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ANALYZING THE SPECTRAL (A)SYMMETRY OF THE MASSLESS DIRAC OPERATOR ON THE 3–TORUS

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Abstract. We analyze the spectrum of the massless Dirac operator on the 3-torus $\mathbb{T}^3$. It is known that it is possible to calculate this spectrum explicitly, that it is symmetric about zero and that each eigenvalue has even multiplicity. However, for a general oriented closed Riemannian 3-manifold (M,g) there is no reason for the spectrum of the massless Dirac operator to be symmetric. Using perturbation theory, we derive the asymptotic formulae for its eigenvalues and prove that by the perturbation of the Euclidean metric on the 3-torus, it is possible to obtain spectral asymmetry of the massless Dirac operator in the axisymmetric case.

1. Introduction

We work on a 3-dimensional connected compact oriented manifold M equipped with a Riemannian metric $g_{\alpha\beta}$. Our aim is to analyze the spectrum of the massless Dirac operator on M which describes a single massless neutrino living in a 3-dimensional compact universe. The relationship between the spectrum of an operator and the manifold geometry is of particular interest in recent research. The eigenvalues of the massless Dirac operator represent the energy levels of the massless neutrino and we are in particular interested in studying the spectrum of that operator, i.e. the set of all eigenvalues of the operator. An advanced review of the theory of the Dirac operator in general can be found in e.g. [15]. It is however very difficult to determine the spectrum of the massless Dirac operator on an arbitrary manifold M and there are only two known examples where the spectrum of the massless Dirac operator can be calculated explicitly: the unit 3-torus $\mathbb{T}^3$ equipped with Euclidean metric (see [12]) and the unit 3-sphere $\mathbb{S}^3$ equipped with metric induced by the natural embedding of $\mathbb{S}^3$ in $\mathbb{R}^4$ (see [5, 26]).

In this paper we choose to work on the unit 3-torus $\mathbb{T}^3$ parameterized by cyclic coordinates x^α, $\alpha = 1,2,3$ of period 2π. Under the assumption that the metric is

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Euclidean, the massless Dirac operator corresponding to the standard spin structure reads

$$W = -i \begin{pmatrix} \frac{\partial}{\partial x^3} & \frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2} \\ \frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2} & -\frac{\partial}{\partial x^3} \end{pmatrix}, \tag{1}$$

see e.g. [9]. Note that we denote the massless Dirac operator by W as it is often also referred to as the *Weyl* operator, especially in theoretical physics, whereas we previously examined in some detail solutions of Einstein-Weyl theory, which incorporates the massless Dirac equation and operator (see [6, 17, 18, 19, 20, 21, 22, 23]), where we suggested spacetimes with torsion as metric-affine models for the massless neutrino, which in turn motivated our current work in spectral analysis.

The eigenvalues and the eigenfunctions of the operator (1) can be calculated explicitly. The spectrum of the massless Dirac operator on the unit torus $\mathbb{T}^3$ equipped with the Euclidean metric is as follows: zero is an eigenvalue of multiplicity two and for each $m \in \mathbb{Z}^3 \setminus \{0\}$, the eigenvalues are $\pm\|m\|$, which in general have even multiplicity. Even multiplicity of eigenvalues is a consequence of the fact that the massless Dirac operator (1) and the operator of charge conjugation

$$C \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} -\overline{v_2} \\ \overline{v_1} \end{pmatrix}. \tag{2}$$

commute, see [9].

It is very important to note that the spectrum of the operator (1) is symmetric about zero, as was demonstrated in [5, 8, 26]. However, as emphasized in [1, 2, 3, 4] for a general oriented Riemannian 3-manifold (M,g) there is no justifiable reason for the spectrum of the massless Dirac operator to be symmetric, as it would imply that in these two cases there is no difference between the properties of the massless neutrino and the massless antineutrino. Therefore the primary objective of our study is to break the spectral symmetry of the massless Dirac operator.

2. Perturbing the massless Dirac operator

As we work on the 3–torus $\mathbb{T}^3$, which has trivial topology, spectral asymmetry will be obtained by perturbing the Euclidean metric itself, i.e. we consider a metric $g_{\alpha\beta}(x;\epsilon)$, whose components are smooth functions of coordinates $x^\alpha, \alpha = 1,2,3$ and small real parameter ϵ, which satisfies $g_{\alpha\beta}(x;0) = \delta_{\alpha\beta}$. This is the approach that was similarly successfully used in [9] for the eigenvalue with smallest modulus $\lambda = 0$, where spectral asymmetry was achieved and two particular families of Riemannian metrics were presented for which the eigenvalue with smallest modulus can be evaluated explicitly. The main goal of this paper is to derive the asymptotic formulae for all the other eigenvalues and to see under which conditions we can create spectral asymmetry. Note that the behavior of eigenvalues of the massless Dirac operator under perturbations of the metric was studied in [7], on a much more abstract level.

Our first goal is to write down explicitly the massless Dirac operator. To do this, we use the concepts of *frame* and *coframe*, whose differential geometrical definition and properties were given in [8]. According to [14, 25], a 3-dimensional oriented manifold is parallelizable and consequently, there exist smooth real vector fields $e_j(x)$, $j=1,2,3$ that are linearly independent in every point x of the manifold M, which we call a *frame*. We can assume that the vector fields $e_j(x)$ are orthonormal, and if not, the orthonormality can always be achieved using the Gram-Schmidt process. The coordinate components of the vector $e_j(x)$ are $e_j{}^\alpha(x)$, $\alpha=1,2,3$, where the so-called anholonomic or *frame index*, denoted by the Latin letter j, enumerates the vector field and the holonomic or *tensor index*, denoted by Greek letter α, enumerates their components. The *coframe* is defined as the triple of covector fields $e^k(x)$, $k=1,2,3$ and the coordinate components of the vector $e^k(x)$ are $e^k{}_\alpha(x)$, $\alpha=1,2,3$, where $e^k{}_\beta := \delta^{kj} g_{\beta\gamma} e_j{}^\gamma$. The frame is completely determined by the coframe, and vice versa, by the relation $e_j{}^\alpha e^k{}_\alpha = \delta_j{}^k$.

We consider the *perturbed* coframe to be a smooth real-valued matrix function $e^j{}_\alpha(x;\epsilon)$, $j,\alpha=1,2,3$ satisfying the conditions

$$g_{\alpha\beta}(x;\epsilon) = \delta_{jk} e^j{}_\alpha(x;\epsilon) e^k{}_\beta(x;\epsilon), \quad e^j{}_\alpha(x;0) = \delta^j{}_\alpha,$$

as was explained in [9]. The perturbed frame is considered to be the smooth real-valued matrix-function $e_j{}^\alpha(x;\epsilon), j,\alpha=1,2,3$ defined by

$$e_j{}^\alpha(x;\epsilon) e^k{}_\alpha(x;\epsilon) = \delta_j{}^k.$$

REMARK 1. In this paper we choose the frame and the coframe so that they depend on the coordinate x^1 only, so that the original eigenvalue problem for a partial differential operator reduces to an eigenvalue problem for an ordinary differential operator. This case we call the *axisymmetric case.*

Consider therefore the perturbed metric $g_{\alpha\beta}(x^1;\epsilon)$, the components of which are smooth functions of the coordinate x^1 and a small real parameter ϵ which satisfies

$$g_{\alpha\beta}(x^1;0) = \delta_{\alpha\beta}. \tag{3}$$

For a given function $f : \mathbb{T}^3 \to \mathbb{C}$ we denote its Fourier coefficients by

$$\widehat{f}(m_1) := \frac{1}{2\pi} \int_{\mathbb{T}^3} e^{-im_1 x^1} f(x^1) dx^1, \ \ m_1 \in \mathbb{Z}, \tag{4}$$

and we let

$$h_{\alpha\beta}(x^1) := \left. \frac{\partial g_{\alpha\beta}(x^1;\epsilon)}{\partial \epsilon} \right|_{\epsilon=0}, \quad k_{\alpha\beta}(x^1) := 4 \left. \frac{\partial^2 g_{\alpha\beta}(x^1;\epsilon)}{\partial \epsilon^2} \right|_{\epsilon=0}. \tag{5}$$

We choose to deal with the massless Dirac operator on half-densities denoted by $W_{1/2}$ which is the operator $W_{1/2} := (\det g_{\kappa\lambda})^{1/4} W (\det g_{\mu\nu})^{-1/4}$, which therefore differs from the massless Dirac operator W only by "scalar" factors on the left and on the right, as was explained in [8]. It is known that these operators are equivalent, i.e. have the same spectrum, see [8, 9].

The axisymmetric massless Dirac operator on half-densities corresponding to the perturbed metric $g(x^1;\epsilon)$ reads

$$W_{1/2}(\epsilon) = -\frac{i}{2}\begin{pmatrix} e_3{}^1 & e_1{}^1 - ie_2{}^1 \\ e_1{}^1 + ie_2{}^1 & -e_3{}^1 \end{pmatrix}\frac{d}{dx^1} - \frac{i}{2}\frac{d}{dx^1}\begin{pmatrix} e_3{}^1 & e_1{}^1 - ie_2{}^1 \\ e_1{}^1 + ie_2{}^1 & -e_3{}^1 \end{pmatrix}$$
$$+\frac{\delta_{jk}}{4\sqrt{\det g_{\alpha\beta}}}\left(e^j{}_3\left(\frac{de^k{}_2}{dx^1}\right) - e^j{}_2\left(\frac{de^k{}_3}{dx^1}\right)\right)I, \tag{6}$$

where $e^i{}_j$ and $e_i{}^j$ are the components of the perturbed coframe $e^j{}_\alpha(x^1;\epsilon)$ and perturbed frame $e_j{}^\alpha(x^1;\epsilon)$ respectively, I is identity matrix is 2×2 and

$$\sqrt{\det g_{\alpha\beta}} = \frac{1}{\sqrt{\det g^{\alpha\beta}}} = \det e^j{}_\alpha = \frac{1}{\det e_j{}^\alpha},$$

see Section 7 of [9], where it was explained that the operator $W_{1/2}(\epsilon)$ acts on 2-columns $v = (v_1,\ v_2)^T$ of complex-valued half-densities. Our Hilbert space is the vector space of 2-columns of square integrable half-densities equipped with inner product

$$\langle v, w\rangle := \int_M w^* v dx\,. \tag{7}$$

The domain of the operator $W_{1/2}(\epsilon)$ is $H^1(M;\mathbb{C}^2)$, which is the Sobolev space of 2-columns of half-densities that are square integrable together with their first partial derivatives.

REMARK 2. The operator $W_{1/2}(\epsilon): H^1(M;\mathbb{C}^2)\to L^2(M;\mathbb{C}^2)$ is self-adjoint and its spectrum is discrete, with eigenvalues accumulating to $\pm\infty$, see [9].

One of the main reasons we choose to work with the massless Dirac operator on half-densities $W_{1/2}(\epsilon)$ rather than with the massless Dirac operator $W(\epsilon)$ is that we do not want our Hilbert space to depend on ϵ. According to formulae (3) and (5) we have that

$$g_{\alpha\beta}(x^1;\epsilon) = \delta_{\alpha\beta} + \epsilon h_{\alpha\beta}(x^1) + \frac{\epsilon^2}{4}k_{\alpha\beta}(x^1) + O(\epsilon^3) \tag{8}$$

and hence, using Taylor's formula for the function $\sqrt{1+z}$, the coframe is given by

$$e^j{}_\alpha(x^1;\epsilon) = \delta^j{}_\alpha + \frac{\epsilon}{2}h^j{}_\alpha - \frac{\epsilon^2}{8}\left(h^2\right)^j{}_\alpha + \frac{\epsilon^2}{8}k^j{}_\alpha + O(\epsilon^3) \tag{9}$$

and, using Taylor's formula for the function $(1+z)^{-1}$, the frame is given by

$$e_j{}^\alpha(x^1;\epsilon) = \delta_j{}^\alpha - \frac{\epsilon}{2}h_j{}^\alpha + \frac{3\epsilon^2}{8}\left(h^2\right)_j{}^\alpha - \frac{\epsilon^2}{8}k_j{}^\alpha + O(\epsilon^3). \tag{10}$$

REMARK 3. Note that h^2 denotes the square of the perturbation matrix h, i.e. $\left(h^2\right)_j{}^\alpha = h_j{}^\beta h_\beta{}^\alpha$, where the summation is performed over the repeated index β.

Note that for a given metric $g_{\alpha\beta}(x^1;\epsilon)$ the coframe $e^j{}_\alpha(x^1;\epsilon)$ and the frame $e_j{}^\alpha(x^1;\epsilon)$ are not defined uniquely. We can multiply the matrix functions $e^j{}_\alpha(x^1;\epsilon)$ and $e_j{}^\alpha(x^1;\epsilon)$ from the left by an arbitrary smooth 3×3 special orthogonal matrix-function $O(x^1;\epsilon)$ satisfying the condition $O(x^1;0)=I$. It is important to stress that this choice of the coframe does not affect the spectrum of the massless Dirac operator, see [8, 9] for more details.

It is also known that the eigenfunctions of the operator $W_{1/2}(\epsilon)$ are infinitely smooth, see [8, 9, 10]. We get directly that for any integer eigenvalue $\lambda\in\mathbb{Z}$, see Remark 2, the corresponding eigenvector of the massless Dirac operator $W_{1/2}(0)$ is

$$v_\lambda(x^1)=\frac{1}{2\sqrt{\pi}}\begin{pmatrix}1\\1\end{pmatrix}e^{i\lambda x^1}. \tag{11}$$

REMARK 4. As each eigenvalue of massless Dirac operator in the axisymmetric case has multiplicity two, see [9] and we also have that the vector

$$w_\lambda(x^1)=C(v_\lambda(x^1))=\frac{1}{2\sqrt{\pi}}\begin{pmatrix}-1\\1\end{pmatrix}e^{-i\lambda x^1} \tag{12}$$

is also an eigenvector of the massless Dirac operator corresponding to eigenvalue $\lambda\in\mathbb{Z}$, where C is the operator of charge conjugation (2).

Let $W_{1/2}(\epsilon)$ therefore be the massless Dirac operator on half-densities (6) corresponding to the perturbed metric $g_{\alpha\beta}(x;\epsilon)$ and let

$$W_{1/2}(\epsilon)=W_{1/2}^{(0)}+\epsilon W_{1/2}^{(1)}+\epsilon^2W_{1/2}^{(2)}+\cdots \tag{13}$$

be the asymptotic expansion of the perturbed massless Dirac operator in powers of the small parameter ϵ. The operator $W_{1/2}^{(0)}=W_{1/2}(0)$ is the unperturbed massless Dirac operator on half-densities (6). We denote by $\lambda^{(0)}$ the eigenvalue of this operator and by $v^{(0)}$ the corresponding eigenvector.

The perturbation of an isolated eigenvalue of finite multiplicity of a bounded operator was described in [24] and that procedure can be applied in our case with some additional conditions.

The operators $W_{1/2}^{(k)}$, $k=0,1,2,\ldots$ are formally self-adjoint first order differential operators which also commute with the antilinear operator of charge conjugation (2).

We need to solve the eigenvalue problem

$$W_{1/2}(\epsilon)v(\epsilon)=\lambda(\epsilon)v(\epsilon).$$

We seek the eigenvalue and eigenfunction of the perturbed operator $W_{1/2}(\epsilon)$ in the form of asymptotic expansions

$$\lambda(\epsilon)=\lambda^{(0)}+\epsilon\lambda^{(1)}+\epsilon^2\lambda^{(2)}+\cdots, \tag{14}$$

$$v(\epsilon)=v^{(0)}+\epsilon v^{(1)}+\epsilon^2v^{(2)}+\cdots. \tag{15}$$

In general, these asymptotic series do *not* converge and the formal procedure for this perturbation process was described in Section 4 in [9]. However, the asymptotic expansion can be justified, as was described in Section 5 in [9], by showing that by taking a finite number of terms one gets the expected estimate for the remainder. This argument holds for any simple eigenvalue.

3. Main results

Let $n \in \mathbb{N}$ be a positive eigenvalue of the unperturbed massless Dirac operator. Throughout the rest of this paper we denote by $\lambda_{+n}(\epsilon)$ the asymptotic expansion of the eigenvalue n and by $\lambda_{-n}(\epsilon)$ the asymptotic expansion of the eigenvalue $-n$. By $\lambda_{+n}^{(i)}$ and $\lambda_{-n}^{(i)}$, $i = 1,2,3,\ldots$, we denote their respective asymptotic coefficients.

The main result of this paper is the following

THEOREM 1. *Under an arbitrary perturbation of the metric (8), the asymptotic expansion of the eigenvalues n and $-n$ are*

$$\lambda_{\pm n}(\epsilon) = \pm n + \lambda_{\pm n}^{(1)}\epsilon + \lambda_{\pm n}^{(2)}\epsilon^2 + O(\epsilon^3) \quad \text{as} \quad \epsilon \to 0, \tag{16}$$

where the constants $\lambda_{+n}^{(1)}$, $\lambda_{-n}^{(1)}$, $\lambda_{+n}^{(2)}$ and $\lambda_{-n}^{(2)}$ appearing in (16) *are given by*

$$\lambda_{\pm n}^{(1)} = \mp \frac{n}{2}\widehat{h}_{11}(0), \tag{17}$$

$$\begin{aligned}\lambda_{\pm n}^{(2)} = &\pm \frac{3n}{8}\widehat{(h^2)}_{11}(0) \mp \frac{n}{8}\widehat{k}_{11}(0) - \frac{i}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m)\\ &-\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{\pm n\}}(m\mp n)\left(\widehat{h}_{31}(m\pm n)+i\widehat{h}_{21}(m\pm n)\right)\left(\overline{\widehat{h}_{31}(m\pm n)}-\overline{i\widehat{h}_{21}(m\pm n)}\right),\\ &-\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{\pm n\}}\frac{1}{m\mp n}(m\pm n)^2\,\widehat{h}_{11}(m\mp n)\overline{\widehat{h}_{11}(m\mp n)},\end{aligned} \tag{18}$$

REMARK 5. Note that, as usual, $\varepsilon_{\alpha\beta\gamma}$ denotes the totally antisymmetric quantity, $\varepsilon_{123} := +1$, while overline stands for complex conjugation.

REMARK 6. Note that (17) directly implies that spectral asymmetry cannot be achieved in the linear term.

REMARK 7. Formula (18) implies that

$$\begin{aligned}
\lambda_{+n}^{(2)}+\lambda_{-n}^{(2)} = &-\frac{i}{8}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m) \\
&-\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}(m+n)^2\,\widehat{h}_{11}(m-n)\overline{\widehat{h}_{11}(m-n)} \\
&-\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{n\}}(m-n)\left(\widehat{h}_{31}(m+n)+i\widehat{h}_{21}(m+n)\right)\left(\overline{\widehat{h}_{31}(m+n)}-i\overline{\widehat{h}_{21}(m+n)}\right) \\
&-\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-n\}}\frac{1}{m+n}(m-n)^2\,\widehat{h}_{11}(m+n)\overline{\widehat{h}_{11}(m+n)} \\
&-\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{-n\}}(m+n)\left(\widehat{h}_{31}(m-n)+i\widehat{h}_{21}(m-n)\right)\left(\overline{\widehat{h}_{31}(m-n)}-i\overline{\widehat{h}_{21}(m-n)}\right).
\end{aligned}$$

We conclude that one way to create spectral asymmetry is to choose the perturbation matrix $h_{\alpha\beta}(x^1)$ such that $\widehat{h}_{11}(m\pm n)=\widehat{h}_{21}(m\pm n)=\widehat{h}_{31}(m\pm n)=0$ for all $m\in\mathbb{Z}$ such that the term

$$\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m)\neq 0.$$

REMARK 8. The length of the unit x^1 circle is $2\pi\left(1+\frac{1}{2}\widehat{h}_{11}(0)\epsilon\right)+O(\epsilon^2)$. Therefore the coefficients $\lambda_{+n}^{(1)}(\epsilon)$ and $\lambda_{-n}^{(1)}(\epsilon)$ are determined by the change of the length of the unit x^1 circle.

Proof of Theorem 1. In order to prove Theorem 1 we need to write down explicitly the differential operators $W^{(1)}$ and $W^{(2)}$ appearing in the asymptotic expansion of the perturbed massless Dirac operator on half-densities

$$W_{1/2}(\epsilon)=W^{(0)}+\epsilon W^{(1)}+\epsilon^2 W^{(2)}+O(\epsilon^3).$$

Substituting formulae (9) and (10) into (6), we get that

$$W^{(1)}=\frac{i}{4}\begin{pmatrix} {h_3}^1 & {h_1}^1-i{h_2}^1 \\ {h_1}^1+i{h_2}^1 & -{h_3}^1\end{pmatrix}\frac{d}{dx^1}+\frac{i}{4}\frac{d}{dx^1}\begin{pmatrix} {h_3}^1 & {h_1}^1-i{h_2}^1 \\ {h_1}^1+i{h_2}^1 & -{h_3}^1\end{pmatrix}, \tag{19}$$

$$\begin{aligned}
W^{(2)}= &-\frac{3i}{16}\begin{pmatrix} {(h^2)_3}^1 & {(h^2)_1}^1-i{(h^2)_2}^1 \\ {(h^2)_1}^1+i{(h^2)_2}^1 & -{(h^2)_3}^1\end{pmatrix}\frac{d}{dx^1} \\
&-\frac{3i}{16}\frac{d}{dx^1}\begin{pmatrix} {(h^2)_3}^1 & {(h^2)_1}^1-i{(h^2)_2}^1 \\ {(h^2)_1}^1+i{(h^2)_2}^1 & -{(h^2)_3}^1\end{pmatrix}-\frac{1}{16}\varepsilon_{\beta\gamma 1}h_{\alpha\beta}\frac{dh_{\alpha\gamma}}{dx^1}I. \\
&+\frac{i}{16}\begin{pmatrix} {k_3}^1 & {k_1}^1-i{k_2}^1 \\ {k_1}^1+i{k_2}^1 & -{k_3}^1\end{pmatrix}\frac{d}{dx^1}+\frac{i}{16}\frac{d}{dx^1}\begin{pmatrix} {k_3}^1 & {k_1}^1-i{k_2}^1 \\ {k_1}^1+i{k_2}^1 & -{k_3}^1\end{pmatrix}
\end{aligned} \tag{20}$$

We calculate the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ from equation (14) using

$$\lambda^{(1)} = \langle W^{(1)}v^{(0)}, v^{(0)}\rangle, \tag{21}$$

$$\lambda^{(2)} = \langle W^{(2)}v^{(0)}, v^{(0)}\rangle - \langle (W^{(1)} - \lambda^{(1)})Q(W^{(1)} - \lambda^{(1)})v^{(0)}, v^{(0)}\rangle, \tag{22}$$

see Appendix B for details, where $v^{(0)}$ is the eigenvector corresponding to the eigenvalue $\lambda^{(0)}$. Note that Q denotes the *pseudoinverse operator* (see Appendix A) of the unperturbed operator $W_{1/2}(0)$ corresponding to the integer eigenvalue λ. It is given by the formula

$$Q = \frac{1}{4\pi} \sum_{m\in\mathbb{Z}\setminus\{\lambda\}} \frac{1}{m-\lambda} \left[e^{imx^1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \int_0^{2\pi} e^{-imy^1} (\,\cdot\,) dy^1 \right.$$
$$\left. + e^{-imx^1} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \int_0^{2\pi} e^{imy^1} (\,\cdot\,) dy^1 \right], \tag{23}$$

see e.g. [9, 24] and Appendix A for the general details on the pseudoinverse operator. Using formula (4) for the Fourier coefficients, substituting formulae (11), (19), (20) and (23) into (21) and (22), we obtain equations (17) and (18), which describe the linear and quadratic terms in the asymptotic expansion of eigenvalues $\pm n$. Note that we get the same formulae if we use the eigenvector (12). □

REMARK 9. Appendix C contains very detailed explicit derivations of equations (17) and (18).

4. Explicit examples of spectral asymmetry

In this section we also give two explicit families of the perturbed Euclidean metric for which we obtain spectral asymmetry of the massless Dirac operator in the axisymmetric case.

EXAMPLE 1. If we consider

$$h_{\alpha\beta}(x^1) = 2\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix}, \quad k_{\alpha\beta}(x^1) = \begin{pmatrix} \sin x^1 & \cos x^1 & 0 \\ \cos x^1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then by using formulae (19) and (20), we get that the corresponding perturbed massless Dirac operator is given by

$$W(\epsilon) = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{d}{dx^1} + \frac{\epsilon^2}{16}\begin{pmatrix} 0 & \cos x^1 + i\sin x^1 \\ -\cos x^1 + i\sin x^1 & 0 \end{pmatrix}\frac{d}{dx^1}$$
$$+ \frac{\epsilon^2}{16}\frac{d}{dx^1}\begin{pmatrix} 0 & \cos x^1 + i\sin x^1 \\ -\cos x^1 + i\sin x^1 & 0 \end{pmatrix} - \frac{\epsilon^2}{2}I.$$

Using formula (23) for the pseudoinverse operator in the case of eigenvalues $n = \pm 1$, as well as (7), (21) and (22), we get that

$$\lambda_{+1}(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3), \tag{24}$$

$$\lambda_{-1}(\epsilon) = -1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3). \tag{25}$$

Applying the Fourier transform (4) to matrices h and k gives us

$$\widehat{h}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -i \\ 0 & -i & -1 \end{pmatrix} & \text{for } m = 1, \\ \\ 0 & \text{for } m = 2, 3, \dots, \end{cases} \tag{26}$$

$$\widehat{k}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} -i/2 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for } m = 1, \\ \\ 0 & \text{for } m = 2, 3, \dots, \end{cases} \tag{27}$$

$$\widehat{(h^2)}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} & \text{for } m = 0, \\ \\ 0 & \text{for } m = 1, 2, 3, \dots \end{cases} \tag{28}$$

Substituting (26), (27) and (28) into (17) and (18), we get that $\lambda_{\pm 1}^{(1)} = 0$ and $\lambda_{\pm 1}^{(2)} = -\frac{1}{2}$, which is in accordance with (24) and (25).

EXAMPLE 2. If we consider

$$h_{\alpha\beta}(x^1) = \begin{pmatrix} 1 & \cos x^1 & \sin x^1 \\ \cos x^1 & \cos x^1 & \sin x^1 \\ \sin x^1 & \sin x^1 & -\cos x^1 \end{pmatrix}, \quad k_{\alpha\beta}(x^1) = \begin{pmatrix} \sin x^1 & \cos x^1 & 0 \\ \cos x^1 & -\sin x^1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then using formulae (19) and (20), we get that the corresponding perturbed massless Dirac operator is given by

$$\begin{aligned} W(\epsilon) = {} & -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx^1} + \frac{\epsilon}{4} \begin{pmatrix} i\sin x^1 & i + \cos x^1 \\ i - \cos x^1 & -i\sin x^1 \end{pmatrix} \frac{d}{dx^1} + \frac{\epsilon}{4} \frac{d}{dx^1} \begin{pmatrix} i\sin x^1 & i + \cos x^1 \\ i - \cos x^1 & -i\sin x^1 \end{pmatrix} \\ & + \frac{\epsilon^2}{16} \begin{pmatrix} -3i\sin x^1 & -6i - 3 - 2\cos x^1 + i\sin x^1 \\ -6i + 3 + 2\cos x^1 + i\sin x^1 & 3i\sin x^1 \end{pmatrix} \frac{d}{dx^1} \\ & + \frac{\epsilon^2}{16} \frac{d}{dx^1} \begin{pmatrix} -3i\sin x^1 & -6i - 3 - 2\cos x^1 + i\sin x^1 \\ -6i + 3 + 2\cos x^1 + i\sin x^1 & 3i\sin x^1 \end{pmatrix} - \frac{3\epsilon^2}{16} I. \end{aligned}$$

Using formula (23) for the pseudoinverse operator in the case of eigenvalues $n=\pm 1$, as well as (7), (21) and (22), we get that

$$\lambda_{+1}(\epsilon) = 1 - \frac{1}{2}\epsilon + \frac{3}{4}\epsilon^2 + O(\epsilon^3), \tag{29}$$

$$\lambda_{-1}(\epsilon) = -1 + \frac{1}{2}\epsilon - \epsilon^2 + O(\epsilon^3). \tag{30}$$

Application of the Fourier transform (4) gives us

$$\widehat{h}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for } m = 0, \\ \begin{pmatrix} 0 & 1/2 & -i/2 \\ 1/2 & 1/2 & -i/2 \\ -i/2 & -i/2 & -1/2 \end{pmatrix} & \text{for } m = 1, \\ 0 & \text{for } m = 2, 3, \ldots, \end{cases} \tag{31}$$

$$\widehat{k}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} -i/2 & 1/2 & 0 \\ 1/2 & i/2 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \text{for } m = 1, \\ 0 & \text{for } m = 2, 3, \ldots, \end{cases} \tag{32}$$

$$\widehat{(h^2)}_{\alpha\beta}(m) = \begin{cases} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3/2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix} & \text{for } m = 0, \\ \begin{pmatrix} 0 & 1/2 & -i/2 \\ 1/2 & 0 & 0 \\ -i/2 & 0 & 0 \end{pmatrix} & \text{for } m = 1, \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -i/4 \\ 0 & -i/4 & -1/4 \end{pmatrix} & \text{for } m = 2, \\ 0 & \text{for } m = 3, 4, \ldots. \end{cases} \tag{33}$$

Substituting (31), (32) and (33) into (17) and (18), we get that $\lambda^{(1)}_{\pm 1} = \mp\frac{1}{2}$, $\lambda^{(2)}_{+1} = \frac{3}{4}$ and $\lambda^{(2)}_{-1} = -1$ which is in accordance with (29) and (30).

5. Conclusions and future goals

In this paper we obtained spectral asymmetry of the massless Dirac operator in the axisymmetric case for an arbitrary eigenvalue different from zero. We also developed general methods and performed calculations which we can apply in order to obtain spectral asymmetry not only in the axisymmetric case but also in the general case (1), which we were not able to do at this time, but hope to achieve in the near future.

The eigenvalues with smallest modulus of the massless Dirac operator on the unit torus $\mathbb{T}^3$ equipped with Euclidean metric and standard spin structure and the unit sphere $\mathbb{S}^3$ equipped with Riemannian metric were considered in [9] and [11], for which the asymptotic formulae were derived. In this paper we worked on the unit torus $\mathbb{T}^3$ and for the axisymmetric case we derived the asymptotic formulae for the eigenvalues $\pm n$.

Our future goal is to consider eigenvalues, which are not with smallest modulus, of the massless Dirac operator when the metric not only depends on one spatial coordinate and to derive asymptotic formulae for these eigenvalues in order to obtain spectral asymmetry.

Appendices

A. Pseudoinverse operator construction

Let $v^{(0)}$ be a normalized eigenvector of the operator A corresponding to the eigenvalue $\lambda^{(0)}$. Then the vector $C(v^{(0)})$ is also a normalized eigenvector corresponding to the eigenvalue $\lambda^{(0)}$, see Remark 4. Consider the problem

$$\left(A-\lambda^{(0)}\right)v=f, \tag{34}$$

for a given function $f\in L^2(M;\mathbb{C}^2)$ where we need to find the function $v\in H^1(M;\mathbb{C}^2)$. Suppose that the function f satisfies the conditions

$$\langle f,v^{(0)}\rangle=\langle f,C(v^{(0)})\rangle=0,$$

where C is the charge conjugation operator (2). The problem (34) can be resolved for the function v and the uniqueness of this function is achieved by the conditions

$$\langle v,v^{(0)}\rangle=\langle v,C(v^{(0)})\rangle=0.$$

We define the operator Q as $Q:f\mapsto v$. The operator Q is a bounded linear operator acting on the orthogonal complement of the eigenspace of the operator A corresponding to the eigenvalue $\lambda^{(0)}$. Also, the bounded linear operator Q is self-adjoint and commutes with the antilinear operator of charge conjugation (2). We can extend the acting of the pseudoinverse operator Q to the whole Hilbert space $L^2(M;\mathbb{C}^2)$ in accordance with $Qv^{(0)}=QC(v^{(0)})=0$.

Now we present the construction of the pseudoinverse operator Q itself, largely following the exposition of [24]. Let λ be an eigenvalue of multiplicity k of a Hermitian operator A. Then the homogenous equation $(A-\lambda^{(0)})v=0$ has k linearly independent solutions $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(k)}$ for which we can assume orthonormality, i.e.

$$\langle \phi^{(i)}, \phi^{(j)} \rangle = \delta_{ij}, \quad (i, j = 1, 2, \dots, k).$$

The operator $A - \lambda^{(0)}$ has no inverse, but there is a unique bounded Hermitian operator Q such that $Q\phi^{(i)} = 0, \ (i = 1, 2, \dots, k)$ and

$$Q(A - \lambda^{(0)})u = u - \sum_{i=1}^{k} \langle \phi^{(i)}, u \rangle \phi^{(i)}.$$

Define the projector operator P by

$$Pu := \sum_{i=1}^{k} \langle \phi^{(i)}, u \rangle \phi^{(i)}.$$

Then the properties of the above operator Q can be written as $QP = 0$ and $Q(A - \lambda^{(0)}) = I - P$. The operator Q is called the pseudoinverse operator of the operator $A - \lambda$. We can complete the eigenvectors $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(k)}$ with eigenvectors $\phi^{(k+1)}, \phi^{(k+2)}, \dots, \phi^{(n)}$, which correspond to eigenvalues $\lambda_{k+1}, \lambda_{k+2}, \dots, \lambda_n$, respectively, such that

$$\langle \phi^{(i)}, \phi^{(j)} \rangle = \delta_{ij}, \quad (i, j = 1, 2, \dots, n).$$

Expanding v and f as

$$v = \sum_{i=1}^{n} \langle \phi^{(i)}, v \rangle \phi^{(i)}, \quad f = \sum_{i=1}^{n} \langle \phi^{(i)}, f \rangle \phi^{(i)},$$

from equation (34) we get that

$$\langle \phi^{(i)}, f \rangle = 0, \quad (i = 1, 2, \dots, k),$$
$$\langle \phi^{(i)}, f \rangle = (\lambda_i - \lambda) \langle \phi^{(i)}, v \rangle, \quad (i = k+1, k+2, \dots, n).$$

If the equation for v has a solution, it is necessary for f to be orthogonal to all solutions of the homogeneous equation. Hence we set

$$v = \sum_{i=1}^{k} v_i \phi^{(i)} + \sum_{\lambda_i \neq \lambda} \frac{\langle \phi^{(i)}, f \rangle}{\lambda_i - \lambda} \phi^{(i)},$$

where v_i are arbitrary constants. The vector v defines the complete solution of the equation.

Let the operator P be the projector operator A into the space spanned by the vectors $\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(k)}$ and P_i the projector into the one-dimensional space spanned by $\phi^{(i)}, \ (i = k+1, k+2, \dots, n)$.

DEFINITION 1. The operator

$$Q = \sum_{\lambda_i \neq \lambda^{(0)}} \frac{P_i}{\lambda_i - \lambda^{(0)}}$$

is the *pseudoinverse operator* of the operator $A - \lambda^{(0)}$.

B. Explicit formulae for the asymptotic coefficients

Now we will derive the explicit formulae for the coefficients $\lambda^{(1)}$ and $\lambda^{(2)}$ in the asymptotic expansion (14). Consider the perturbed eigenvalue problem

$$W_{1/2}(\epsilon)v(\epsilon) = \lambda(\epsilon)v(\epsilon).$$

Using formulae (13), (14) and (15), we get that

$$\begin{aligned}(W^{(0)}_{1/2} + \epsilon W^{(1)}_{1/2} + \epsilon^2 W^{(2)}_{1/2} + \cdots)(v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots)\\ = (\lambda^{(0)} + \epsilon\lambda^{(1)} + \epsilon^2\lambda^{(2)} + \cdots)(v^{(0)} + \epsilon v^{(1)} + \epsilon^2 v^{(2)} + \cdots).\end{aligned}$$

By grouping together the elements not containing ϵ, we get that

$$W^{(0)}_{1/2}v^{(0)} = \lambda^{(0)}v^{(0)},$$

which is the unperturbed eigenvalue problem. By grouping together the elements containing ϵ, we get that

$$W^{(0)}_{1/2}v^{(1)} + W^{(1)}_{1/2}v^{(0)} = \lambda^{(0)}v^{(1)} + \lambda^{(1)}v^{(0)}$$

and hence $(W^{(0)}_{1/2} - \lambda^{(0)})v^{(1)} = (\lambda^{(1)} - W^{(1)}_{1/2})v^{(0)}$, i.e. $v^{(1)} = Q(\lambda^{(1)} - W^{(1)}_{1/2})v^{(0)}$ where Q is the pseudoinverse operator of the operator $W^{(0)}_{1/2} - \lambda^{(0)}$. We denote by

$$f^{(1)} = (\lambda^{(1)} - W^{(1)}_{1/2})v^{(0)}. \tag{35}$$

By grouping together the elements containing ϵ^2, we get that

$$W^{(0)}_{1/2}v^{(2)} + W^{(1)}_{1/2}v^{(1)} + W^{(2)}_{1/2}v^{(0)} = \lambda^{(0)}v^{(2)} + \lambda^{(1)}v^{(1)} + \lambda^{(2)}v^{(0)}$$

and hence

$$(W^{(0)}_{1/2} - \lambda^{(0)})v^{(2)} = (\lambda^{(2)} - W^{(2)}_{1/2})v^{(0)} + (\lambda^{(1)} - W^{(1)}_{1/2})v^{(1)}.$$

We denote by

$$\begin{aligned}f^{(2)} &= (\lambda^{(2)} - W^{(2)}_{1/2})v^{(0)} + (\lambda^{(1)} - W^{(1)}_{1/2})v^{(1)}\\ &= (\lambda^{(2)} - W^{(2)}_{1/2})v^{(0)} + (\lambda^{(1)} - W^{(1)}_{1/2})Q(\lambda^{(1)} - W^{(1)}_{1/2})v^{(0)}.\end{aligned} \tag{36}$$

Continuing this process, the vectors $f^{(k)}$ and the coefficients $\lambda^{(k)}$ are obtained from the conditions

$$\langle f^{(k)}, v^{(0)}\rangle = 0, \tag{37}$$

$$\langle f^{(k)}, C(v^{(0)})\rangle = 0. \tag{38}$$

REMARK 10. The eigenvalues have even multiplicity, so the condition (38) is an additional condition which need to be satisfied. This is the part where our perturbation process differs from the standard perturbation process for single eigenvalues.

The components $v^{(k)}$ are given by

$$v^{(k)} = Qf^{(k)},$$

where Q is the pseudoinverse operator of the operator $W^{(0)}_{1/2} - \lambda^{(0)}$. Substituting formulae (35), (36) into formulae (37), (38) we obtain formulae (21), (22).

C. Detailed calculations of the asymptotic coefficients

In this appendix we provide detailed calculations of the formulae for the asymptotic coefficients (17) and (18) from Theorem 1 which correspond to the eigenvalues $\pm n$. We will use the perturbation theory which is described in Section 2 and the explicitly derived formulae for the asymptotic coefficients given by (21), (22). We will also use the concept of the pseudoinverse of the massless Dirac operator whose construction is given in Appendix A.

C.1. The calculations of the $\lambda^{(1)}_{\pm n}$ coefficients

We first want to derive equation (17) from Theorem 1. Using the formula for the eigenvector (11) corresponding to the eigenvalues n and $-n$ respectively, as well as formula (19) for the differential operator $W^{(1)}_{1/2}$, integrating by parts, we get that the equation (21) for the eigenvalues $\pm n$ becomes

$$\begin{aligned}
\lambda^{(1)}_{\pm n} &= \langle W^{(1)}_{1/2} v^{(0)}, v^{(0)} \rangle = \int_0^{2\pi} [v^{(0)}]^* W^{(1)}_{1/2} v^{(0)} dx^1 \\
&= \frac{i}{4} \int_0^{2\pi} [v^{(0)}]^* \begin{pmatrix} {h_3}^1 & {h_1}^1 - i{h_2}^1 \\ {h_1}^1 + i{h_2}^1 & -{h_3}^1 \end{pmatrix} \frac{d}{dx^1} v^{(0)} dx^1 \\
&\quad - \frac{i}{4} \int_0^{2\pi} \frac{d}{dx^1} [v^{(0)}]^* \begin{pmatrix} {h_3}^1 & {h_1}^1 - i{h_2}^1 \\ {h_1}^1 + i{h_2}^1 & -{h_3}^1 \end{pmatrix} v^{(0)} dx^1 \\
&= \mp \frac{n}{4\pi} \int_0^{2\pi} {h_1}^1(x^1) dx^1 = \mp \frac{n}{2} \widehat{h}_{11}(0),
\end{aligned} \tag{39}$$

where $\widehat{h}(m)$ denotes the Fourier coefficient (4) for the function $h(x^1)$.

C.2. The calculations of the $\lambda^{(2)}_{\pm n}$ coefficients

First, we will derive equation (18) from Theorem 1 for the coefficient $\lambda^{(2)}_{+n}$ in the asymptotic expansion (16) of the eigenvalue n. We will separate the calculation of this coefficient into several parts, for sake of simplicity and readability, using the explicit formula (22). Let us therefore first evaluate the term $\langle W^{(2)}_{1/2} v^{(0)}, v^{(0)} \rangle$. Using the formula for the eigenvector (11) corresponding to the eigenvalue n, as well as formula (20) for the differential operator $W^{(2)}_{1/2}$, integrating by parts we get that

$$\begin{aligned}\langle W^{(2)}_{1/2}v^{(0)},v^{(0)}\rangle &= \int_0^{2\pi}[v^{(0)}]^* W^{(2)}_{1/2}v^{(0)}dx^1\\ &= \frac{3n}{8}\widehat{(h^2)}_{11}(0)-\frac{n}{8}\widehat{k}_{11}(0)-\frac{i}{16}\varepsilon_{\beta\gamma 1}\sum_{m\in\mathbb{Z}\setminus\{0\}} m\overline{\widehat{h}_{\alpha\beta}(m)}\widehat{h}_{\alpha\gamma}(m).\end{aligned}\tag{40}$$

We simplified equation (40) using the Fourier coefficients (4) and Parseval's formula

$$\frac{1}{2\pi}\int_0^{2\pi}p(x)\overline{q(x)}dx=\sum_{m\in\mathbb{Z}}\widehat{p}(m)\overline{\widehat{q}(m)}.$$

Secondly, we will evaluate the term $\langle (W^{(1)}_{1/2}-\lambda^{(1)})Q(W^{(1)}_{1/2}-\lambda^{(1)})v^{(0)},v^{(0)}\rangle$. The pseudoinverse operator (23) corresponding to the operator $W_{1/2}-nI$ is given by

$$\begin{aligned}Q_{+n}=\frac{1}{4\pi}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}\Bigg[&e^{imx^1}\begin{pmatrix}1&1\\1&1\end{pmatrix}\int_0^{2\pi}e^{-imy^1}(\,\cdot\,)dy^1\\ &+e^{-imx^1}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\int_0^{2\pi}e^{imy^1}(\,\cdot\,)dy^1\Bigg].\end{aligned}\tag{41}$$

Using (19), (11) (for eigenvalue n) and (17), we have that

$$\begin{aligned}(W^{(1)}_{1/2}-\lambda^{(1)}_{+n})v^{(0)}&=-\frac{n}{4\sqrt{\pi}}\begin{pmatrix}{h_1}^1-i{h_2}^1+{h_3}^1\\{h_1}^1+i{h_2}^1-{h_3}^1\end{pmatrix}e^{inx^1}\\ &+\frac{i}{8\sqrt{\pi}}e^{inx^1}\frac{d}{dx^1}\begin{pmatrix}{h_1}^1-i{h_2}^1+{h_3}^1\\{h_1}^1+i{h_2}^1-{h_3}^1\end{pmatrix}+\frac{n\widehat{h}_{11}(0)}{4\sqrt{\pi}}\begin{pmatrix}1\\1\end{pmatrix}e^{inx^1}.\end{aligned}\tag{42}$$

Using the formula for the pseudoinverse (41), we evaluate $Q_{+n}((W^{(1)}_{1/2}-\lambda^{(1)}_{+n})v^{(0)})$ in three parts. First we will act with the pseudoinverse Q_{+n} on the first term on the RHS of equation (42). Using the well known property of the Fourier coefficient that $\widehat{h}(-m)=\overline{\widehat{h}(m)}$, we obtain

$$\begin{aligned}&Q_{+n}\left(-\frac{n}{4\sqrt{\pi}}\begin{pmatrix}{h_1}^1-i{h_2}^1+{h_3}^1\\{h_1}^1+i{h_2}^1-{h_3}^1\end{pmatrix}e^{inx^1}\right)=-\frac{n}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}\\ &\Bigg[e^{imx^1}\begin{pmatrix}1&1\\1&1\end{pmatrix}\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}{h_1}^1-i{h_2}^1+{h_3}^1\\{h_1}^1+i{h_2}^1-{h_3}^1\end{pmatrix}e^{-i(m-n)y^1}dy^1\\ &\quad+e^{-imx^1}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}\frac{1}{2\pi}\int_0^{2\pi}\begin{pmatrix}{h_1}^1-i{h_2}^1+{h_3}^1\\{h_1}^1+i{h_2}^1-{h_3}^1\end{pmatrix}e^{-i(-(m+n))y^1}dy^1\Bigg]\\ &=-\frac{n}{4\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}\begin{pmatrix}\widehat{h}_{11}(m-n)e^{imx^1}-i\,\overline{\widehat{h}_{21}(m+n)}e^{-imx^1}+\overline{\widehat{h}_{31}(m+n)}e^{-imx^1}\\ \widehat{h}_{11}(m-n)e^{imx^1}+i\,\overline{\widehat{h}_{21}(m+n)}e^{-imx^1}-\overline{\widehat{h}_{31}(m+n)}e^{-imx^1}\end{pmatrix}.\end{aligned}$$

Secondly, we act with the pseudoinverse Q_{+n} on the second term on the RHS of (42). Integrating by parts, we obtain

$$Q_{+n}\left(\frac{i}{8\sqrt{\pi}}e^{inx^1}\frac{d}{dx^1}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix}\right) = -\frac{1}{16\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}$$
$$\left[e^{imx^1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\frac{m-n}{2\pi}\int_0^{2\pi} e^{-i(m-n)y^1}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} dy^1\right.$$
$$\left. - e^{-imx^1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\frac{m+n}{2\pi}\int_0^{2\pi} e^{-i(-(m+n))y^1}\begin{pmatrix} {h_1}^1 - i{h_2}^1 + {h_3}^1 \\ {h_1}^1 + i{h_2}^1 - {h_3}^1 \end{pmatrix} dy^1\right]$$
$$= -\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\begin{pmatrix} \widehat{h}_{11}(m-n) \\ \widehat{h}_{11}(m-n) \end{pmatrix} e^{imx^1}$$
$$-\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{m+n}{m-n}\begin{pmatrix} \overline{i\widehat{h}_{21}(m+n)} - \overline{\widehat{h}_{31}(m+n)} \\ -\overline{i\widehat{h}_{21}(m+n)} + \overline{\widehat{h}_{31}(m+n)} \end{pmatrix} e^{-imx^1}.$$

Thirdly, we act with Q_{+n} on the third and final term on the RHS of (42)

$$Q_{+n}\left(\frac{\widehat{h}_{11}(0)}{4\sqrt{\pi}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{inx^1}\right) = \frac{\widehat{h}_{11}(0)}{16\pi\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}\left[e^{imx^1}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\int_0^{2\pi}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(n-m)y^1} dy^1\right.$$
$$\left. + e^{-imx^1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\int_0^{2\pi}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i(m+n)y^1} dy^1\right]. \tag{43}$$

For $m \in \mathbb{Z}$ we have that

$$\int_0^{2\pi} e^{i(n-m)y^1} dy^1 = \begin{cases} 2\pi, & m = n, \\ 0, & m \neq n, \end{cases}$$
$$\int_0^{2\pi} e^{i(m+n)y^1} dy^1 = \begin{cases} 2\pi, & m = -n, \\ 0, & m \neq -n. \end{cases}$$

We get that the sum (43) only makes sense if $m = -n$. Hence

$$Q_{+n}\left(\frac{\widehat{h}_{11}(0)}{4\sqrt{\pi}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{inx^1}\right) = -\frac{\widehat{h}_{11}(0)}{32\pi\sqrt{\pi}}e^{inx^1}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2\pi \\ 2\pi \end{pmatrix} = 0.$$

Putting the above calculations together, we get that

$$Q_{+n}((A^{(1)} - \lambda_{+n}^{(1)})v^{(0)}) = -\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{m+n}{m-n}\begin{pmatrix} \widehat{h}_{11}(m-n) \\ \widehat{h}_{11}(m-n) \end{pmatrix} e^{imx^1}$$
$$-\frac{1}{8\sqrt{\pi}}\sum_{m\in\mathbb{Z}\setminus\{n\}}\begin{pmatrix} \overline{i\widehat{h}_{21}(m+n)} - \overline{\widehat{h}_{31}(m+n)} \\ -\overline{i\widehat{h}_{21}(m+n)} + \overline{\widehat{h}_{31}(m+n)} \end{pmatrix} e^{-imx^1}. \tag{44}$$

Calculating $\frac{d(Q_+((A^{(1)}-\lambda_+^{(1)})v^{(0)}))}{dx^1}$ and using equations (19), (17) and (44), we get the term $(W_{1/2}^{(1)}-\lambda_+^{(1)})Q_+((W_{1/2}^{(1)}-\lambda_+^{(1)})v^{(0)})$ and finally, the second term $\langle(W_{1/2}^{(1)}-\lambda_+^{(1)})Q_+(W_{1/2}^{(1)}-\lambda_+^{(1)})v^{(0)},v^{(0)}\rangle$, using (7), becomes

$$\begin{aligned}
&\langle(W_{1/2}^{(1)}-\lambda_{+n}^{(1)})Q_{+n}((W_{1/2}^{(1)}-\lambda_{+n}^{(1)})v^{(0)}),v^{(0)}\rangle\\
&\quad=\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{n\}}\frac{1}{m-n}(m+n)^2\,\widehat{h}_{11}(m-n)\overline{\widehat{h}_{11}(m-n)}\\
&\qquad+\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{n\}}(m-n)\,\widehat{h}_{31}(m+n)\left(\overline{\widehat{h}_{31}(m+n)}-i\overline{\widehat{h}_{21}(m+n)}\right)\\
&\qquad+\frac{1}{16}\sum_{m\in\mathbb{Z}\setminus\{n\}}i(m-n)\,\widehat{h}_{21}(m+n)\left(\overline{\widehat{h}_{31}(m+n)}-i\overline{\widehat{h}_{21}(m+n)}\right).
\end{aligned}\tag{45}$$

Combining equations (40) and (45), we get the formula (22), for the coefficient $\lambda_+^{(2)}$ is given by (18).

REMARK 11. Using the eigenvector (11) corresponding to the eigenvalue $-n$ of the massless Dirac operator and the pseudoinverse operator (23) (corresponding to $-n$) of the operator $W_{1/2}+nI$, analogously to the above calculations performed for the eigenvector (11) corresponding to the eigenvalue n, we get the formula (22), for the coefficient $\lambda_-^{(2)}$.

D. Numerical analysis of our results

This appendix deals with the operator (6) and here we numerically analyze its spectrum. Using Galerkin's method (see e.g. [13]), we discretize the eigenvalue problem of this operator.

Consider the $2m+1$ eigenvalues $\lambda_i=i,\ (i=0,\pm1,\dots,\pm m)$ of the unperturbed massless Dirac operator on half-densities $W_{1/2}(0)$. Each eigenvalue λ_i has multiplicity two and the corresponding eigenvectors $v_i(x^1)$ and $w_i(x^1)$, $(i=0,\pm1,\dots,\pm m)$ are given by (11) and (12). Hence, we have that

$$W_{1/2}(0)v_i(x^1)=\lambda_i v_i(x^1),\tag{46}$$

$$W_{1/2}(0)w_i(x^1)=\lambda_i w_i(x^1),\tag{47}$$

where $i=0,\pm1,\dots,\pm m$. The eigenvectors $v_i(x^1)$ and $w_i(x^1)$ are orthonormal with respect to the inner product (7), i.e.

$$\langle v_i,v_j\rangle=\langle w_i,w_j\rangle=\delta_{ij},\quad\langle v_i,w_j\rangle=\langle w_i,v_j\rangle=0,\quad(i,j=0,\pm1,\dots,\pm m).\tag{48}$$

According to (48), from equations (46) and (47), for $i,j=0,\pm1,\dots,\pm m$ we have that

$$\lambda_i=\langle W_{1/2}(0)v_i(x^1),v_i(x^1)\rangle=\langle W_{1/2}(0)w_i(x^1),w_i(x^1)\rangle$$

and

$$\langle W_{1/2}(0)v_i(x^1), w_j(x^1)\rangle = \langle W_{1/2}(0)v_j(x^1), w_i(x^1)\rangle = 0,$$
$$\langle W_{1/2}(0)w_i(x^1), v_j(x^1)\rangle = \langle W_{1/2}(0)w_j(x^1), v_i(x^1)\rangle = 0.$$

Let us now construct the matrices

$$H_{i,j} := \begin{pmatrix} \langle W_{1/2}(0)v_i, v_j\rangle & \langle W_{1/2}(0)v_i, w_j\rangle \\ \langle W_{1/2}(0)w_i, v_j\rangle & \langle W_{1/2}(0)w_i, w_j\rangle \end{pmatrix} \tag{49}$$

where $i,j = 0,\pm 1,\dots,\pm m$. Using matrices (49), we can construct the block matrix H as follows

$$H := \begin{pmatrix} H_{-m,m} & & & H_{0,m} & & & H_{m,m} \\ & \ddots & & \vdots & & \iddots & \\ & & & H_{0,1} & & & \\ & \cdots & H_{-1,0} & H_{0,0} & H_{1,0} & \cdots & \\ & & & H_{0,-1} & & & \\ & \iddots & & \vdots & & \ddots & \\ H_{-m,-m} & & & H_{0,-m} & & & H_{m,-m} \end{pmatrix}.$$

The matrix H is a quadratic matrix of order $2(2m+1)$ and by construction it is a Hermitian matrix. The eigenvalues of the matrix H are $\lambda = 0,\pm 1,\dots,\pm m$ and each eigenvalue has multiplicity two. Therefore, the analysis of the eigenvalue problem (46)-(47) and the analysis of the eigenvalue problem of the matrix H are equivalent.

Now we choose to consider the matrix $H(\epsilon)$ with the perturbed massless Dirac operator $W_{1/2}(\epsilon)$ instead of the unperturbed operator $W_{1/2}(0)$. Then the matrix $H(\epsilon)$ is a Hermitian matrix whose entries depend on the parameter ϵ and $H(0) = H$. Using the perturbation process described in [16] for perturbed Hermitian matrices, we can get the asymptotic expansions of the eigenvalues of the perturbed matrix $H(\epsilon)$ and specially the asymptotic expansions of the eigenvalues $\lambda = \pm 1$. The eigenvalues of the matrix $H(\epsilon)$ will converge to the eigenvalues of the matrix H as $\epsilon \to 0$.

The examination of the spectrum of the perturbed massless Dirac operator will reduce to the examination of the spectrum of the Hermitian matrix $H(\epsilon)$. The numerical calculations were performed using *Wolfram Mathematica*.

EXAMPLE 3. Consider the coframe

$$e^j{}_\alpha = \delta^j{}_\alpha + \epsilon \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos x^1 & \sin x^1 \\ 0 & \sin x^1 & -\cos x^1 \end{pmatrix}. \tag{50}$$

which was also considered in [9]. The explicit formula for the perturbed massless Dirac operator corresponding to the coframe (50) reads

$$W(\epsilon) = -i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx^1} - \frac{\epsilon^2}{2(1-\epsilon^2)} I. \tag{51}$$

The eigenvalues of the operator (51) are explicitly given by

$$\lambda_n(\epsilon) = n - \frac{\epsilon^2}{2(1-\epsilon^2)} = n - \frac{\epsilon^2}{2} - \frac{\epsilon^4}{2} + O(\epsilon^6), \quad n \in \mathbb{Z} \tag{52}$$

and all eigenvalues have multiplicity two.

Now we will use the coframe (50) to analyze the spectrum of the massless Dirac operator using the Galerkin method described above in order to numerically confirm these results. We explicitly constructed the matrix $H(\epsilon)$ of order 102×102 and numerically analyzed the part of its spectrum. The eigenvalues $0, \pm 1, \pm 2$ of the matrix $H(\epsilon)$ are perturbed as follows

	-2	-1	0	1	2
$\epsilon = 0.2$	-2.02083	-1.02083	-0.0208333	0.979167	1.97917
$\epsilon = 0.1$	-2.00505	-1.00505	-0.00505051	0.994949	1.994950
$\epsilon = 0.01$	-2.00005	-1.00005	-0.000050005	0.99995	1.99995

and each eigenvalue has multiplicity two. Analyzing the data given in the above table we see that for this choice of the coframe the spectral symmetry of the matrix $H(\epsilon)$ is broken and consequently we obtain spectral asymmetry of the massless Dirac operator in the axisymmetric case.

Using the perturbation process for the matrices with double eigenvalues described in [16], we get that the asymptotic formulae for the eigenvalues ± 1 are given by

$$\lambda_+(\epsilon) = 1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3),$$
$$\lambda_-(\epsilon) = -1 - \frac{1}{2}\epsilon^2 + O(\epsilon^3).$$

which is in accordance with (52).

Now, we will consider the coframe which is not symmetric to show that in this case it is also possible to obtain spectral asymmetry.

EXAMPLE 4. Consider the coframe

$$e^j{}_\alpha = \delta^j{}_\alpha + \epsilon \begin{pmatrix} 0 & \cos x^1 - \cos 2x^1 + \cos 3x^1 & \sin x^1 + \sin 2x^1 - \sin 3x^1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Analyzing the spectrum of the matrix $H(\epsilon)$ of order 102×102 we get that the eigenvalues $0, \pm 1, \pm 2$ of the matrix are perturbed as follows

	-2	-1	0	1	2
$\epsilon = 0.2$	-2.10913	-1.05372	0.00169489	1.0571	2.11252
$\epsilon = 0.1$	-2.02923	-1.01456	0.000119453	1.0148	2.02947
$\epsilon = 0.01$	-2.0003	-1.00015	1.24941×10^{-8}	1.00015	2.0003

and each eigenvalue has multiplicity two. Analyzing the data given in the above table we see that the spectral symmetry is broken. Using the method described in [16], we obtain that the asymptotic formulae for the eigenvalues ± 1 are given by

$$\lambda_+(\epsilon) = 1 + \frac{3}{2}\epsilon^2 - \frac{17}{8}\epsilon^4 + O(\epsilon^5),$$
$$\lambda_-(\epsilon) = -1 - \frac{3}{2}\epsilon^2 + \frac{37}{8}\epsilon^4 + O(\epsilon^5),$$

hence we see that spectral asymmetry is achieved in the quartic term.

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