COMPLEX SYMMETRY OF WEIGHTED COMPOSITION
OPERATORS ON A HILBERT SPACE OF DIRICHLET SERIES

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(*Communicated by R. Curto*)

Abstract. In this paper, some characterizations of symbols ϕ and φ are provided when the weighted composition operators $C_{\phi,\varphi}$ are complex symmetric on the Hilbert space of Dirichlet series with square summable coefficients. As an application, it is proved that there are non-trivial complex symmetric and non-normal weighted composition operators, although this phenomenon happens for neither composition operators nor multiplication operators.

1. Introduction

Hedenmalm-Lindqvist-Seip [12] introduced and started studying the Hilbert space of Dirichlet series with square summable coefficients:

$$\mathscr{H} = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\| = \left(\sum_{n=1}^{\infty} |a_n|^2 \right)^{1/2} < \infty \right\}.$$

By the Cauchy-Schwarz inequality, the functions in $\mathscr{H}$ are all analytic on the half-plane $\mathbb{C}_{1/2}$ (where $\mathbb{C}_\theta = \{s \in \mathbb{C} : \operatorname{Re} s > \theta\}$ for $\theta \in \mathbb{R}$). A characteristic feature of the space $\mathscr{H}$ is that of its reproducing kernel function being essentially the Riemann zeta function from number theory: $K_w(s) = \zeta(\overline{w} + s)$ for $w, s \in \mathbb{C}_{1/2}$, where $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$. As usual, we denote by $\mathscr{H}^\infty$ the space of all Dirichlet series which are bounded and analytic on $\mathbb{C}_0$. In particular, Theorem 3.1 of [12] asserted that the set of multipliers of $\mathscr{H}$ is $\mathscr{H}^\infty$.

In this paper, we consider the problem of describing all weighted composition operators on $\mathscr{H}$ which are complex symmetric. A weighted composition operator $C_{\phi,\varphi}$ on $\mathscr{H}$ takes an analytic function $f \in \mathscr{H}$ to the analytic function $\phi \cdot f \circ \varphi$, where the analytic map $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$, and ϕ is an analytic function on the half-plane $\mathbb{C}_{1/2}$. It should be mentioned that, if the weighted composition operator $C_{\phi,\varphi}$ is bounded on $\mathscr{H}$, then $\phi \in \mathscr{H}$. Clearly, there are two particularly interesting special cases of such operators: the composition operator C_φ by taking $\phi \equiv 1$ and the multiplication operator T_ϕ by putting $\varphi = id$, the identity function of $\mathbb{C}_{1/2}$. Composition operators acting on $\mathscr{H}$ (or other spaces of Dirichlet series) have been extensively studied in many papers in recent years (see [1, 2, 3, 5, 11, 18, 24]). Following [11, 18], it is convenient to

call φ a member of the Gordon-Hedenmalm class denoted by $\mathscr{G}$, if the analytic map $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded composition operator C_φ on $\mathscr{H}$.

DEFINITION. The *Gordon-Hedenmalm class* $\mathscr{G}$ consists of the maps $\varphi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ of the form
$$\varphi(s) = c_0 s + \varphi_0(s),$$
where c_0 is a non-negative integer (called the character of φ), and $\varphi_0(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ converges uniformly in $\mathbb{C}_\varepsilon$ for every $\varepsilon > 0$ and has the following properties:

(a) If $c_0 = 0$, then $\varphi_0(\mathbb{C}_0) \subseteq \mathbb{C}_{1/2}$.

(b) If $c_0 \geqslant 1$, then either $\varphi_0 \equiv 0$ or $\varphi_0(\mathbb{C}_0) \subseteq \mathbb{C}_0$.

To study the complex symmetry of weighted composition operators on the Hilbert space $\mathscr{H}$, we first recall that a bounded linear operator T on a separable Hilbert space H is complex symmetric if T has a self-transpose (i.e., symmetric) matrix representation with respect to some orthogonal basis of H. This is equivalent to the existence of a conjugation (i.e., a conjugate-linear, isometric involution) $\mathscr{C}$ on H such that $T = \mathscr{C} T^* \mathscr{C}$. (Here the notation of involution and isometry means $\mathscr{C}^2 = I$ and $\langle \mathscr{C} x, \mathscr{C} y \rangle = \langle y, x \rangle$ for all $x, y \in H$, respectively.) In this case T is called $\mathscr{C}$-symmetric. The general study of complex symmetric operators on Hilbert spaces was initiated by Garcia, Putinar and Wogen ([7, 8, 9, 10]).

We refer to monographs written by Cowen-MacCluer [4], Shapiro [19] and Zhu [25] for general study of composition operators on classical spaces of analytic functions. In recent decades, the complex symmetry of weighted composition operators acting on the classical Hilbert space of analytic functions has been studied intensively, for example see [6, 13, 16]. Narayan-Sievewright-Thompson [15] gave complex symmetric composition operators on the Hardy-Hilbert space $H^2(\mathbb{D})$ over the unit disk $\mathbb{D}$ whose symbols are linear-fractional but not automorphic. As an application, that answered the question of Noor [17]: *Does there exist a non-constant and non-automorphic symbol ϕ for which C_ϕ is complex symmetric but not normal on $H^2(\mathbb{D})$?* Coming back to the case of Dirichlet series, it is shown that the only complex symmetric composition operators with non-constant symbols on the Hilbert space $\mathscr{H}$ are normal in [23].

In this paper, we investigate Hermitian, normal, coisometric or complex symmetric weighted composition operators on the Hilbert space $\mathscr{H}$. Especially, if $\phi \in \mathscr{H}$ is defined in larger domains and $C_{\phi,\varphi} : \mathscr{H} \to \mathscr{H}$ is complex symmetric, then φ is either a constant function or of the form $s + c$ with $\operatorname{Re} c \geqslant 0$. We also characterize completely complex symmetric weighted composition operators with respect to special conjugations, and study the Hermitianness, normality, isometry and spectra of such operators. As an application, it is interesting for us to show that there are non-trivial complex symmetric and non-normal weighted composition operators on $\mathscr{H}$, although this phenomenon happens for neither composition operators nor multiplication operators.

2. Complex symmetry

For simplicity, we always assume that the following symbols ϕ and φ induce a bounded weighted composition operator $C_{\phi,\varphi}$ on $\mathscr{H}$, and ϕ is not identically zero, throughout the work. As is well-known, span$\{K_w : w \in \mathbb{C}_{1/2}\}$ is dense in $\mathscr{H}$. If we denote by $C_{\phi,\varphi}^*$ the adjoint of the bounded linear operator $C_{\phi,\varphi}$, then it is easy to see that

$$C_{\phi,\varphi}^*(K_w) = \overline{\phi(w)}K_{\varphi(w)}, \ w \in \mathbb{C}_{1/2}.$$

In order to determine the conditions under which the weighted composition operators become complex symmetric, we need the following preliminary lemmas related to their kernels. The first one can be easily obtained by the Identity Theorem.

LEMMA 2.1. *Let $\phi \in \mathscr{H}$ have no zeros in $\mathbb{C}_{1/2}$, and let $\varphi \in \mathscr{G}$ be non-constant. Then* $\ker C_{\phi,\varphi} = \{0\}$.

LEMMA 2.2. *Let $\phi \in \mathscr{H}$ be defined in $\mathbb{C}_\theta$ for some $\theta < 1/2$, and let $\varphi \in \mathscr{G}$ be not of the form $s + c$. Then* $\dim \ker C_{\phi,\varphi}^* = +\infty$.

Proof. Suppose that $\varphi(s) \neq s + c$ with $\operatorname{Re} c \geqslant 0$. Let $g \in \operatorname{Ran} C_{\phi,\varphi}$, then $g = \phi \cdot h \circ \varphi$ for some $h \in \mathscr{H}$. Since $\phi \in \mathscr{H}$ is defined in $\mathbb{C}_\theta$ for some $\theta < 1/2$, then any function $g \in \operatorname{Ran} C_{\phi,\varphi}$ is defined and bounded in $\mathbb{C}_{1/2-\varepsilon}$ for some $\varepsilon < 1/2$, by Lemma 11 in [2]. Also, there exists a function $f \in \mathscr{H}$ does not extend to any domain larger than $\mathbb{C}_{1/2}$. We set $F = \operatorname{span}\{n^{-s}f : n \geqslant 1\}$, which is an infinite dimensional subspace of $\mathscr{H}$. Using the similar proof of Theorem 14 in [2], it follows that $F \cap \operatorname{Ran} C_{\phi,\varphi} = \{0\}$ and $\operatorname{codim} \operatorname{Ran} C_{\phi,\varphi} = +\infty$. Thus $\dim \ker C_{\phi,\varphi}^* = \operatorname{codim} \operatorname{Ran} C_{\phi,\varphi} = +\infty$. $\square$

We begin with characterizing when the weighted composition operator $C_{\phi,\varphi}$ is Hermitian on $\mathscr{H}$. Especially, that improves the corresponding results in [22].

THEOREM 2.3. *If $\phi(s) = \sum_{k=1}^{\infty} b_k k^{-s} \in \mathscr{H}$ and $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$, then $C_{\phi,\varphi}$ is Hermitian if and only if one of the following statements holds:*

(i) *ϕ is a nonzero real-valued constant function and $\varphi(s) = s + c_1$ with $c_1 \geqslant 0$.*

(ii) *$\phi(s) = b_1 \zeta(\overline{c_1} + s)$ and $\varphi \equiv c_1$, with $b_1 \in \mathbb{R} \setminus \{0\}$, $\operatorname{Re} c_1 > 1/2$.*

Proof. If (i) holds, clearly $C_{\phi,\varphi}$ is Hermitian by [22, Theorem 3.4]. If $\phi(s) = b_1 \zeta(\overline{c_1} + s)$ with $b_1 \in \mathbb{R}$, and $\varphi \equiv c_1$ a constant function with $\operatorname{Re} c_1 > 1/2$, then by Cauchy-Schwartz inequality,

$$\|C_{\phi,\varphi}f\| = |f(c_1)| \|\phi\| \leqslant (\zeta(2\operatorname{Re} c_1)^{1/2} \|\phi\|) \|f\|, \ \text{for all } f \in \mathscr{H}$$

which implies that $C_{\phi,\varphi}$ is bounded on $\mathscr{H}$. Note that

$$C_{\phi,\varphi}K_w(s) = \phi(s)K_w(\varphi(s)) = b_1 \zeta(\overline{c_1} + s)\zeta(\overline{w} + c_1)$$

and

$$C_{\phi,\varphi}^* K_w(s) = \overline{\phi(w)} K_{\varphi(w)}(s) = b_1 \zeta(c_1 + \overline{w}) \zeta(\overline{c_1} + s).$$

Then $C_{\phi,\varphi} K_w = C_{\phi,\varphi}^* K_w$ for each $w \in \mathbb{C}_{1/2}$, and thus $C_{\phi,\varphi}$ is Hermitian.

Conversely, suppose that $C_{\phi,\varphi}$ is Hermitian. If $c_0 \geqslant 1$, then it follows from [22, Theorem 3.4] that ϕ is a nonzero real-valued constant function on $\mathbb{C}_{1/2}$ and $\varphi(s) = s + c_1$ with $c_1 \geqslant 0$.

If $c_0 = 0$, we claim that φ must be a constant function. If this is not the case, assume that φ is not a constant function. Since $C_{\phi,\varphi}$ is Hermitian, it follows from [22, Theorem 3.2] that $\phi(s) = b_1 \zeta(\overline{c_1} + s)$ with $b_1 \in \mathbb{R} \setminus \{0\}$. Note that $\mathrm{Re}\, c_1 > 1/2$ by [11, P.319]. So $\phi(s)$ is defined in some $\mathbb{C}_\theta$ for $\theta < 1/2$, and it has no zeros in $\mathbb{C}_{1/2}$. Then $\ker C_{\phi,\varphi} = \{0\}$ by Lemma 2.1, and $\dim \ker C_{\phi,\varphi}^* = +\infty$ by Lemma 2.2. That leads to a contradiction with $C_{\phi,\varphi} = C_{\phi,\varphi}^*$. So the proof is complete. $\square$

Following from [2, Theorem 15], C_φ is a normal composition operator if and only if $\varphi(s) = s + c_1$ with $\mathrm{Re}\, c_1 \geqslant 0$. To characterize the normality of weighted composition operators, we also recall the following simple observation concerning the images of 1 of the adjoints of composition operators on $\mathcal{H}$ from [21, Lemma 3.1].

LEMMA 2.4. *Let* $\varphi(s) = c_0 s + \sum_{k=1}^{\infty} c_k k^{-s}$ *be in* $\mathcal{G}$. *Then the following statements hold:*

(1) *If* $c_0 = 0$, *then* $C_\varphi^* 1 = \sum_{n=1}^{\infty} n^{-\overline{c_1}} n^{-s}$.
(2) *If* $c_0 \geqslant 1$, *then* $C_\varphi^* 1 = 1$.

Now the normality of the weighted composition operator $C_{\phi,\varphi}$ can be characterized as follows.

PROPOSITION 2.5. *Let* $\phi(s) = \sum_{k=1}^{\infty} b_k k^{-s} \in \mathcal{H}^\infty$ *and* $\varphi(s) = c_0 s + \varphi_0(s) \in \mathcal{G}$. *Then the following hold:*

(i) *If* $c_0 \geqslant 1$, *then* $C_{\phi,\varphi}$ *is normal if and only if* ϕ *is a nonzero constant function and* $\varphi(s) = s + c_1$ *with* $\mathrm{Re}\, c_1 \geqslant 0$.

(ii) *If* $c_0 = 0$ *and* $C_{\phi,\varphi}$ *is normal, then the norm of* ϕ *is a nonzero multiple of* $\|K_{c_1}\|$, *and* $\phi \cdot K_\alpha \circ \varphi = \phi(\alpha) K_\alpha$, *where* α *is the fixed point of* φ *in* $\mathbb{C}_{1/2}$.

Proof. (i) The sufficiency is due to [2, Theorem 15]. Conversely, note that $C_{\phi,\varphi}$ is normal if and only if $\|C_{\phi,\varphi} f\| = \|C_{\phi,\varphi}^* f\|$ for all $f \in \mathcal{H}$. Then $\|C_{\phi,\varphi} 1\| = \|C_{\phi,\varphi}^* 1\|$, and thus $\|\phi\| = \|C_\varphi^* T_\phi^* 1\| = \|C_\varphi^* \overline{b_1}\| = |b_1|$ by Lemma 2.4. That implies that $b_2 = b_3 = \cdots = 0$, i.e., $\phi \equiv b_1$ for some $b_1 \neq 0$. Therefore normality of $C_{\phi,\varphi}$ implies that C_φ is normal, and then $\varphi(s) = s + c_1$ with $\mathrm{Re}\, c_1 \geqslant 0$ by [2, Theorem 15]. So the proof is complete.

(ii) Let $\phi \in \mathcal{H}^\infty$ and $\varphi(s) = \sum_{k=1}^{\infty} c_k k^{-s} \in \mathcal{G}$. If $C_{\phi,\varphi}$ is normal, then again $\|C_{\phi,\varphi}^* 1\| = \|C_{\phi,\varphi} 1\|$. Note that $C_{\phi,\varphi} 1 = \phi$, and $C_{\phi,\varphi}^* 1 = C_\varphi^* T_\phi^* 1 = \overline{b_1} C_\varphi^* 1 = \overline{b_1} K_{c_1}$ by Lemma 2.4. So $\|\phi\| = |b_1| \|K_{c_1}\|$ with $b_1 \neq 0$.

Note that φ has a fixed point $\alpha \in \mathbb{C}_{1/2}$, i.e., $\varphi(\alpha) = \alpha$. So $C_{\phi,\varphi}^* K_\alpha = \overline{\phi(\alpha)} K_\alpha$, which shows that K_α is an eigenvector of $C_{\phi,\varphi}^*$ with eigenvalue $\overline{\phi(\alpha)}$. If $C_{\phi,\varphi}$ is normal, then $C_{\phi,\varphi} K_\alpha = \phi(\alpha) K_\alpha$, i.e., $\phi \cdot K_\alpha \circ \varphi = \phi(\alpha) K_\alpha$. $\square$

Next we aim to characterize the coisometric weighted composition operators.

PROPOSITION 2.6. *If $\phi \in \mathscr{H}$ is defined on a larger domain than $\mathbb{C}_{1/2}$ and $\varphi \in \mathscr{G}$, then $C_{\phi,\varphi}$ is coisometric on $\mathscr{H}$ if and only if ϕ is a unimodule constant function, and $\varphi(s) = s + it$ with $t \in \mathbb{R}$.*

Proof. The sufficiency can be easily obtained, so it is needed to show the necessity. For this, assume that $C_{\phi,\varphi}$ is coisometric on $\mathscr{H}$. Then we have

$$\|C_{\phi,\varphi}^* f\| = \|f\|, \quad \text{for all } f \in \mathscr{H}.$$

So $\|C_{\phi,\varphi}^* K_w\| = \|K_w\|$, that is, $|\phi(w)| \|K_{\varphi(w)}\| = \|K_w\|$, for each $w \in \mathbb{C}_{1/2}$. Thus

$$|\phi(w)|^2 \zeta(2\mathrm{Re}\,\varphi(w)) = \zeta(2\mathrm{Re}\,w). \tag{2.1}$$

Now we claim that $\varphi(s) = s + it$ with $t \in \mathbb{R}$. Otherwise, if φ is not of the form $s + it$ with $t \in \mathbb{R}$. Following from [2], there exist $\eta > 0$ and $\varepsilon > 0$ such that $\varphi(\mathbb{C}_{1/2-\varepsilon}) \subset \mathbb{C}_{1/2+\eta}$. Since $\phi \in \mathscr{H}$ is defined on a larger domain than $\mathbb{C}_{1/2}$, there is a point $w_0 \in \partial\mathbb{C}_{1/2}$ such that $\phi(w_0)$ is well-defined. Now take a sequence of points w_n in $\mathbb{C}_{1/2}$ satisfying that $w_n \to w_0$ with $\mathrm{Re}\,w_0 = 1/2$, as $n \to \infty$. And set $\varphi(w_n) \to w_0'$ as $n \to \infty$. So $\mathrm{Re}\,w_0' > 1/2 + \eta$. Replacing w by w_n and letting $n \to \infty$ in the above equality (2.1) yield that the right hand side tends to the infinity, but the left hand side is

$$|\phi(w_0)| \zeta(2\mathrm{Re}\,w_0') < +\infty.$$

This contradiction implies that $\varphi(s) = s + it$ for some constant $t \in \mathbb{R}$. Again by (2.1), $|\phi(w)| = 1$ for all $w \in \mathbb{C}_{1/2}$. Thus ϕ is a unimodule constant function. $\square$

Now we give some characterizations of symbols ϕ and φ when the weighted composition operators $C_{\phi,\varphi}$ are complex symmetric. The next characterizes complex symmetric multiplication operators by Proposition 2.5 and using the similar argument as in Theorem 2.1 in [14].

PROPOSITION 2.7. *For $\phi \in \mathscr{H}^\infty$, the following are equivalent:*

(i) *T_ϕ is complex symmetric.*

(ii) *T_ϕ is normal.*

(iii) *ϕ is a constant function on $\mathbb{C}_{1/2}$.*

The following can be given similarly as in [6, 13].

LEMMA 2.8. *Let $\phi \in \mathscr{H}$, and let $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$ be non-constant. If $C_{\phi,\varphi}$ is $\mathscr{C}$-symmetric with the conjugation $\mathscr{C}$, then the following hold:*

(i) ϕ never vanishes on $\mathbb{C}_{1/2}$.

(ii) φ is univalent and $\phi = \dfrac{\phi(w)\mathscr{C}K_{\varphi(w)}}{(\mathscr{C}K_w)\circ\varphi}$ for any $w \in \mathbb{C}_{1/2}$.

As a consequence, we can give some necessary conditions under which $C_{\phi,\varphi}:\mathscr{H} \to \mathscr{H}$ is complex symmetric as follows.

THEOREM 2.9. *Let* $\phi \in \mathscr{H}$ *be defined in* $\mathbb{C}_\theta$ *for some* $\theta < 1/2$, *and* $\varphi \in \mathscr{G}$. *If* $C_{\phi,\varphi}:\mathscr{H} \to \mathscr{H}$ *is complex symmetric, then* φ *is either a constant function or of the form* $s+c$ *with* $\operatorname{Re} c \geqslant 0$.

Proof. Here we argue by contradiction. For this, suppose that $\varphi \in \mathscr{G}$ is neither of the form $s+c$ nor a constant function. Since $\phi \in \mathscr{H}$ is defined in $\mathbb{C}_\theta$ for some $\theta < 1/2$, then $\dim \ker C^*_{\phi,\varphi} = +\infty$ by Lemma 2.2. If $C_{\phi,\varphi}:\mathscr{H} \to \mathscr{H}$ is complex symmetric with respect to some conjugation $\mathscr{C}$, then for each $g \in \ker C^*_{\phi,\varphi}$, we conclude that $C_{\phi,\varphi}\mathscr{C}(g) = \mathscr{C}C^*_{\phi,\varphi}(g) = 0$, i.e., $\mathscr{C}(g) \in \ker C_{\phi,\varphi}$. Thus $\mathscr{C}(\ker C^*_{\phi,\varphi}) \subseteq \ker C_{\phi,\varphi}$, which in turn implies that $\dim \ker C_{\phi,\varphi} = +\infty$ because of $\dim \ker C^*_{\phi,\varphi} = +\infty$ and the isometry of $\mathscr{C}$ (i.e., $\mathscr{C}^2 = I$). However $\ker C_{\phi,\varphi} = \{0\}$ following from Lemma 2.1, since ϕ has no zeros in $\mathbb{C}_{1/2}$ by Lemma 2.8 and φ is a non-constant function. That contradiction implies the desired result. $\square$

The following gives some necessary conditions when weighted composition operators $C_{\phi,\varphi}:\mathscr{H} \to \mathscr{H}$, for $\phi \in \mathscr{H}^\infty$ and $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$ with $c_0 \geqslant 1$, are complex symmetric.

COROLLARY 2.10. *Let* $\phi(s) = \sum_{k=1}^{\infty} b_k k^{-s} \in \mathscr{H}^\infty$, *and* $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$ *with* $c_0 \geqslant 1$. *If* $C_{\phi,\varphi}$ *is* $\mathscr{C}$*-symmetric, then* $b_1 \neq 0$, *and* $\mathscr{C}1$ *is the eigenfunction of* $C_{\phi,\varphi}$ *with respect to the eigenvalue* b_1. *Moreover, if* $\mathscr{C}1$ *is a constant function, then* ϕ *is a nonzero constant function and* $\varphi(s) = s+c_1$ *with* $\operatorname{Re} c_1 \geqslant 0$.

Proof. Assume that $C_{\phi,\varphi}$ is $\mathscr{C}$-symmetric with the conjugation $\mathscr{C}$. It follows from Theorem 2.9 that $\varphi(s) = s+c_1$ with $\operatorname{Re} c_1 \geqslant 0$. We next claim that the constant term of ϕ is not zero. Otherwise, if the constant term $b_1 = 0$, we may write $\phi(s) = \sum_{n=\ell}^{\infty} b_n n^{-s}$ with $b_\ell \neq 0$ for some integer $\ell > 1$. Then $C^*_{\phi,\varphi}1 = C^*_\varphi T^*_\phi 1 = 0$, and thus $\mathscr{C}1 \in \ker C_{\phi,\varphi}$ following from the equality $C_{\phi,\varphi}\mathscr{C}1 = \mathscr{C}C^*_{\phi,\varphi}1$. Noticing that φ is non-constant and ϕ never vanishes on $\mathbb{C}_{1/2}$ by Lemma 2.8, it follows from Lemma 2.1 that $\mathscr{C}1 = 0$. By the involution of $\mathscr{C}$, $1 = \mathscr{C}^2 1 = 0$, which is impossible. This contradiction shows that $b_1 \neq 0$.

Thus $C_{\phi,\varphi}\mathscr{C}1 = \mathscr{C}C^*_{\phi,\varphi}1 = \mathscr{C}C^*_\varphi T^*_\phi 1 = b_1 \mathscr{C}1$, i.e., $\mathscr{C}1$ is an eigenfunction of $C_{\phi,\varphi}$ with respect to the eigenvalue b_1, where b_1 is the constant term of ϕ. Moreover, assume that $\mathscr{C}1 = \lambda$ for some constant $\lambda \neq 0$. Then

$$\lambda\phi = C_{\phi,\varphi}\mathscr{C}1 = \mathscr{C}C^*_{\phi,\varphi}1 = \mathscr{C}C^*_\varphi T^*_\phi 1 = \lambda b_1,$$

which implies that $\phi \equiv b_1$, i.e., ϕ is a nonzero constant function. $\square$

It is well known that the isometry and co-isometry are not equivalent for general operators such as the unilateral shift on ℓ^2. But for the complex symmetric weighted composition operators on $\mathscr{H}$, we have the following:

COROLLARY 2.11. *Let* $\phi \in \mathscr{H}$ *be defined on a larger domain than* $\mathbb{C}_{1/2}$ *and* $\varphi \in \mathscr{G}$. *If* $C_{\phi,\varphi}$ *is complex symmetric, then the following are equivalent:*

(i) $C_{\phi,\varphi}$ *is isometric.*

(ii) $C_{\phi,\varphi}$ *is coisometric.*

(iii) ϕ *is a unimodule constant function, and* $\varphi(s) = s + it$ *with* $t \in \mathbb{R}$.

(iv) $C_{\phi,\varphi}$ *is unitary.*

Proof. (i)$\Rightarrow$(ii) If $C_{\phi,\varphi}$ is complex symmetric with respect to the conjugation $\mathscr{C}$ and $C_{\phi,\varphi}$ is isometric, then $\|\mathscr{C}C_{\phi,\varphi}^* f\| = \|C_{\phi,\varphi}\mathscr{C}f\|$, $\|C_{\phi,\varphi}f\| = \|f\|$ for all $f \in \mathscr{H}$. Using the isometry of $\mathscr{C}$, we have

$$\|C_{\phi,\varphi}^* f\| = \|f\|, \quad \text{for all } f \in \mathscr{H}.$$

So $C_{\phi,\varphi}$ is coisometric.

(ii)$\Rightarrow$(iii) is Proposition 2.6. (iii)$\Rightarrow$(iv) and (iv)$\Rightarrow$(i) are obvious. $\square$

In the sequel, we aim to characterize when the weighted composition operators $C_{\phi,\varphi}$ are complex symmetric with respect to certain conjugations $J_{\mu,\sigma}$ defined as follows. Given $|\mu| = 1$ and a sequence $\{\sigma_n\}$ of real numbers, define the following conjugation-linear operator

$$(J_{\mu,\sigma}f)(s) = \mu \overline{\sum_{n=1}^{\infty} a_n n^{-\bar{s}-i\sigma_n}}, \tag{2.2}$$

for any $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$. For each $\lambda \in \mathbb{C}$ and $f, g \in \mathscr{H}$, $J_{\mu,\sigma}(\lambda f) = \bar{\lambda}J_{\mu,\sigma}f$, $J_{\mu,\sigma}^2 f = f$ and $\langle J_{\mu,\sigma}f, J_{\mu,\sigma}g\rangle_{\mathscr{H}} = \langle g,f\rangle_{\mathscr{H}}$ by a simple computation. So $J_{\mu,\sigma}$ is a conjugation on $\mathscr{H}$. Note that

$$(J_{\mu,\sigma}K_w)(s) = \mu \sum_{n=1}^{\infty} n^{-(w-i\sigma_n)}n^{-s}.$$

For simplicity, denote $\zeta_\sigma(z) := \sum_{n=1}^{\infty} n^{i\sigma_n}n^{-z}$. Then $(J_{\mu,\sigma}K_w)(s) = \zeta_\sigma(w+s)$. Now we characterize the $J_{\mu,\sigma}$-symmetric weighted composition operators, which strengthens the result given by Theorem 2.7 in [23].

THEOREM 2.12. *If* $\phi \in \mathscr{H}$ *and* $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$, *then* $C_{\phi,\varphi}$ *is* $J_{\mu,\sigma}$-*symmetric on* $\mathscr{H}$ *if and only if one of the following holds:*

(i) ϕ *is a nonzero constant function and* $\varphi(s) = s + c_1$ *with* $\operatorname{Re}c_1 \geqslant 0$.

(ii) $\phi(s) = b_1 \zeta_\sigma(s + c_1)$ and $\varphi \equiv c_1$, with $b_1 \neq 0$ and $\operatorname{Re} c_1 > 1/2$.

Proof. Note that $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric if and only if

$$J_{\mu,\sigma} C_{\phi,\varphi}^* f = C_{\phi,\varphi} J_{\mu,\sigma} f, \quad \text{for all } f \in \mathscr{H}. \tag{2.3}$$

That is equivalent to

$$J_{\mu,\sigma} C_{\phi,\varphi}^* K_w = C_{\phi,\varphi} J_{\mu,\sigma} K_w,$$

i.e.,

$$J_{\mu,\sigma} \overline{\phi(w)} K_{\varphi(w)} = C_{\phi,\varphi} J_{\mu,\sigma} K_w.$$

for all $w \in \mathbb{C}_{1/2}$, since $\{K_w : w \in \mathbb{C}_{1/2}\}$ spans a dense set of $\mathscr{H}$. Also

$$\phi(w) J_{\mu,\sigma} K_{\varphi(w)}(s) = \phi(w) \mu \sum_{n=1}^{\infty} n^{-(\varphi(w) - i\sigma_n)} n^{-s},$$

and

$$C_{\phi,\varphi} J_{\mu,\sigma} K_w(s) = \phi(s) \mu \sum_{n=1}^{\infty} n^{-(w - i\sigma_n)} n^{-\varphi(s)},$$

which imply that (2.3) is equivalent to

$$\phi(w) \sum_{n=1}^{\infty} n^{-(\varphi(w) - i\sigma_n)} n^{-s} = \phi(s) \sum_{n=1}^{\infty} n^{-(w - i\sigma_n)} n^{-\varphi(s)}. \tag{2.4}$$

Now the sufficiency immediately follows from a tedious computation, so it is needed to prove the necessity. To finish the proof, we divide it into the following two cases.

(i) If $c_0 \geqslant 1$, taking $\operatorname{Re} w \to +\infty$ in (2.4), it follows that $\phi \equiv b_1 \neq 0$, a nonzero constant function, by the similar argument as in the proof of [23, Theorem 2.7]. Thus $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric if and only if C_φ is $J_{\mu,\sigma}$-symmetric. By Theorem 2.5 in [23], $\varphi(s) = s + c_1$ with $\operatorname{Re} c_1 \geqslant 0$.

(ii) If $c_0 = 0$, it follows from taking $\operatorname{Re} w \to +\infty$ in (2.4) that

$$\phi(s) = b_1 \sum_{n=1}^{\infty} n^{-(c_1 - i\sigma_n)} n^{-s}, \quad b_1 \neq 0,$$

by the similar argument as in the proof of [23, Theorem 2.7]. Also $\operatorname{Re} c_1 > 1/2$, so $\phi(s)$ is defined in some $\mathbb{C}_\theta$ for $\theta < 1/2$. Then by Theorem 2.9, we have $\varphi \equiv c_1$, a constant function. Therefore the proof is complete. $\square$

Comparing Theorem 2.3 and Theorem 2.12 can describe completely $J_{\mu,\sigma}$-symmetric and Hermitian weighted composition operators as follows.

COROLLARY 2.13. *Let* $\phi \in \mathscr{H}$ *and* $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$. *Then* $C_{\phi,\varphi}$ *is* $J_{\mu,\sigma}$-*symmetric and Hermitian if and only if one of the following holds:*

(i) ϕ *is a nonzero real-valued constant function and* $\varphi(s) = s + c_1$, $\operatorname{Re} c_1 \geqslant 0$.

(ii) $\phi(s) = b_1\zeta(\overline{c_1}+s)$ and $\varphi \equiv c_1$, with $b_1 \in \mathbb{R}\backslash\{0\}$, $\mathrm{Re}\,c_1 > 1/2$, and $k^{i(\sigma_k - 2\mathrm{Im}\,c_1)} = 1$ for all positive integers k.

The normality of $J_{\mu,\sigma}$-symmetric weighted composition operators can be easily obtained by Proposition 2.5 and Theorem 2.12.

COROLLARY 2.14. *Let $\phi \in \mathscr{H}$ and $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$.*

(i) *If $c_0 \geqslant 1$, then $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric if and only if it is normal on $\mathscr{H}$.*

(ii) *If $c_0 = 0$, $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric and normal, then $k^{i(\sigma_k - 2\mathrm{Im}\,c_1)} = \frac{\zeta_\sigma(2c_1)}{\zeta(2\mathrm{Re}\,c_1)}$ for each positive integer k.*

Now we can give non-trivial complex symmetric but non-normal weighted composition operators following from the above corollary.

EXAMPLE. Let $|\mu| = 1$ and $\sigma_n = 0$ for all positive integers n. Let $\varphi(s) \equiv c$ with $\mathrm{Re}\,c > 1/2$ and $\mathrm{Im}\,c \neq 0$, and $\phi(s) = \zeta(s+c)$. Then $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric by Theorem 2.12. But it is not normal. Indeed, if it is normal, then $\frac{\zeta(2\mathrm{Re}\,c_1)}{\zeta(2c_1)} = k^{i2\mathrm{Im}\,c_1}$ for each positive integer k. That leads to a contradiction, since $k^{i2\mathrm{Im}\,c_1}$ is dense in the unit circle for $\mathrm{Im}\,c_1 \neq 0$.

As is well-known, the understanding of the spectra of (weighted) composition operators is far from complete. However the spectra of $J_{\mu,\sigma}$-symmetric operator $C_{\phi,\varphi}$ will be characterized completely in the following corollary.

COROLLARY 2.15. *Let $\phi(s) = \sum_{k=1}^{\infty} b_k k^{-s} \in \mathscr{H}$ and $\varphi(s) = c_0 s + \varphi_0(s) \in \mathscr{G}$. If $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric, then the following hold:*

(i) $\mathrm{Sp}\,(C_{\phi,\varphi}) = \overline{\{b_1 n^{-c_1} : n \in \mathbb{N}\}}$ *when $c_0 \geqslant 1$.*

(ii) $C_{\phi,\varphi}$ *is compact and* $\mathrm{Sp}\,(C_{\phi,\varphi}) = \{0, b_1\zeta_\sigma(2c_1)\}$ *when $c_0 = 0$.*

Proof. (i) If $c_0 \geqslant 1$ and $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric, it follows from Theorem 2.12 that $\phi \equiv b_1$ and $\varphi(s) = s + c_1$ with $b_1 \neq 0$, $\mathrm{Re}\,c_1 \geqslant 0$. Now it will be shown that $\mathrm{Sp}\,(C_{\phi,\varphi}) = \overline{\{b_1 n^{-c_1} : n \in \mathbb{N}\}}$ by the same argument as Theorem 3.1 in [20].

(ii) If $c_0 = 0$ and $C_{\phi,\varphi}$ is $J_{\mu,\sigma}$-symmetric, then by Theorem 2.12, $\phi(s) = b_1\zeta_\sigma(s + c_1)$ with $b_1 \neq 0$, and $\varphi \equiv c_1$ with $\mathrm{Re}\,c_1 > 1/2$. Thus $C_{\phi,\varphi}$ is compact. Indeed, given an arbitrary sequence $\{f_n\} \subseteq \mathscr{H}$ which converges weakly to zero, $f_n(c_1) = \langle f_n, K_{c_1}\rangle_{\mathscr{H}}$ tends to zero. Thus $\|C_{\phi,\varphi}f_n\| = |f_n(c_1)|\zeta(2\mathrm{Re}\,c_1)^{1/2}$ tends to zero. Then $C_{\phi,\varphi}f_n$ converges strongly to zero, which implies the desired result. Now it follows from Theorem 5.2 [22] that the spectrum of $C_{\phi,\varphi}$ is $\mathrm{Sp}\,(C_{\phi,\varphi}) = \{0, b_1\zeta_\sigma(2c_1)\}$, which completes the proof. $\square$

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