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ADDITIVE LOCAL MULTIPLICATIONS AND ZERO–PRESERVING MAPS ON $C(X)$

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Dedicated to Fuying Zhang, my primary school math teacher

(*Communicated by N.-C. Wong*)

Abstract. Suppose X is a compact Hausdorff space. In terms of topological properties of X, we find topological conditions on X that are equivalent to each of the following: 1. Every additive local multiplication on $C(X)$ is a multiplication, 2. Every additive local multiplication on $C_R(X)$ is a multiplication, 3. Every additive map on $C(X)$ that is zero-preserving (i.e., $f(x)=0$ implies $(Tf)(x)=0$) has the form $T(f)=T(1)\operatorname{Re} f+T(i)\operatorname{Im} f$.

1. Introduction

Suppose X is a topological space. Let $C(X)$ and $C_R(X)$ be the set of all complex continuous functions and real continuous functions on X, respectively. This paper studies local multiplications and zero-preserving maps on the algebra $C(X)$ and $C_R(X)$ when X is a compact Hausdorff space. We find an interesting interplay between the algebraic or linear-algebraic conditions and unusual topological properties of the space X.

Suppose $\mathscr{A}$ is a ring with identity. A map T on $\mathscr{A}$ is a *local left* (*right, respectively*) *multiplication* if, for each $x\in\mathscr{A}$, there is an $a_x\in\mathscr{A}$ such that

$$T(x)=a_x x\ (xa_x, \text{respectively}).$$

The map T is a *left* (*right, respectively*) *multiplication* if there is an $a\in\mathscr{A}$ such that, for every $x\in\mathscr{A}$,

$$T(x)=ax\ (xa, \text{respectively}).$$

In this case we must have

$$a=T(1),$$

and we write

$$T=L_{T(1)}\ (R_{T(1)}, \text{respectively}),$$

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i.e., left (right, respectively) multiplication by the element $T(1)$. When the algebra $\mathscr{A}$ is commutative, there is no difference between left and right, and we write $M_{T(1)}$ for $L_{T(1)}$. A map T on $\mathscr{A}$ is an *additive map* if for every x and y in $\mathscr{A}$, $T(x+y) = T(x)+T(y)$.

There has been a lot of work characterizing cases in which every local multiplication of a certain type is a multiplication. In 1983 D. Hadwin [5] proved an early result on local multiplications. In 1994 the so-called "Hadwin Lunch Bunch" [6] gave necessary conditions for local multiplications in rings with many idempotents to be multiplications.

In 1997 D. Hadwin and J. W. Kerr [7] studied $\mathscr{R}$-linear local multiplications on an algebra $\mathscr{A}$ over commutative ring $\mathscr{R}$ with identity, and gave conditions that implied that every local multiplication on $\mathscr{A}$ is a multiplication. When $\mathscr{R}$ is the ring $\mathbb{Z}$ of integers, the $\mathscr{R}$-linear maps are simply the additive maps. When $\mathscr{A}$ is a vector space over the rational numbers $\mathbb{Q}$, the additive maps are precisely the $\mathbb{Q}$-linear maps. When $\mathscr{A}$ is a topological vector space over $\mathbb{R}$, the continuous additive maps are precisely the $\mathbb{R}$-linear ones. In [7] D. Hadwin and J. W. Kerr proved, for a large class of unital C*-algebras every additive local left (right) multiplication is a left (right) multiplication. These algebras include ones for which the set of finite-dimensional (dimension greater than 1) irreducible representations that separate the points of the algebra. However, for additive maps, the commutative algebras, i.e., ones for which every irreducible representation is 1-dimensional, were not considered. A unital C*-algebra is commutative if and only if it is isomorphic to $C(X)$ for some compact Hausdorff space X. Note that Hadwin and Kerr's results imply that every local left (right) multiplication on the algebra $\mathbb{M}_2(C(X))$ of 2×2 matrices over $C(X)$ is a left (right) multiplication.

There is a vast literature on local derivations and local automorphisms, e.g., [1], [2], [8], [9], [10], [11], [14], [16].

Another related active area of research is the study of maps that preserve a particular property. According to MATHSCINET there have been almost 200 papers studying linear preservers and at least 48 papers studying additive maps that preserve some property.

In this paper we restrict ourselves to the case where our algebra is the space $C(X)$ of complex continuous functions on a compact Hausdorff space X. It was shown in [5, Theorem 6] that every $\mathbb{C}$-linear map on $C(X)$ that is a local multiplication is a multiplication. We study additive or $\mathbb{R}$-linear local multiplications on $C(X)$. We also study maps T that are *zero-preserving*, i.e., for every $f \in C(X)$ and every $x \in X$,

$$f(x) = 0 \text{ implies } (Tf)(x) = 0.$$

In Section 2 we consider those compact Hausdorff spaces X for which every additive local multiplication on $C(X)$ must be a multiplication, and we call such X an *η-space*. If every local multiplication on $C_R(X)$ is a multiplication, we call X a *real η-space*. There is a vast difference between these two concepts. We prove (Theorem 2) that if the set of points $x \in X$ that are a limit of a *sequence* in $X \setminus \{x\}$ is dense in X, then X is an η-space. In particular, if X is first countable, then X is an η-space if and only if X has no isolated points. We also prove (Theorem 3) that the closure of the union of

a collection of η-subspaces of X is also an η-space. Hence every compact Hausdorff space has a unique maximal compact subspace that is an η-space. It follows (Theorem 4) that the Cartesian product of an η-space with any compact Hausdorff space is an η-space. We also construct many spaces that are not η-spaces. The conjugation map on $C(X)$ is $\mathbb{R}$-linear and is never a multiplication. We prove (Theorem 6) that the conjugation map is a local multiplication on $C(X)$ if and only if X is an F-space in the sense of L. Gillman and M. Jerison [4]. This is also equivalent to the set of $\mathbb{R}$-linear local multiplications on $C(X)$ being precisely the maps of the form

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

We also prove (Theorem 7) that there is a nonzero function g such that the map $T(f) = g\overline{f}$ is a local multiplication if and only if there is a nonempty open F_σ-subset (i.e., a countable union of closed sets) of X that is an F-space. As a consequence, we prove (Corollary 5) that if no nonempty open F_σ-subset of X is an F-space, then every $\mathbb{R}$-linear (or continuous additive) local multiplication on $C(X)$ is a multiplication. We also characterize (Proposition 1) the additive local multiplications on $\ell^\infty = C_b(\mathbb{N}) = C(\beta(\mathbb{N}))$, where $\beta(\mathbb{N})$ is the Stone-Čech compactification of $\mathbb{N}$.

In Section 3 we focus on the additive maps on $C(X)$ that are zero-preserving. Since the conjugation map satisfies this property, we can't expect all of these maps to be multiplications. But we can hope for them to be of the form

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

We call a space X for which every additive zero-preserving map has the above form an υ*-space*. We show (Theorem 10) that this property is equivalent to that every additive zero-preserving map on $C_R(X)$ is a multiplication. Thus every υ-space is a real η-space. This characterization allows us to carry over results on η-spaces to those of υ-spaces. In particular we prove (Theorem 12) that, if the set of sequential limit points is dense in X or X is the closure of the union of a family of compact υ-subspaces, or X is the product of a compact Hausdorff space and an υ-space, then X is an υ-space. This also shows that every compact Hausdorff space has a unique maximal compact subspace that is an υ-space. We also show (Theorem 9) that every $\mathbb{R}$-linear zero-preserving map on $C_R(X)$ is a multiplication.

In Section 4 We introduce the notions of $\mathfrak{q}$-point and strong $\mathfrak{q}$-point, which are generalizations of a sequential limit point. We prove (Theorem 13) that if the set of strong $\mathfrak{q}$-points is dense, then X is an η-space, and if the set of $\mathfrak{q}$-points is dense, then X is an υ-space. It turns out (Lemma 3) that x is a $\mathfrak{q}$-point if and only if x is not a P-point in the sense of L. Gillman and M. Henriksen [3], and we show that the set of $\mathfrak{q}$-points of X is dense if and only if X has no isolated points.

In Section 5, we present our main theorems. Our first main theorem (Theorem 14) characterizes υ-spaces and real η-spaces: Suppose X is a compact Hausdorff space. The following are equivalent:

1. X is an υ-space
2. X is a real η-space

3. The set of $\mathfrak{q}$-points of X is dense in X

4. X has no isolated points.

Our second main theorem (Theorem 15) topologically characterizes η-spaces: Suppose X is a compact Hausdorff space. Then X is an η-space if and only if no nonempty open F_σ set in X is an F-space.

In Remark 4 we describe how to construct the maximal η-subspace and the maximal υ-subspace of a compact Hausdorff space X.

We conclude with remarks about $\beta(\mathbb{N})\setminus\mathbb{N}$, where $\beta(\mathbb{N})$ denotes the Stone-Čech compactification of the set $\mathbb{N}$ of positive integers. We have $\beta(\mathbb{N})\setminus\mathbb{N}$ is an υ-space since $\beta(\mathbb{N})\setminus\mathbb{N}$ has no isolated points. However, $\beta(\mathbb{N})\setminus\mathbb{N}$ is not an η-space, since it is an F-space. We also remark that W. Rudin [13] and S. Shelah (see [16]) have proved that the assertion that every point in $\beta(\mathbb{N})\setminus\mathbb{N}$ is a $\mathfrak{q}$-point is independent from the axioms of set theory (ZFC).

For topological notions we refer the reader to [15] and [4].

2. η-spaces

Suppose Y is any completely regular Hausdorff space, then $\beta(Y)$ denotes the Stone-Čech Compactification of Y. For any real number r, $[r]$ is the largest integer not more than r.

DEFINITION 1. Suppose X is a compact Hausdorff space. If every additive local multiplication on $C(X)$ is a multiplication, then we call X an *η-space*. If every additive local multiplication on $C_R(X)$ is a multiplication, we call X a *real η-space.*

EXAMPLE 1. Suppose $X=\{a\}$ is a singleton. Thus $C(X)\cong\mathbb{C}$. Every additive map ($\mathbb{Q}$-linear map) on $\mathbb{C}$ is a local multiplication. Given a linear basis B for $\mathbb{C}$ over $\mathbb{Q}$, we can get $card(B)=card(\mathbb{R})$. Thus the cardinality of the set of all $\mathbb{Q}$-linear (i.e., additive) maps on $\mathbb{C}$ is $card(\mathbb{C})^{card(B)}=2^{(2^{\aleph_0})}$. But the cardinality of the set of all multiplications on $\mathbb{C}$ is $2^{\aleph_0}$. Thus no singleton is an η-space.

DEFINITION 2. Suppose $\{Y_i\}$ is a family of topology spaces. Let Y be the disjoint union of the Y_i's (If there are two sets intersecting, then let Y_i be $Y_i\times\{i\}$). Define a subset U of Y to be open if and only if the intersection of U and each Y_i is open in Y_i. Thus we defined the *disjoint union topology* on Y.

THEOREM 1. *Suppose X is the disjoint union of compact Hausdorff spaces Y and Z. Then X is an η-space if and only if Y and Z are η-spaces.*

Proof. We have $C(X)$ is isomorphic to the direct sum of $C(Y)$ and $C(Z)$. A map T is a local multiplication on $C(X)$ if and only if T is the direct sum of local multiplications on $C(Y)$ and $C(Z)$. $\square$

COROLLARY 1. *An η-space has no isolated points.*

Proof. If an η-space X has an isolated point x, then X is the disjoint union of $\{x\}$ and $X \setminus \{x\}$. Since $\{x\}$ is not an η-space, from Theorem 1 it follows that X is not an η-space. $\square$

DEFINITION 3. Suppose X is a topological space and $x \in X$. If there is a sequence $\{x_n\}$ in $X \setminus \{x\}$ such that $x_n \to x$, then we call x a *sequential limit point* of X.

Note that if X is a T_1 space, then $x \in X$ is a sequential limit point if and only if there is a sequence $\{x_n\}$ in $X \setminus \{x\}$ whose terms are different from each other such that $x_n \to x$.

THEOREM 2. *Suppose X is a compact Hausdorff space, and let A be the set of all sequential limit points of X. If $X = \overline{A}$, then X is an η-space.*

Proof. Suppose T is an additive local multiplication on $C(X)$, thus T is $\mathbb{Q}$-linear. Since the set of local multiplications on $C(X)$ is closed under linear combinations and compositions, T is a (local) multiplication if and only if $T - M_{T(1)}$ is a (local) multiplication. We may suppose $T(1) = 0$, and prove $T = 0$.

First we prove $T(a \cdot 1) = T(a) = 0$ for every $a \in \mathbb{R}$. Since T is $\mathbb{Q}$-linear, we know, for every $r \in \mathbb{Q}$, that

$$T(r \cdot 1) = rT(1) = 0.$$

Suppose $a \in \mathbb{R}$. Assume, via contradiction that $T(a)(y) \neq 0$ for some $y \in X$. Since A is dense in X, there is an $x \in A$ such that $T(a)(x) \neq 0$. Since $x \in A$, there is a sequence $\{x_n\}$ in $X \setminus \{x\}$ whose terms are different from each other such that $x_n \to x$. Let $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Clearly, K is a closed subset of X. Define a function $f : K \to [0,1]$ by

$$f(x_n) = a - \frac{[10^n a]}{10^n} \text{ for every } n \in \mathbb{N}, \text{ and } f(x) = 0.$$

Clearly, f is continuous on K. Since X is compact and Hausdorff, X is normal. By the Tietze extension theorem, there is a continuous function F from X to $[0,1]$ such that $F|_K = f$. Since $F(x) = 0$ and T is a local multiplication, we know that $T(F)(x) = 0$. Thus

$$\begin{aligned} T(a - F)(x) &= T(a)(x) - T(F)(x) \\ &= T(a)(x) - 0 = T(a)(x) \neq 0, \end{aligned}$$

i.e.,

$$T(a - F)(x) \neq 0.$$

Since $T(a-F)\in C(X)$, and $x_n\to x$, there exists a $k\in\mathbb{N}$ such that $T(a-F)(x_k)\neq 0$. But

$$(a-F)(x_k)=a-f(x_k)=\frac{[10^k a]}{10^k}=r_k\in\mathbb{Q}.$$

Thus

$$(a-F-r_k)(x_k)=0.$$

$$T(a-F)(x_k)=T(a-F-r_k)(x_k)=0,$$

since $T(r_k)=r_k\cdot T(1)=0$. This is a contradiction. Thus for every $y\in X$, $T(a)(y)=0$, whence $T(a)=0$.

For every $g\in C_R(X)$ and every $x\in X$, $(g-g(x)\cdot 1)(x)=0$, and since T is a local multiplication,

$$0=T(g-g(x)\cdot 1)(x)=T(g)(x)-T(g(x)\cdot 1)(x)=T(g)(x).$$

Thus $T(g)=0$.

Now suppose $h\in C(X)$, and let $h=h_1+ih_2$, where h_1, $h_2\in C_R(X)$. We have

$$T(h)=T(h_1)+T(ih_2)=T(ih_2)=T(i)h_2.$$

(Set $L(f)=T(if)$ for every $f\in C(X)$. Thus L is an additive local multiplication on $C(X)$. From above, if $L(1)=0$, then $L(g)=0$ for every $g\in C_R(X)$. Thus $L(g)=L(1)g$, i.e., $T(ig)=T(i)g$ for any $g\in C_R(X)$.) To prove $T(h)=0$, it's enough to show $T(i)=0$.

For every $x\in A$, there is a sequence $\{x_n\}$ in $X\setminus\{x\}$ whose terms are different from each other such that $x_n\to x$. Let $K=\{x_n:n\in\mathbb{N}\}\cup\{x\}$. Then K is a closed subset of X. Define a function $f:K\to[0,1]$ by

$$f(x_n)=\frac{1}{n}\cdot\left|\sin\left(\frac{n\pi}{2}\right)\right| \text{ for every } n\in\mathbb{N}, \text{ and } f(x)=0.$$

Thus f is continuous on K. By the Tietze extension theorem, there is a continuous function F from X to $[0,1]$ such that $F|_K=f$. Similarly, define $g:K\to[0,1]$ by

$$g(x_n)=\frac{1}{n} \text{ for every } n\in\mathbb{N}, \text{ and } g(x)=0.$$

We get a continuous function G from X to $[0,1]$ such that $G|_K=g$. Set $H=F+iG\in C(X)$. Thus

$$T(H)=T(F+iG)=T(i)G=h_H\cdot(F+iG)$$

for some $h_H\in C(X)$. Thus

$$\begin{aligned}
T(i)(x_n)G(x_n)&=h_H(x_n)\cdot(F(x_n)+iG(x_n))\\
T(i)(x_n)\cdot\frac{1}{n}&=h_H(x_n)\cdot\left(\frac{1}{n}\cdot\left|\sin\left(\frac{n\pi}{2}\right)\right|+i\frac{1}{n}\right)\\
T(i)(x_n)&=h_H(x_n)\cdot\left|\sin\left(\frac{n\pi}{2}\right)\right|+ih_H(x_n)
\end{aligned}$$

for every $n \in \mathbb{N}$.

Clearly, we have $T(i)(x_n) = ih_H(x_n)$ when n is even, and $T(i)(x_n) = h_H(x_n) + ih_H(x_n)$ when n is odd. Since $T(i)(x_n) \to T(i)(x)$ as $n \to \infty$, we have

$$T(i)(x) = ih_H(x) = h_H(x) + ih_H(x).$$

Thus $h_H(x) = 0$. Thus $T(i)(x) = 0$ for every $x \in A$. Since $X = \overline{A}$, $T(i) = 0$. Thus $T(h) = 0$ for every $h \in C(X)$, i.e., $T = 0$. Thus every additive local multiplication on $C(X)$ is a multiplication, i.e., X is an η-space. $\square$

REMARK 1. The preceding theorem tells us whether X is an η-space has nothing to do with the connectedness of X. From the theorem, we know the Cantor set and the closed interval $[0,1]$ are η-spaces but the former is totally disconnected and the latter is connected. Note that in this theorem $X = \overline{A}$ implies that X has no isolated points.

COROLLARY 2. *Suppose X is a compact Hausdorff space. If X is first countable, then X is an η-space if and only if X has no isolated points.*

COROLLARY 3. *Suppose Y is a completely regular Hausdorff space, and let A be the set of all sequential limit points of Y. If A is dense in Y, then $\beta(Y)$ is an η-space.*

Next theorem is another version of [7, Theorem 6].

THEOREM 3. *Suppose X is a compact Hausdorff space, and $\{K_i\}$ is a collection of closed subset of X. If each K_i is an η-space, then the closure of the union of K_i's is an η-space.*

Proof. Let K be the closure of the union of K_i's. Suppose T is an additive local multiplication on $C(K)$ and $T(1) = 0$. For each i in I, define T_i on $C(K_i)$ by

$$T_i(f) = T\left(\widetilde{f}\right)|_{K_i} \text{ for every } f \in C(K_i),$$

where $\widetilde{f} \in C(K)$ is a Tietze extension of f. The definition is well-defined. In fact, suppose $\widetilde{f}_1, \widetilde{f}_2 \in C(K)$ and $\widetilde{f}_1|_{K_i} = \widetilde{f}_2|_{K_i}$. Thus

$$T\left(\widetilde{f}_1\right) - T\left(\widetilde{f}_2\right) = T\left(\widetilde{f}_1 - \widetilde{f}_2\right) = h \cdot \left(\widetilde{f}_1 - \widetilde{f}_2\right)$$

for some $h \in C(K)$. Then

$$\begin{aligned} T\left(\widetilde{f}_1\right)|_{K_i} - T\left(\widetilde{f}_2\right)|_{K_i} &= T\left(\widetilde{f}_1 - \widetilde{f}_2\right)|_{K_i} = h|_{K_i} \cdot \left(\widetilde{f}_1 - \widetilde{f}_2\right)|_{K_i} \\ &= 0. \end{aligned}$$

Thus $T\left(\widetilde{f}_1\right)|_{K_i} = T\left(\widetilde{f}_2\right)|_{K_i}$. Clearly, T_i is an additive local multiplication on $C(K_i)$. Since K_i is an η-space and $T_i(1) = 0$, we have $T_i = 0$. Thus for every $g \in C(K)$, $T(g)|_{K_i} = 0$ for each i in I. Since K is the closure of the union of K_i's, $T(g) = 0$. Since g is arbitrary, we have $T = 0$. Thus K is an η-space. $\square$

THEOREM 4. *Suppose Y is a compact Hausdorff space, and K is an η-space. Then $Y \times K$ is an η-space.*

Proof. For any $y \in Y$, $\{y\} \times K$ is a closed η-subspace of $Y \times K$. Thus from the last theorem, $Y \times K = \cup_{y \in Y} \{y\} \times K$ is an η-space. □

COROLLARY 4. *Every compact Hausdorff space is homeomorphic to a subspace of an η-space.*

In the next theorem if $X = \{a\}$, then the maximal η-subspace of X is the empty set.

THEOREM 5. *Every compact Hausdorff space has a unique maximal η-subspace.*

Proof. Suppose X is a compact Hausdorff space, and let K be the closure of the union of all η-subspaces of X. Thus K is an η-space. Any η-subspace of X is a subset of K. Thus K is the unique maximal η-subspace of X. □

DEFINITION 4. Suppose X is a completely regular Hausdorff space. For every $f \in C(X)$, let $Z(f) = \{x \in X : f(x) = 0\}$ be the *zero-set of* f. Any set that is a zero-set of some function in $C(X)$ is called a *zero-set in* X. We call the complement of a zero-set a *cozero-set*. We say a subspace S of X is *R-embedded* if every bounded function in $C_R(S)$ can be extended to a bounded function in $C_R(X)$. If every cozero-set in X is R-embedded, we call X an *F-space* [4, Theorem 14.25(6)].

If $g \in C(X)$ is nonzero, then the map $T(f) = g\overline{f}$ is not a multiplication. We now characterize the spaces X for which T is a local multiplication.

For any a, $b \in \mathbb{R}$, the symbol $a \vee b$ denotes $\sup\{a, b\}$. Likewise, $a \wedge b$ denotes $\inf\{a, b\}$. For any f, $g \in C_R(X)$, define $(f \vee g)(x) = f(x) \vee g(x)$, for every $x \in X$. Thus $f \vee g \in C_R(X)$. Dually, we defined $f \wedge g \in C_R(X)$. In the proof of the next theorem, we use an equivalent condition for X to be an F-space [4, Theorem 14.25(5)], i.e., given an $f \in C_R(X)$, there exists a $k \in C_R(X)$ such that $f = k \cdot |f|$.

LEMMA 1. *Suppose X is a compact Hausdorff space, and $0 \neq g \in C(X)$. Let $A = X \setminus Z(g)$. Define $T(f) = g \cdot \overline{f}$ for every $f \in C(X)$. Then T is a local multiplication if and only if A is an F-space.*

Proof. $\Leftarrow$: Suppose A is an F-space. For every $f \in C(X)$, $f|_A \in C(A)$. Denote $f|_A$ by f_A. Set $f_A = u + iv$, where u, $v \in C_R(A)$. Thus on $A \setminus Z(f_A)$, we have

$$\frac{\overline{f}_A}{f_A} = \frac{u^2 - v^2}{u^2 + v^2} - i\frac{2uv}{u^2 + v^2}.$$

Let $h_1 = \frac{u^2-v^2}{u^2+v^2}$ and $h_2 = \frac{2uv}{u^2+v^2}$. Clearly, h_1, h_2 are bounded real continuous functions on $A \setminus Z(f_A) = A \setminus Z\left(u^2+v^2\right)$, where $u^2+v^2 \in C_R(A)$. Since A is an F-space, h_1, h_2 have bounded continuous extensions on A, say $\widetilde{h_1}$, $\widetilde{h_2}$. Define

$$h = g \cdot \left(\widetilde{h_1} - i\widetilde{h_2}\right) \text{ on } A, \text{and } h = 0 \text{ on } Z(g).$$

Thus $h \in C(X)$ and $h \cdot f = g \cdot \overline{f}$. Thus $T(f) = g \cdot \overline{f}$ is a local multiplication.

$\Longrightarrow$: Suppose T is a local multiplication. For each $h \in C_R(A)$, if h is not bounded, choose an $r > 0$ and let $h_r = (-r \vee h) \wedge r$, then h_r is a bounded real continuous function on A. Define

$$k = g \cdot (h_r + i|h_r|) \text{ on } A, \text{ and } k = 0 \text{ on } Z(g).$$

Thus $k \in C(X)$. Since T is a local multiplication, there is an $h_k \in C(X)$ such that $T(k) = g \cdot \overline{k} = h_k \cdot k$. Thus on $A \setminus Z(h_r)$, we have

$$h_k = g \cdot \frac{\overline{k}}{k} = g \cdot \frac{\overline{g} \cdot (h_r - i|h_r|)}{g \cdot (h_r + i|h_r|)} = -i\overline{g} \cdot \frac{h_r}{|h_r|},$$

i.e., $h_k = -i\overline{g} \cdot \frac{h_r}{|h_r|}$. Since $\overline{g} \neq 0$ on A, we have

$$\frac{ih_k}{\overline{g}} = \frac{h_r}{|h_r|}$$

on $A \setminus Z(h_r)$. Set $\frac{ih_k}{\overline{g}} = f_1 + if_2$, where f_1, $f_2 \in C_R(A)$. From above, $f_1 = \frac{h_r}{|h_r|}$, and $f_2 = 0$ on $A \setminus Z(h_r)$. Thus $h_r = f_1 \cdot |h_r|$ on A.

We say $h = f_1 \cdot |h|$ on A. Since if $h(x) < -r$ for some $x \in A$, we have $h_r(x) = -r$. Then $f_1(x) = -1$ and $h(x) = -|h(x)| = f_1(x) \cdot |h(x)|$. For $h(x) > r$, we get the same result. Thus we proved that for every $h \in C_R(A)$, there exists an $f_1 \in C_R(A)$ such that $h = f_1 \cdot |h|$, i.e., A is an F-space. $\square$

THEOREM 6. *Suppose X is a compact Hausdorff space. The following are equivalent:*

1. *The conjugation map is a local multiplication on $C(X)$.*
2. *X is an F-space.*
3. *The set of $\mathbb{R}$-linear local multiplications on $C(X)$ consists of all maps of the form*
$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

Proof. The equivalent of (1) and (2) is immediately derived from Lemma 1.

$(2) \Leftrightarrow (3)$: Suppose T is an $\mathbb{R}$-linear local multiplication on $C(X)$. Thus $T(a \cdot 1) = aT(1)$ for every $a \in \mathbb{R}$. From the proof in Theorem 2 we have that T has the form

$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f)$ for any $f \in C(X)$. Conversely, if T is a map on $C(X)$ which has the form $T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f)$, then of course it is $\mathbb{R}$-linear. Also we have

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f) = \left[\frac{T(1) - iT(i)}{2}\right] \cdot f + \left[\frac{T(1) + iT(i)}{2}\right] \cdot \overline{f}.$$

Thus T is a local multiplication if and only if $\left[\frac{T(1)+iT(i)}{2}\right] \cdot \overline{f}$ is a local multiplication. If X is an F-space, then every cozero-set in X is an F-space [4, 14.26]. From Lemma 1, we have $\left[\frac{T(1)+iT(i)}{2}\right] \cdot \overline{f}$ is a local multiplication, i.e., T is a local multiplication. Thus we prove $(2) \Rightarrow (3)$.

If every T of the form $T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f)$ is a local multiplication, then $\left[\frac{T(1)+iT(i)}{2}\right] \cdot \overline{f}$ is a local multiplication. From Lemma 1, $X \setminus Z\left(\frac{T(1)+iT(i)}{2}\right)$ is an F-space. Choose $T(1)$ and $T(i)$ such that $Z\left(\frac{T(1)+iT(i)}{2}\right)$ is empty. Then X is an F-space. $\square$

The following theorem is an immediate consequence of Lemma 1 and the fact that $A \subseteq X$ is an open F_σ if and only if there is a $g \in C(X)$ such that $A = X \setminus Z(g)$ (see [4].)

THEOREM 7. *Suppose X is a compact Hausdorff space. The following are equivalent:*

1. *There is a nonzero function $g \in C(X)$ such that the map $T(f) = g\overline{f}$ is a local multiplication on $C(X)$.*

2. *There is a nonempty open F_σ subset of X that is an F-space.*

COROLLARY 5. *No nonempty open F_σ-subset of X is an F-space if and only if every $\mathbb{R}$-linear local multiplication on X is a multiplication.*

Proof. The sufficiency follows from Theorem 7, since a map $T(f) = g\overline{f}$ with $g \neq 0$ is never a multiplication. Conversely, suppose no nonempty open F_σ-subset of X is an F-space. Suppose T is a real linear local multiplication on $C(X)$. We may suppose $T(1) = 0$. Thus $T(u) = 0$ for every $u \in C_R(X)$. Thus $T(f) = T(i)\operatorname{Im} f$ for any $f \in C(X)$. If $T(i) \neq 0$, we have

$$\frac{i}{2}T(i)\overline{f} = T(i)(\operatorname{Im} f) + M_{\frac{i}{2}T(i)}f$$

is a local multiplication. It follows from Theorem 7 that $X \setminus Z(T(i))$ is a nonempty F-space, which is a contradiction.

Thus $T(i) = 0$ and $T = 0$, i.e., T is a multiplication. $\square$

EXAMPLE 2. If X is an F-space, then the maximal η-subspace is the empty set. If X is the union of a compact η-space A and a compact F-space Y, then A is the maximal η-subspace of X. To see this, suppose $A \subseteq K$ and $A \neq K$ and K is compact. Choose $b \in K \setminus A$ and choose a continuous function $g : X \to [0,1]$ such that $g|_A = 0$ and $g(b) = 1$. Suppose $f \in C(X)$. Since Y is an F-space, we know from Theorem 6 and the Tietze extension theorem that there is an $h \in C(X)$ such that $\overline{f} = hf$ on Y. Since $g|_A = 0$, we see that

$$g\overline{f} = ghf$$

on X. Thus $T(f) = g\overline{f}$ is a local multiplication that is not a multiplication. Thus $T(f|_K) = (g|_K)\left(\overline{f}|_K\right)$ defines a local multiplication on $C(K)$ that is not a multiplication.

If Y is a completely regular Hausdorff space, let $C_b(Y)$ denote the bounded continuous functions on Y. Then $C_b(Y) \cong C(\beta(Y))$ (see [4]). We can say Y is an η-space when $\beta(Y)$ is an η-space.

THEOREM 8. *Suppose I is an infinite set. Suppose Y_i is a nonempty compact η-space for each i in I, and Y is the disjoint union of Y_i's. Let $K = \beta(Y) \setminus Y$. Then K is an F-space if and only if I is countable.*

Proof. A function f in $C_b(Y)$ corresponds to a uniformly bounded family $\{f_i\}$ with f_i in $C(Y_i)$. The function f is 0 on K if and only if for every $r > 0$, there is a finite subset D of I so that $|f_i| < r$ whenever i is not in D. Let J be the set of all functions in $C_b(Y)$ which vanish on K. Thus $C(K) \cong C_b(Y)/J$.

$\Longrightarrow$: Suppose K is an F-space, thus the conjugation map is a local multiplication on $C(K) \cong C_b(Y)/J$. For every $f \in C_b(Y)$, we have $\overline{f} - h_f \cdot f \in J$ for some $h_f \in C_b(Y)$. Thus for every $n \in \mathbb{N}$, there is a finite subset $D_{(f,n)}$ of I so that $|\overline{f}_i - h_{f_i} \cdot f_i| < \frac{1}{n}$ whenever i is not in $D_{(f,n)}$. Set $D_f = \bigcup D_{(f,n)}$. Thus D_f is countable.

Since each Y_i is a compact η-space, the conjugation map is not a local multiplication on $C(Y_i)$. Thus there is a $g_i \in C(Y_i)$ such that $\overline{g}_i - h \cdot g_i \neq 0$ for every $h \in C(Y_i)$. Let $g = \bigcup g_i \in C(Y)$. For every $n \in \mathbb{N}$, set

$$g(n) = (-n \vee \operatorname{Re} g) \wedge n + i(-n \vee \operatorname{Im} g) \wedge n.$$

Thus $g(n) \in C_b(Y)$ for every $n \in \mathbb{N}$. Set $D = \bigcup D_{g(n)}$ which $D_{g(n)}$ is defined similarly with D_f for every $n \in \mathbb{N}$. Since each $D_{g(n)}$ is countable, we have D is countable. Also $D \subseteq I$.

If there exists an $m \in I$ and $m \notin D$, we have $g_m \in C(Y_m) = C_b(Y_m)$. There is a $k \in \mathbb{N}$ so that $g_m = g(k)_m$. Since $g(k) \in C_b(Y)$, we have $\overline{g(k)} - h \cdot g(k) \in J$ for some $h \in C_b(Y)$. Thus for every $n \in \mathbb{N}$, we have $|\overline{g(k)}_m - h_m \cdot g(k)_m| < \frac{1}{n}$, since $m \notin D \supseteq D_{g(k)}$. Thus $|\overline{g(k)}_m - h_m \cdot g(k)_m| = 0$. Since $g_m = g(k)_m$, we have $\overline{g}_m = h_m \cdot g_m$ where $h_m \in C(Y_m)$. This contradicts our choice of g_m. Thus $I = D$, i.e., we get I is countable.

$\Longleftarrow$: Suppose I is countable. Let $I = \mathbb{N}$. For every $f \in C_b(Y)$, $f_n \in C(Y_n)$ for every $n \in \mathbb{N}$. Let $F_n = \left\{x \in Y_n \colon |f_n(x)| \geqslant \frac{1}{n}\right\}$, and F_n is a closed subset of Y_n. Thus

$\frac{\bar{f}_n}{f_n}$ is continuous on F_n and $|\frac{\bar{f}_n}{f_n}| = 1$. By the Tietze extension theorem, there is a continuous function h_n from Y_n to $\mathbb{C}$ such that $h_n|_{F_n} = \frac{\bar{f}_n}{f_n}$ and $|h_n| \leqslant 2$. Thus on $Y_n \setminus F_n$, we have

$$|\bar{f}_n - h_n \cdot f_n| \leqslant |\bar{f}_n| + |h_n| \cdot |f_n| < \frac{1}{n} + 2 \cdot \frac{1}{n} = \frac{3}{n}.$$

On F_n, we have $|\bar{f}_n - h_n \cdot f_n| = 0$. Thus $|\bar{f}_n - h_n \cdot f_n| < \frac{3}{n}$ on Y_n.

Set $h = \bigcup h_n$. Since $|h_n| \leqslant 2$ for every $n \in \mathbb{N}$, we have $h \in C_b(Y)$. Thus $\bar{f} - h \cdot f \in J$. Thus the conjugation map is a local multiplication on $C_b(Y)/J \cong C(K)$, i.e., we get K is an F-space. $\square$

REMARK 2. Suppose X is a compact η-space. We know that every clopen subset of X is an η-space. From Theorem 8, we get a closed subset $K = \beta(Y) \setminus Y$ of a compact η-space $\beta(Y)$ which is not an η-space, in fact, an F-space, where Y is the one in Theorem 8.

COROLLARY 6. *Suppose I is uncountable and Y is the disjoint union of nonempty compact Hausdorff spaces Y_i's with i in I. Let $K = \beta(Y) \setminus Y$. Then K is an F-space if and only if all but countable many Y_i's are F-spaces.*

Let $\mathbb{N}$ be all positive integers with discrete topology. Since the conjugation map is a local multiplication on $C_b(\mathbb{N}) \cong C(\beta(\mathbb{N}))$, we get that $\beta(\mathbb{N})$ is an F-space. We have characterized all real linear local multiplications on $C(\beta(\mathbb{N}))$. Next we will see how an additive local multiplication on $C(\beta(\mathbb{N}))$ relates to a real linear local multiplication.

PROPOSITION 1. *Suppose T is an additive local multiplication on $C(\beta(\mathbb{N})) \cong C_b(\mathbb{N})$. Then $T(f) = T(1) \cdot f$ for every $f \in C_R(\beta(\mathbb{N}))$ except finite many points of $\mathbb{N}$.*

Proof. Clearly, $C_b(\mathbb{N})$ is the set of all bounded complex sequences. Let $C_0(\mathbb{N})$ be the set of all sequences which converge to 0. We may suppose $T(1) = 0$. Thus $T(r \cdot 1) = T(r) = 0$ for every $r \in \mathbb{Q}$. Also for every bounded rational sequences $\{r_n\}$, $T(\{r_n\}) = 0$, since for every $n \in \mathbb{N}$, $T(\{r_n\})(n) = T(\{r_n\} - r_n \cdot 1)(n) = h(n) \cdot (r_n - r_n) = 0$ for some $h \in C_b(\mathbb{N})$. For every $a \in \mathbb{R}$, there is a sequence $\{a_n\}$ in $\mathbb{Q}$ such that $a_n \to a$. Thus

$$T(\{a_n\} - a) = -T(a) = h \cdot (\{a_n\} - a)$$

for some $h \in C_b(\mathbb{N})$. Thus for every $n \in \mathbb{N}$, $-T(a)(n) = h(n) \cdot (a_n - a)$. Let $n \to \infty$,

$$|T(a)(n)| = |h(n)| \cdot |a_n - a| \to 0,$$

since h is a bounded sequence. Thus $T(a) \in C_0(\mathbb{N})$ for every $a \in \mathbb{R}$.

Let f be any bounded real sequence. It has a convergent subsequence $\{f(n_k)\}$ and $f(n_k) \to b$ as $k \to \infty$ for some $b \in \mathbb{R}$. Thus

$$T(f - b) = T(f) - T(b) = h_b \cdot (f - b)$$

for some $h_b \in C_b(\mathbb{N})$. For every $k \in \mathbb{N}$,

$$T(f)(n_k) - T(b)(n_k) = h_b(n_k) \cdot (f(n_k) - b).$$

Let $k \to \infty$, we get $T(f)(n_k) \to 0$.

Thus every subsequence $\{T(f)(n_k)\}$ of $T(f)$ has a subsequence $\left\{T(f)\left(n_{k_j}\right)\right\}$ that converges to 0. Then $T(f) \in C_0(\mathbb{N})$ for every bounded $f \in C_R(\mathbb{N})$. Thus $T(f)$ is 0 on $\beta(\mathbb{N}) \setminus \mathbb{N}$ for every $f \in C_R(\beta(\mathbb{N}))$.

For every $n \in \mathbb{N}$, let e_n be the sequence with 1 in the n-th place and 0 other places. For every $n \in \mathbb{N}$, define $T_n : \mathbb{R} \to \mathbb{R}$ by

$$T_n(b) = T(b \cdot e_n)(n) \text{ for every } b \in \mathbb{R}.$$

If $k \neq n$, then $T(b \cdot e_n)(k) = h_b(k) \cdot b \cdot e_n(k) = 0$. Thus we have $T_n(b) \cdot e_n = T(b \cdot e_n)$.

We have known $T_n(1) = 0$ for every $n \in \mathbb{N}$. Next we prove there is an $n_T \in \mathbb{N}$ such that $T_n = 0$ for every $n \geqslant n_T$.

Else, we can get a subsequence $\{n_k\}$ of $\mathbb{N}$ such that $\left\{b_{n_k}\right\} \subseteq \mathbb{R}$ and $T_{n_k}\left(b_{n_k}\right) \neq 0$ for every $k \in \mathbb{N}$. For every $k \in \mathbb{N}$, choose an $q_k \in \mathbb{Q}$ such that $q_k > \frac{1}{|T_{n_k}(b_{n_k})|}$. We know $q_k \cdot b_{n_k} \in \mathbb{R}$ and choose an $r_k \in \mathbb{Q}$ such that $|q_k \cdot b_{n_k} - r_k| < 1$. Define a sequence

$$a(n) = q_k \cdot b_{n_k} - r_k \text{ when } n = n_k, \text{ and } a(n) = 0 \text{ when } n \neq n_k,$$

for every $n \in \mathbb{N}$. Thus $\{a(n)\}$ is a real bounded sequence.

For every $k \in \mathbb{N}$,

$$\begin{aligned} T(a) &= T\left(a - a(n_k) \cdot e_{n_k}\right) + T\left(a(n_k) \cdot e_{n_k}\right) \\ T(a)(n_k) &= 0 + T\left(a(n_k) \cdot e_{n_k}\right)(n_k) \\ &= T\left(\left(q_k \cdot b_{n_k} - r_k\right) \cdot e_{n_k}\right)(n_k) \\ &= q_k \cdot T\left(b_{n_k} \cdot e_{n_k}\right)(n_k) \\ &= q_k \cdot T_{n_k}\left(b_{n_k}\right), \end{aligned}$$

since q_k, $r_k \in \mathbb{Q}$. But $|q_k \cdot T_{n_k}\left(b_{n_k}\right)| > 1$ for every $k \in \mathbb{N}$. Thus $T(a) \notin C_0(\mathbb{N})$. This is a contradiction.

Thus there is an $n_T \in \mathbb{N}$ such that $T_n = 0$ for every $n \geqslant n_T$. Now for any bounded $f \in C_R(\mathbb{N})$,

$$\begin{aligned} T(f)(n) &= T(f - f(n) \cdot e_n)(n) + T(f(n) \cdot e_n)(n) \\ &= 0 + 0 = 0 \end{aligned}$$

for every $n \geqslant n_T$. Let $D = \{n \in \mathbb{N} \colon 1 \leqslant n \leqslant n_T\}$. Thus $T(f) = 0$ on $\beta(\mathbb{N}) \setminus D$ for every $f \in C_R(\beta(\mathbb{N}))$.

Thus $T(f) = T(1) \cdot f$ on $\beta(\mathbb{N}) \setminus D$ for every $f \in C_R(\beta(\mathbb{N}))$. $\square$

REMARK 3. Note that the n_T is in terms of T, so we cannot say that there is a finite subset D of $\mathbb{N}$ such that for every additive local multiplication T on $C(\beta(\mathbb{N}))$, $T(f) = T(1) \cdot f$ on $\beta(\mathbb{N}) \setminus D$ for every $f \in C_R(\beta(\mathbb{N}))$. In fact, we can get that for every additive local multiplication T on $C(\beta(\mathbb{N}))$, $T(f) = T(1) \cdot f$ on $\beta(\mathbb{N}) \setminus \mathbb{N}$ for every $f \in C_R(\beta(\mathbb{N}))$.

3. υ-spaces

DEFINITION 5. Suppose X is a compact Hausdorff space. Call a map T on $C(X)$ ($C_R(X)$, respectively) *zero-preserving* if for every $f \in C(X)$ ($C_R(X)$, respectively) and every $x \in X$, $f(x) = 0$ implies $T(f)(x) = 0$, i.e., $Z(f) \subseteq Z(T(f))$.

The zero-preserving property is weaker than being a local multiplication. Every local multiplication has this property. However, in general, not every zero-preserving additive map is a local multiplication. For example, any map T of the form

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

For every additive T on $C(X)$ there corresponds to four additive maps on $C_R(X)$, namely,

$$\begin{aligned}
T_1(u) &= \operatorname{Re}(T(u)) \\
T_2(u) &= \operatorname{Im}(T(u)) \\
T_3(u) &= \operatorname{Re}(T(iu)) \\
T_4(u) &= \operatorname{Im}(T(iu)) \text{ for every } u \in C_R(X),
\end{aligned}$$

and it turns out that T is zero-preserving if and only if each of these four maps is zero-preserving and T has the form

$$T(f) = T_1(\operatorname{Re}(f)) + iT_2(\operatorname{Re}(f)) + T_3(\operatorname{Im}(f)) + iT_4(\operatorname{Im}(f)).$$

DEFINITION 6. We call a compact Hausdorff space X an υ*-space* if every zero-preserving additive map on $C_R(X)$ is a multiplication. Equivalently X is an υ-space if and only if every zero-preserving map on $C(X)$ has the form

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

A careful reading of the proof of Theorem 2 shows that a compact space X in which the set of sequential limit points is dense, e.g., a metric space with no isolated points, must be an υ-space. Theorem 14 gives a very simple characterization of all υ-spaces.

The relationship between local multiplications and zero-preserving maps is given in the following lemma.

LEMMA 2. *Suppose T is any map on $C(X)$. Then*

1. T is a local multiplication if and only if T leaves invariant every ideal of $C(X)$.
2. T is zero-preserving if and only if T leaves invariant every closed ideal of $C(X)$.

Proof. (1). Suppose T is a local multiplication and I is an ideal of $C(X)$. Then for every $f \in I$, $T(f) = h_f \cdot f \in I$. Now suppose T leaves invariant every ideal of

$C(X)$. Suppose $f \in C(X)$ and I_f is an ideal generated by f. Since $T(f) \in I_f$, $T(f) = h \cdot f$ for some $h \in C(X)$. T is a local multiplication.

(2). Suppose T is zero-preserving and I is a closed ideal of $C(X)$. Then there is a closed subset K of X such that I is the set of all functions in $C(X)$ which vanish on K. For every $f \in I$, $f = 0$ on K. Thus $T(f) = 0$ on K, i.e., $T(f) \in I$. Now suppose T leaves invariant every closed ideal of $C(X)$. For every $f \in C(X)$ and every $x \in X$, if $f(x) = 0$, let I_x be the ideal of all functions in $C(X)$ which vanish on $\{x\}$. Thus I_x is closed. Thus $T(f) \in I_x$, $T(f)(x) = 0$, i.e., T is zero-preserving. □

Here we prove a purely algebraic result that relates to zero-preserving maps, since the evaluation maps at points of X are algebra homomorphisms of $C(X)$ to $\mathbb{C}$ and $C_R(X)$ into $\mathbb{R}$, respectively.

THEOREM 9. *Suppose $\mathscr{A}$ is an algebra with identity 1 over a field $\mathbb{F}$. Suppose also that, whenever $x \in \mathscr{A}$ and $x \neq 0$, there is an algebra homomorphism h from $\mathscr{A}$ to $\mathbb{F}$ such that $h(x) \neq 0$. Let S be the set of unital algebra homomorphism from $\mathscr{A}$ to $\mathbb{F}$. Suppose T is a linear map from $\mathscr{A}$ to $\mathscr{A}$ such that, for every a in $\mathscr{A}$ and every s in S, $s(a) = 0$ implies $s(T(a)) = 0$. Then T is left multiplication by $T(1)$.*

Proof. If there exists an $a \in \mathscr{A}$ such that $T(a) \neq T(1)a$, i.e., $T(a) - T(1)a \neq 0$. From the assumption, there is an algebra homomorphism h from $\mathscr{A}$ to $\mathbb{F}$ such that $h(T(a) - T(1)a) \neq 0$. But

$$\begin{aligned} h(T(a) - T(1)a) &= h(T(a)) - h(T(1)) \cdot h(a) \\ &= h(T(a)) - h(h(a) \cdot T(1)) \\ &= h(T(a) - h(a) \cdot T(1)) \\ &= h(T(a) - T(h(a) \cdot 1)) \\ &= h(T(a - h(a) \cdot 1)). \end{aligned}$$

Since $h \in S$ and

$$h(a - h(a) \cdot 1) = h(a) - h(a) \cdot h(1) = 0,$$

we have

$$h(T(a - h(a) \cdot 1)) = 0.$$

This is a contradiction. □

COROLLARY 7. *Suppose X is a compact Hausdorff space. Then any linear map on $C(X)$ ($\mathbb{R}$-linear map on $C_R(X)$) that is zero-preserving is a multiplication.*

COROLLARY 8. *Suppose X is a compact Hausdorff space. Then T is an $\mathbb{R}$-linear zero-preserving map on $C(X)$ if and only if T has the form*

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

The equivalence of (1) and (5) in the following theorem shows that if X is an υ-space, then X is like a "real" η-space.

THEOREM 10. *Suppose X is a compact Hausdorff space. The following are equivalent:*

1. *X is an υ-space.*
2. *Every additive zero-preserving map on $C(X)$ is $\mathbb{R}$-linear.*
3. *Every additive zero-preserving map on $C(X)$ is continuous.*
4. *Every additive zero-preserving map on $C(X)$ has the form*
$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f) \text{ for every } f \in C(X).$$
5. *Every additive zero-preserving map on $C_R(X)$ is a multiplication.*

Proof. (5) $\Rightarrow$ (4): We have already known that for every additive T on $C(X)$ there corresponds to four additive maps on $C_R(X)$, namely,

$$\begin{aligned}
T_1(u) &= \operatorname{Re}(T(u)) \\
T_2(u) &= \operatorname{Im}(T(u)) \\
T_3(u) &= \operatorname{Re}(T(iu)) \\
T_4(u) &= \operatorname{Im}(T(iu)) \text{ for every } u \in C_R(X),
\end{aligned}$$

and it turns out that T is zero-preserving if and only if each of these four maps is zero-preserving and T has the form

$$T(f) = T_1(\operatorname{Re}(f)) + iT_2(\operatorname{Re}(f)) + T_3(\operatorname{Im}(f)) + iT_4(\operatorname{Im}(f)).$$

Since every additive zero-preserving map on $C_R(X)$ is a multiplication, We have

$$\begin{aligned}
T(f) &= T_1(1)\operatorname{Re}(f) + iT_2(1)\operatorname{Re}(f) + T_3(1)\operatorname{Im}(f) + iT_4(1)\operatorname{Im}(f) \\
&= T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).
\end{aligned}$$

(4) $\Rightarrow$ (3): Suppose $\{f_n\}$ is a net in $C(X)$ and $f_n \to f$. Then $T(f_n) = T(1)\operatorname{Re}(f_n) + T(i)\operatorname{Im}(f_n) \to T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f) = T(f)$. Thus T is continuous.

(3) $\Rightarrow$ (2): Suppose $a \in \mathbb{R}$ and $\{r_n\}$ is a rational sequence which converges to a. Then $r_n \cdot f \to a \cdot f$. Thus

$$T(a \cdot f) = \lim T(r_n \cdot f) = \lim r_n T(f) = aT(f).$$

Thus T is $\mathbb{R}$-linear.

(2) $\Rightarrow$ (1): Let T be an additive zero-preserving map on $C_R(X)$. Suppose T is not $\mathbb{R}$-linear. Define T_0 on $C(X)$ by $T_0(f) = T(\operatorname{Re}(f)) + iT(\operatorname{Im}(f))$. Thus T_0 is an additive zero-preserving map on $C(X)$ which is not $\mathbb{R}$-linear. This is a contradiction. Then T is $\mathbb{R}$-linear. From Corollary 7, T is a multiplication. Thus X is an υ-space.

(1) $\Rightarrow$ (5): This follows immediately from the definition of υ-space. $\square$

EXAMPLE 3. Suppose $X = \{a\}$. Then $C_R(X)$ is isomorphic to $\mathbb{R}$. Every additive map on $\mathbb{R}$ is zero-preserving. The number of additive (i.e., $\mathbb{Q}$-linear) maps on $\mathbb{R}$ is $2^{2^{\aleph_0}}$ but the number of multiplications is $2^{\aleph_0}$, so X is not an υ-space.

As in the η-space case, we get the following for free.

THEOREM 11. *Suppose X is the disjoint union of compact Hausdorff spaces Y and Z. Then X is an υ-space if and only if Y and Z are υ-spaces.*

COROLLARY 9. *An υ-space has no isolated points.*

If we examine the proofs of Theorems in the preceding section, we immediately obtain the following results.

THEOREM 12. *Suppose X is a compact Hausdorff space. Then*

1. *If the set of sequential limit points is dense in X, then X is an υ-space.*
2. *If X is first countable, then X is an υ-space if and only if X has no isolated points.*
3. *If $\{K_i : i \in I\}$ is a collection of closed subsets of X and if each K_i is an υ-space, then the closure of the union of K_i's is an υ-space.*
4. *X has a unique maximal compact υ-subspace.*
5. *If Y is a compact Hausdorff υ-space, then $X \times Y$ is an υ-space.*

4. P-points and $\mathfrak{q}$-points

We now want to generalize Theorem 2. We call a point $x \in X$ a $\mathfrak{q}$-point if and only if there is a disjoint sequence $\{K_n\}$ of compact subsets of X such that

1. Each K_n is disjoint from the closure of $\cup_{m\in\mathbb{N}, m\neq n} K_m$;
2. $x \in (\cup_{n\in\mathbb{N}} K_n)^- \setminus (\cup_{n\in\mathbb{N}} K_n)$.

We say that x is a *strong $\mathfrak{q}$-point* if there is a disjoint sequence $\{K_n\}$ of compact sets also satisfying

3. $x \in (\cup_{n\in\mathbb{N}} K_{2n})^- \cap (\cup_{n\in\mathbb{N}} K_{2n-1})^-$.

It is clear that these conditions on the sequence $\{K_n\}$ is precisely what is needed to ensure that if $\{z_n\}$ is a sequence of complex numbers converging to z, then the function f on $\cup_{n=1}^{\infty} K_n$ defined by $f|_{K_n} = z_n$ extends to $(\cup_{n\in\mathbb{N}} K_n)^-$, which by the Tietze extension theorem extends to a continuous function on X. By examining the proof of Theorem 2, we easily obtain the following.

THEOREM 13. *Suppose X is a compact Hausdorff space.*

1. *If the set of $\mathfrak{q}$-points is dense in X, then every additive local multiplication T on $C(X)$ has the form $T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f)$ for some $T(1), T(i) \in C(X)$.*

2. *If the set of strong $\mathfrak{q}$-points is dense in X, then X is an η-space.*

The notion of a $\mathfrak{q}$-point is very closely related to the classical notion of a P-point defined by L. Gillman and M. Henriksen [3], which can be defined as follows:

$x \in X$ is a *P-point* in X if and only if every continuous function in $C(X)$ is constant on some open set containing x.

LEMMA 3. *Suppose X is a compact Hausdorff space. Then*

1. *If $x \in X$, then x is a $\mathfrak{q}$-point if and only if x is not a P-point.*

2. *If $K \subseteq X$ is compact and every point of K is a P-point of X, then K is finite.*

3. *If A is the set of all $\mathfrak{q}$-points of X, then $X \setminus \bar{A}$ is the set of isolated points of X.*

Proof. (1). Suppose x is a $\mathfrak{q}$-point. We can choose a disjoint collection $\{K_1, K_2, \ldots\}$ of compact sets such that each is disjoint from the closure of the union of the others, and $x \in (\cup_{n=1}^{\infty} K_n)^- \setminus (\cup_{n=1}^{\infty} K_n)$. We can define a continuous function $f : (\cup_{n=1}^{\infty} K_n)^- \to [0,1]$ so that $f(a) = 1/n$ when $a \in K_n$ and $f(a) = 0$ on $(\cup_{n=1}^{\infty} K_n)^- \setminus (\cup_{n=1}^{\infty} K_n)$. By the Tietze extension theorem, we can assume $f \in C(X)$. It is clear that $f(x) = 0$ but there is no neighborhood of x on which f is 0. Thus x is not a P-point.

Conversely, suppose x is not a P-point and suppose $g \in C(X)$ and $g(x) = 0$ but g is not 0 on any neighborhood of x. Thus

$$x \in [X \setminus Z(g)]^- .$$

By replacing g with $|g| / (1 + |g|)$ we can assume that $0 \leqslant g \leqslant 1$. For each $n \in \mathbb{N}$, let $E_n = \left\{x \in X : \frac{1}{n+1} \leqslant g(x) \leqslant \frac{1}{n}\right\}$. We know that x is in the closure of the union of the E_n's. Thus x is either in the closure of $\cup_{n\in\mathbb{N}} E_{2n}$ or the closure of $\cup_{n\in\mathbb{N}} E_{2n-1}$. In the former case we let $K_n = E_{2n}$ for each n, and in the latter case we let $K_n = E_{2n-1}$ for each n. In either case we see that x is a $\mathfrak{q}$-point.

(2). Suppose $K \subseteq X$ is compact and every point of K is a P-point of X. It follows from the Tietze extension theorem that every point of K is a P-point of K. It follows from Proposition 4.1 in [12] that K is finite.

(3). Let $E = \bar{A}$ be the closure of the set of all $\mathfrak{q}$-points of X. Suppose $x_0 \in X \setminus E$. By Urysohn's lemma we can find a continuous $h : X \to [0,1]$ such that $h(x_0) = 1$ and $h|_E = 0$. Thus $\{x \in X : h(x) \geqslant 1/2\}$ is a compact subset of X for which every point is a P-point of X, which means it is finite. Thus $\{x \in X : h(x) > 1/2\}$ is a finite open set containing x_0. Hence x_0 is an isolated point of X. Also no isolated point is in E, so $X \setminus E$ is the set of isolated points of X. $\square$

5. Main results

We can now give complete characterizations of η-spaces, real η-spaces and υ-spaces. Here is our first main theorem, which shows that being a real η-space and being an υ-space are the same as having no isolated points.

THEOREM 14. *Suppose X is a compact Hausdorff space. Then the following are equivalent:*

1. *X is an υ-space.*
2. *X is a real η-space.*
3. *The set of $\mathfrak{q}$-points of X is dense in X.*
4. *X has no isolated points.*

Proof. We already proved that $(3) \Rightarrow (1) \Rightarrow (2) \Rightarrow (4)$. The proof of $(4) \Rightarrow (3)$ follows from part (3) of Lemma 3. □

The following result characterizes η-spaces.

THEOREM 15. *Suppose X is a compact Hausdorff space. Then the following are equivalent:*

1. *X is an η-space.*
2. *No nonempty open F_σ set in X is an F-space.*
3. *X has no isolated points and, for every $0 \neq g \in C(X)$, the map $T(f) = g\overline{f}$ is not a local multiplication.*

Proof. $(1) \Rightarrow (2)$ If X is an η-space, it follows from Theorem 7 that no nonempty open F_σ set in X is an F-space.

$(2) \Rightarrow (3)$. Suppose that no nonempty open F_σ set in X is an F-space. It follows that X has no isolated points. Also, by Lemma 1, for every $0 \neq g \in C(X)$, the map $T(f) = g\overline{f}$ is not a local multiplication.

$(3) \Rightarrow (1)$. By Theorem 14, X is an υ-space. Suppose T is an additive local multiplication on $C(X)$. Since X is an υ-space, T must have the form

$$T(f) = T(1)\operatorname{Re}(f) + T(i)\operatorname{Im}(f).$$

Thus T is $\mathbb{R}$-linear. It follows from Corollary 5 that T is a multiplication. □

We now describe how the maximal subsets of a compact Hausdorff space X that are η-spaces or υ-spaces can be constructed.

REMARK 4. One might guess that the maximal υ-subspace of X is the closure of the set of $\mathfrak{q}$-points of X. However, if $X = \{0, 1, 1/2, 1/3, 1/4, \ldots\}$, then the closure of the set of $\mathfrak{q}$-points of X is precisely $\{0\}$. However, this is not an υ-space. We have to use a transfinite construction argument. If K is a compact Hausdorff space, we define

$$\begin{aligned}\mathfrak{n}(K) &= K \setminus \{x \in K : x \text{ is an isolated point}\} \\ &= \{x \in K : x \text{ is a } \mathfrak{q}\text{-point of } K\}^{-}.\end{aligned}$$

We let $E_0 = X$, and suppose $\alpha > 0$ is an ordinal such that, for all $\beta < \alpha$, E_β is defined. We define E_α by

$$E_\alpha = \begin{cases} \mathfrak{n}(E_\beta) & \text{if } \alpha = \beta + 1 \\ \cap_{\beta<\alpha} E_\beta & \text{if } \alpha \text{ is a limit ordinal} \end{cases}.$$

Since $\{E_\beta \setminus E_{\beta+1} : \beta \text{ is an ordinal}\}$ is a disjoint collection of subsets of X, there is a smallest ordinal α such that $E_\alpha = E_{\alpha+1}$. It is clear that if K is a compact subset of X having no isolated points, then $K \subset E_\beta$ for every ordinal β; in particular, $K \subset E_\alpha$. Since $E_\alpha = E_{\alpha+1} = \mathfrak{n}(E_\alpha)$, E_α has no isolated points. Thus E_α is the maximal compact subset of X that has no isolated points, i.e., the maximal compact subset of X that is an υ-space.

The maximal η-subspace is a little more complicated. First suppose Y is a compact subset of X and Y is an η-space. Suppose also that V is an open F_σ subset of X and that V is an F-space. It follows from Lemma 1 that there is a $g \in C(X)$ such that $X \setminus Z(g) = V$ and the map $T(f) = g\overline{f}$ is a local multiplication on $C(X)$. If $g_Y = g|_Y \neq 0$, then the map $S(f) = g_Y \overline{f}$ is a local multiplication on $C(Y)$ that is not a multiplication. Since Y is an η-space, $g|_Y = 0$, or $Y \cap V = \varnothing$.

We now imitate the process above. If K is a compact Hausdorff space, define

$$\mathfrak{m}(K) = K \setminus \cup\{V \subset X : V \text{ is an open } \mathrm{F}_\sigma \text{ and an F-space}\}.$$

We define $F_0 = X$, and for $\alpha > 0$ define

$$F_\alpha = \begin{cases} \mathfrak{m}(F_\beta) & \text{if } \alpha = \beta + 1 \\ \cap_{\beta<\alpha} F_\beta & \text{if } \alpha \text{ is a limit ordinal} \end{cases}$$

Then we can choose α to be the smallest ordinal such that $F_\alpha = F_{\alpha+1}$. Then F_α is the maximal compact η-subspace of X.

We conclude with remarks on the space $\beta(\mathbb{N}) \setminus \mathbb{N}$.

REMARK 5. Since $\beta(\mathbb{N}) \setminus \mathbb{N}$ is an F-space, we know that $\beta(\mathbb{N}) \setminus \mathbb{N}$ is not an η-space. However, $\beta(\mathbb{N}) \setminus \mathbb{N}$ has no isolated points. Thus, by Theorem 14, $\beta(\mathbb{N}) \setminus \mathbb{N}$ is an υ-space. Thus every local multiplication on $C_R(\beta(\mathbb{N}) \setminus \mathbb{N})$ is a multiplication, but the same is not true for $C(\beta(\mathbb{N}) \setminus \mathbb{N})$.

REMARK 6. Another interesting fact is that the question of whether every point in $\beta(\mathbb{N}) \setminus \mathbb{N}$ is a $\mathfrak{q}$-point is independent from the Zermelo-Fraenkel axioms of set theory

plus the axiom of choice(ZFC). W. Rudin [13] proved that if you assume the continuum hypothesis(which is independent from (ZFC)), then not every point in $\beta(\mathbb{N})\setminus\mathbb{N}$ is a $\mathfrak{q}$-point. Thus there is a model of (ZFC) in which not every point in $\beta(\mathbb{N})\setminus\mathbb{N}$ is a $\mathfrak{q}$-point. Later, S. Shelah (see [16]) showed that there is a model of (ZFC) in which every point of $\beta(\mathbb{N})\setminus\mathbb{N}$ is a $\mathfrak{q}$-point.

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