& -A_{r,j+1}B_{r,j}^{*} + B_{r,j}^{*}A_{r,j}. \quad \square\end{aligned}$$

Observe that (4.3) and (4.4) for $r=1$ were proven in [33, Proposition 5] by using different identities.

DEFINITION 4.1. A sequence of matrices $(A_{r,j}, B_{r,j})_{j\geqslant 0}$ that satisfies (4.2) through (4.4) is called the matrix Toda sequence.

In the following theorem, we express the first (resp. second) derivative with respect to t of the matrix polynomials $P_{r,j}$ and $Q_{r,j}$ in terms of $P_{r,j}$, $A_{r,j}$, $B_{r,j}$ and $\dot{P}_{r,j}$ for $t\in(0,+\infty)$.

THEOREM 4.2. *Let $P_{r,j}$, $Q_{r,j}$, $A_{r,j}$, $B_{r,j}$, be as in Definition 2.2 and Definition 2.3, respectively. Furthermore, let $\Sigma_{r,j}$, $L_{1,j}$ and $u_{2,j}$ be defined as in (3.2), the first equality of (3.1) and the third equality of (2.2). The following equalities are then valid:*

$$\dot{P}_{r,j}(z) = B_{r,j-1}^{*}P_{r,j-1}(z), \quad j\geqslant 1, \tag{4.5}$$
$$\ddot{P}_{r,j+1}(z) = B_{r,j}^{*}A_{r,j}P_{r,j}(z) - A_{r,j+1}\dot{P}_{r,j+1}(z) + B_{r,j}^{*}\dot{P}_{r,j}(z), \quad j\geqslant 0. \tag{4.6}$$

Moreover,

$$\dot{Q}_{1,1}(z) = -s_1, \qquad \dot{Q}_{2,1}(z) = B_{2,0}^{*}Q_{1,1}(z) + \Sigma_{2,1}^{*}R_1(z)L_{1,2}^{*}u_{2,2}, \tag{4.7}$$
$$\dot{Q}_{1,j}(z) = B_{1,j-1}^{*}Q_{1,j-1}(z) + P_{1,j}(z)s_0 + \Sigma_{1,j}^{*}R_j(z)u_{2,j}, \quad j\geqslant 2, \tag{4.8}$$
$$\dot{Q}_{2,j}(z) = B_{2,j-1}^{*}Q_{2,j-1}(z) + \Sigma_{2,j}^{*}R_j(z)L_{1,j+1}^{*}u_{2,j+1}, \quad j\geqslant 2. \tag{4.9}$$

Proof. We prove (4.5) for $r=1$ and $j=1$, and we have

$$\begin{aligned}\dot{P}_{1,1}(z) =& (-s_1 s_0^{-1}, I_q)^{\cdot} R_1(z)v_1 = (-(s_1 s_0^{-1})^{\cdot}, 0_q)R_1(z)v_1\\ =& ((s_2 - s_1 s_0^{-1}s_1)s_0^{-1}, 0_q)R_1(z)v_1 = \widehat{H}_{1,1}\widehat{H}_{1,0}^{-1}(I_q, 0_q)\begin{pmatrix} I_q \\ zI_q\end{pmatrix} = B_{1,0}^{*}P_{1,0}(z).\end{aligned}$$

For $j \geqslant 2$, we have

$$\dot{P}_{1,j}(z) = \dot{\Sigma}_{1,j}^{*} R_j(z) v_j = \widehat{H}_{r,j} \widehat{H}_{1,j-1} (\Sigma_{r,j-1}, 0_q) \begin{pmatrix} R_{j-1}(z) v_{j-1} \\ z^j I_q \end{pmatrix} = \widehat{H}_{1,j} \widehat{H}_{1,j-1}^{-1} P_{1,j-1}(z).$$

Equality (4.6) is proved by using (4.5) and (4.4). Both equalities of relations (4.7) can be readily calculated. To prove (4.8) and (4.9), we use (3.28), (2.1), equalities $\dot{u}_{1,j} = -u_{2,j} - v_j s_0$, $\dot{u}_{2,j} = -L_{1,j+1} u_{2,j+1}$ and (2.9). □
Note that (4.5) for $r = 1$ was proved in [33, Proposition 5] by using different identities related to those we presented in Section 3.

In the next corollary, we express the second derivative of $P_{r,j+1}$ in terms of $P_{r,j}$, $A_{r,j}$ and $B_{r,j}$ with different values for j, respectively.

COROLLARY 4.1. *The following identities are valid:*

$$\begin{aligned} \ddot{P}_{r,j+1}(z) =& (B_{r,j-1} A_{r,j} - A_{r,j+1} B_{r,j}^{*}) P_{r,j}(z) + B_{r,j}^{*} B_{r,j-1}^{*} P_{r,j-1}(z), \\ \dddot{P}_{r,j+1}(z) =& (A_{r,j+1} B_{r,j}^{*} - B_{r,j}^{*} A_{r,j}) P_{r,j+1} + (B_{r,j+1}^{*} B_{r,j}^{*} - A_{r,j+1} B_{r,j}^{*} A_{r,j} + A_{r,j+1}^{2} B_{r,j}^{*} \\ & + z B_{r,j}^{*} A_{r,j} - z A_{r,j+1} B_{r,j}^{*} - 2 B_{r,j}^{*2}) P_{r,j} + (z I_q + A_{r,j+1}) B_{r,j}^{*} B_{r,j-1}^{*} P_{r,j-1}. \end{aligned}$$

5. Matrix Volterra sequence

In this section, we treat the matrix generalization of the Volterra lattice also called Langmuir lattice or discrete Korteweg-de Vries (1.6). See also [5], [23], [4] and references therein.

The following definition appeared within the statement of [13, Proposition 9.4].

DEFINITION 5.1. For $t \in [0, +\infty)$, the sequence $(\boldsymbol{\zeta}_j(t))_{j \geqslant 1}$ of $q \times q$ matrices defined by

$$\boldsymbol{\zeta}_1(t) := 0_q, \tag{5.1}$$

$$\boldsymbol{\zeta}_{2j}(t) := \widehat{H}_{2,j-1}(t) \widehat{H}_{1,j-1}^{-1}(t), \quad j \geqslant 1, \tag{5.2}$$

$$\boldsymbol{\zeta}_{2j+1}(t) := \widehat{H}_{1,j}(t) \widehat{H}_{2,j-1}^{-1}(t), \quad j \geqslant 1, \tag{5.3}$$

is called the matrix Volterra sequence.

The next remark reproduces Proposition 9.4 of [13].

REMARK 5.1. Let $A_{r,j}$, $B_{r,j}$ be as in Definition 2.3, and let $\boldsymbol{\zeta}_j$ be as in Definition (5.1)-(2.6). For $j \geqslant 0$, the following identities are valid:

$$A_{1,j} = \boldsymbol{\zeta}_{2j+1} + \boldsymbol{\zeta}_{2j+2}, \qquad B_{1,j}^{*} = \boldsymbol{\zeta}_{2j+3} \boldsymbol{\zeta}_{2j+2}, \tag{5.4}$$

$$A_{2,j} = \boldsymbol{\zeta}_{2j+3} + \boldsymbol{\zeta}_{2j+2}, \qquad B_{2,j}^{*} = \boldsymbol{\zeta}_{2j+4} \boldsymbol{\zeta}_{2j+3}. \tag{5.5}$$

The proof of the next lemma readily follows from Remark 5.1 and Theorem 4.1.

LEMMA 5.1. *Let* $\boldsymbol{\zeta}_j$ *be as in (5.1)-(2.6). If* $(\boldsymbol{\zeta}_j)_{j\geqslant 1}$ *is a Volterra sequence, then* $(\boldsymbol{\zeta}_{2,j+1}+\boldsymbol{\zeta}_{2j+2},\,\boldsymbol{\zeta}_{2j+3}\boldsymbol{\zeta}_{2j+2})_{j\geqslant 1}$ *and* $(\boldsymbol{\zeta}_{2j+3}+\boldsymbol{\zeta}_{2j+2},\,\boldsymbol{\zeta}_{2j+4}\boldsymbol{\zeta}_{2j+3})_{j\geqslant 1}$ *are matrix Toda sequences.*

The next theorem is a generalization of [12, Equations (2.4) and(2.3)], where the orthogonality properties of the scalar version of $P_{1,j}$ and $P_{2,j}$ are used. Conversely, the proof of the following theorem is based on the identities of Section 3.

THEOREM 5.1. *Let* $P_{r,j}$ *for* $r=1,2$ *be as in (2.9) and (2.10). Furthermore, let* $\boldsymbol{\zeta}_j$ *be as in (5.1)-(2.6). The following equalities are valid:*

$$P_{1,j+1}(z)-zP_{2,j}(z)+\boldsymbol{\zeta}_{2j+2}P_{1,j}(z)=0_q \tag{5.6}$$

and

$$P_{2,j+1}(z)-P_{1,j+1}(z)+\boldsymbol{\zeta}_{2j+3}P_{2,j}(z)=0_q. \tag{5.7}$$

Proof. We use the next identities:

$$R_j(z)v_j=L^*_{2,j+1}R_{j+1}(z)v_{j+1}, \tag{5.8}$$

$$zR_j(z)v_j=L^*_{1,j+1}R_{j+1}(z)v_{j+1}. \tag{5.9}$$

By replacing (5.8), (5.9) in (5.6), and (5.7), we have

$$P_{1,j+1}(z)-zP_{2,j}(z)+\boldsymbol{\zeta}_{2j+2}P_{1,j}(z)=(\Sigma^*_{1,j+1}-\Sigma^*_{2,j}L^*_{1,j+1}+\boldsymbol{\zeta}_{2j+2}\Sigma^*_{1,j}L^*_{2,j+1})R_{j+1}(z)v_{j+1},$$
$$P_{2,j+1}(z)-P_{1,j+1}(z)+\boldsymbol{\zeta}_{2j+3}P_{2,j}(z)=(\Sigma^*_{2,j+1}-\Sigma^*_{1,j+1}+\boldsymbol{\zeta}_{2j+3}\Sigma^*_{2,j}L^*_{2,j+1})R_{j+1}(z)v_{j+1}.$$

Equalities (5.6) and (5.7) readily follow by employing (3.13) and (3.14), respectively. □
Define

$$\widetilde{R}_{2j}(z,t):=P_{1,j}(z^2,t),\qquad \widetilde{R}_{2j+1}(z,t):=zP_{2,j}(z^2,t),\quad j\geqslant 0. \tag{5.10}$$

Note that the scalar version of $\widetilde{R}_j$ for $t=0$ was introduced in [12, Equation (2.2)]. Equalities (5.6) and (5.7) readily imply the following corollary.

COROLLARY 5.1. *The polynomials* $(\widetilde{R}_j)_{j\geqslant 0}$ *satisfy the following three-term recursive relation:*

$$\widetilde{R}_j(z)=z\widetilde{R}_{j-1}(z)-\boldsymbol{\zeta}_j\widetilde{R}_{j-2}(z),\quad j\geqslant 2.$$

The following result is a matrix generalization of the scalar Volterra lattice (1.6).

PROPOSITION 5.1. *Let* $\boldsymbol{\zeta}_j$ *be as in Definition 5.1. For* $t\in(0,+\infty)$*, the following differential equation holds:*

$$\dot{\boldsymbol{\zeta}}_j=\boldsymbol{\zeta}_j\boldsymbol{\zeta}_{j-1}-\boldsymbol{\zeta}_{j+1}\boldsymbol{\zeta}_j,\qquad j\geqslant 2. \tag{5.11}$$

Proof. Let j be an even number (resp. odd number) greater than 2. By taking the derivative with respect to t and by using (3.26) (resp. (3.27)), we have

$$\dot{\zeta}_{2j} = -\widehat{H}_{1,j}\widehat{H}_{1,j-1}^{-1} + \widehat{H}_{2,j-1}\widehat{H}_{2,j-2}^{-1}. \tag{5.12}$$

Moreover, we obtain

$$\dot{\zeta}_{2j+1} = \widehat{H}_{1,j}\widehat{H}_{1,j-1}^{-1} - \widehat{H}_{2,j}\widehat{H}_{2,j-1}^{-1}. \tag{5.13}$$

The right-hand side of (5.12) is equivalent to

$$-\widehat{H}_{1,j}\widehat{H}_{2,j-1}^{-1}\widehat{H}_{2,j-1}^{-1}\widehat{H}_{1,j-1}^{-1} + \widehat{H}_{2,j-1}\widehat{H}_{1,j-1}^{-1}\widehat{H}_{1,j-1}\widehat{H}_{2,j-2}^{-1}.$$

Furthermore, by using (2.5) and (5.3), equality (5.11) is proved for even j. In a similar manner, equality (5.11) is proved for odd j. $\square$

5.1. Proof of Proposition 2.1

Now we come to the proof of Proposition 2.1, which was alluded to at the end of Section 2. We prove (1.3) for $r = 1$. We omit the arguments z and t of $P_{1,j}$, $A_{1,j}$ and $B_{1,j}$, then we have

$$\begin{aligned} P_{1,j+1} - (zI_q - A_{1,j})P_{1,j} + B_{1,j-1}^{*}P_{1,j-1} &= zP_{2,j} - zP_{1,j} + \zeta_{2j+1}P_{1,j} + B_{1,j-1}^{*}P_{1,j-1} \\ &= \zeta_{2j+1}(P_{1,j} - zP_{2,j} + \zeta_{2j}P_{1,j-1}) = 0_q. \end{aligned}$$

In this chain of equalities, the first equality follows from (5.6) and first equality of (5.4). The second yields from equality (5.7), and the second equality of (5.4). The last equality is attained by using (5.6).

Now we prove (1.3) for $r = 2$. By using (5.7) and the first equality of (5.5), we have

$$\begin{aligned} P_{2,j+1} - (zI_q - A_{2,j})P_{2,j} + B_{2,j-1}^{*}P_{2,j-1} &= P_{1,j+1} - zP_{2,j} + \zeta_{2j+2}P_{2,j} + B_{2,j-1}^{*}P_{2,j-1} \\ &= \zeta_{2j+2}(-P_{1,j} + P_{2,j} + \zeta_{2j+1}P_{2,j-1}) = 0_q. \end{aligned}$$

The second equality follows from (5.6) and the second equality of (5.5). The last equality is attained by using (5.7).

Note that the proof of Proposition 2.1 is usually given by using the orthogonality properties of $P_{1,j}$ and $P_{2,j}$, as well as the integral representation of $A_{r,j}$ and $B_{r,j}$; see for example [19] and [29]. It should be mentioned that the orthogonal matrix polynomials were studied by [32], [2], [26], [25], [24], [22], [18] and the references therein.

6. Associated orthogonal matrix polynomials of order k

In this section, we consider the matrix generalization of the so-called associated orthogonal polynomials of order k studied in [35] and [36].

For $r=1,2$ and $t\in[0,\infty)$, let $P_{r,j}^{(k)}$ denote the associated matrix polynomials of order k, $k\in\mathbb{N}_0$, defined by

$$P_{r,0}^{(k)}(z,t):=I_q,\quad P_{r,1}^{(k)}(z,t):=zI_q-A_{r,k}(t),\tag{6.1}$$

$$P_{r,j+1}^{(k)}(z,t):=(zI_q-A_{r,j+k}(t))P_{r,j}^{(k)}(z,t)-B_{r,j+k-1}^{*}(t)P_{r,j-1}^{(k)}(z,t),\tag{6.2}$$

where $A_{r,j}$ and $B_{r,j}$ are as in (2.13)-(2.15). We will usually omit the argument t in the notation for $P_{r,j}$, $A_{r,j}$ and $B_{r,j}$.

The next lemma is a matrix generalization of [35, equality (2.4)].

LEMMA 6.1. *Let $P_{r,j}^{(k)}$ be as in (6.1) and (6.2). Furthermore, let $A_{r,j}$ and $B_{r,j}$ be as in (2.13)-(2.15). For $j\geqslant 1$ and $k\geqslant 0$, the following identity holds:*

$$P_{r,j}^{(k)}(z)=P_{r,j-1}^{(k+1)}(z)(zI_q-A_{r,k})-P_{r,j-2}^{(k+2)}(z)B_{r,k}^{*}.\tag{6.3}$$

Proof. We prove (6.3) for $j=2$:

$$\begin{aligned}P_{r,2}^{(k)}(z)&=(zI_q-A_{r,k+1})P_{r,1}^{(k)}(z)-B_{r,k}^{*}P_{r,0}^{(k)}(z)=(zI_q-A_{r,k+1})(zI_q-A_{r,k})-B_{r,k}^{*}\\&=P_{r,1}^{(k+1)}(z)(zI_q-A_{r,k})-P_{r,0}^{(k+2)}(z)B_{r,k}^{*}.\end{aligned}$$

We use the mathematical induction; let (6.3) be true for j. We prove (6.3) for $j+1$:

$$\begin{aligned}P_{r,j+1}^{(k)}(z)=&(zI_q-A_{r,j+k})[P_{r,j-1}^{(k+1)}(z)(zI_q-A_{r,k})-P_{r,j-2}^{(k+2)}(z)B_{r,k}^{*}]\\&-B_{j+k-1}^{*}[P_{r,j-2}^{(k+1)}(z)(zI_q-A_{r,k})-P_{r,j-3}^{(k+2)}(z)B_{r,k}^{*}]\\=&[(zI_q-A_{r,j+k})P_{r,j-1}^{(k+1)}(z)-B_{j+k-1}^{*}P_{r,j-2}^{(k+1)}(z)](zI_q-A_{r,k})\\&-[(zI_q-A_{r,j+k})P_{r,j-2}^{(k+2)}(z)-B_{r,j+k-1}^{*}P_{r,j-3}^{(k+2)}(z)]B_{r,k}^{*}\\=&P_{r,j}^{(k+1)}(z)(zI_q-A_{r,k})-P_{r,j-1}^{(k+2)}(z)B_{r,k}^{*}.\quad\square\end{aligned}$$

In the following proposition, we generalize [35, Lemma 2] for the matrix case.

PROPOSITION 6.1. *Let $P_{r,j}^{(k)}$ and $B_{r,j}$ be as in (6.1), (6.2) and (2.16), respectively. Assume that $B_{r,-1}=0_q$. The sequence $(A_{r,j},B_{r,j})$ is a matrix Toda sequence if and only if the following equality holds:*

$$\dot{P}_{r,j}^{(k)}(z)=B_{r,j+k-1}^{*}P_{r,j-1}^{(k)}(z)-P_{r,j-1}^{(k+1)}(z)B_{r,k-1}^{*},\tag{6.4}$$

for $r=1,2,$ $j\geqslant 1,$ $k\geqslant 0$ and $t\in(0,\infty)$.

Proof. For the case $k=0$ and arbitrary j, as well as for $j=1$ and arbitrary k, the statement is immediately verified by using (4.2) and (6.1). Furthermore, for $k\geqslant 1$ we follow the proof of [35, Lemma 2]. For the sufficiency part, equalities (4.2) and (4.3)

follow from (6.4) for $j=1$. Equality (4.4) is a consequence of (6.4) for $j=2$, equalities (4.2), (4.3) and the identity $\dot{A}_{r,k+1}A_{r,k}+A_{r,k+1}\dot{A}_{r,k}+B^*_{r,k+1}A_{r,k}-A_{r,k+1}B^*_{r,k-1}=B^*_{r,k}A_{r,k}-A_{r,k+1}B^*_{r,k}$.
Now we prove the necessity part. Let $(A_{r,j},B_{r,j})$ be a matrix Toda sequence. Here and in the sequel, we omit the subscript r, the arguments z and t of $P^{(k)}_{r,j}$, $A_{r,j}$ and $B_{r,j}$. By taking the derivative of equality (6.2) for $j=1$ and by employing (4.3), (4.4) with the second equality of (6.1), we have

$$\begin{aligned}\dot{P}^{(k)}_2 &= (B^*_{k+1}-B^*_k)P^{(k)}_1+(zI_q-A_{k+1})(B^*_k-B^*_{k-1})-B^*_kA_k+A_{k+1}B^*_k\\ &= B^*_{k+1}P^{(k)}_1-(zI_q-A_{k+1})B^*_{k-1}+B^*_k(-P^{(k)}_1+zI_q-A_k)=B^*_{k+1}P^{(k)}_1-P^{(k+1)}_1B^*_{k-1}.\end{aligned}$$

By employing mathematical induction, let us assume that (6.4) is valid for $j-1$ and j. We take the derivative of (6.2), and we have:

$$\begin{aligned}\dot{P}^{(k)}_{j+1} =&(B^*_{j+k}-B^*_{j+k-1})P^{(k)}_j+(zI_q-A_{j+k})(B^*_{j+k-1}P^{(k)}_{j-1}-P^{(k)}_{j-1}B^*_{k-1})\\ &-(B^*_{j+k-1}A_{j+k-1}-A_{j+k}B^*_{j+k-1})P^{(k)}_{j-1}-B^*_{j+k-1}(B^*_{j+k-2}P^{(k)}_{j-2}-P^{(k+1)}_{j-2}B^*_{k-1})\\ =&B^*_{j+k}P^{(k)}_j+B^*_{j+k-1}(-P^{(k)}_j+zP^{(k)}_{j-1}-A_{j+k-1}P^{(k)}_{j-1}-B^*_{j+k-2}P^{(k)}_{j-2})\\ &+(-zP^{(k+1)}_{j-1}+A_{j+k}P^{(k+1)}_{j-1}+B^*_{j+k-1}P^{(k+1)}_{j-2})B^*_{k-1}=B^*_{j+k}P^{(k)}_j-P^{(k+1)}_jB^*_{k-1}.\end{aligned}$$

The proposition is proved. $\square$

7. Example

Consider the 2×2 matrix distribution on $[0,+\infty)$, $\sigma(x)=\begin{pmatrix}4-2e^{-\frac{1}{2}x} & e^{-x}\\ e^{-x} & 2-e^{-x}\end{pmatrix}$, which corresponds to a positive matrix measure on $[0,+\infty)$. For details on positive matrix measures, see [8]. The matrices $s_j(t)=j!\begin{pmatrix}\frac{1}{(\frac{1}{2}+t)^{j+1}} & -\frac{1}{(1+t)^{j+1}}\\ -\frac{1}{(1+t)^{j+1}} & \frac{1}{(1+t)^{j+1}}\end{pmatrix}$ for $j\geqslant 0$ are the corresponding moments (1.4). One can immediately verify that the block matrices $H_{1,j}$ and $H_{2,j}$ for $j=0,\dots,2$ are positive definite matrices. The first Schur complements are the following:

$$\widehat{H}_{1,0}=\begin{pmatrix}\frac{2}{2t+1} & -\frac{1}{t+1}\\ -\frac{1}{t+1} & \frac{1}{t+1}\end{pmatrix},\quad \widehat{H}_{2,0}=\begin{pmatrix}\frac{1}{\left(t+\frac{1}{2}\right)^2} & -\frac{1}{(t+1)^2}\\ -\frac{1}{(t+1)^2} & \frac{1}{(t+1)^2}\end{pmatrix},$$
$$\widehat{H}_{1,1}=\begin{pmatrix}\frac{2\left(4t^2+6t+3\right)}{(t+1)^2(2t+1)^3} & -\frac{1}{(t+1)^3}\\ -\frac{1}{(t+1)^3} & \frac{1}{(t+1)^3}\end{pmatrix},\quad \widehat{H}_{2,1}=\begin{pmatrix}\frac{16\left(8t^3+18t^2+16t+5\right)}{(t+1)^2(2t+1)^4(4t+3)} & -\frac{2}{(t+1)^4}\\ -\frac{2}{(t+1)^4} & \frac{2}{(t+1)^4}\end{pmatrix}.$$

The first three-term recurrence relation coefficients are given by

$$A_{1,0} = \begin{pmatrix} \frac{4t+3}{2t^2+3t+1} & \frac{2}{2t+1} \\ 0 & \frac{1}{t+1} \end{pmatrix}, \quad A_{2,0} = \begin{pmatrix} \frac{24t^2+36t+14}{8t^3+18t^2+13t+3} & \frac{8(t+1)}{8t^2+10t+3} \\ 0 & \frac{2}{t+1} \end{pmatrix},$$

$$A_{1,1} = \begin{pmatrix} \frac{(4t+3)\left(16t^2+24t+11\right)}{16t^4+48t^3+54t^2+27t+5} & \frac{2\left(8t^2+16t+9\right)}{16t^3+32t^2+22t+5} \\ 0 & \frac{3}{t+1} \end{pmatrix},$$

$$A_{2,1} = \begin{pmatrix} \frac{4\left(960t^6+4320t^5+8304t^4+8712t^3+5244t^2+1710t+235\right)}{(t+1)(2t+1)(4t+3)\left(96t^4+288t^3+336t^2+180t+37\right)} & \frac{16\left(48t^5+216t^4+396t^3+366t^2+169t+31\right)}{(2t+1)(4t+3)\left(96t^4+288t^3+336t^2+180t+37\right)} \\ 0 & \frac{4}{t+1} \end{pmatrix},$$

$$B_{1,0} = \begin{pmatrix} \frac{8t^2+12t+5}{\left(2t^2+3t+1\right)^2} & 0 \\ \frac{4}{(2t+1)^2} & \frac{1}{(t+1)^2} \end{pmatrix}, \quad B_{2,0} = \begin{pmatrix} \frac{2\left(96t^4+288t^3+336t^2+180t+37\right)}{\left(8t^3+18t^2+13t+3\right)^2} & 0 \\ \frac{8\left(8t^2+16t+7\right)}{\left(8t^2+10t+3\right)^2} & \frac{2}{(t+1)^2} \end{pmatrix},$$

$$B_{1,1} = \begin{pmatrix} \frac{4\left(384t^6+1728t^5+3360t^4+3600t^3+2232t^2+756t+109\right)}{\left(2t^2+3t+1\right)^2\left(8t^2+12t+5\right)^2} & 0 \\ \frac{16\left(32t^4+112t^3+152t^2+92t+21\right)}{\left(16t^3+32t^2+22t+5\right)^2} & \frac{4}{(t+1)^2} \end{pmatrix}$$

and

$$B_{2,1} = \begin{pmatrix} \frac{6(4t+3)^2\left(3072t^8+18432t^7+49920t^6+79488t^5+81120t^4+54144t^3+23008t^2+5676t+621\right)}{\left(2t^2+3t+1\right)^2\left(96t^4+288t^3+336t^2+180t+37\right)^2} & 0 \\ \frac{24\left(3072t^8+21504t^7+66048t^6+115968t^5+127104t^4+88992t^3+38872t^2+9688t+1055\right)}{(2t+1)^2\left(96t^4+288t^3+336t^2+180t+37\right)^2} & \frac{6}{(t+1)^2} \end{pmatrix}.$$

With these matrices, the identities of Remark 4.1 and Theorem 4.1 can be readily verified.
The orthogonal matrix polynomials are the following: $P_{1,0} = I_2$, $P_{2,0} = I_2$,

$$P_{1,1} = \begin{pmatrix} z - \frac{4t+3}{2t^2+3t+1} & -\frac{2}{2t+1} \\ 0 & z - \frac{1}{t+1} \end{pmatrix}, P_{2,1} = \begin{pmatrix} z - \frac{2\left(12t^2+18t+7\right)}{8t^3+18t^2+13t+3} & -\frac{8(t+1)}{8t^2+10t+3} \\ 0 & z - \frac{2}{t+1} \end{pmatrix},$$

$$P_{1,2} = \begin{pmatrix} z^2 - \frac{4\left(24t^3+54t^2+43t+12\right)z}{16t^4+48t^3+54t^2+27t+5} + \frac{2\left(96t^4+288t^3+336t^2+180t+37\right)}{\left(2t^2+3t+1\right)^2\left(8t^2+12t+5\right)} & -\frac{4\left(16zt^4+4(13z-8)t^3+8(8z-9)t^2+(35z-58)t+7z-16\right)}{(2t+1)^2\left(8t^3+20t^2+17t+5\right)} \\ 0 & z^2 - \frac{4z}{t+1} + \frac{2}{(t+1)^2} \end{pmatrix},$$

$$P_{2,2} = \begin{pmatrix} P_{2,2}^{(1,1)} & P_{2,2}^{(1,2)} \\ 0 & z^2 - \frac{6z}{t+1} + \frac{6}{(t+1)^2} \end{pmatrix},$$

where

$$P_{2,2}^{(1,1)} := z^2 - \frac{6\left(4t^2+6t+3\right)(4t+3)^3 z}{192t^6+864t^5+1632t^4+1656t^3+950t^2+291t+37} + \frac{6\left(768t^6+3456t^5+6656t^4+7008t^3+4240t^2+1392t+193\right)}{\left(2t^2+3t+1\right)^2\left(96t^4+288t^3+336t^2+180t+37\right)},$$

$$P_{2,2}^{(1,2)} := -\frac{24\left(32zt^5+32(4z-3)t^4+16(13z-18)t^3+2(85z-172)t^2+(69z-188)t+11z-39\right)}{(2t+1)^2\left(96t^4+288t^3+336t^2+180t+37\right)}.$$

The second kind polynomials are given by $Q_{1,0} = 0_2$, $Q_{2,0} = \begin{pmatrix} \frac{2}{2t+1} & -\frac{1}{t+1} \\ -\frac{1}{t+1} & \frac{1}{t+1} \end{pmatrix}$,

$$Q_{1,1} = \begin{pmatrix} \frac{2}{2t+1} & -\frac{1}{t+1} \\ -\frac{1}{t+1} & \frac{1}{t+1} \end{pmatrix}, Q_{2,1} = \begin{pmatrix} \frac{2\left(8zt^3+2(9z-4)t^2+(13z-10)t+3z-4\right)}{(2t+1)^2\left(4t^2+7t+3\right)} & -\frac{z(t+1)-1}{(t+1)^2} \\ -\frac{z(t+1)-1}{(t+1)^2} & \frac{z(t+1)-1}{(t+1)^2} \end{pmatrix},$$

$$Q_{1,2} = \begin{pmatrix} \frac{2\left(16zt^4+48(z-1)t^3+2(27z-52)t^2+(27z-82)t+5z-24\right)}{(2t+1)^2\left(8t^3+20t^2+17t+5\right)} & -\frac{z(t+1)-3}{(t+1)^2} \\ -\frac{z(t+1)-3}{(t+1)^2} & \frac{z(t+1)-3}{(t+1)^2} \end{pmatrix},$$

$$Q_{2,2} = \begin{pmatrix} \frac{\text{num}}{(t+1)^2(2t+1)^3\left(96t^4+288t^3+336t^2+180t+37\right)} & \frac{(t+1)^2z^2-5(t+1)z+2}{(t+1)^3} \\ \frac{(t+1)^2z^2-5(t+1)z+2}{(t+1)^3} & \frac{-(t+1)^2z^2+5(t+1)z-2}{(t+1)^3} \end{pmatrix},$$

where

$$\begin{aligned} \text{num} :=& 2\left(384t^8+2304t^7+6048t^6+9072t^5+8500t^4+5088t^3+1897t^2+402t+37\right)z^2 \\ & -4\left(960t^7+4896t^6+10848t^5+13560t^4+10330t^3+4789t^2+1247t+140\right)z \\ & +4\left(384t^6+1536t^5+2688t^4+2640t^3+1540t^2+506t+73\right). \end{aligned}$$

With $P_{r,j}$ and $Q_{r,j}$ for $r = 1,2$ and $j = 1,2$, the identities of Theorem 4.2 and Corollary 4.1 can be readily verified.

The first matrices of the Volterra sequence are the following:

$$\xi_1 = 0_2, \quad \xi_2 = \begin{pmatrix} \frac{4t+3}{2t^2+3t+1} & \frac{2}{2t+1} \\ 0 & \frac{1}{t+1} \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} \frac{8t^2+12t+5}{8t^3+18t^2+13t+3} & \frac{2}{8t^2+10t+3} \\ 0 & \frac{1}{t+1} \end{pmatrix},$$

$$\xi_4 = \begin{pmatrix} \frac{2\left(96t^4+288t^3+336t^2+180t+37\right)}{(t+1)(2t+1)(4t+3)\left(8t^2+12t+5\right)} & \frac{4\left(16t^3+40t^2+36t+11\right)}{64t^4+176t^3+184t^2+86t+15} \\ 0 & \frac{2}{t+1} \end{pmatrix},$$

$$\xi_5 = \begin{pmatrix} \xi_5^{(1,1)} & \frac{4\left(192t^5+768t^4+1248t^3+1024t^2+424t+71\right)}{(2t+1)\left(8t^2+12t+5\right)\left(96t^4+288t^3+336t^2+180t+37\right)} \\ 0 & \frac{2}{t+1} \end{pmatrix},$$

where $\xi_5^{(1,1)} := \frac{2\left(1536t^7+8064t^6+18624t^5+24480t^4+19728t^3+9720t^2+2704t+327\right)}{(t+1)(2t+1)\left(8t^2+12t+5\right)\left(96t^4+288t^3+336t^2+180t+37\right)}$.

With the given matrices, the identities of Theorem 5.1 and Proposition 5.1 are immediately verified.

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